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COHOMOLOGY OF $\mathsf{U}(2,1)$ REPRESENTATION VARIETIES OF SURFACE GROUPS

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Abstract. In this paper we use the Morse theory of the Yang-Mills-Higgs functional on the singular space of Higgs bundles on Riemann surfaces to compute the equivariant cohomology of the space of semistable $\mathsf{U}(2,1)$ and $\mathsf{SU}(2,1)$ Higgs bundles with fixed Toledo invariant. In the non-coprime case this gives new results about the topology of the $\mathsf{U}(2,1)$ and $\mathsf{SU}(2,1)$ character varieties of surface groups. The main results are a calculation of the equivariant Poincaré polynomials, a Kirwan surjectivity theorem in the non-fixed determinant case, and a description of the action of the Torelli group on the equivariant cohomology of the character variety. This builds on earlier work for stable pairs and rank 2 Higgs bundles.

1. Introduction

Let X be a closed Riemann surface of genus $g \geq 2$. Choose complex hermitian vector bundles E_1, E_2 on X with $\operatorname{rank} E_i = i$ and degree $\deg E_i = d_i$. Let $\mathcal{B}(d_1, d_2)$ denote the space of $\mathsf{U}(2,1)$-Higgs bundle structures on $E_2 \oplus E_1$ (see Section 2.1), and let $\mathcal{G}$ denote the group of $\mathsf{U}(2) \times \mathsf{U}(1)$ gauge transformations. For a holomorphic line bundle $\Lambda \to X$ of degree $d_1 + d_2$, let $\mathcal{B}_\Lambda(d_1, d_2)$ be the subspace defined by restricting to holomorphic structures with fixed holomorphic isomorphism $E_1 \otimes \det E_2 \cong \Lambda$, and let $\mathcal{G}_0$ denote the group of $\mathsf{S}(\mathsf{U}(2) \times \mathsf{U}(1))$ gauge transformations. Denote the corresponding moduli spaces of semistable Higgs bundles by

$$\begin{aligned} \mathcal{M}(d_1, d_2) &= \mathcal{B}^{ss}(d_1, d_2) /\!\!/ \mathcal{G}^{\mathbb{C}} \\ \mathcal{M}_\Lambda(d_1, d_2) &= \mathcal{B}^{ss}_\Lambda(d_1, d_2) /\!\!/ \mathcal{G}_0^{\mathbb{C}} \end{aligned} \tag{1.1}$$

The main result of this paper is a computation of the $\mathcal{G}$ and $\mathcal{G}_0$-equivariant Betti numbers of $\mathcal{B}^{ss}(d_1, d_2)$ and $\mathcal{B}^{ss}_\Lambda(d_1, d_2)$.

Tensoring by line bundles and dualizing give equivariant isomorphisms of these spaces. The distinct cases are therefore enumerated by the mod 3 values $d_1 + d_2 \equiv 0, 1$, which we will refer to as the *non-coprime* and *coprime* cases, respectively. The moduli spaces are nonempty only if $\tau = \tau(d_1, d_2) = \frac{2}{3}(2d_1 - d_2)$ satisfies $|\tau| \leq 2g - 2$. By duality, we will assume without loss of generality that $\tau \geq 0$. For a rank 2 hermitian vector bundle $E \to X$ of degree d, we also introduce the space $\mathcal{C}(E)$ of *holomorphic pairs* consisting of holomorphic structures on E plus a choice of holomorphic section. Given a real number σ, $d/2 \leq \sigma \leq d$, let $\mathcal{C}_\sigma(E) \subset \mathcal{C}(E)$ denote the space of σ-semistable pairs in the sense of Bradlow [3, 4]. We denote the corresponding moduli

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space $\mathcal{N}_\sigma(E) = \mathcal{C}_\sigma(E)/\!\!/\mathcal{G}^{\mathbb{C}}(E)$, where $\mathcal{G}^{\mathbb{C}}(E)$ is the complexification of the group $\mathcal{G}(E)$ of unitary gauge transformations of E. For generic σ (generic means semistable implies stable, which occurs at noninteger values in $(d/2, d)$), the Poincaré polynomials of $\mathcal{N}_\sigma(E)$ were computed in [17]. For general values of σ (not necessarily generic), the $\mathcal{G}(E)$-equivariant cohomology of $\mathcal{C}_\sigma(E)$ was computed in [20].

To state the main results, set

$$\sigma(d_1, d_2) = 2g - 2 + (d_2 - 2d_1)/3 \tag{1.2}$$

We also let $J(X)$ and $S^m X$ denote the Jacobian variety and m-th symmetric product of X, respectively. With this background we have

Theorem 1.1 ($\mathsf{U}(2,1)$ Higgs bundles)**.** *Fix (d_1, d_2) such that $0 \leq \tau(d_1, d_2) \leq 2g-2$ and $d_1 + d_2 \equiv 0 \mod 3$. Then the $\mathcal{G}$-equivariant Poincaré polynomial is given by*

$$\begin{aligned} P_t^{\mathcal{G}}(\mathcal{B}^{ss}(d_1, d_2)) &= \frac{1}{(1-t^2)} P_t^{\mathcal{G}(E)}(\mathcal{C}_{\sigma(d_1,d_2)}(E)) P_t(J(X)) \\ &+ \sum_{\frac{1}{3}(d_1+d_2) < \ell \leq d_2 - d_1 + 2g - 2} \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)} P_t(J(X)) P_t(S^{d_2-d_1+2g-2-\ell}X) P_t(S^{d_1-\ell+2g-2}X) \end{aligned} \tag{1.3}$$

where $\deg E = d_2 - 2d_1 + 4g - 4$*. For* $d_1 + d_2 \equiv 1 \mod 3$*,*

$$\begin{aligned} P_t(\mathcal{M}(d_1, d_2)) = (1-t^2) P_t^{\mathcal{G}}(\mathcal{B}^{ss}(d_1, d_2)) &= P_t(\mathcal{N}_{\sigma(d_1,d_2)}(E)) P_t(J(X)) \\ &+ \sum_{\frac{1}{3}(d_1+d_2) < \ell \leq d_2 - d_1 + 2g - 2} t^{2(g-1+2\ell-d_2)} P_t(J(X)) P_t(S^{d_2-d_1+2g-2-\ell}X) P_t(S^{d_1-\ell+2g-2}X). \end{aligned} \tag{1.4}$$

In order to state the result for fixed determinant, let $\widetilde{S}(m_1, m_2)$ denote the pullback by the 3^{2g}-fold cover $J(X) \to J(X) : L \mapsto L^3$ of the product $S^{m_1}X \times S^{m_2}X$, where the map to $J(X)$ factors through $(L_1, L_2) \mapsto L_1^* L_2 \Lambda$. The Poincaré polynomial of $\widetilde{S}(m_1, m_2)$ was computed by Gothen [11] (see also Corollary 5.2 below).

Theorem 1.2 ($\mathsf{SU}(2,1)$ Higgs bundles)**.** *Fix (d_1, d_2) such that $0 \leq \tau(d_1, d_2) \leq 2g-2$ and $d_1 + d_2 \equiv 0 \mod 3$. Then the $\mathcal{G}_0$-equivariant Poincaré polynomial is given by*

$$\begin{aligned} P_t^{\mathcal{G}_0}(\mathcal{B}_\Lambda^{ss}(d_1, d_2)) &= \frac{1}{(1-t^2)} P_t^{\mathcal{G}(E)}(\mathcal{C}_{\sigma(d_1,d_2)}(E)) P_t(J(X)) \\ &+ \sum_{\frac{1}{3}(d_1+d_2) < \ell \leq d_2 - d_1 + 2g - 2} t^{2(g-1+2\ell-d_2)} P_t(\widetilde{S}(d_2 - d_1 + 2g - 2 - \ell, d_1 - \ell + 2g - 2)) \end{aligned} \tag{1.5}$$

where $\deg E = d_2 - 2d_1 + 4g - 4$*. For* $d_1 + d_2 \equiv 1 \mod 3$*,*

$$\begin{aligned} P_t(\mathcal{M}_\Lambda(d_1, d_2)) = P_t^{\mathcal{G}_0}(\mathcal{B}_\Lambda^{ss}(d_1, d_2)) &= P_t(\mathcal{N}_{\sigma(d_1,d_2)}(E)) \\ &+ \sum_{\frac{1}{3}(d_1+d_2) < \ell \leq d_2 - d_1 + 2g - 2} t^{2(g-1+2\ell-d_2)} P_t(\widetilde{S}(d_2 - d_1 + 2g - 2 - \ell, d_1 - \ell + 2g - 2)) \end{aligned} \tag{1.6}$$

Eq.'s (1.4) and (1.6) have been previously obtained by Gothen [11]. Gothen's results use slightly different notation to that given here; to obtain (1.4) and (1.6) from [11, Theorem 3.3, Theorem 4.1] one has to make the substitutions $m_2 = 2g - 2 + d_2 - d_1 - \ell$ and $d = d_1 + d_2$ and then interchange d_1 and d_2.

In the coprime case the moduli space is smooth, and one may use the moment map associated to Hitchin's S^1-action as a Morse-Bott function. Critical points correspond to fixed points of the S^1-action, and the cohomology of these critical sets (as well as their Morse indices) can be computed. As outlined below, the derivation of the Poincaré polynomials in this paper is different from that of [11]. Indeed, showing that the two results agree in the coprime case actually depends on the results of [17, 20]. The stable pairs moduli space that occurs in Gothen's calculations has a different stability parameter σ to that which occurs in the calculations of this paper, and one needs to look at different critical sets for the terms corresponding to the flips that relate the two different Bradlow spaces (cf. (4.2)). Therefore the connection between the two pictures is somewhat complicated and is not merely a comparison of critical sets.

We also point out the following special case (see Section 4).

Corollary 1.3. *In the maximal case $\tau(d_1, d_2) = 2g - 2$,*

$$P_t^{\mathcal{G}}(\mathcal{B}^{ss}(d_1, d_2)) = \frac{1}{(1-t^2)^2} P_t(J(X))^2$$

This is exactly what one would expect from the result in [19] (see also [5]).

We now describe the relationship with representation varieties. Fix $p \in X$, and let $\pi = \pi_1(X, p)$ denote the fundamental group acting by deck transformations on the universal cover $\widetilde{X}$ of X. Let $\omega_{\mathbb{B}^2}$ denote the complete $\mathsf{PU}(2,1)$-invariant Kähler metric on the complex ball $\mathbb{B}^2 \subset \mathbb{C}^2$, normalized to have constant holomorphic sectional curvature -1. Given $\rho : \pi \to \mathsf{PU}(2,1)$, choose a ρ-equivariant map $f : \widetilde{X} \to \mathbb{B}^2$. Then $f^*\omega_{\mathbb{B}^2}$ is a π-invariant form, and the *Toledo invariant* of ρ is by definition

$$\tau(\rho) = \frac{1}{2\pi} \int_X f^*\omega_{\mathbb{B}^2} \tag{1.7}$$

By [18], $\tau(\rho)$ is an integer that is constant on connected components of the representation variety, and which satisfies the bound $|\tau(\rho)| \leq 2g-2$. Extend the definition of $\tau(\rho)$ to representations of π to $\mathsf{SU}(2,1)$ and $\mathsf{U}(2,1)$ by projection to $\mathsf{PU}(2,1)$. Let $\mathrm{Hom}_\tau(\pi, G)$, $G = \mathsf{SU}(2,1)$, $\mathsf{U}(2,1)$, or $\mathsf{PU}(2,1)$, denote the subset of representations $\pi \to G$ with Toledo invariant $= \tau$, and let $\mathrm{Hom}_\tau(\pi, G)/\!\!/G$ be the corresponding moduli space of conjugacy classes of semisimple representations. By work of Hitchin, Simpson, Corlette and Donaldson ([13, 6, 9, 16]; see also [5]) we have

$$\mathrm{Hom}_\tau(\pi, \mathsf{U}(2,1))/\!\!/\mathsf{U}(2,1) \simeq \mathcal{M}(d_1, d_2)$$
$$\mathrm{Hom}_\tau(\pi, \mathsf{SU}(2,1))/\!\!/\mathsf{SU}(2,1) \simeq \mathcal{M}_\Lambda(d_1, d_2)$$

as real algebraic varieties, where $d_1 + d_2 = 0$, and we have $\tau = \tau(\rho) = \tau(d_1, d_2)$. As explained in [8], the results of this paper also compute the equivariant cohomology of these representation varieties (in this paper we take rational coefficients unless otherwise indicated).

Theorem 1.4. *Let $d_1 + d_2 = 0$ and $\tau = \frac{2}{3}(2d_1 - d_2)$. Then there are isomorphisms of equivariant cohomologies*

$$H^*_{\mathsf{U}(2,1)}(\mathrm{Hom}_\tau(\pi, \mathsf{U}(2,1))) \simeq H^*_{\mathcal{G}}(\mathcal{B}^{ss}(d_1, d_2))$$
$$H^*_{\mathsf{SU}(2,1)}(\mathrm{Hom}_\tau(\pi, \mathsf{SU}(2,1))) \simeq H^*_{\mathcal{G}_0}(\mathcal{B}^{ss}_\Lambda(d_1, d_2))$$

Tensoring a rank-n bundle by the n-torsion points in the Jacobian variety $J(M)$ leaves the determinant unchanged. Hence, the group $\Gamma_n = H^1(M, \mathbb{Z}/n)$ acts on fixed determinant moduli spaces, and the study of its induced action on the cohomology of moduli spaces goes back to Harder-Narasimhan [12]. In terms of representations, this action corresponds to the different possible lifts of $\mathsf{PU}(n)$ bundles to $\mathsf{SU}(n)$. More precisely, in our situation $\mathcal{M}_\Lambda(d_1, d_2)$ is a Γ_3-covering of a connected component of $\mathrm{Hom}(\pi, \mathsf{PU}(2,1))/\!\!/\mathsf{PU}(2,1)$. Furthermore, by a theorem of Xia [22] the connected components of the space of $\mathsf{PU}(2,1)$ representations are in 1-1 correspondence with the mod 3 values of $d = \deg \Lambda$ and the possible values of the Toledo invariant $|\tau(d_1, d_2)| \le 2g - 2$. As in the theorem above we have

$$H^*_{\mathsf{PU}(2,1)}(\mathrm{Hom}_{\tau,d}(\pi, \mathsf{PU}(2,1))) = \left[H^*_{\mathcal{G}_0}(\mathcal{B}^{ss}_\Lambda(d_1, d_2))\right]^{\Gamma_3} \tag{1.8}$$

where $d = d_1 + d_2$, $\tau(d_1, d_2) = \tau$, and the superscript indicates the Γ_3-invariant part of the cohomology.

It was shown in Atiyah-Bott [1], and illustrated further in [7] for $\mathsf{SL}(2, \mathbb{C})$, that the action of Γ_n is also the key to understanding *Kirwan surjectivity*, which we now define. Since the spaces $\mathcal{B}(d_1, d_2)$ and $\mathcal{B}_\Lambda(d_1, d_2)$ are contractible, the inclusions $\mathcal{B}^{ss}(d_1, d_2) \hookrightarrow \mathcal{B}(d_1, d_2)$ and $\mathcal{B}^{ss}_\Lambda(d_1, d_2) \hookrightarrow \mathcal{B}_\Lambda(d_1, d_2)$ give maps, which we call *Kirwan maps*,

$$\begin{aligned} \kappa &: H^*(B\mathcal{G}) \longrightarrow H^*_{\mathcal{G}}(\mathcal{B}^{ss}(d_1, d_2)) \\ \kappa_0 &: H^*(B\mathcal{G}_0) \longrightarrow H^*_{\mathcal{G}_0}(\mathcal{B}^{ss}_\Lambda(d_1, d_2)) \end{aligned} \tag{1.9}$$

where $B\mathcal{G}$ and $B\mathcal{G}_0$ are the classifying spaces of $\mathcal{G}$ and $\mathcal{G}_0$, respectively. We say that Kirwan surjectivity holds if κ (or κ_0) is surjective. For $\mathsf{U}(n)$ and $\mathsf{SU}(n)$ bundles, it turns out that the Kirwan maps are always surjective [1]. This is a consequence of the perfection of the Harder-Narasimhan (and Morse) stratification. It is also the case that Γ_n acts trivially on $H^*(B\mathcal{G}_0)$, and so surjectivity implies the same for the cohomology of the representation varieties. On the other hand, for $\mathsf{SL}(2, \mathbb{C})$ Higgs bundles, κ_0 is not in general surjective (cf. [7]).

Continuing in this vein, we show in this paper that a certain modification of the Harder-Narasimhan stratification for $\mathsf{U}(2,1)$ Higgs bundles is $\mathcal{G}$-equivariantly perfect (Theorem 2.6), and hence Kirwan surjectivity holds in this case. We also show that Γ_3 acts trivially on the equivariant cohomology of the moduli space of $\mathsf{SU}(2,1)$ Higgs bundles if and only if Kirwan surjectivity holds. In the fixed determinant case, surjectivity holds for only about a third of the components.

Theorem 1.5. *Kirwan surjectivity holds for the moduli spaces of* $\mathsf{U}(2,1)$ *and* $\mathsf{PU}(2,1)$ *Higgs bundles. Kirwan surjectivity holds for the moduli spaces of* $\mathsf{SU}(2,1)$ *Higgs bundles if and only if the Toledo invariant satisfies* $|\tau| > \frac{4}{3}(g-1)$.

The action of Γ_n is also closely intertwined with the action of the Torelli group $\mathcal{I}(X)$, defined as the subgroup of the mapping class group that acts trivially on the homology of X (see [8]). Since $\mathcal{I}(X)$ is a subgroup of the outer automorphism group of π, it acts on representation varieties by precomposition, and the induced action on equivariant cohomology commutes with Γ_n. On the other hand, by results of Looijenga [15] characters of Γ_n give rise to projective unitary representations of $\mathcal{I}(X)$ over cyclotomic fields. In Theorem 5.3, we explicitly determine the representations that appear for the action of $\Gamma_3 \times \mathcal{I}(X)$ on the moduli space of $\mathsf{SU}(2,1)$ Higgs bundles. As a consequence, we prove

Theorem 1.6. *The group* $\Gamma_3 \times \mathcal{I}(X)$ *acts trivially on the equivariant cohomology of the moduli spaces of* $\mathsf{U}(2,1)$ *and* $\mathsf{PU}(2,1)$ *representations of* π. *The Torelli group* $\mathcal{I}(X)$ *(resp. the group* Γ_3*) acts trivially on the equivariant cohomology of the moduli spaces of* $\mathsf{SU}(2,1)$ *representations if and only if the Toledo invariant satisfies* $|\tau| \geq \frac{4}{3}(g-1)$ *(resp.* $|\tau| > \frac{4}{3}(g-1)$*).*

The borderline case $\tau = \frac{4}{3}(g-1)$ (which occurs only for $g \equiv 1 \mod 3$) gives further examples in higher genus of representation varieties where Kirwan surjectivity fails but where the Torelli group nevertheless acts trivially on equivariant cohomology (this also occurs for $\mathsf{SL}(2,\mathbb{C})$ bundles, but only when $g = 2$). Gothen also studies the action of Γ_3 on the cohomology of the moduli space and shows in [11, Proposition 4.2] that in general it acts non-trivially on $H^*(\mathcal{M}_\Lambda(d_1, d_2))$ in the coprime case.

The method of proof for the results above is an extension of the equivariant Morse theory techniques of Atiyah-Bott and Kirwan from [1] and [14] to the singular space of Higgs bundles. This continues a program begun in [7, 8] (for rank 2 Higgs bundles) and [20] (for rank 2 stable pairs), and we use these results as part of our calculations for the $\mathsf{U}(2,1)$ case. The basic strategy is to use the Yang-Mills-Higgs functional as an equivariant Morse function on the spaces of $\mathsf{U}(2,1)$ Higgs bundles (resp. $\mathsf{SU}(2,1)$ Higgs bundles), where equivariance is defined with respect to the group of gauge transformations in the maximal compact subgroup of $\mathsf{U}(2,1)$ (resp. $\mathsf{SU}(2,1)$).

In Section 2 we describe the stratification of the space of Higgs bundles by the gradient flow of the Yang-Mills-Higgs functional and assert that the gradient flow of the Yang-Mills-Higgs functional on the space of $\mathsf{U}(2,1)$ Higgs bundles induces a Morse stratification identical to the Harder-Narasimhan stratification. Another result of this section is that the equivariant cohomology of the critical sets can be computed inductively in terms of lower rank Higgs bundles.

The major subtlety induced by the singularities in the space of Higgs bundles occurs in the study of the change in cohomology when attaching each of the Morse/Harder-Narasimhan strata to the union of lower strata. The Morse index is not constant on each connected component of the set of critical points, and so instead of attaching a bundle over the critical set (as in the usual Morse-Kirwan theory) one has to attach a more general space that fibers over the critical set. Section 3 contains a detailed analysis of these spaces and a calculation of their cohomology.

The Poincaré polynomial calculations are summarized in Section 4. The key point is that the spaces described in the previous paragraph appear as extra terms in the Poincaré polynomials. In Section 5 we prove Theorem 1.6 and describe the relationship between Kirwan surjectivity and the action of the finite group Γ_3 on the cohomology of the space of semistable points.

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2. Stratifications

2.1. **Critical points of the Yang-Mills-Higgs functional.** The goal of this section is to describe the stratification of the space $\mathcal{B}(d_1, d_2)$. There are in fact two natural stratifications: the Morse stratification given by the gradient flow of the Yang-Mills-Higgs functional which is detailed in this subsection, and the algebraic stratification according to Harder-Narasimhan type which is discussed in the next subsection. In Proposition 2.5 we claim that, as in [7] and [20], these stratifications coincide.

We begin with the classification of the critical sets of the Yang-Mills-Higgs functional in the general case. Fix smooth complex hermitian vector bundles $E_p, E_q \to X$, with $\operatorname{rank} E_p = p$, $\operatorname{rank} E_q = q$, $\deg E_i = d_i$. Without loss of generality (see the remark in [11, p731]) we always assume that $pd_q \geq qd_p$. A $\mathsf{U}(p,q)$ *Higgs bundle* consists of a split holomorphic structure on $V = E_p \oplus E_q$, and a Higgs field of the type $H^0(E_p^* E_q \otimes K) \oplus H^0(E_q^* E_p \otimes K)$, where K is the canonical bundle of X. In other words, a pair

$$(\bar{\partial}_A, \Phi) = \left(\bar{\partial}_{A_p} \oplus \bar{\partial}_{A_q}, \begin{pmatrix} 0 & c \\ b & 0 \end{pmatrix} \right)$$

Let $\mathcal{A}(E_i)$ denote the infinite dimensional affine space of $\bar{\partial}$-operators on E_i. By the Chern connection, these spaces are identical to the space of unitary connections. Then the set of pairs $(\bar{\partial}_A, \Phi)$ as above is a subspace

$$\mathcal{B}(E_p, E_q) \subset (\mathcal{A}(E_p) \times \mathcal{A}(E_q)) \times (\Omega^0(E_p^* E_q \otimes K) \oplus \Omega^0(E_q^* E_p \otimes K)) \tag{2.1}$$

which we call the space of $\mathsf{U}(p,q)$ Higgs bundles (for more details, see [5]). The gauge group $\mathcal{G} = \mathcal{G}(E_p, E_q)$ of $\mathsf{U}(p) \times \mathsf{U}(q)$ and its complexification $\mathcal{G}^{\mathbb{C}}$ act on $\mathcal{B}(E_p, E_q)$. We note two facts that are important for the discussion here: the first is that $\mathcal{B}(E_p, E_q)$ is $\mathcal{G}$-equivariantly contractible. The second is that $\mathcal{B}(E_p, E_q)$ is a singular space in general. To see this, we introduce the deformation complex for Higgs bundles

$$\Omega^0(E_p E_p^* \oplus E_q E_q^*) \xrightarrow{D''} \Omega^{0,1}(E_p E_p^* \oplus E_q E_q^*) \oplus \Omega^{1,0}(E_p^* E_q \oplus E_q^* E_p) \xrightarrow{D''} \Omega^{1,1}(E_p^* E_q \oplus E_q^* E_p) \tag{2.2}$$

where $D'' = \bar{\partial}_A + \Phi$. The Zariski tangent space to $\mathcal{B}(E_p, E_q)/\mathcal{G}^{\mathbb{C}}$ is given by H^1 of the complex. Singularities occur when H^2 (or equivalently H^0) of the complex is bigger than the generic value

$\mathbb{C}$. This happens, for example, when there is a nonzero endomorphism $\phi : E_p \to E_p$, $\operatorname{rank}\phi \leq p-q$, such that $c(E_q) \subset \ker\phi \otimes K$.

Equivalently, one may view a $\mathsf{U}(p,q)$ Higgs bundle as a *twisted quiver bundle*, with diagram

$$\begin{tikzcd} \bullet_{E_q} \arrow[r, bend left, "c"] & \bullet_{E_p} \arrow[l, bend left, "b"] \end{tikzcd} \tag{2.3}$$

where $b : E_p \to E_q \otimes K$ and $c : E_q \to E_p \otimes K$ are holomorphic. The Yang-Mills-Higgs functional on the space of Higgs bundles is defined by

$$\mathrm{YMH}(\bar\partial_A, \Phi) = \|F_A + [\Phi, \Phi^*]\|^2$$

where F_A denotes the curvature of the Chern connection associated to $\bar\partial_A$ and the hermitian structure on $E_p \oplus E_q$, and $\|\cdot\|$ is the L^2-norm taken with respect to a choice of conformal metric on X. On restriction to the space $\mathcal{B}(E_p, E_q)$ this becomes

$$\mathrm{YMH}(\bar\partial_{A_p}, \bar\partial_{A_q}, b, c) = \left\|F_{A_p} + bb^* + c^*c\right\|^2 + \left\|F_{A_q} + b^*b + cc^*\right\|^2$$

Let $\mathcal{B}_{min}(E_p, E_q)$ denote the set of absolute minima of YMH. We also study the non-minimal critical sets of YMH, however the usual definition of critical point does not make sense due to the singularities of $\mathcal{B}(E_p, E_q)$ and so we define the critical sets as follows. The singular space $\mathcal{B}(E_p, E_q)$ is defined in (2.1) to be a subset of an infinite dimensional manifold defined by imposing the holomorphicity condition on the Higgs field. The gradient flow of YMH is defined on this manifold, and when the initial conditions are in $\mathcal{B}(E_p, E_q)$ the flow is generated by $g(t) \in \mathcal{G}^{\mathbb{C}}$ satisfying

$$\frac{\partial g}{\partial t} g^{-1} = *(F_A + [\Phi, \Phi^*]) \tag{2.4}$$

(see [21]). Since $\mathcal{G}^{\mathbb{C}}$ preserves $\mathcal{B}$, then the gradient flow also preserves $\mathcal{B}(E_p, E_q)$, and we define the critical points of YMH to be the stationary points of the gradient flow. Equation (2.4) shows that these are the pairs $(\bar\partial_A, \Phi)$ for which the infinitesimal action of $*(F_A + [\Phi, \Phi^*]) \in \mathrm{Lie}(\mathcal{G}^{\mathbb{C}})$ is trivial. More precisely, the critical point equations for YMH on $\mathcal{B}(E_p, E_q)$ are

$$\bar\partial_{A_q} * \left(F_{A_q} + bb^* + c^*c\right) = 0 \tag{2.5}$$

$$\bar\partial_{A_p} * \left(F_{A_p} + b^*b + cc^*\right) = 0 \tag{2.6}$$

$$b * \left(F_{A_p} + b^*b + cc^*\right) - * \left(F_{A_q} + bb^* + c^*c\right) b = 0 \tag{2.7}$$

$$c * \left(F_{A_q} + bb^* + c^*c\right) - * \left(F_{A_p} + b^*b + cc^*\right) c = 0 \tag{2.8}$$

Using (2.5) and (2.6) and the same method of proof for holomorphic bundles in [1, Section 5], we conclude that the eigenvalues of $*\left(F_{A_q} + bb^* + c^*c\right)$ and $*\left(F_{A_p} + b^*b + cc^*\right)$ are constant and the holomorphic structures on E_p and E_q split according to these eigenvalues. We can therefore write

$*\left(F_{A_q}+bb^*+c^*c\right)$ and $*\left(F_{A_p}+b^*b+cc^*\right)$ in the following block-diagonal form

$$*(F_{A_q}+bb^*+c^*c)=\begin{pmatrix}\lambda_1^q & 0 & \cdots & 0\\ 0 & \lambda_2^q & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{n_q}^q\end{pmatrix} \quad\text{and}\quad *\left(F_{A_p}+b^*b+cc^*\right)=\begin{pmatrix}\lambda_1^p & 0 & \cdots & 0\\ 0 & \lambda_2^p & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{n_p}^p\end{pmatrix}$$

(recall that these expressions are skew-Hermitian with respect to the metrics on E_p and E_q, and hence diagonalizable). The bundles E_p and E_q then split with respect to this decomposition as follows

$$E_p=E_p^{(\lambda_1^p)}\oplus\cdots\oplus E_p^{(\lambda_{n_p}^p)}\ ,\ E_q=E_q^{(\lambda_1^q)}\oplus\cdots\oplus E_q^{(\lambda_{n_q}^q)}$$

where $E_p^{(\lambda_k^p)}$ (resp. $E_q^{(\lambda_k^q)}$) is the holomorphic sub-bundle of E_p (resp. E_q) corresponding to the eigenvalue λ_k^p (resp. λ_k^q). The Higgs fields b and c also decompose with respect to this splitting, and it follows from equations (2.7) and (2.8) that, if $\lambda_j^p\neq\lambda_k^q$, then the component of b mapping $E_p^{(\lambda_j^p)}$ to $E_q^{(\lambda_k^q)}$ is zero and the component of c mapping $E_q^{(\lambda_k^q)}$ to $E_p^{(\lambda_j^q)}$ is zero.

Therefore, the critical point equations define a splitting of $(\bar\partial_{A_p},\bar\partial_{A_q},b,c)$ into $\mathsf{U}(p',q')$ sub-bundles

$$(\bar\partial_{A_p},\bar\partial_{A_q},b,c)=\bigoplus_\ell(\bar\partial_{A_p^\ell},\bar\partial_{A_q^\ell},b_\ell,c_\ell) \tag{2.9}$$

where ℓ ranges over the set of all eigenvalues of $*\left(F_{A_q}+bb^*+c^*c\right)$ and $*\left(F_{A_p}+b^*b+cc^*\right)$, and $q'=\operatorname{rank}(E_q^{(\ell)})$, $p'=\operatorname{rank}(E_p^{(\ell)})$ (note that it is possible for one of p' or q' to be zero). Moreover, the usual Chern-Weil technique shows that the eigenvalues are determined by the slope of the bundles $E_p^\ell\oplus E_q^\ell$, and that $(\bar\partial_{A_q^\ell},\bar\partial_{A_p^\ell},b_\ell,c_\ell)$ minimizes the Yang-Mills-Higgs functional on $\mathcal{B}(E_p^\ell,E_q^\ell)$.

These results are summarized in the following proposition.

Proposition 2.1. *A $\mathsf{U}(p,q)$ Higgs structure $(\bar\partial_{A_p},\bar\partial_{A_q},b,c)$ is a critical point for the Yang-Mills-Higgs functional if and only if it splits into the direct sum of $U(p',q')$ sub-bundles, each of which is a minimizer for the associated Yang-Mills-Higgs functional on the sub-bundles. The splitting is determined by the eigenvalues and eigenspaces of $*\left(F_{A_q}+bb^*+c^*c\right)$ and $*\left(F_{A_p}+b^*b+cc^*\right)$.*

We now specialize to the case $p=2$ and $q=1$, and use the notation $\mathcal{B}(d_1,d_2)$ for $\mathcal{B}(E_2,E_1)$, where $d_i=\deg E_i$. In this case, there are only three types of decomposition that can occur at nonminimal critical points, one for each possible configuration of distinct eigenspaces for $*\left(F_{A_q}+bb^*+c^*c\right)$ and $*\left(F_{A_p}+b^*b+cc^*\right)$. The first is where the Higgs field is zero and the bundle E_2 is polystable. In terms of Proposition 2.1, the bundles E_1 and E_2 are distinct eigenspaces for $*\left(F_{A_q}+bb^*+c^*c\right)$ and $*\left(F_{A_p}+b^*b+cc^*\right)$ and we have a splitting of the structure into a $\mathsf{U}(1)$ and a $\mathsf{U}(2)$ Higgs bundle. Call these critical points *Type A* and let $\mathcal{C}_a$ denote the set of all critical points of Type A.

The second type of decomposition is where the Higgs field is zero and the structure splits into three $\mathsf{U}(1)$ Higgs bundles. Call these critical points *Type B*. In this case, the bundle E_2 is the direct sum of holomorphic line bundles $E_2\cong S\oplus Q$ and, in the language of Proposition 2.1, the bundles E_1, S and Q are distinct eigenspaces. Without loss of generality, assume that $\deg S>\deg Q$, and note that the Higgs field is necessarily zero when $\deg S \neq d_1$. Also, use the notation $d_S = \deg S$, $d_Q = \deg Q$.

For convenience, when $d_S = d_1$ we also include the possibility that the Higgs field can be nonzero (see also Remark 2.3). The critical point equations imply that such a Higgs bundle must take the following form

$$\begin{tikzcd} & \bullet Q \\ \bullet E_1 \arrow[dr, bend left, "c"] & \\ & \bullet S \arrow[ul, bend left, "b"] \end{tikzcd}$$

where b and c are related by $\|b\|^2 = \|c\|^2$.

The connected components of the space of Type B critical points are in one-to-one correspondence with the range of values for d_S. Moreover, there are three different cases for d_S that lead to different contributions to the Morse theory calculations of Section 3. For each value of ℓ in the range $\frac{1}{2}d_2 < \ell < d_1$, let $\mathcal{C}_{b_1}^{\ell}$ denote the set of Type B critical points for which $d_S = \ell$, define $\mathcal{C}_{b_2}^{d_1}$ to be the set of Type B critical points for which $d_S = d_1$, and for $d_1 < \ell$ define $\mathcal{C}_{b_3}^{\ell}$ to be the set of Type B critical points for which $d_S = \ell$.

The third type of decomposition is where the $\mathsf{U}(2,1)$ structure splits into the direct sum of a stable $U(1,1)$ structure and a $U(1)$ structure. Equivalently, the bundle E_2 splits into line bundles, $E_2 \cong S \oplus Q$, and, depending on the degree of S and Q, the Higgs field takes on one of the following forms.

(i) $d_S > \frac{1}{2}(d_Q + d_1)$. In this case the maximal semistable subobject of the Higgs bundle $(E_2 \oplus E_1, b, c)$ is a line subbundle of S, which does not interact with the Higgs field. Define $\ell = d_S$. Since we have assumed that $d_2 \leq 2d_1$ (see the Introduction), then the condition $\ell > \frac{1}{2}(d_Q + d_1)$ implies that $d_1 > d_2 - \ell = d_Q$. Minimality of the Yang-Mills-Higgs functional on the subobject $(Q \oplus E_1, b, c)$ then implies that b and c are related by $\frac{1}{\pi}\left(\|c\|^2 - \|b\|^2\right) = d_1 - d_Q > 0$ and therefore $c \neq 0$. Label these critical sets $\mathcal{C}_{c_1}^{\ell}$. A graphical representation of the Higgs field at these critical points is as follows.

$$\begin{tikzcd} & \bullet Q \arrow[dl, bend left, "b"] \\ \bullet E_1 \arrow[ur, bend left, "c \neq 0"] & \\ & \bullet S \end{tikzcd}$$

The section c can only be nonzero if $\deg(E_1^* Q \otimes K) \geq 0$, and so these critical points only exist for values of ℓ such that $d_2 - \ell - d_1 + 2g - 2 \geq 0$ and $\ell > \frac{1}{2}(d_2 - \ell + d_1)$. This is equivalent to the condition that ℓ is in the range $\frac{1}{3}(d_1 + d_2) < \ell \leq d_2 - d_1 + 2g - 2$.

(ii) $d_Q < \frac{1}{2}(d_S + d_1)$ and $d_1 > d_S$. In this case the maximal semistable subobject of (E_1, E_2, b, c) is $(E_1 \oplus S, b, c)$, and so we define $\ell = d_S$ and $d_Q = d_2 - \ell$. Then the same analysis as before shows that b and c are related by $\frac{1}{\pi}\left(\|c\|^2 - \|b\|^2\right) = d_1 - d_S > 0$, and therefore $c \neq 0$. Call these critical sets $\mathcal{C}_{c_2}^{\ell}$. The corresponding picture is

$$\begin{tikzcd} & \bullet Q \\ \bullet E_1 \arrow[dr, bend left, "c\neq 0"] & \\ & \bullet S \arrow[ul, bend left, "b"] \end{tikzcd}$$

Critical sets of this type can only exist if $c \neq 0$, and so we must have $\deg(E_1^* S \otimes K) \geq 0$. Combining this with the conditions that $d_Q < \frac{1}{2}(d_S + d_1)$ and $d_1 > d_S$ gives

$$\max\big(d_1 - 2g + 1, \tfrac{1}{3}(2d_2 - d_1)\big) < \ell < d_1 \tag{2.10}$$

The bound on the Toledo invariant $2d_1 - d_2 \leq 3g - 3$ is equivalent to $d_1 - (2g - 2) \leq \frac{1}{3}(2d_2 - d_1)$. Therefore, the inequality (2.10) reduces to $\frac{1}{3}(2d_2 - d_1) < \ell < d_1$.

(iii) $d_Q < \frac{1}{2}(d_S + d_1)$ and $d_1 < d_S$. In this case the maximal semistable subobject of $(E_2 \oplus E_1, b, c)$ is $(E_1 \oplus S, b, c)$, and so we define $\ell = d_S$ and $d_Q = d_2 - \ell$. An analysis of the critical point equations shows that now $b \neq 0$ and that b and c are related by $\frac{1}{\pi}\left(\|b\|_{L^2}^2 - \|c\|_{L^2}^2\right) = d_S - d_1 > 0$. Call these critical sets $\mathcal{C}_{c_3}^{\ell}$, and note that the quiver bundle picture reduces to

$$\begin{tikzcd} & \bullet Q \\ \bullet E_1 \arrow[dr, bend left, "c"] & \\ & \bullet S \arrow[ul, bend left, "b\neq 0"] \end{tikzcd}$$

These critical sets can only exist if $b \neq 0$, and so $\deg(S^* E_1 \otimes K) \geq 0$. Note that $d_1 < \ell$ implies that $d_Q < \frac{1}{2}(d_S + d_1)$, and so we have the inequalities $d_1 - \ell + 2g - 2 \geq 0$ and $\ell > d_1$. Therefore $d_1 < \ell \leq d_1 + 2g - 2$.

Remark 2.2. From the above diagrams one can also read off the eigenspaces of $*\left(F_{A_q} + bb^* + c^*c\right)$ and $*\left(F_{A_p} + b^*b + cc^*\right)$. For critical sets of type $\mathcal{C}_{c_1}^{\ell}$, the bundles $E_1 \oplus Q$ and S form eigenspaces of with distinct eigenvalues, and for critical sets of type $\mathcal{C}_{c_2}^{\ell}$ and $\mathcal{C}_{c_3}^{\ell}$ the bundles $E_1 \oplus S$ and Q are the distinct eigenspaces. The reason for the difference between the cases will become apparent in the next section when we study the Harder-Narasimhan filtration: the bundle S always forms part of the subobject of maximal slope, the bundle Q always forms part of the quotient and we take the direct sum of E_1 with either S or Q depending on the degrees of E_1, S and Q.

Remark 2.3. Note that there are two possible values of ℓ that have not been included in the above list. The first is $\ell = d_1$, for which the critical points have already been classified as type $\mathcal{C}_{b_2}^{d_1}$. The second is $\ell = \frac{1}{3}(2d_2 - d_1)$, in which case the critical point minimizes the Yang-Mills-Higgs functional.

Using the descriptions above, a standard calculation gives the following results for the equivariant Poincaré polynomial of each nonminimal critical set. In Table 1 we have used the notation

TABLE 1. Classification of the critical sets and their topology

Critical set	Range of values of ℓ	Equivariant Poincaré polynomial
$\mathcal{C}_a$	n/a	$\frac{1}{(1-t^2)^2}P_t(J(X))P_t^{\mathcal{G}(E_2)}(\mathcal{A}^{ss}(E_2))$
$\mathcal{C}_{b_1}^\ell$	$\frac{1}{2}d_2 < \ell < d_1$	$\frac{1}{(1-t^2)^3}P_t(J(X))^3$
$\mathcal{C}_{b_2}^\ell$	$\ell = d_1$	$\frac{1}{(1-t^2)^3}P_t(J(X))^3$
$\mathcal{C}_{b_3}^\ell$	$d_1 < \ell$	$\frac{1}{(1-t^2)^3}P_t(J(X))^3$
$\mathcal{C}_{c_1}^\ell$	$\frac{1}{3}(d_1+d_2) < \ell \leq d_2 - d_1 + 2g - 2$	$\frac{1}{(1-t^2)^2}P_t(J(X))^2P_t(S^{\ell-d_1+2g-2}X)$
$\mathcal{C}_{c_2}^\ell$	$\frac{1}{3}(2d_2-d_1) < \ell < d_1$	$\frac{1}{(1-t^2)^2}P_t(J(X))^2P_t(S^{\ell-d_1+2g-2}X)$
$\mathcal{C}_{c_3}^\ell$	$d_1 < \ell \leq d_1 + 2g - 2$	$\frac{1}{(1-t^2)^2}P_t(J(X))^2P_t(S^{d_1-\ell+2g-2}X)$

$\mathcal{A}^{ss}(E_2) \subset \mathcal{A}(E_2)$ for the subset of semistable bundles. We denote the ordered set of possible values in the labeling of the critical sets above by

$$\Delta_{d_1,d_2} = \left\{\tfrac{1}{2}d_2\right\} \cup \left\{\ell \in \mathbb{Z} : \ell > \tfrac{1}{3}(2d_2 - d_1)\right\} \tag{2.11}$$

We will express the various components as $\mathcal{C}_a$, $\mathcal{C}_b^\ell$, and $\mathcal{C}_c^\ell$.

2.2. **Harder-Narasimhan and Morse stratifications.** We now describe the algebraic stratification of the space of $\mathsf{U}(2,1)$ Higgs bundles. As in the previous section, let

$$V = E_2 \oplus E_1\ ,\ \ \Phi = \begin{pmatrix} 0 & c \\ b & 0 \end{pmatrix}$$

Recall that (V, Φ) is *stable* (resp. *semistable*) if

$$\mu(F) = \frac{\deg F}{\operatorname{rank} F} < \mu(V) = \frac{\deg V}{\operatorname{rank} V} \qquad (\text{resp. } \leq)$$

for every Φ-invariant subsheaf $0 \neq \operatorname{rank} F \neq \operatorname{rank} V$. If (V, Φ) is not semistable, a *maximally destabilizing subbundle* is a subsheaf $0 \neq F \subsetneq V$, satisfying the following:

- F is Φ-invariant;
- $\mu(F) > \mu(V)$;
- F is maximal in the sense that for any $F' \neq F$ satisfying the first two conditions, then $\mu(F') \leq \mu(F)$, and if equality, then $\operatorname{rank} F' < \operatorname{rank} F$.

If F satisfies these conditions then F must be saturated, i.e. V/F is torsion-free. Unstable Higgs bundles have a unique (Harder-Narasimhan) filtration by sub-Higgs bundles. The associated graded of this filtration will be denoted by $\operatorname{Gr}(V, \Phi)$.

Recall that by assumption, $2d_1 \geq d_2$. Below we determine all the possible Harder-Narasimhan filtrations of unstable $\mathsf{U}(2,1)$ Higgs bundles. Let F be a maximally destabilizing subbundle of (V, Φ).

Case I: $\operatorname{rank} F = 1$. Let f_i be the induced maps $F \to E_i$. Then $f_2 \equiv 0$ implies f_1 is an isomorphism, and $f_1 \equiv 0$ implies f_2 is everywhere injective, and we claim that one of these two possibilities occurs. For suppose neither $f_i \equiv 0$, and let $F_2 \subset E_2$ be the saturation of $\operatorname{im} f_2$. Then $E_1 \oplus F_2$ is a subbundle with slope at least $\deg F$, contradicting the assumption that F is maximal. It follows that there are two possibilities according to whether F lies in E_1 or E_2.

(i) If $F = E_1$, then $c \equiv 0$. If E_2 is semistable then the stratum is defined by the condition $c \equiv 0$, and we label it by $\mathcal{S}_a$. The quiver diagram in this case is

$$\begin{tikzcd} \bullet_{E_1} & \bullet_{E_2} \arrow[l, dashed, "b"'] \end{tikzcd}$$

and the associated graded is $(E_2, 0) \oplus (E_1, 0)$ (In this diagram and the others below, we use a *dashed arrow* to represent a component of the Higgs field that may or may not be zero and a *solid arrow* to represent a component of the Higgs field that must be nonzero. If a component of the Higgs field must be zero then there is no arrow between the vertices). If the bundle E_2 is unstable, let $S \subset E_2$ be the maximal destabilizing line bundle, and write $0 \to S \to E_2 \to Q \to 0$, with extension class $[a] \in H^1(X, Q^*S)$. Notice that $d_S = \deg S < d_1$, since either $S \subset \ker b$ and S is a subobject of (V, Φ), or S is not in $\ker b$ and $S \oplus E_1$ is a subobject. The associated graded $\operatorname{Gr}(V, \Phi) = (E_1, 0) \oplus (S, 0) \oplus (Q, 0)$, and we label this stratum by $\mathcal{S}_{b_1}^{\ell}$, where $\ell = \deg S$. The quiver diagram for this case is

$$(\mathcal{S}_{b_1}^{\ell}) \qquad \begin{tikzcd} & \bullet_Q \arrow[dl, dashed, "b_Q"'] \arrow[dd, dashed, "a"] \\ \bullet_{E_1} & \\ & \bullet_S \arrow[ul, dashed, "b_S"] \end{tikzcd}$$

(ii) If $F = S \subset E_2$ with quotient Q, then $S \subset \ker b$, E_2 is unstable, and the graded object of the Harder-Narasimhan filtration of E_2 is precisely $S \oplus Q$. If $c_Q \equiv 0$, then we also require $d_S > d_1$, for otherwise E_1 would be invariant with slope at least d_S. In this case, $\operatorname{Gr}(V, \Phi) = (S, 0) \oplus (E_1, 0) \oplus (Q, 0)$. If $c_Q \neq 0$, then the only requirement is that $d_S > \frac{1}{3}(d_1 + d_2)$ (otherwise $E_1 \oplus Q$ would be invariant with slope at least d_S), and $\operatorname{Gr}(V, \Phi) = (S, 0) \oplus (E_1 \oplus Q, b_Q, c_Q)$, where (b_Q, c_Q) is the induced Higgs field on $E_1 \oplus Q$ coming from b and the projection c_Q of c to Q. We label the strata $\mathcal{S}_{b_3}^{\ell}$ and $\mathcal{S}_{c_1}^{\ell}$, respectively. The quiver diagrams for the two cases are

$$(\mathcal{S}_{b_3}^{\ell}) \qquad \begin{tikzcd} & \bullet_Q \arrow[dl, dashed, "b_Q"'] \arrow[dd, dashed, "a"] \\ \bullet_{E_1} \arrow[dr, dashed, "c_S"'] & \\ & \bullet_S \end{tikzcd} \qquad\qquad (\mathcal{S}_{c_1}^{\ell}) \qquad \begin{tikzcd} & \bullet_Q \arrow[dl, dashed, bend left, "b_Q"] \arrow[dd, dashed, "a"] \\ \bullet_{E_1} \arrow[ur, bend left, "c_Q \neq 0"] \arrow[dr, dashed, "c_S"'] & \\ & \bullet_S \end{tikzcd}$$

Case II: $\operatorname{rank} F = 2$. The projection $F \to E_1$ cannot vanish. Indeed, if if did, then $F = E_2$ and $d_2/2 > (1/3)(d_1 + d_2)$. But this contradicts the assumption $d_2 \leq 2d_1$. Let S be the kernel of the projection $F \to E_1$. Then $\deg(P = F/S) \le d_1$. We also have $S \subset E_2$. Since E_2/S is a subsheaf of V/F which we assume to be torsion-free, we conclude that S is a subbundle of E_2. Let $[a_F]$ and $[a]$ denote the extension classes for the sequences

$$0 \longrightarrow S \longrightarrow F \longrightarrow P \longrightarrow 0 \tag{2.12}$$

$$0 \longrightarrow S \longrightarrow E_2 \longrightarrow Q \longrightarrow 0 \tag{2.13}$$

In terms of the smooth splittings $E_2 \oplus E_1 = S \oplus Q \oplus E_1$ and $F = S \oplus P$, we can write the inclusion $F \hookrightarrow V$ and the Higgs field as

$$f = \begin{pmatrix} 1 & f_1 \\ 0 & f_2 \\ 0 & f_P \end{pmatrix} \ , \ \Phi = \begin{pmatrix} 0 & 0 & c_S \\ 0 & 0 & c_Q \\ b_S & b_Q & 0 \end{pmatrix}$$

where $f_P : P \to E_1$ is nonzero (since the projection of F to E_1 cannot vanish), and $f_1 : P \to S$, $f_2 : P \to Q$ are induced by the projection from F to E_2. Since f has everywhere rank 2, f_2 and f_P have no common zeros. Holomorphicity of f implies f_2, f_P holomorphic, and f_1 satisfies

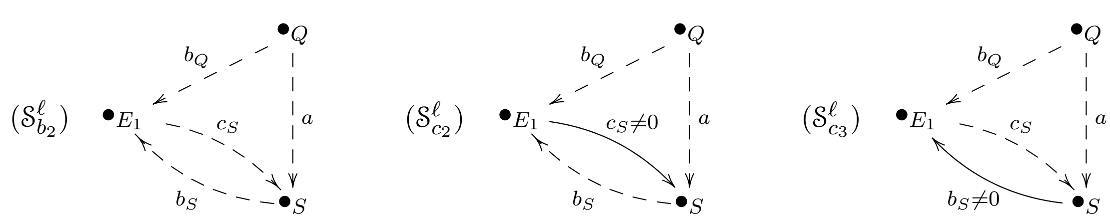
$$\bar\partial f_1 + a f_2 - a_F = 0 \tag{2.14}$$

where $\bar\partial$ is the induced holomorphic structure on $P^* \otimes S$. On the other hand, since F is destabilizing

$$\begin{aligned} \deg(QP^*) &= d_Q - \deg P = d_2 - d_S - \deg P = d_2 - \deg F \\ &< d_2 - \tfrac{2}{3}(d_1 + d_2) = -\tfrac{1}{3}(2d_1 - d_2) \le 0 \end{aligned}$$

by the assumption on degrees. It follows that $f_2 \equiv 0$, f_P gives an isomorphism $P \cong E_1$, and by (2.14) the sequence (2.12) splits. The condition that $F \cong E_1 \oplus S$ be invariant under the Higgs field is equivalent to $c_Q \equiv 0$. Moreover, S is invariant if and only if $b_S \equiv 0$. These are the only conditions coming from invariance.

A splitting of $F \subset V$ gives a splitting of (2.13). In this case, $\mathrm{Gr}(V, \Phi) = (E_1 \oplus S, b_S, c_S) \oplus (Q, 0)$. and the condition on degrees is $\ell > \frac{1}{3}(2d_2 - d_1)$. By the assumption that F is maximally destabilizing, $\ell < d_1 \Rightarrow c_S \neq 0$, and $\ell > d_1 \Rightarrow b_S \neq 0$. We label the former case $\mathcal{S}^\ell_{c_2}$ and the latter case $\mathcal{S}^\ell_{c_3}$. When $\ell = d_1$ then there are no conditions on b_S and c_S. We label this stratum $\mathcal{S}_{b_2}$. The quiver diagrams for these strata are



We conclude that there is a 1-1 correspondence between the associated graded objects listed above and the critical sets of the YMH functional. The collection $\{\mathcal{S}_a, \mathcal{S}^\ell_b, \mathcal{S}^\ell_c\}$, where $\ell \in \Delta_{d_1,d_2}$ has its natural ordering, combine to form the *Harder-Narasimhan stratification* of $\mathcal{B}(d_1, d_2)$. As in [20], however, it turns out that this stratification is too fine a structure for an equivariantly perfect Morse theory. The main reason is that unlike the situation in [7], we cannot prove the Morse-Bott lemma for *all* critical sets (see Proposition 3.12). Instead, there is a cancellation that occurs between certain B and C strata (cf. Remark 3.8) that makes the combination of these strata more suitable for the Morse theory. For $k \in \Delta_{d_1,d_2}$ define

$$X_k = \begin{cases} \mathcal{B}^{ss}(d_1,d_2) \cup \bigcup_{\ell\in\Delta_{d_1,d_2},\,\ell\leq k} \mathcal{S}_c^\ell & k < d_2/2 \\ \mathcal{B}^{ss}(d_1,d_2) \cup \mathcal{S}_a \cup \bigcup_{\ell\in\Delta_{d_1,d_2},\,\ell\leq d_2/2} \mathcal{S}_c^\ell & k = d_2/2 \\ X^{ss} \cup \mathcal{S}_a \cup \bigcup_{\ell\in\Delta_{d_1,d_2},\,\ell\leq k} \mathcal{S}_c^\ell \cup \bigcup_{\ell\in\Delta_{d_1,d_2},\ \ell\leq k} \mathcal{S}_b^\ell & k > d_2/2 \end{cases} \tag{2.15}$$

$$X_k^* = \begin{cases} \mathcal{B}^{ss}(d_1,d_2) \cup \bigcup_{\ell\in\Delta_{d_1,d_2},\,\ell< k} \mathcal{S}_c^\ell & k \leq d_2/2 \\ \mathcal{B}^{ss}(d_1,d_2) \cup \mathcal{S}_a \cup \bigcup_{\ell\in\Delta_{d_1,d_2},\,\ell< k} \mathcal{S}_c^\ell \cup \bigcup_{\ell\in\Delta_{d_1,d_2},\ \ell< k} \mathcal{S}_b^\ell & k > d_2/2 \end{cases} \tag{2.16}$$

In the notation above, $\mathcal{S}_c^\ell$ means the union over all possible C-strata with index ℓ. This gives a $\mathcal{G}$-invariant stratification of $\mathcal{B}(d_1,d_2)$ which we refer to as the *modified Harder-Narasimhan stratification.*

Remark 2.4. As in [20], the ordering of the set Δ_{d_1,d_2} does not in general coincide with the one coming from values of the YMH functional. This is irrelevant for the calculations in this paper.

As in [21], we have

Proposition 2.5. *The Morse stratification of the YMH flow coincides with the Harder-Narasimhan stratification of $\mathcal{B}(d_1,d_2)$. In particular, the gradient flow of the YMH functional defines $\mathcal{G}$-equivariant deformation retractions $\mathcal{B}_{min}(d_1,d_2) \hookrightarrow \mathcal{B}^{ss}(d_1,d_2)$, $\mathcal{C}_a \hookrightarrow \mathcal{S}_a$, $\mathcal{C}_b^\ell \hookrightarrow \mathcal{S}_b^\ell$, and $\mathcal{C}_c^\ell \hookrightarrow \mathcal{S}_c^\ell$.*

We now state one of the main results of this paper. The proof occupies the next section.

Theorem 2.6 (Perfect stratification)**.** *The modified Harder-Narasimhan stratification $\{X_k\}_{k\in\Delta_{d_1,d_2}}$ of $\mathcal{B}(d_1,d_2)$ is $\mathcal{G}$-equivariantly perfect in the sense that the inclusions $\mathcal{B}^{ss}(d_1,d_2) \subset X_k \subset X_\ell$ induce surjections $H_{\mathcal{G}}^*(X_k) \to H_{\mathcal{G}}^*(\mathcal{B}^{ss}(d_1,d_2))$ and $H_{\mathcal{G}}^*(X_\ell) \to H_{\mathcal{G}}^*(X_k)$ for all $k \leq \ell$ in Δ_{d_1,d_2}.*

Corollary 2.7 (Kirwan surjectivity)**.** *The Kirwan map κ in* (1.9) *is surjective.*

In order to prove Theorem 2.6 we shall need to bootstrap an intermediary stratification lying between the HN and modified HN strata. Define $\{X_k'\}_{k\in\Delta_{d_1,d_2}}$ by setting $X_k' = X_k^* \cup \mathcal{S}_c^k$. There are three crucial regions, essentially depending upon the number of C-strata. We refer to these by the following:

- **(I)** where $\frac{1}{3}(d_1+d_2) < k \leq d_2 - d_1 + 2g - 2$;
- **(II)** where $\frac{1}{3}(2d_2-d_1) < k \leq \frac{1}{3}(d_1+d_2)$, or where $d_2 - d_1 + 2g - 2 < k \leq d_1$ (if possible);
- **(III)** where $\max\{d_1, d_2 - d_1 + 2g - 2\} < k$.

3. Singular Morse Theory

In this section we develop the necessary machinery to perform the Morse theory calculations on the singular space $\mathcal{B}(E_1, E_2)$. Recall Kirwan's result [14]. For a Hamiltonian action of a compact connected Lie group K on a compact smooth symplectic manifold M, there is a compatible Riemannian structure such that induced Morse stratification $\{S_\mu\}_{\mu \in I}$ is smooth, where I is a partially ordered set labeling the critical sets. Let

$$X_\mu = \cup_{\nu \leq \mu} S_\nu \ , \ X_\mu^* = \cup_{\nu < \mu} S_\nu$$

Then Kirwan shows that the long exact sequence

$$\cdots \longrightarrow H_K^p(X_\mu, X_\mu^*) \xrightarrow{\alpha^p} H_K^p(X_\mu) \xrightarrow{\beta^p} H_K^p(X_\mu^*) \longrightarrow \cdots \tag{3.1}$$

splits into short exact sequences. Moreover, the Thom isomorphism implies that $H_K^p(X_\mu, X_\mu^*) \cong H_K^{p-\lambda_\mu}(C_\mu)$, where C_μ is the critical set at the minimum of the stratum S_μ and λ_μ is the Morse index. The splitting of (3.1) is a consequence of the fact that α^p is always injective, which in turn follows from the Atiyah-Bott lemma [1]. Therefore, to compute the change in cohomology that occurs when attaching the stratum S_μ, it is sufficient to know the cohomology and the Morse index of each critical set. Moreover, α^p injective for all p implies that β^p is surjective for all p, and so inclusion $X_\mu^* \hookrightarrow X_\mu$ induces a surjective map $H_K^*(X_\mu) \to H_K^*(X_\mu^*)$.

When the ambient space is singular, the idea behind the calculation is an extension of the one described above. We still study the long exact sequence (3.1), however the calculation of $H_K^p(X_\mu, X_\mu^*)$ is much more complicated than an application of the Thom isomorphism, and in fact α^p is not always injective for $\mathsf{SU}(2,1)$ Higgs bundles.

In order to compute $H_\mathcal{G}^*(X_\mu, X_\mu^*)$ from (3.1), we first compute the relative cohomology groups $H_\mathcal{G}^*(\nu_\mu^-, \nu_\mu^- \setminus \{0\})$ (where ν_μ^- denotes the negative normal space to the critical set C_μ). The strategy is to compute these groups by a series of excisions and a diagram chase that reduces the problem to computing the cohomology of lower-rank moduli spaces that are explicitly known (see for example the proof of Lemma 3.6). The spaces X_μ (unions of strata) also have an analogous collection of spaces defined using excision, where we construct the excisions using the algebraic characterization of the strata by Harder-Narasimhan type (see for example the proof of Proposition 3.12), and to complete the picture we need to compute the cohomology of these spaces using the explicit results for the negative normal space to each critical set. This is the Morse-Bott isomorphism that is the main result of Section 3.3. Once this process is complete then we can compute $H_\mathcal{G}^*(X_\mu, X_\mu^*)$ and study the analog of (3.1) in our case.

An essential part of the above procedure is the result of Proposition 2.5, which gives us the ability to switch between the algebraic description of the strata (which we use to define the spaces constructed from the strata by excision) and the analytic description (which we use to relate the computations on the strata to those on the critical sets).

This section is divided into four subsections. In the first subsection we describe the negative eigenspace of the Hessian at each critical point. In the second we compute the relevant cohomology groups needed to compute $H^*_{\mathcal{G}}(\nu_\mu^-, \nu_\mu^- \setminus \{0\})$. In the third subsection we prove the isomorphism $H^*_{\mathcal{G}}(X_\mu, X_\mu^*) \cong H^*_{\mathcal{G}}(\nu_\mu^-, \nu_\mu^- \setminus \{0\})$ (in certain cases), and in the final section we show that the modified Harder-Narasimhan stratification defined in the previous section is equivariantly perfect for $\mathsf{U}(2,1)$ Higgs bundles (i.e. our analog of (3.1) splits into short exact sequences).

Finally, it is worth mentioning here that *a priori* we should do our cohomology computations on a small neighborhood of the zero section in the negative normal space ν_μ^-. The proofs of the relevant results (e.g. Proposition 3.12) decompose all of the necessary calculations to calculations where the ambient space is a manifold, which allows us to study the whole space ν_μ^- instead of a neighborhood of the zero section. This observation simplifies some of the definitions and calculations in this section.

3.1. **Indices of critical sets.** First, recall the following result for Higgs vector bundles.

Lemma 3.1. *Let (A, Φ) be a critical point of* YMH *on the space of Higgs bundles. A pair $(\alpha, \varphi) \in \Omega^{0,1}(\operatorname{End} V) \oplus \Omega^{1,0}(\operatorname{End} V)$ is in the negative eigenspace of the Hessian at (A, Φ) if*

(i) *The pair (α, φ) is orthogonal to the $\mathcal{G}^{\mathbb{C}}$-orbit through (A, Φ). Equivalently, $\bar{\partial}_A^* \alpha - \bar{*}[\Phi, \bar{*}\varphi] = 0$.*

(ii) *The pair $(A+\alpha, \Phi+\varphi)$ is a Higgs pair. Equivalently, the following equation is satisfied*

$$\bar{\partial}_A \varphi + [\alpha, \Phi] + [\alpha, \varphi] = 0$$

(iii) *The pair (α, φ) is an eigenvector for the operator $i \operatorname{ad} *(F_A + [\Phi, \Phi^*])$ with negative eigenvalue. Equivalently, the following equations are satisfied*

$$i\,[*(F_A + [\Phi, \Phi^*]), \alpha] = \lambda \alpha$$
$$i\,[*(F_A + [\Phi, \Phi^*]), \varphi] = \lambda \varphi$$

for some $\lambda < 0$. (Note that the eigenvalues are necessarily real since $i * (F_A + [\Phi, \Phi^*])$ is self-adjoint.)

To translate this into a statement for $\mathsf{U}(p,q)$ Higgs bundles $V = E_p \oplus E_q$, we use the following inclusions

$$\Omega^{0,1}(\operatorname{End} E_p) \oplus \Omega^{0,1}(\operatorname{End} E_q) \hookrightarrow \Omega^{0,1}(\operatorname{End} V)$$
$$\Omega^0(E_p^* E_q \otimes K) \oplus \Omega^0(E_q^* E_p \otimes K) \hookrightarrow \Omega^0((\operatorname{End} V) \otimes K)$$

Corollary 3.2. *Let (A_p, A_q, b, c) be a critical point of* YMH *on $\mathcal{B}(E_p, E_q)$. Then*

$$(\alpha_p, \alpha_q, \beta, \gamma) \in \Omega^{0,1}(\operatorname{End} E_p) \oplus \Omega^{0,1}(\operatorname{End} E_q) \oplus \Omega^0(E_p^* E_q \otimes K) \oplus \Omega^0(E_q^* E_p \otimes K)$$

is in the negative eigenspace of the Hessian at (A_p, A_q, b, c) if

(i) $(\alpha_p, \alpha_q, \beta, \gamma)$ *is orthogonal to the* $\mathcal{G}^{\mathbb{C}}$*-orbit through* (A_p, A_q, b, c)*. Equivalently, the following equations are satisfied*

$$\bar{\partial}^*_{A_q}\alpha_q - \bar{*}(b(\bar{*}\beta)) - \bar{*}((\bar{*}\gamma)c) = 0$$
$$\bar{\partial}^*_{A_p}\alpha_p - \bar{*}((\bar{*}\beta)b) - \bar{*}(c(\bar{*}\gamma)) = 0$$

(ii) $(A_p + \alpha_p, A_q + \alpha_q, b + \beta, c + \gamma)$ *is a Higgs pair. Equivalently, the following equations are satisfied*

$$\bar{\partial}_A\beta + (\alpha_q)(b+\beta) + (b+\beta)(\alpha_p) = 0$$
$$\bar{\partial}_A\gamma + (\alpha_p)(c+\gamma) + (c+\gamma)(\alpha_q) = 0$$

where $\bar{\partial}_A$ *denotes the holomorphic structure induced by* $\bar{\partial}_{A_p}$ *and* $\bar{\partial}_{A_q}$ *on both* $E_p^*E_q \otimes K$ *and* $E_q^*E_p \otimes K$*.*

(iii) *The pair* (α, φ) *is an eigenvector for the operator* $i \operatorname{ad} *(F_A + [\Phi, \Phi^*])$ *with negative eigenvalue. Equivalently, the following equations are satisfied*

$$i\left[*(F_{A_q} + bb^* + c^*c), \alpha_q\right] = \lambda\alpha_q$$
$$i\left[*(F_{A_p} + b^*b + cc^*), \alpha_p\right] = \lambda\alpha_p$$
$$i\left(*(F_{A_q} + bb^* + c^*c)\beta - \beta * (F_{A_p} + b^*b + cc^*)\right) = \lambda\beta$$
$$i\left(*(F_{A_p} + b^*b + cc^*)\gamma - \gamma * (F_{A_q} + bb^* + c^*c)\right) = \lambda\gamma$$

for some $\lambda < 0$*.*

Now specialize again to $\mathsf{U}(2,1)$, i.e. $\operatorname{rank} E_i = i$.

(1) $\mathcal{C}_a$. The negative eigenspace ν_a^- of the Hessian consists of holomorphic sections

$$\gamma \in H^0(E_1^*E_2 \otimes K)$$

and the quiver bundle picture is

$$(A) \qquad \begin{tikzcd} \bullet_{E_1} \arrow[r, dashed, "\gamma"] & \bullet_{E_2} \end{tikzcd}$$

(2) The critical points where $E_2 = S \oplus Q$ and the Higgs field is zero have negative eigenspace as follows.

(i) $\mathcal{C}^\ell_{b_3}$. Since $\ell = d_S > d_1$, then the negative eigenspace ν_ℓ^- of the Hessian consists of sections

$$(\alpha, \beta_S, \gamma_Q) \in H^{0,1}(S^*Q) \oplus H^0(S^*E_1 \otimes K) \oplus H^0(E_1^*Q \otimes K).$$

These show up in the quiver bundle picture as dashed arrows in the diagram below

$$(B_3) \qquad \begin{tikzcd} & \bullet_Q \\ \bullet_{E_1} \arrow[ur, dashed, "\gamma_Q"] & \\ & \bullet_S \arrow[uu, dashed, "\alpha"'] \arrow[ul, dashed, "\beta_S"] \end{tikzcd}$$

(ii) $\mathcal{C}^{\ell}_{b_1}$. Since $d_2/2 < \ell = d_S < d_1$, then the negative eigenspace of the Hessian is as follows

$$\nu_\ell^- = \left\{ (\alpha, \gamma_S, \gamma_Q) \in H^{0,1}(S^*Q) \oplus H^0(E_1^*S \otimes K) \oplus \Omega^0(E_1^*Q \otimes K) \,:\, \bar\partial_A \gamma_Q + \alpha\gamma_S = 0 \right\}$$

The quiver bundle diagram is

$$(B_1) \qquad \begin{tikzcd} & & Q \\ E_1 \arrow[urr, dashed, "\gamma_Q"] \arrow[drr, dashed, "\gamma_S"'] & & \\ & & S \arrow[uu, dashed, "\alpha"'] \end{tikzcd}$$

This is the case where the negative eigenspace of the Hessian is a singular space, and not a vector space.

As for the example of stable pairs (see for example [20, Lemma 8.3.12]), the idea is to further decompose the negative eigenspace of the Hessian. We have the equations

$$\bar\partial_{A_1}^* \alpha = 0 \ , \ \bar\partial_A \gamma_Q + \alpha\gamma_S = 0 \ , \ \bar\partial_A \gamma_S = 0. \tag{3.2}$$

Consider the projection from the solutions of (3.2) to the set $\{\gamma_S \,:\, \bar\partial_A \gamma_S = 0\}$. The remaining two equations are linear in (α, γ_Q), and therefore the fibers of this projection are vector spaces. The goal is to compute the dimension of these fibers (which will depend on γ_S).

The case where $\gamma_S = 0$ is easy, since the equations decouple and the space of solutions is $H^{0,1}(S^*Q) \oplus H^{1,0}(E_1^*Q)$. When $\gamma_S \neq 0$ then the equations do not decouple, and we need to consider the following deformation complex (cf. (2.2))

$$\Omega^{0,1}(S^*Q) \oplus \Omega^{1,0}(E_1^*Q) \xrightarrow{\ D\ } \Omega^0(S^*Q) \oplus \Omega^{1,1}(E_1^*Q), \tag{3.3}$$

where $D(\alpha, \gamma_Q) = (\bar\partial_A^* \alpha, \bar\partial \gamma_Q + \alpha\gamma_S)$. The adjoint is $D^*(u, \eta) = \left(\bar\partial_A u - \bar\ast\left(\gamma_S \bar\ast \eta\right), \bar\partial_A^* \eta\right)$. We claim that the kernel of D^* is trivial. To see this, note that $(u, \eta) \in \ker D^*$ implies that $\bar\partial_A^* \eta = 0$, and $\bar\partial_A u - \bar\ast\left(\gamma_S \bar\ast \eta\right) = 0$. The following calculation shows that this second equation decouples

$$\begin{aligned} \left\langle \bar\partial_A u, \bar\ast\left(\gamma_S \bar\ast \eta\right)\right\rangle &= \left\langle u, \bar\partial_A^* \bar\ast\left(\gamma_S \bar\ast \eta\right)\right\rangle = \left\langle u, -\bar\ast \bar\partial_A \bar\ast\bar\ast\left((\gamma_S)\bar\ast \eta\right)\right\rangle \\ &= \left\langle u, \bar\ast \bar\partial_A\left(\gamma_S \bar\ast \eta\right)\right\rangle = \left\langle u, \bar\ast\left(\bar\partial_A \gamma_S \bar\ast \eta - \gamma_S \bar\partial_A \bar\ast \eta\right)\right\rangle = 0 \end{aligned}$$

since $\bar\partial_A \gamma_S = 0$ and $-\bar\ast \bar\partial_A \bar\ast \eta = \bar\partial_A^* \eta = 0$ by assumption.

Therefore $(u, \eta) \in \ker D^*$ implies that $\bar\partial_A u = 0$, $\bar\ast\left(\gamma_S \bar\ast \eta\right) = 0$, and $\bar\partial_A^* \eta = 0$. Since $\deg S^*Q = 0$ then the first equation implies that $u = 0$. Since $\gamma_S \neq 0$ then the second equation implies that $\eta = 0$, and together this shows that the kernel of D^* is trivial.

Therefore we can compute $\dim_{\mathbb{C}} \ker D$ from the index of the complex and Riemann-Roch

$$\begin{aligned}
\dim_{\mathbb{C}} \ker D &= h^{0,1}(S^*Q) - h^0(S^*Q) + h^{1,0}(S^*Q) - h^{1,1}(E_1^*Q) \\
&= g - 1 - \deg(S^*Q) + h^{0,1}(Q^*E_1) - h^0(Q^*E_1) \\
&= g - 1 - \deg(S^*Q) + g - 1 - \deg(Q^*E_1) \\
&= 2g - 2 + d_S - d_1
\end{aligned}$$

We have proven the following

Lemma 3.3. *The projection map from the space of solutions to* (3.2) *to the set* $\{\gamma_S : \bar{\partial}_A \gamma_S = 0\}$ *has linear fibers. The fiber over zero is isomorphic to* $H^{0,1}(S^*Q) \oplus H^{1,0}(E_1^*Q)$*, and the fiber over any nonzero point has dimension* $2g - 2 + d_S - d_1$*.*

(iii) When $\ell = d_1$ then the homomorphism γ_S in the diagram (B_1) no longer corresponds to a negative eigenvalue of the Hessian (the eigenvalue is now zero). Therefore we have

$$\nu_\ell^- = H^{0,1}(S^*Q) \oplus H^0(E_1^*Q \otimes K)$$

and the quiver bundle picture is

$$(B_2) \qquad \begin{tikzcd} & \bullet Q \\ \bullet E_1 \arrow[ur, dashed, "\gamma_Q"] & \\ & \bullet S \arrow[uu, dashed, "\alpha"'] \end{tikzcd}$$

(3) The critical points where $E_2 = S \oplus Q$ and the Higgs field is nonzero have negative eigenspace as follows.

(i) $\mathcal{C}_{c_1}^\ell$. Since $\ell = d_S > \frac{1}{2}(d_Q + d_2)$ then the negative eigenspace η_ℓ^- of the Hessian is

$$\eta_\ell^- = H^{0,1}(S^*Q) \oplus H^0(S^*E_1 \otimes K)$$

The quiver bundle picture is

$$(C_1) \qquad \begin{tikzcd} & \bullet Q \\ \bullet E_1 \arrow[ur, "c_Q"] & \\ & \bullet S \arrow[uu, dashed, "\alpha"'] \arrow[ul, dashed, "\beta_S"] \end{tikzcd}$$

(ii) $\mathcal{C}_{c_2}^\ell$. Since $d_Q < \frac{1}{2}(d_S + d_1)$ then the negative eigenspace ζ_ℓ^- of the Hessian consists of pairs $(\alpha, \gamma_Q) \in \Omega^{0,1}(S^*Q) \oplus \Omega^0(E_1^*Q \otimes K)$ such that

$$\bar{\partial}_{A_1}^* \alpha - \bar{*}(c_S(\bar{*}\gamma_Q)) = 0 \ , \ \bar{\partial}_A \gamma_Q + \alpha c_S = 0$$

The quiver bundle picture is

$$(C_2) \qquad \begin{tikzcd} & \bullet Q \\ \bullet E_1 \arrow[ur, dashed, "\gamma_Q"] \arrow[dr, "c_S"'] & \\ & \bullet S \arrow[uu, dashed, "\alpha"'] \end{tikzcd}$$

Note that these equations are both linear in (α, γ_Q), and that they correspond to the harmonic forms in the middle term of the following deformation complex

$$\Omega^0(S^*Q) \xrightarrow{D_1} \Omega^{0,1}(S^*Q) \oplus \Omega^{1,0}(E_1^*Q) \xrightarrow{D_2} \Omega^{1,1}(E_1^*Q)$$

where the maps D_1 and D_2 are

$$D_1(u) = \left(\bar{\partial}_A u, -uc_S\right) \ , \ D_2(\alpha, \gamma) = \bar{\partial}_A \gamma_Q + \alpha c_S$$

(A calculation shows that $D_2 \circ D_1 = 0$.) The corresponding adjoints are

$$D_1^*(\alpha, \gamma) = \bar{\partial}_{A_1}^* \alpha - \bar{*}(c_S \bar{*} \gamma_Q) \ , \ D_2^*(\eta) = \left(\bar{*}(c_S \bar{*} \eta), \bar{\partial}_A^* \eta\right)$$

If $c_S \neq 0$, then the maps $u \mapsto -uc_S$ and $\eta \mapsto \bar{*}(c_S \bar{*} \eta)$ both have trivial kernel, and hence the dimension of the harmonic forms in the middle term is equal to the index of the complex.

$$\begin{aligned} \dim_{\mathbb{C}} (\ker D_1^* \cap \ker D_2) &= h^{0,1}(L_1^* L_2) - h^0(L_1^* L_2) + h^{1,0}(E_1^* L_2) - h^{1,1}(E_1^* L_2) \\ &= g - 1 + \ell - \ell_2 + h^{0,1}(L_2^* E_1) - h^0(L_2^* E_1) \\ &= g - 1 + \ell - \ell_2 + g - 1 + \ell_2 - d_2 \\ &= 2g - 2 + \ell - d_2 \end{aligned}$$

Therefore the negative eigenspace of the Hessian at these critical points has constant (complex) dimension $2g - 2 + \ell - d_2$.

(iii) $\mathcal{C}_{c_3}^\ell$. Since $d_Q < \frac{1}{2}(d_S + d_2)$ then the negative eigenspace ζ_ℓ^- of the Hessian consists of pairs $(\alpha, \gamma_Q) \in \Omega^{0,1}(S^*Q) \oplus \Omega^0(E_1^*Q \otimes K)$ such that $\bar{\partial}_{A_1}^* \alpha = 0$, $\bar{\partial}_A \gamma_Q = 0$, and so the space of solutions is isomorphic to $H^{0,1}(S^*Q) \oplus H^0(E_1^*Q \otimes K)$. The quiver bundle picture is

$$(C_3) \qquad \begin{tikzcd} & \bullet Q \\ \bullet E_1 \arrow[ur, dashed, "\gamma_Q"] & \\ & \bullet S \arrow[uu, dashed, "\alpha"'] \arrow[ul, "b_S"] \end{tikzcd}$$

3.2. **Cohomology of negative normal directions.** We now compute the relative cohomology groups of the negative normal spaces given in the previous section.

(1) Consider the case of the A-stratum, where $\ell = d_2/2$. Let

$$\begin{aligned} \nu_a^- &= \left\{(E_1, E_2, 0, \gamma) : \gamma \in H^0(E_1^* E_2 \otimes K) \ , \ E_2 \text{ semistable}\right\} \\ \nu_a' &= \left\{(E_1, E_2, 0, \gamma) \in \nu_a^- : \gamma \neq 0\right\} \end{aligned}$$

We also fix

$$\sigma_{min} := 2g - 2 - d_1 + d_2/2 + 1/4 \tag{3.4}$$

The important point is that $\frac{1}{2}\deg(E_1^* E_2 \otimes K) < \sigma_{min} < \lfloor \frac{1}{2}\deg(E_1^* E_2 \otimes K)\rfloor + 1$.

Lemma 3.4. *For the A stratum:*

(i) $H^*_{\mathcal{G}}(\nu_a^-) \simeq H^*_{\mathcal{G}}(\mathcal{A}(E_1) \times \mathcal{A}^{ss}(E_2))$

(ii) *If d_2 is odd, then* $H^*_{\mathcal{G}}(\nu_a') \simeq H^*(\mathcal{N}_{\sigma_{min}}(E_1^* E_2 \otimes K))$

(iii) *If d_2 is even, then*

$$H^*_{\mathcal{G}}(\nu_a') \simeq H^*(\mathcal{N}_{\sigma_{min}}(E_1^* E_2 \otimes K)) \oplus H^{*-2(2g-2-d_1+d_2/2)}_{S^1 \times S^1}(J(X) \times J(X) \times S^{2g-2-d_1+d_2/2} X)$$

Proof. Part (i) follows by the deformation retraction $\gamma \mapsto 0$. Let $E = E_1^* E_2 \otimes K$. Part (ii) follows because $\mathcal{G}(E)$ acts freely with $\nu_a'/\mathcal{G} = \mathcal{N}_{\sigma_{min}}(E)$. Part (iii) is slightly more subtle. A σ_{min}-Bradlow stable pair is a nonvanishing section $\gamma \in H^0(E)$ with the additional assumption, in case E is strictly semistable, that γ does not lie in the maximally destabilizing subbundle. Hence, the space ν_a' is obtained by attaching the first nonminimal stratum to the Bradlow semistable stratum in the σ_{min}-YMH stratification of the space of pairs given in [20, Section 8.2.1]. Then part (iii) follows from the computation in [20, Theorem 8.4.1]. □

(2) $\frac{1}{3}(d_1 + d_2) < \ell \leq d_2 - d_1 + 2g - 2$. This is the case of the C_1-stratum; see the quiver diagram (C_1). Define the following spaces

$$\begin{aligned}
\eta_\ell^- &= \left\{(\alpha, \beta_S) : \bar\partial^*\alpha = 0\ ,\ \bar\partial\beta_S = 0\right\}\\
\eta_\ell' &= \left\{(\alpha, \beta_S) \in \eta_\ell^- : (\alpha, \beta_S) \neq 0\right\}\\
\eta_\ell'' &= \left\{(\alpha, \beta_S) \in \eta_\ell^- : \alpha \neq 0\right\}
\end{aligned}$$

Then by the argument in [7] we have

Lemma 3.5. *For the C_1 stratum,*

$$H^*_{\mathcal{G}}(\eta_\ell^-, \eta_\ell'') = H^{*-2(d_2-2\ell+g-1)}_{S^1 \times S^1}\left(J(X) \times J(X) \times S^{\ell-d_1+2g-2} X\right) \tag{3.5}$$

$$H^*_{\mathcal{G}}(\eta_\ell', \eta_\ell'') = H^{*-2(d_2-2\ell+g-1)}_{S^1}\left(J(X) \times S^{d_2-d_1+2g-2-\ell} X \times S^{d_1-\ell+2g-2} X\right) \tag{3.6}$$

$$H^*_{\mathcal{G}}(\eta_\ell^-, \eta_\ell'') = H^*_{\mathcal{G}}(\eta_\ell^-, \eta_\ell') \oplus H^*_{\mathcal{G}}(\eta_\ell', \eta_\ell'') \tag{3.7}$$

(3) $\frac{1}{3}(2d_2 - d_1) < \ell < d_1$. These are the B_1 and C_2 strata. Consider first the diagram (B_1). Define the following spaces

$$\begin{aligned}
\nu_\ell^- &= \left\{(\alpha, \gamma_S, \gamma_Q) : \bar\partial^*\alpha = 0\ ,\ \bar\partial\gamma_Q + \alpha\gamma_S = 0\ ,\ \bar\partial\gamma_S = 0\right\}\\
\nu_\ell' &= \left\{(\alpha, \gamma_S, \gamma_Q) \in \nu_\ell^- : (\alpha, \gamma_S, \gamma_Q) \neq 0\right\}\\
\nu_\ell'' &= \left\{(\alpha, \gamma_S, \gamma_Q) \in \nu_\ell^- : \alpha \neq 0\right\}\\
\omega_\ell &= \left\{(\alpha, \gamma_S, \gamma_Q) \in \nu_\ell^-\ :\ (\alpha, \gamma_Q) \neq 0\right\}
\end{aligned}$$

Lemma 3.6. *For the B_1 stratum, $\ell \leq d_2 - d_1 + 2g - 2$,*

$$H^*_{\mathcal{G}}(\nu_\ell^-, \nu_\ell'') = H^{*-2(2\ell-d_2+g-1)}_{S^1\times S^1\times S^1}\left(J(X)\times J(X)\times J(X)\right) \tag{3.8}$$

$$H^*_{\mathcal{G}}(\nu_\ell', \nu_\ell'') = H^*_{\mathcal{G}}(\nu_\ell', \omega_\ell) \oplus H^*_{\mathcal{G}}(\omega_\ell, \nu_\ell'') \tag{3.9}$$

$$H^*_{\mathcal{G}}(\nu_\ell', \omega_\ell) = H^{*-2(\ell-d_1+2g-2)}_{S^1\times S^1}\left(J(X)\times J(X)\times S^{\ell-d_1+2g-2}X\right) \tag{3.10}$$

$$H^*_{\mathcal{G}}(\omega_\ell, \nu_\ell'') = H^{*-2(2\ell-d_2+g-1)}_{S^1\times S^1}\left(J(X)\times J(X)\times S^{d_2-d_1+2g-2-\ell}X\right) \tag{3.11}$$

If $d_2 - d_1 + 2g - 2 < \ell < d_1$, then (3.8) *holds, with*

$$H^*_{\mathcal{G}}(\nu_\ell^-, \nu_\ell'') = H^*_{\mathcal{G}}(\nu_\ell^-, \nu_\ell') \oplus H^*_{\mathcal{G}}(\nu_\ell', \nu_\ell'') \tag{3.12}$$

$$H^*_{\mathcal{G}}(\nu_\ell', \nu_\ell'') = H^{*-2(\ell-d_1+2g-2)}_{S^1\times S^1}\left(J(X)\times J(X)\times S^{\ell-d_1+2g-2}X\right) \tag{3.13}$$

Proof. Notice that (3.8) follows by retracting $(\gamma_S, \gamma_Q) \mapsto 0$ and using the Atiyah-Bott argument. Consider the following commutative diagram.

$$\begin{tikzcd}
 & & \vdots \arrow[d] & & & \\
 & \cdots \arrow[r] & H^p_{\mathcal{G}}(\nu_\ell^-, \nu_\ell') \arrow[r] \arrow[d] & H^p_{\mathcal{G}}(\nu_\ell^-) \arrow[r] & H^p_{\mathcal{G}}(\nu_\ell') \arrow[r] & \cdots \\
 & & H^p_{\mathcal{G}}(\nu_\ell^-, \nu_\ell'') \arrow[ur, "\xi"] \arrow[dr, "\xi''"] \arrow[d] & & & \\
\cdots \arrow[r] & H^p_{\mathcal{G}}(\nu_\ell', \omega_\ell) \arrow[r] & H^p_{\mathcal{G}}(\nu_\ell', \nu_\ell'') \arrow[r, "\beta"] \arrow[d] & H^p_{\mathcal{G}}(\omega_\ell, \nu_\ell'') \arrow[r] & \cdots & \\
 & & \vdots & & &
\end{tikzcd} \tag{3.14}$$

By the argument in [7] and assuming (3.11), the map ξ'' is surjective. It follows that the lower horizontal exact sequence splits. Thus, (3.9) follows from (3.11). Define the following spaces

$$\begin{array}{lcl lcl}
W_\ell & = & \left\{(\alpha,\gamma_S,\gamma_Q)\in\nu_\ell^- : \gamma_S = 0\right\} & W_\ell' & = & \{(\alpha,\gamma_S,\gamma_Q)\in\nu_\ell : \gamma_S = 0, (\alpha,\gamma_Q)\neq 0\} \\
W_\ell'' & = & \left\{(\alpha,\gamma_S,\gamma_Q)\in\nu_\ell^- : \gamma_S = 0, \alpha\neq 0\right\} & Z_\ell & = & \left\{(\alpha,\gamma_S,\gamma_Q)\in\nu_\ell^- : (\gamma_S,\gamma_Q) = 0, \alpha\neq 0\right\} \\
R_\ell & = & \left\{(\alpha,\gamma_S,\gamma_Q)\in\nu_\ell^- : \gamma_Q\neq 0, (\alpha,\gamma_S) = 0\right\} & Y_\ell'' & = & \left\{(\alpha,\gamma_S,\gamma_Q)\in\nu_\ell^- : \gamma_S\neq 0, (\alpha,\gamma_Q)\neq 0\right\} \\
T_\ell & = & \left\{(\alpha,\gamma_S,\gamma_Q)\in\nu_\ell^- : \gamma_S\neq 0, (\alpha,\gamma_Q) = 0\right\} & Y_\ell' & = & \left\{(\alpha,\gamma_S,\gamma_Q)\in\nu_\ell^- : \gamma_S\neq 0\right\} \\
Y_\ell'' & = & \left\{(\alpha,\gamma_S,\gamma_Q)\in\nu_\ell^- : \gamma_S\neq 0, (\alpha,\gamma_Q)\neq 0\right\} & T_\ell & = & \left\{(\alpha,\gamma_S,\gamma_Q)\in\nu_\ell^- : \gamma_S\neq 0, (\alpha,\gamma_Q) = 0\right\}
\end{array}$$

Note that $Y_\ell' = \nu_\ell^- \setminus W_\ell = \nu_\ell' \setminus W_\ell'$ and $Y_\ell'' = \omega_\ell \setminus W_\ell'$. By the retraction $\gamma_S \mapsto 0$, the pair $(\omega_\ell, \nu_\ell'') \simeq (W_\ell', W_\ell'')$. By excision,

$$H^*_{\mathcal{G}}(W_\ell', W_\ell'') \simeq H^*_{\mathcal{G}}(W_\ell'\setminus Z_\ell, W_\ell''\setminus Z_\ell)$$

Now $W_\ell' \setminus Z_\ell$ fibers over R_ℓ with fiber dimension $d_S - d_Q + g - 1$. Hence, (3.11) follows from the Thom isomorphism. Finally, for (3.10) we need the following lemma, whose proof is straightforward.

Lemma 3.7. *For fixed $\gamma_S \neq 0$, the space of solutions (α, γ_Q) to $\bar\partial^*\alpha = 0$, $\bar\partial\gamma_Q + \alpha\gamma_S = 0$, has dimension $= \ell - d_1 + 2g - 2$.*

Excision of W'_ℓ gives $H^*_{\mathcal{G}}(\nu'_\ell, \omega_\ell) \simeq H^*_{\mathcal{G}}(\nu'_\ell \setminus W'_\ell, \omega_\ell \setminus W'_\ell) = H^*_{\mathcal{G}}(Y'_\ell, Y''_\ell)$. Now by the lemma, Y'_ℓ fibers over T_ℓ with fiber dimension $\ell - d_1 + 2g - 2$, and (3.10) again follows from Thom isomorphism. In case $d_2 - d_1 + 2g - 2 < \ell < d_1$, then notice that W'_ℓ is closed in ν''_ℓ. Hence, (3.13) follows by Lemma 3.7 and excision. Eq. (3.12) follows by the argument in [7]. □

For C_2, the normal directions are given by

$$\zeta^-_\ell = \{(\alpha, \gamma_Q) : \bar\partial^*\alpha = 0\ ,\ \bar\partial\gamma_Q + \alpha c_S = 0\}$$
$$\zeta'_\ell = \{(\alpha, \gamma_Q) \in \zeta^-_\ell : (\alpha, \gamma_Q) \neq 0\}$$

where $c_S \neq 0$. It follows from Lemma 3.7 that

$$H^*_{\mathcal{G}}(\zeta^-_\ell, \zeta'_\ell) \simeq H^{*-2(\ell-d_1+2g-2)}_{S^1\times S^1}(J(X) \times J(X) \times S^{\ell-d_1+2g-2}X) \tag{3.15}$$

Remark 3.8. For the B_1 and C_2 strata, $\ell \leq d_2 - d_1 + 2g - 2$, $H^*_{\mathcal{G}}(\nu'_\ell, \omega_\ell) \simeq H^*_{\mathcal{G}}(\zeta^-_\ell, \zeta'_\ell)$. In case $d_2 - d_1 + 2g - 2 < \ell < d_1$, then $H^*_{\mathcal{G}}(\nu'_\ell, \nu''_\ell) \simeq H^*_{\mathcal{G}}(\zeta^-_\ell, \zeta'_\ell)$.

(4) $d_1 \leq \ell$. These are the B_2, B_3, and C_3 strata. Consider first the the (C_3) diagram. Define the following spaces

$$\zeta^-_\ell = \{(\alpha, \gamma_Q) : \bar\partial^*\alpha = 0\ ,\ \bar\partial\gamma_Q = 0\}$$
$$\zeta'_\ell = \{(\alpha, \gamma_Q) \in \zeta^-_\ell : (\alpha, \gamma_Q) \neq 0\}$$
$$\zeta''_\ell = \{(\alpha, \gamma_Q) \in \zeta^-_\ell : \alpha \neq 0\}$$

Then by the argument in [7] we have

Lemma 3.9. *For the C_3 stratum, if $\ell \leq d_2 - d_1 + 2g - 2$ then*

$$H^*_{\mathcal{G}}(\zeta^-_\ell, \zeta''_\ell) = H^{*-2(2\ell-d_2+g-1)}_{S^1\times S^1}\left(J(X) \times J(X) \times S^{d_1-\ell+2g-2}X\right) \tag{3.16}$$

$$H^*_{\mathcal{G}}(\zeta'_\ell, \zeta''_\ell) = H^{*-2(2\ell-d_2+g-1)}_{S^1}\left(J(X) \times S^{d_2-d_1-\ell+2g-2}X \times S^{d_1-\ell+2g-2}X\right) \tag{3.17}$$

$$H^*_{\mathcal{G}}(\zeta^-_\ell, \zeta''_\ell) = H^*_{\mathcal{G}}(\zeta^-_\ell, \zeta'_\ell) \oplus H^*_{\mathcal{G}}(\zeta'_\ell, \zeta''_\ell) \tag{3.18}$$

If $d_2 - d_1 + 2g - 2 < \ell \leq d_1 + 2g - 2$ then

$$H^*_{\mathcal{G}}(\zeta^-_\ell, \zeta'_\ell) \simeq H^{*-2(2\ell-d_2+2g-2)}_{S^1\times S^1}(J(X) \times J(X) \times S^{d_1-\ell+2g-2}X) \tag{3.19}$$

The B_2 case is exactly the same as the C_3 case. We define the spaces $\nu^-_{d_1}$, ν'_{d_1}, and ν''_{d_1} by analogy to ζ^-_ℓ, ζ'_ℓ, and ζ''_ℓ above.

Lemma 3.10. *For the B_2 stratum, if $d_1 \leq d_2 - d_1 + 2g - 2$, then*

$$H^*_{\mathcal{G}}(\nu^-_{d_1}, \nu''_{d_1}) = H^{*-2(2d_1-d_2+g-1)}_{S^1\times S^1}(J(X) \times J(X) \times J(X)) \tag{3.20}$$

$$H^*_{\mathcal{G}}(\nu'_{d_1}, \nu''_{d_1}) = H^{*-2(2d_1-d_2+g-1)}_{S^1}\left(J(X) \times J(X) \times S^{d_2-2d_1+2g-2}X\right) \tag{3.21}$$

$$H^*_{\mathcal{G}}(\nu^-_{d_1}, \nu''_{d_1}) = H^*_{\mathcal{G}}(\nu^-_{d_1}, \nu'_{d_1}) \oplus H^*_{\mathcal{G}}(\nu'_{d_1}, \nu''_{d_1}) \tag{3.22}$$

If $d_2 - d_1 + 2g - 2 < d_1$*, then*

$$H^*_{\mathcal{G}}(\nu^-_{d_1}, \nu'_{d_1}) \simeq H^{*-2(2d_1-d_2+d-1)}_{S^1\times S^1\times S^1}(J(X)\times J(X)\times J(X)) \tag{3.23}$$

Finally, consider the (B_3) diagram. There are three cases. First, if $d_1 < \ell \le d_2 - d_1 + 2g - 2$, define the following spaces

$$\begin{aligned}
\nu^-_\ell &= \left\{(\alpha, \beta_S, \gamma_Q) : \bar\partial^*\alpha = 0\ ,\ \bar\partial\beta_S = 0\ ,\ \bar\partial\gamma_Q = 0\right\} \\
\nu'_\ell &= \left\{(\alpha, \beta_S, \gamma_Q) \in \nu^-_\ell : (\alpha, \beta_S, \gamma_Q) \neq 0\right\} \\
\nu''_\ell &= \left\{(\alpha, \beta_S, \gamma_Q) \in \nu^-_\ell : \alpha \neq 0\right\} \\
\omega_\ell &= \left\{(\alpha, \beta_S, \gamma_Q) \in \nu^-_\ell\ :\ (a, \gamma_Q) \neq 0\right\}
\end{aligned}$$

Lemma 3.11. *For the* B_3 *stratum,* $d_1 < \ell \le d_2 - d_1 + 2g - 2$*,*

$$H^*_{\mathcal{G}}(\nu^-_\ell, \nu''_\ell) = H^{*-2(2\ell-d_2+g-1)}_{S^1\times S^1\times S^1}\,(J(X)\times J(X)\times J(X)) \tag{3.24}$$

$$H^*_{\mathcal{G}}(\nu'_\ell, \nu''_\ell) = H^*_{\mathcal{G}}(\nu'_\ell, \omega_\ell) \oplus H^*_{\mathcal{G}}(\omega_\ell, \nu''_\ell) \tag{3.25}$$

$$H^*_{\mathcal{G}}(\nu'_\ell, \omega_\ell) = H^*_{\mathcal{G}}(\zeta^-_\ell, \zeta'_\ell) \tag{3.26}$$

$$H^*_{\mathcal{G}}(\omega_\ell, \nu''_\ell) = H^{*-2(2\ell-d_2+g-1)}_{S^1\times S^1}\left(J(X)\times J(X)\times S^{d_2-d_1-\ell+2g-2}X\right) \tag{3.27}$$

Proof. (3.24) follows as before, and (3.25) follows from (3.27). For (3.26), use excision on the set $\{\beta_S = 0\}$. Finally, for (3.27), first retract $\beta_S \mapsto 0$ and then excise the set $\{\gamma_S = 0\}$. The rest fibers over $\{\gamma_S \neq 0\}$, and the result follows from the Thom isomorphism. □

In case $d_2 - d_1 + 2g - 2 < \ell \le d_1 + 2g - 2$, then $\gamma_Q \equiv 0$. Eq. (3.24) holds as before, but now

$$\begin{aligned}
H^*_{\mathcal{G}}(\nu'_\ell, \nu''_\ell) &= H^*_{\mathcal{G}}(\zeta^-_\ell, \zeta'_\ell) \\
H^*_{\mathcal{G}}(\nu^-_\ell, \nu''_\ell) &= H^*_{\mathcal{G}}(\nu^-_\ell, \nu'_\ell) \oplus H^*_{\mathcal{G}}(\nu'_\ell, \nu''_\ell)
\end{aligned} \tag{3.28}$$

If $d_1 + 2g - 2 < \ell$, then both $\beta, \gamma \equiv 0$, and by the Atiyah-Bott isomorphism

$$H^*_{\mathcal{G}}(\nu^-_\ell, \nu'_\ell) \simeq H^{*-2(2\ell-d_2+g-1)}_{S^1\times S^1\times S^1}(J(X)\times J(X)\times J(X)) \tag{3.29}$$

3.3. **The Morse-Bott Lemma.** The goal of this section is prove the validity of the Morse-Bott isomorphism, which relates the equivariant cohomology of the pair of successive strata to the equivariant cohomology of the pair consisting of negative normal directions and nonzero negative normal directions. Because of singularities, Bott's argument in [2] does not apply, and as in [7] and [20] we need to circumvent this. In fact, we do not prove the Morse-Bott lemma for all critical sets. Nevertheless, the results below are sufficient for the cohomological calculations in the next section.

We begin with particular regions of the parameter $\ell \in \Delta_{d_1,d_2}$ using the definition on page 14.

Proposition 3.12. *For regions* (**II**) *and* (**III**)*,*

$$H^*_{\mathcal{G}}(X^*_{d_2/2} \cup \mathcal{S}_a, X^*_{d_2/2}) \simeq H^*_{\mathcal{G}}(\nu^-_a, \nu'_a) \tag{3.30}$$

$$H^*_{\mathcal{G}}(X_\ell, X'_\ell) \simeq H^*_{\mathcal{G}}(\nu^-_\ell, \nu'_\ell) \tag{3.31}$$

$$H^*_{\mathcal{G}}(X'_\ell, X^*_\ell) \simeq H^*_{\mathcal{G}}(\zeta^-_\ell, \zeta'_\ell) \qquad (\ell \le d_1 + 2g - 2) \tag{3.32}$$

Moreover, in these regions the inclusions $X'_\ell \subset X_\ell$ and $X^*_\ell \subset X'_\ell$ induce surjections $H^*_{\mathcal{G}}(X_\ell) \longrightarrow H^*_{\mathcal{G}}(X'_\ell)$ and $H^*_{\mathcal{G}}(X'_\ell) \longrightarrow H^*_{\mathcal{G}}(X^*_\ell)$.

We will need the following result. Consider $\mathsf{U}(2,1)$ bundles where $b \equiv 0$, i.e. quiver bundles of the form

$$\bullet_{E_1} \xrightarrow{c} \bullet_{E_2} \tag{3.33}$$

The data is clearly equivalent to a choice of holomorphic section (also denoted c) of the bundle $E_1^* E_2 \otimes K$. We have the following

Lemma 3.13. *For quivers of the type above, Higgs (semi)stability of $(E_2 \oplus E_1, 0, c)$ is equivalent to Bradlow (semi)stability of the pair $(E_1^* E_2 \otimes K, c)$ for $\sigma = \sigma(d_1, d_2)$ as defined in* (1.2).

Proof. Set $\Phi = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$ and $E = E_1^* E_2 \otimes K$. Any line subbundle $S \subset E_2$ is automatically Φ-invariant, so Higgs semistability implies $d_S \leq \frac{1}{3}(d_1 + d_2)$. If moreover $c(E_1) \subset S \otimes K$, then Higgs semistability implies $\frac{1}{2}(d_S + d_1) \leq \frac{1}{3}(d_1 + d_2)$. On the other hand, $S \subset E_2$ gives a line subbundle $E_1^* S \otimes K \subset E$. Then σ-semistability implies $\deg(E_1^* S \otimes K) \leq \sigma$, or $d_S - d_1 + 2g - 2 \leq \sigma$. If $c(E_1) \subset S \otimes K$, then the corresponding section of E lies in $E_1^* S \otimes K \subset E$, so σ-semistability implies

$$\sigma \leq \deg E - \deg(E_1^* S \otimes K \subset E) \leq d_2 - d_1 - d_S + 2g - 2$$

Now for the given choice $\sigma = \sigma(d_1, d_2)$ as in (1.2), the conditions for Higgs and σ-semistability are equivalent. □

Proof of Proposition 3.12. For the C_2 stratum, the C_3 stratum $d_2 - d_1 + 2g - 2 < \ell < d_1 + 2g - 2$, the B_3 stratum $d_1 + 2g - 2 < \ell$, or the B_2 stratum when $d_2 - d_1 + 2g - 2 < d_1$, the negative normal directions are vector bundles. The result then follows from a standard argument. To prove (3.31) for the portion of the B_1 stratum where $d_2/2 < \ell \leq \frac{1}{3}(d_1 + d_2)$ (or $d_2 - d_1 + 2g - 2 < \ell < d_1$), define the map $\mathrm{pr} : \mathcal{B}(d_1, d_2) \to \mathcal{A}(E_2)$ by projection to the holomorphic structure on E_2. Let

$$K_\ell = \bigcup_{j > \ell} X_\ell \cap \mathrm{pr}^{-1}(\mathcal{A}_j(E_2))$$

where for a rank 2 bundle $E_2 \to X$ of degree d and $j > d/2$, we let $\mathcal{A}_j(E) \subset \mathcal{A}(E)$ be the subset of unstable holomorphic structures on E_2 whose maximally destabilizing line subbundle has degree j. Then $K_\ell \subset X'_\ell$ is closed in X_ℓ. Hence, by excision,

$$H^*_{\mathcal{G}}(X_\ell, X'_\ell) \simeq H^*_{\mathcal{G}}(X_\ell \setminus K_\ell, X'_\ell \setminus K_\ell)$$

Moreover, the pair $(X_\ell \setminus K_\ell, X'_\ell \setminus K_\ell)$ is invariant under the scaling $b \mapsto 0$. The same is true of the pair (ν^-_ℓ, ν'_ℓ). Eq.'s (3.30) and (3.31) therefore reduce to the corresponding result for pairs (3.33), and hence they follow from Lemma 3.13 and [20, eq. (8.28) and Sect. 8.3.6].

It remains to prove (3.31) for the portion of the B_3 stratum where $\max\{d_1, d_2 - d_1 + 2g - 2\} < \ell \leq d_1 + 2g - 2$. For all integers $\ell > d_2/2$, let $X''_\ell = X_\ell \setminus \mathrm{pr}^{-1}(\mathcal{A}_\ell(E_2))$. Then it follows as in [7, eq.

(21)] that

$$H^*_{\mathcal{G}}(X_\ell, X''_\ell) \simeq H^*_{\mathcal{G}}(\nu^-_\ell, \nu''_\ell) \tag{3.34}$$

and by the Atiyah-Bott lemma, $H^*_{\mathcal{G}}(X_\ell, X''_\ell) \to H^*_{\mathcal{G}}(X_\ell)$ is injective. We claim that for $k > d_2 - d_1 + 2g - 2$, $X''_\ell = X^*_\ell$. Indeed, it suffices to show that if $(E_2 \oplus E_1, b, c)$ is semistable, then the Harder-Narasimhan type of E_2 is at most $d_2 - d_1 + 2g - 2$. Suppose not and let $0 \to S \to E_2 \to Q \to 0$ be the Harder-Narasimhan filtration with $\deg S = \ell$. Then if $\ell > d_2 - d_1 + 2g - 2$, the induced map $c : E_1 \to Q$ vanishes and $S \oplus E_1$ is Φ-invariant. Hence,

$$\tfrac{1}{2}(\ell + d_1) \leq \tfrac{1}{3}(d_1 + d_2) \implies \tfrac{1}{2}(d_2 + 2g - 2) < \tfrac{1}{3}(d_1 + d_2) \implies 2g - 2 < \tfrac{1}{3}(2d_1 - d_2) \leq g - 1$$

where the last inequality comes from the bound on the Toledo invariant. This contradicts the assumption on the genus, and the claim follows. Now the proof of (3.31) follows from the fact that $H^*_{\mathcal{G}}(X'_\ell, X^*_\ell) \simeq H^*_{\mathcal{G}}(\nu'_\ell, \nu''_\ell)$ by (3.28), and the Five Lemma applied to the long exact sequence of the triple $(X_\ell, X'_\ell, X^*_\ell)$. □

Corollary 3.14. *For the B_1 stratum in the portion of region* (**II**) *where $d_2/2 < \ell \leq \frac{1}{3}(d_1 + d_2)$, $H^*_{\mathcal{G}}(X_\ell, X^*_\ell) \simeq \ker \xi''$. If $d_2 - d_1 + 2g - 2 < \ell < d_1$, $H^*_{\mathcal{G}}(X_\ell, X^*_\ell) \simeq H^*_{\mathcal{G}}(\nu^-_\ell, \nu''_\ell)$.*

Proof. By the exact sequence of the triple $(X_\ell, X'_\ell, X^*_\ell)$, Remark 3.8, and Proposition 3.12,

$$\begin{tikzcd}
\cdots \arrow[r] & H^*_{\mathcal{G}}(X_\ell, X'_\ell) \arrow[r] \arrow[d, "\wr"] & H^*_{\mathcal{G}}(X_\ell, X^*_\ell) \arrow[r] \arrow[d] & H^*_{\mathcal{G}}(X'_\ell, X^*_\ell) \arrow[r] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r] & H^*_{\mathcal{G}}(\nu^-_\ell, \nu'_\ell) \arrow[r] & \ker \xi'' \arrow[r] & H^*_{\mathcal{G}}(\nu'_\ell, \omega_\ell) \arrow[r] & \cdots
\end{tikzcd}$$

The first statement follows from the Five Lemma. The proof of the second statement is similar. □

Now consider the region (**I**), which involves the C_1 stratum. We have the following

Lemma 3.15. *For all $\frac{1}{3}(d_1 + d_2) < \ell \leq d_2 - d_1 + 2g - 2$, $H^*_{\mathcal{G}}(X^*_\ell, X''_\ell) \simeq H^*_{\mathcal{G}}(\eta'_\ell, \eta''_\ell)$.*

Proof. The argument is similar to the one in [7, Section 3.1]. Note that the set $(X^*_\ell \setminus \mathcal{B}^{ss}(d_1, d_2)) \subset X''_\ell$ is closed in X^*_ℓ. Hence, by excision

$$H^*_{\mathcal{G}}(X^*_\ell, X''_\ell) \simeq H^*_{\mathcal{G}}(\mathcal{B}^{ss}(d_1, d_2), \mathcal{B}^{ss}(d_1, d_2) \setminus \mathrm{pr}^{-1}(\mathcal{A}_\ell(E_2)))$$

By [21], the YMH flow defines a $\mathcal{G}$-equivariant deformation retraction of the pair

$$(\mathcal{B}^{ss}(d_1, d_2), \mathcal{B}^{ss}(d_1, d_2) \setminus \mathrm{pr}^{-1}(\mathcal{A}_\ell(E_2)))$$

with $(\mathcal{B}_{min}(d_1, d_2), \mathcal{B}_{min}(d_1, d_2) \setminus \mathrm{pr}^{-1}(\mathcal{A}_\ell(E_2)))$. Note that $\mathcal{B}_{min}(d_1, d_2) \cap \mathrm{pr}^{-1}(\mathcal{A}_\ell(E_2))$ lies in the smooth locus on which $\mathcal{G}$ acts freely. Excision then reduces the computation to Gothen's calculation in [11]. □

By (3.34), Lemma 3.15, and (3.6), and the argument in [7], we have

Corollary 3.16. *For all $\frac{1}{3}(d_1 + d_2) < \ell \leq d_2 - d_1 + 2g - 2$, the map $H^*_{\mathcal{G}}(X_\ell, X''_\ell) \to H^*_{\mathcal{G}}(X^*_\ell, X''_\ell)$ is surjective.*

3.4. **Proof of Theorem 2.6.**

Lemma 3.17. *The map* $H^*_{\mathcal{G}}(X^*_{d_2/2}\cup \mathcal{S}_a, X^*_{d_2/2}) \to H^*_{\mathcal{G}}(X^*_{d_2/2})$ *is injective.*

Proof. By Proposition 3.12, it suffices to show that $H^*_{\mathcal{G}}(\nu_a^-, \nu_a') \to H^*_{\mathcal{G}}(\nu_a^-)$ is injective, or equivalently, that $H^*_{\mathcal{G}}(\nu_a^-) \to H^*_{\mathcal{G}}(\nu_a')$ is surjective. Consider the following commutative diagram:

$$\begin{tikzcd} H^*(B\mathcal{G}) \arrow[d] \arrow[dr, "\pi^*"] & \\ H^*_{\mathcal{G}}(\nu_a^-) \arrow[r, "j"] & H^*_{\mathcal{G}}(\nu_a') \end{tikzcd}$$

By Lemma 3.4 and [20], π^* is surjective. Therefore j is surjective as well. □

Next, we need the following lemma.

Lemma 3.18. *Let (A, B, C) be a triple of topological spaces with inclusions $C \hookrightarrow B \hookrightarrow A$ and suppose that the map $H^*(A, C) \to H^*(A)$ is injective. Then $P_t(A) - P_t(B) = P_t(A, C) - P_t(B, C)$. Moreover, if we suppose in addition that the inclusion of pairs $(B, C) \hookrightarrow (A, C)$ induces a surjection $H^*(A, C) \to H^*(B, C)$ in cohomology, then the map $H^*(A) \to H^*(B)$ is a surjection.*

Remark 3.19. If the inclusions $C \hookrightarrow B \hookrightarrow A$ are inclusions of G-spaces, then the above result is also true in G-equivariant cohomology.

Proof. We have the following commutative diagram of exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & H^*(A,C) \arrow[r] \arrow[d] & H^*(A) \arrow[r] \arrow[d] & H^*(C) \arrow[r] \arrow[d, "\wr"] & \cdots \\ \cdots \arrow[r] & H^*(B,C) \arrow[r] & H^*(B) \arrow[r] & H^*(C) \arrow[r] & \cdots \end{tikzcd} \tag{3.35}$$

The assumption implies that the top horizontal sequence splits, and therefore the bottom horizontal sequence also splits. The result follows immediately. □

Proof of Theorem 2.6. By Lemma 3.17 and Proposition 3.12, it suffices to consider region (**I**). By the argument in [7, Sect. 3.1], the Atiyah-Bott lemma implies that $H^*_{\mathcal{G}}(X_k, X_k'') \to H^*_{\mathcal{G}}(X_k)$ is injective. By Corollary 3.16, we may then apply Lemma 3.18 to the triple (X_k, X_k^*, X_k'') and conclude that $H^*_{\mathcal{G}}(X_k) \to H^*_{\mathcal{G}}(X_k^*)$ surjects. This completes the proof. We also record that in this case

$$P_t^{\mathcal{G}}(X_\ell) - P_t^{\mathcal{G}}(X_\ell^*) = P_t^{\mathcal{G}}(X_\ell, X_\ell'') - P_t^{\mathcal{G}}(X_\ell^*, X_\ell'') \tag{3.36}$$

□

4. The Equivariant Betti Numbers

4.1. $\mathsf{U}(2,1)$ **bundles.** The calculations in the previous sections lead to the following formula for the equivariant Poincaré polynomial of $\mathcal{B}(d_1, d_2)$. The contributions of individual strata are as follows.

(i) For the A-stratum, use Lemmas 3.4 and 3.17 to conclude

$$\begin{aligned} P_t^{\mathcal{G}}(X^*_{d_2/2} \cup \mathcal{S}_a) - P_t^{\mathcal{G}}(X^*_{d_2/2}) &= \frac{1}{(1-t^2)^2} P_t(J(X)) P_t^{\mathcal{G}}(\mathcal{A}^{ss}(E_2)) \\ &\quad - \frac{1}{(1-t^2)} P_t(\mathcal{N}_{\sigma_{min}}(E_1^* E_2 \otimes K)) P_t(J_{d_1}(X)) \\ &\quad - \begin{cases} 0 & \text{if } d_2 \text{ odd} \\ \dfrac{t^{2(2g-2+d_2/2-d_1)}}{(1-t^2)} P_t(J(X))^2 P_t(S^{2g-2+d_2/2-d_1} X) & \text{if } d_2 \text{ even} \end{cases} \end{aligned}$$

(ii) For $\frac{1}{3}(2d_2 - d_1) < \ell \leq d_2/2$, (3.32) and (3.15) imply

$$P_t^{\mathcal{G}}(X'_\ell) - P_t^{\mathcal{G}}(X^*_\ell) = \frac{t^{4g-4+2\ell-2d_1}}{(1-t^2)^2} P_t(J(X))^2 P_t(S^{\ell-d_1+2g-2} X)$$

(iii) For $d_2/2 < \ell \leq \frac{1}{3}(d_1+d_2)$, Lemma 3.6 and Corollary 3.14 imply (recall that ξ'' is surjective)

$$P_t^{\mathcal{G}}(X_\ell) - P_t^{\mathcal{G}}(X^*_\ell) = \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^3} P_t(J(X))^3 - \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^2} P_t(J(X))^2 P_t(S^{d_2-d_1+2g-2-\ell} X)$$

(iv) For $\frac{1}{3}(d_1 + d_2) < \ell \leq d_2 - d_1 + 2g - 2$, it follows from (3.36), Lemma 3.15, and (3.6) that

$$\begin{aligned} P_t^{\mathcal{G}}(X_\ell) - P_t^{\mathcal{G}}(X^*_\ell) &= \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^3} P_t(J(X))^3 \\ &\quad - \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)} P_t(J(X)) P_t(S^{d_2-d_1+2g-2-\ell} X) P_t(S^{2g-2-\ell+d_1} X) \end{aligned}$$

(v) For $\max\{d_1, d_2 - d_1 + 2g - 2\} < \ell \leq d_1 + 2g - 2$, it follows from Proposition 3.12 and eq.'s (3.19), (3.28), and (3.20) that

$$P_t^{\mathcal{G}}(X'_\ell) - P_t^{\mathcal{G}}(X^*_\ell) = \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^2} P_t(J(X))^2 P_t(S^{d_1-\ell+2g-2} X)$$

$$P_t^{\mathcal{G}}(X_\ell) - P_t^{\mathcal{G}}(X'_\ell) = \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^3} P_t(J(X))^3 - \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^2} P_t(J(X))^2 P_t(S^{d_1-\ell+2g-2} X)$$

(vi) For $d_1 + 2g - 2 < \ell$, or if $d_2 - d_1 + 2g - 2 < \ell \leq d_1$, it follows from Proposition 3.12 and (3.29), from (3.12) and Remark 3.8, or from (3.23), that

$$P_t^{\mathcal{G}}(X_\ell) - P_t^{\mathcal{G}}(X^*_\ell) = \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^3} P_t(J(X))^3$$

Applying Theorem 2.6, we compute

$$P_t(B\mathcal{G}) - P_t^{\mathcal{G}}(\mathcal{B}^{ss}(d_1, d_2)) = \sum_{\ell \in \Delta_{d_1,d_2}} P_t^{\mathcal{G}}(X_\ell) - P_t^{\mathcal{G}}(X^*_\ell)$$

Notice that the last term in (i), which occurs only when d_2 is even, is exactly canceled by one of the terms in (ii). Combining the remaining terms, we obtain

Proposition 4.1. *The $\mathcal{G}$-equivariant Poincaré polynomial of $\mathcal{B}^{ss}(d_1, d_2)$ is given by*

$$\begin{aligned} P_t^{\mathcal{G}}(\mathcal{B}^{ss}(d_1,d_2)) &= P_t(B\mathcal{G}) - \frac{1}{(1-t^2)^2}P_t(J(X))P_t^{\mathcal{G}}(\mathcal{A}^{ss}(E_2)) - \sum_{d_2/2<\ell}\frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^3}P_t(J(X))^3 \\ &+ \frac{1}{(1-t^2)}P_t(\mathcal{N}_{\sigma_{min}}(E_1^*E_2\otimes K))P_t(J_{d_1}(X)) \\ &+ \sum_{d_2/2<\ell\leq\frac{1}{3}(d_1+d_2)}\frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^2}P_t(J(X))^2P_t(S^{d_2-\ell-d_1+2g-2}X) \\ &- \sum_{\frac{1}{3}(2d_2-d_1)<\ell<d_2/2}\frac{t^{4g-4+2\ell-2d_1}}{(1-t^2)^2}P_t(J(X))^2P_t(S^{\ell-d_1+2g-2}X) \\ &+ \sum_{\frac{1}{3}(d_1+d_2)<\ell\leq d_2-d_1+2g-2}\frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)}P_t(J(X))P_t(S^{2g-2+d_2-\ell-d_1}X)P_t(S^{2g-2-\ell+d_1}X) \end{aligned} \tag{4.1}$$

Proof of Theorem 1.1. We need to show that the expression (4.1) agrees with (1.3) and (1.4). By the result of Atiyah-Bott [1],

$$P_t(B\mathcal{G}) - \frac{1}{(1-t^2)^2}P_t(J(X))P_t^{\mathcal{G}}(\mathcal{A}^{ss}(E_2)) - \sum_{d_2/2<\ell}\frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^3}P_t(J(X))^3 = 0$$

eliminating the first line on the right hand side of (4.1). Let $E = E_1^*E_2\otimes K$, and recall the definitions (1.2) and (3.4). By [20, Thm.'s 8.4.1 and 8.4.2],

(4.2)

$$\begin{aligned} &P_t^{\mathcal{G}(E)}(\mathcal{C}_{\sigma(d_1,d_2)}(E)) - P_t(\mathcal{N}_{\sigma_{min}}(E)) = \\ &\quad + \sum_{d_2/2<\ell<\frac{1}{3}(d_1+d_2)}\frac{t^{2(g-1+2\ell-d_2)} - t^{2(g-1+d_2-d_1-\ell)}}{(1-t^2)}P_t(J(X))^2P_t(S^{d_2-\ell-d_1+2g-2}X) \\ &\quad + \begin{cases} 0 & \text{if } d_1+d_2\not\equiv 0 \mod 3 \\ \dfrac{t^{2(g-1+\frac{1}{3}(2d_1-d_2))}}{(1-t^2)}P_t(J(X))^2P_t(S^{2g-2-\frac{2}{3}(2d_1-d_2)}X) & \text{if } d_1+d_2\equiv 0 \mod 3 \end{cases} \end{aligned}$$

Using this, and substituting $\ell\mapsto d_2-\ell$ in the fourth line of (4.1), the result follows. □

Proof of Corollary 1.3. When the Toledo invariant $\frac{2}{3}(2d_1-d_2)$ achieves its maximal value $2g-2$ then the Poincaré polynomial (1.3) simplifies further. Firstly, note that in this case $\frac{1}{3}(d_1+d_2) = d_2-d_1-(2g-2)$, and so the summation on the right hand side of (1.3) vanishes. Secondly, for the Bradlow space, $\deg E = g-1 = \sigma(d_1,d_2)$. Therefore, in the case of maximal Toledo invariant, the stability parameter is maximal (and non-generic). By [20, Thm. 8.4.2], the first term on the right hand side of (1.3) is

$$\begin{aligned}\frac{1}{(1-t^2)}P_t(J(X))^2P_t(\mathbb{C}P^{2g-3})+\frac{t^{4g-4}}{(1-t^2)^2}P_t(J(X))^2\\ &=\frac{1}{(1-t^2)}P_t(J(X))^2\left(1+\cdots+t^{4g-6}+t^{4g-4}(1+t^2+\cdots)\right)\\ &=\frac{1}{(1-t^2)^2}P_t(J(X))^2\end{aligned}$$

□

4.2. $\mathsf{SU}(2,1)$ **bundles.** Many of the constructions for $\mathsf{U}(2,1)$ Higgs bundles described above also carry over to the space $\mathcal{B}_\Lambda(d_1,d_2)$ of $\mathsf{SU}(2,1)$ Higgs bundles. In particular, we have the same indexing set for the stratification, and the index at a critical point can also be computed by an analogous calculation to that in Section 3.2. The major difference between the two cases is that the Kirwan map κ_0 from (1.9) is no longer necessarily surjective. However, repeated application of Lemma 3.18 allows us to compute the contributions from each critical set individually.

Due to the fixed determinant condition, some of the spaces that contribute to the Poincaré polynomial are different to those that appear in the calculation of the previous section: they are finite covers of known spaces (cf. [13], [10], [11]) and so we begin by describing their construction.

Let $\widetilde{S}(m_1,m_2)$ to be the 3^{2g}-fold cover of $S^{m_1}X\times S^{m_2}X$ defined as in the Introduction (see [10, 11]). These spaces appear in (4.6). Recall that the construction is via pullback, as follows

$$\begin{tikzcd} \widetilde{S}(m_1,m_2) \arrow[r, "p_2"] \arrow[d, "p_1"'] & J(X) \arrow[d, "g"] \\ S^{m_1}X\times S^{m_2}X \arrow[r, "f"] & J(X) \end{tikzcd}$$

where $\widetilde{S}(m_1,m_2)\subset S^{m_1}X\times S^{m_2}X\times J(X)$, p_1 is projection onto the first two factors, p_2 is projection onto the third factor, f maps $(M_1,\varphi_1,M_2,\varphi_2)\mapsto M_1^*M_2\Lambda$ and g is the three-fold covering map $L\mapsto L^3$. Note that if $M_1=L_2^*L_3\otimes K=L_1^*(L_2^*)^2\Lambda\otimes K$ and $M_2=L_1^*L_2\otimes K$, then $M_1^*M_2\Lambda=L_2^3$, and so $\widetilde{S}(m_1,m_2)$ is the space of bundles L_1,L_2 together with nonzero sections $\varphi_1\in H^0(L_1^*L_2\otimes K)$ and $\varphi_2\in H^0(L_1^*(L_2^*)^2\otimes K)$, where $m_1=\deg(L_1^*L_2\otimes K)$ and $m_2=\deg(L_1^*(L_2^*)^2\otimes K)$.

As above, let $E=E_1^*E_2\otimes K$. For the Type A stratum (see (4.3)), define $\widetilde{\mathcal{N}}_\sigma(E)$ to be the 3^{2g}-fold cover of the Bradlow space $\mathcal{N}_\sigma(E)$, which is constructed via the following pullback diagram

$$\begin{tikzcd} \widetilde{\mathcal{N}}_\sigma(E) \arrow[r, "p_2"] \arrow[d, "p_1"'] & J(X) \arrow[d, "g"] \\ \mathcal{N}_\sigma(E) \arrow[r, "f"] & J(X) \end{tikzcd}$$

where $p_1(E_1,E_2,\varphi)=(E,\varphi)$, $p_2(E_1,E_2,\varphi)=E_1$, $f(E,\varphi)=\det E$ and $g(L)=(L^*)^3K^2\Lambda$. Note that

$$f\circ p_1(E_1,E_2,\varphi)=g\circ p_2(E_1,E_2,\varphi)\iff (E_1^*)^2(\det E_2)K^2=(E_1^*)^3K^2\Lambda\iff \det(E_2\oplus E_1)=\Lambda$$

The construction is analogous to [10, Proposition 2.9], but the underlying space is different, since we use a different stability parameter in this calculation to that used in Gothen's calculation (σ_{min} as opposed to $\sigma(d_1, d_2)$). Note, however, that by [11], or the methods of [20], it still follows that

$$P_t(\widetilde{\mathcal{N}}_{\sigma_{min}}(E)) = P_t(\mathcal{N}_{\sigma_{min}}(E))$$

The final case to consider is where there are three line bundles L_1, L_2, L_3 satisfying $L_1 L_2 L_3 = \Lambda$, and one section $\varphi \in H^0(L_j^* L_k \otimes K) \setminus \{0\}$, where $j, k \in \{1,2,3\}$ and $j \neq k$. These spaces appear in (4.4), (4.5) and (4.7) as the cohomology of the type C critical sets and also in (4.3) (when d_2 is even). Let $i \in \{1,2,3\} \setminus \{j,k\}$, and note that the fixed determinant condition $L_1 L_2 L_3 = \Lambda$ can be resolved by setting $L_i = \Lambda L_j^* L_k^*$. Then the space under consideration becomes

$$\begin{aligned} &\left\{ (L_1, L_2, L_3, \varphi) \,:\, L_1 L_2 L_3 = \Lambda, \varphi \in H^0(L_j^* L_k \otimes K) \setminus \{0\} \right\} \\ &\qquad\qquad = \left\{ (L_j, L_k, \varphi) \,:\, \varphi \in H^0(L_j^* L_k \otimes K) \setminus \{0\} \right\} \end{aligned}$$

which fibers over $J(X) \times S^{2g-2+\deg L_k - \deg L_j} X$ with fiber $\mathbb{C}^*$. In particular, if S^1 acts freely on the $\mathbb{C}^*$ factor, then the S^1-equivariant Poincaré polynomial is $P_t(J(X)) P_t(S^{2g-2+\deg L_k - \deg L_j} X)$.

In the same way as for the $\mathsf{U}(2,1)$ case, we can calculate the contributions of the individual strata. These contributions are listed below.

(i) For the A-stratum

$$\begin{aligned} P_t^{\mathcal{G}}(X_{d_2/2}^* \cup \mathcal{S}_a) - P_t^{\mathcal{G}}(X_{d_2/2}^*) &= \frac{1}{1-t^2} P_t^{\mathcal{G}}(\mathcal{A}^{ss}(E_2)) - P_t(\mathcal{N}_{\sigma_{min}}(E_1^* E_2 \otimes K)) \\ &\quad - \begin{cases} 0 & \text{if } d_2 \text{ odd} \\ t^{2(2g-2+d_2/2-d_1)} P_t(J(X)) P_t(S^{2g-2+d_2/2-d_1} X) & \text{if } d_2 \text{ even} \end{cases} \end{aligned} \tag{4.3}$$

(ii) For $\frac{1}{3}(2d_2 - d_1) < \ell \leq d_2/2$

$$P_t^{\mathcal{G}}(X_\ell') - P_t^{\mathcal{G}}(X_\ell^*) = \frac{t^{4g-4+2\ell-2d_1}}{(1-t^2)^2} P_t(J(X)) P_t(S^{\ell - d_1 + 2g-2} X) \tag{4.4}$$

(iii) For $d_2/2 < \ell \leq \frac{1}{3}(d_1 + d_2)$ (or $d_2 - d_1 + 2g - 2 < \ell < d_1$)

$$P_t^{\mathcal{G}}(X_\ell) - P_t^{\mathcal{G}}(X_\ell^*) = \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^2} P_t(J(X))^2 - \frac{t^{2(g-1+2\ell-d_2)}}{1-t^2} P_t(J(X)) P_t(S^{d_2 - \ell - d_1 + 2g-2} X) \tag{4.5}$$

(iv) For $\frac{1}{3}(d_1 + d_2) < \ell \leq d_2 - d_1 + 2g - 2$

$$\begin{aligned} P_t^{\mathcal{G}}(X_\ell) - P_t^{\mathcal{G}}(X_\ell^*) &= \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^2} P_t(J(X))^2 \\ &\quad - t^{2(g-1+2\ell-d_2)} P_t(\widetilde{S}(2g-2+d_2-\ell-d_1, 2g-2-\ell+d_1)) \end{aligned} \tag{4.6}$$

(v) For $\max\{d_1, d_2 - d_1 + 2g - 2\} < \ell \leq d_1 + 2g - 2$

$$\begin{aligned} P_t^{\mathcal{G}}(X_\ell') - P_t^{\mathcal{G}}(X_\ell^*) &= \frac{t^{2(g-1+2\ell-d_2)}}{1-t^2} P_t(J(X)) P_t(S^{d_1 - \ell + 2g-2} X) \\ P_t^{\mathcal{G}}(X_\ell) - P_t^{\mathcal{G}}(X_\ell') &= \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^2} P_t(J(X))^2 - \frac{t^{2(g-1+2\ell-d_2)}}{1-t^2} P_t(J(X)) P_t(S^{d_1-\ell+2g-2} X) \end{aligned} \tag{4.7}$$

(vi) For $d_1 + 2g - 2 < \ell$, or if $d_2 - d_1 + 2g - 2 < \ell \leq d_1$,

$$P_t^{\mathcal{G}}(X_\ell) - P_t^{\mathcal{G}}(X_\ell^*) = \frac{t^{2(g-1+2\ell-d_2)}}{(1-t^2)^2} P_t(J(X))^2 \tag{4.8}$$

Theorem 1.2 then follows as in the non-fixed determinant case described in the previous section. We omit the details.

5. Action of Γ_3 and the Torelli group

We first fix the following notation. Recall that $\Gamma_3 = H^1(M, \mathbb{Z}/3)$. Then as elements of Γ_3 are homomorphisms $\pi \to \mathbb{Z}/3$, Γ_3 acts on $\mathrm{Hom}(\pi, \mathsf{SU}(2,1))$ by multiplication. The Torelli group $\mathcal{I}(M)$ acts on $\mathrm{Hom}(\pi, \mathsf{SU}(2,1)) /\!\!/ \mathsf{SU}(2,1)$ by outer automorphisms of π. This induces an action on equivariant cohomology which commutes with Γ_3. In this section we compute the induced action of $\Gamma_3 \times \mathcal{I}(M)$ on the $\mathsf{SU}(2,1)$-equivariant cohomology of $\mathrm{Hom}(\pi, \mathsf{SU}(2,1))$.

Following [15], let $Q(\Gamma_3) = \{\text{cyclic quotients of } \Gamma_3\}$. Then $C \in Q(\Gamma_3)$ is either $\{0\}$ or $\mathbb{Z}/3$. A choice of embedding $C \hookrightarrow \overline{\mathbb{Q}}$ gives a homomorphism $\mathbb{Z}[\Gamma_3] \to \overline{\mathbb{Q}}$, and we let I_C denote the kernel. If $R_C = \mathbb{Z}[\Gamma_3]/I_C$ and $K_C = \mathbb{Q} \otimes R_C$, then $K_C = \mathbb{Q}$ if $C = \{0\}$, and otherwise $K_C \cong \mathbb{Q}(\xi)$, for ξ a nontrivial third root of unity (though not canonically so). The field K_C has a natural "complex conjugation" induced by

$$\overline{\sum_{g \in \Gamma_3} c_g g} = \sum_{g \in \Gamma_3} c_g g^{-1}$$

If W is a K_C-vector space, let $\overline{W}$ denote the vector space with the same underlying $\mathbb{Q}$-structure, but where multiplication by scalars $\lambda \in K_C$ is given by $\lambda \cdot w = \bar{\lambda} w$.

Every $\{0\} \neq C \in Q(\Gamma_3)$ gives rise to a connected, unramified 3-fold covering $X_C \to X$. Namely, the choice of basepoint p gives an Abel mapping $X \hookrightarrow J(X)$. Let $\widetilde{X}_3$ be the Γ_3-covering obtained by pulling back the Γ_3-covering $J(X) \to J(X) : L \mapsto L^3$. Let X_C be the quotient of $\widetilde{X}_3$ by the kernel of $\Gamma_3 \to C$. Then Γ_3 acts on X_C by deck transformations, and there is a decomposition

$$H^1(X_C, \mathbb{Q}) \cong H^1(X, \mathbb{Q}) \oplus \left\{ R_C \otimes_{\mathbb{Z}[\Gamma_3]} H^1(X_C, \mathbb{Q}) \right\}$$

where $W_C(X) = R_C \otimes_{\mathbb{Z}[\Gamma_3]} H^1(X_C, \mathbb{Q})$ is a K_C-vector space of dimension $2g-2$. Lifting elements of the Torelli group then gives a surjection of $\mathcal{I}(X)$ onto the group of projective unitary transformations of $W_C(X)$, where the unitary structure is the extension by K_C of the symplectic pairing (see [15]).

For integers $m_1, m_2 \geq 0$, define the $\mathbb{Q}$-vector space

$$V(m_1, m_2) = \bigoplus_{\{0\} \neq C \in Q(\Gamma_3)} \wedge^{m_1} \overline{W_C(X)} \otimes_{K_C} \wedge^{m_2} W_C(X) \tag{5.1}$$

(the exterior products are over K_C). Also, recall the space $\widetilde{S}(m_1, m_2)$ from the previous section. For $\mathsf{SU}(2,1)$ representations of π, the Toledo invariant is an even integer, and so $m_1 \equiv m_2 \mod 3$. Hence, the diagonal action of Γ_3 is trivial on the terms in $V(m_1, m_2)$. In particular, the projective representation of the Torelli group lifts to a linear one. With this notation we state

Proposition 5.1. *The* Γ_3 *decomposition is given by*

$$H^p(\widetilde{S}(m_1,m_2)) = H^p(S^{m_1}X \times S^{m_2}X) \oplus \begin{cases} \{0\} & \text{if } p \neq m_1+m_2 \\ V(m_1,m_2) & \text{if } p = m_1+m_2 \end{cases}$$

Proof. Let $\widetilde{S}^m X$ be the pull-back of the fibration $S^m X \to J(X)$ under the 3^{2g}-fold covering $J(X) \to J(X) : L \mapsto L^3$. Then Γ_3 acts on $\widetilde{S}^m X$, and by [15] we have

$$H^*(\widetilde{S}^m X, \mathbb{Q}) \cong \bigoplus_{C \in Q(\Gamma_3)} R_C \otimes_{\mathbb{Z}[\Gamma_3]} H^*(\widetilde{S}^m X, \mathbb{Q})$$

For $C = \{0\}$ this amounts to

$$R_C \otimes_{\mathbb{Z}[\Gamma_3]} H^*(\widetilde{S}^m X, \mathbb{Q}) = \left[H^*(\widetilde{S}^m X, \mathbb{Q})\right]^{\Gamma_3} = H^*(S^m X, \mathbb{Q})$$

For $C \neq \{0\}$, we have an identification of K_C-vector spaces

$$R_C \otimes_{\mathbb{Z}[\Gamma_3]} H^*(\widetilde{S}^m X, \mathbb{Q}) \cong H^*(S^m X, \mathcal{F}_C^{(m)})$$

where $\mathcal{F}_C^{(m)} \to S^m X$ is a rank-1 local system. It follows exactly as in Hitchin [13] that there is a rank-1 local system $\mathcal{F}_C \to X$, such that

$$H^p(X, \mathcal{F}_C) \cong \begin{cases} \{0\} & p = 0,2 \\ W_C(X) & p = 1 \end{cases}$$

$$H^p(S^m X, \mathcal{F}_C^{(m)}) \cong \begin{cases} \{0\} & p \neq m \\ \wedge^m H^1(X, \mathcal{F}_C) & p = m \end{cases}$$

Explicitly, if $\mathrm{pr} : X_C \to X$ is the covering, the presheaf $\mathcal{F}_C(U)$ is given by locally constant functions $\varphi : \mathrm{pr}^{-1}(U) \to K_C$ satisfying $\varphi(gx) = g\varphi(x)$ for all $x \in X_C$, $g \in \Gamma_3$. In the case where the map $S^m X \to J(X)$ is factored through $L \mapsto L^*$, then

$$H^m(S^m X, \mathcal{F}_C^{(m)}) \cong \wedge^m H^1(X, \mathcal{F}_C^*)$$

and clearly $H^1(X, \mathcal{F}_C^*) \cong \overline{W_C(X)}$. Applying this argument to $S(m_1,m_2)$, we have

$$H^*(S(m_1,m_2), \mathbb{Q}) = H^*(S^{m_1}X \times S^{m_2}X, \mathbb{Q}) \oplus \bigoplus_{\{0\} \neq C \in Q(\Gamma_3)} R_C \otimes_{\mathbb{Z}[\Gamma_3]} H^*(S(m_1,m_2), \mathbb{Q})$$

Now by the Kunneth formula, for $C \neq \{0\}$,

$$\begin{aligned} R_C \otimes_{\mathbb{Z}[\Gamma_3]} H^p(S(m_1,m_2), \mathbb{Q}) &= H^p(S^{m_1}X \times S^{m_2}X, \mathcal{F}_C^{(m_1)} \boxtimes \mathcal{F}_C^{(m_2)}) \\ &= \bigoplus_{j+k=p} H^j(S^{m_1}X, \mathcal{F}_C^{(m_1)}) \otimes_{K_C} H^k(S^{m_2}X, \mathcal{F}_C^{(m_2)}) \\ &= \begin{cases} \{0\} & p \neq m_1+m_2 \\ V(m_1,m_2) & p = m_1+m_2 \end{cases} \end{aligned}$$

$\square$

Since $[K_C : \mathbb{Q}] = 2$ for $C \neq \{0\}$, and $\#Q(\Gamma_3) = 1 + \frac{1}{2}(3^{2g}-1)$, we have the following

Corollary 5.2 ([10, Proposition 3.11]). *If either m_1 or $m_2 > 2g-2$, then*

$$P_t(\widetilde{S}(m_1,m_2)) = P_t(S^{m_1}X)P_t(S^{m_2}X)$$

If $0 \le m_1, m_2 \le 2g-2$, then

$$P_t(\widetilde{S}(m_1,m_2)) = P_t(S^{m_1}X)P_t(S^{m_2}X) + (3^{2g}-1)\binom{2g-2}{m_1}\binom{2g-2}{m_2}t^{m_1+m_2}$$

We now state the result on the action of the Torelli group.

Theorem 5.3. *Fix a Toledo invariant $0 \le \tau \le 2g-2$. Let*

$$S_\tau = \{6g-6+\tau/2+2\ell : \ell \in \mathbb{Z}\ ,\ \max\{1,\tau/2\} \le \ell \le 2g-2-\tau\}$$

Then the following hold.

(i) *The $\mathsf{SU}(2,1)$-equivariant cohomology of $\mathrm{Hom}_\tau(\pi_1(X), \mathsf{SU}(2,1))$ is $\Gamma_3 \times \mathcal{I}(X)$-invariant in all dimensions $p \notin S_\tau$.*

(ii) *For $p = 6g-6+\tau/2+2\ell \in S_\tau$, the nontrivial part of the action of $\Gamma_3 \times \mathcal{I}(X)$ on the $\mathsf{SU}(2,1)$-equivariant cohomology of $\mathrm{Hom}_\tau(\pi_1(X), \mathsf{SU}(2,1))$ in dimension p is precisely $V(m_1,m_2)$, where $m_1 = 2g-2-\tau-\ell$, $m_2 = 2g-2+\tau/2-\ell$.*

Proof. Using the stratification $\{X_\ell\}$, the argument is the same as in [8]. Note that the action on the cohomology of the Bradlow spaces is trivial, since Kirwan surjectivity holds for these by [20]. □

Proof of Theorem 1.5. Kirwan surjectivity for $\mathsf{U}(2,1)$ is the content of Corollary 2.7. In the fixed determinant case it follows in exactly the same way as in [7] that Kirwan surjectivity also holds on the Γ_3-invariant part of the equivariant cohomology. Hence, the assertion for $\mathsf{PU}(2,1)$ Higgs bundles follows from the expression (1.8). Finally, note that by Theorem 5.3, if $|\tau| > \frac{4}{3}(g-1)$ then the equivariant cohomology for $\mathsf{SU}(2,1)$ Higgs bundles is Γ_3 invariant, and so Kirwan surjectivity follows in this case as well. □

Proof of Theorem 1.6. For $|\tau| > \frac{4}{3}(g-1)$, the statement follows as above. In the borderline case $|\tau| = \frac{4}{3}(g-1)$, note that Γ_3 does not act trivially but rather by permutation of the factors in (5.1); however $m_1 = 0$ and $m_2 = 2g-2$ in this case, so the representation of the Torelli group in (5.1) is trivial. □

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