



Kosta, D. and Thoma, A. (2020) Unboundedness of Markov complexity of monomial curves in A_n for $n \geq 4$. *Journal of Pure and Applied Algebra*, 224(6), 106249. (doi: [10.1016/j.jpaa.2019.106249](https://doi.org/10.1016/j.jpaa.2019.106249))

There may be differences between this version and the published version. You are advised to consult the publisher's version if you wish to cite from it.

<http://eprints.gla.ac.uk/198890/>

Deposited on 4 October 2019

Enlighten – Research publications by members of the University of Glasgow
<http://eprints.gla.ac.uk>

UNBOUNDEDNESS OF MARKOV COMPLEXITY OF MONOMIAL CURVES IN \mathbb{A}^n FOR $n \geq 4$.

DIMITRA KOSTA AND APOSTOLOS THOMA

ABSTRACT. Computing the complexity of Markov bases is an extremely challenging problem; no formula is known in general and there are very few classes of toric ideals for which the Markov complexity has been computed. A monomial curve C in \mathbb{A}^3 has Markov complexity $m(C)$ two or three. Two if the monomial curve is complete intersection and three otherwise. Our main result shows that there is no $d \in \mathbb{N}$ such that $m(C) \leq d$ for all monomial curves C in \mathbb{A}^4 . The same result is true even if we restrict to complete intersections. We extend this result to all monomial curves in \mathbb{A}^n , where $n \geq 4$.

1. INTRODUCTION

Much of the current interest in Markov bases of toric ideals and their complexity began with the seminal paper [9], which constitutes one of the first connections between commutative algebra and statistics. This work proposes algebraic algorithms to construct a connected Markov chain over high-dimensional contingency tables with fixed marginals. A Markov basis is then a set of moves connecting any two contingency tables that have the same column sums and row sums. Equivalently, the fundamental theorem of Markov bases established in [9] states that a Markov basis corresponds to a system of generators of a toric ideal. Motivated by this work, the Markov bases of certain contingency tables were studied in [2] and also the first examples of matrices with finite Markov complexity were provided. The Markov complexity is a notion which captures the complexity of Markov bases for hierarchical models.

In an effort to understand better the Markov basis $\mathcal{M}(\mathcal{A})$ of a toric ideal associated to a matrix \mathcal{A} , the study of auxiliary generating sets, such as the indispensable set $\mathcal{S}(\mathcal{A})$ and the Graver basis $\mathcal{G}(\mathcal{A})$ of \mathcal{A} , is employed. One can think of the indispensable set $\mathcal{S}(\mathcal{A})$ as the set of moves which belong to every Markov basis, since Markov bases are not unique. The Graver basis, on the other hand, is a superset of a Markov basis and is essentially another way of computing a generating set of the ideal. The Graver complexity evaluates the complexity of Graver bases for hierarchical models (see [2] and [16]). Building on the work by [2], it was proven in [16] that the Markov complexity is bounded above by the Graver complexity, and since the latter one is finite, the Markov complexity is also finite.

In [4], a geometric description is given for the elements of the Markov basis $\mathcal{M}(\mathcal{A})$ and the indispensable set $\mathcal{S}(\mathcal{A})$. This description uses the correspondence between fibers of \mathcal{A} , (i.e. set of contingency tables with the same marginal sums) and certain connected components of a simplicial complex associated to \mathcal{A} . At the same time, the notion of proper semiconformal decomposition was introduced in a more algebraic approach to describe the indispensable set $\mathcal{S}(\mathcal{A})$ (see [12]). Building on the results of [12], a complete algebraic characterization for the elements of the indispensable set $\mathcal{S}(\mathcal{A})$ and the Markov basis $\mathcal{M}(\mathcal{A})$

2010 *Mathematics Subject Classification.* 14M25, 13P10, 62H17, 05C90.

Key words and phrases. Toric ideals, Markov basis, monomial curves, Lawrence liftings.

is provided in [5] using extended notions of conformality, i.e. conformal, semiconformal, strongly semiconformal (see Section 2 for definitions).

The study of Markov and Graver bases was motivated, not only by applications to contingency tables in statistics, but also by applications to integer programming problems in operational research, such as high-dimensional transportation problems and packing problems [7]. In particular, Graver bases and their complexity have important applications in integer programming, where considerable effort has been put into estimating the growth of the Graver complexity, as it specifies the time complexity of various n -fold integer programmes (see [14, Chapter 4] and references therein). Most efforts in the integer programming community have focused on proving exponential lower bounds for the Graver complexity of complete bipartite graphs, as in [3], [13] and [11]. It is still a conjecture that the Graver complexity of the complete bipartite graph $K_{3,m}$ is equal to 3^{m-1} .

For $\mathcal{A} = (n_1, n_2, \dots, n_d)$ an $1 \times d$ matrix of positive integers, we call the toric ideal $I_{\mathcal{A}}$ the toric ideal of a monomial curve, since the set of zeroes $V(I_{\mathcal{A}})$ in the d -dimensional affine space over an algebraically closed field K is the set $\{(t^{n_1}, \dots, t^{n_d}) | t \in K\}$. In [5], it is shown that the Markov complexity of the monomial curve $\mathcal{A} = (n_1, n_2, n_3)$ is equal to two if the toric ideal $I_{\mathcal{A}}$ is complete intersection, and equal to three otherwise, answering a question posed by Santos and Sturmfels (see [16, Example 6]). However, computing the complexity $m(\mathcal{A})$ of Markov bases is an extremely challenging problem; no formula for the $m(\mathcal{A})$ is known in general and there are very few classes of toric ideals in the literature for which the complexity has been computed [2, 12, 16, 5].

The purpose of this paper is two fold: (a) to study the Markov complexity $m(\mathcal{A})$ of monomial curves in $\mathbb{A}^m, m \geq 4$ and (b) to demonstrate that the result of [5], which bounds the Markov complexity of complete intersection monomial curves in \mathbb{A}^3 by their codimension, is a special property of monomial curves in \mathbb{A}^3 and cannot be generalised to higher dimension. In particular, we obtain that complete intersection monomial curves in \mathbb{A}^4 may have arbitrary large Markov complexity; this is a corollary of the following Theorem which is the main result of this paper.

Theorem 4.1. *Monomial curves in \mathbb{A}^4 may have arbitrary large Markov complexity.*

To prove this, we need to find a family $\mathcal{A}_r = (a_1(r), a_2(r), a_3(r), a_4(r))$ of monomial curves in \mathbb{A}^4 , where the numbers $a_1(r), a_2(r), a_3(r), a_4(r)$ depend on a parameter r , such that the Markov complexity of \mathcal{A}_n , the n^{th} member of the family, is at least n . After several months working with the computational commutative algebra package 4ti2 [1], we did find one such family, $\mathcal{A}_n = (1, n, n^2 - n, n^2 - 1)$ and an element of type n in $\mathcal{L}(\mathcal{A}_n^{(n)})$, which we proved that it belongs to every Markov basis of $\mathcal{A}_n^{(n)}$.

The paper is organised in the following manner. Section 2 contains all the necessary definitions and properties of different types of decompositions. It also features Theorem 2.4, which states that Markov bases of higher Lawrence liftings behave well with respect to elimination and implies necessary conditions for the Markov complexity to be equal to 2. In Section 3, we provide the guiding example of a family of monomial curves in \mathbb{A}^4 with arbitrary large Markov complexity. Then, the final Section 4 includes the proof of our main result Theorem 4.1, which we also generalise to monomial curves in $\mathbb{A}^m, m \geq 4$.

2. PRELIMINARIES

Consider a set of vectors $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{N}^m, n, m \in \mathbb{N}$, and the corresponding matrix $A \in \mathcal{M}_{m \times n}(\mathbb{N})$, whose columns are the vectors of \mathcal{A} . We let $\mathcal{L}(\mathcal{A}) := \text{Ker}_{\mathbb{Z}}(A)$ be the corresponding sublattice of \mathbb{Z}^n and denote by $I_{\mathcal{A}}$ the corresponding toric ideal of \mathcal{A}

in $\mathbb{k}[x_1, \dots, x_n]$, where \mathbb{k} is a field. We recall that $I_{\mathcal{A}}$ is generated by all binomials of the form $x^{\mathbf{u}} - x^{\mathbf{w}}$, where $\mathbf{u} - \mathbf{w} \in \mathcal{L}(\mathcal{A})$.

A *Markov basis* \mathcal{M} of \mathcal{A} is a finite subset of $\text{Ker}_{\mathbb{Z}}(A)$ such that, whenever $\mathbf{w}, \mathbf{u} \in \mathbb{N}^n$ with $\mathbf{w} - \mathbf{u} \in \text{Ker}_{\mathbb{Z}}(A)$, there exists a subset $\{\mathbf{v}_i : i = 1, \dots, s\}$ of \mathcal{M} that connects \mathbf{w} to \mathbf{u} . This means that $(\mathbf{w} - \sum_{i=1}^p \mathbf{v}_i) \in \mathbb{N}^n$ for all $1 \leq p \leq s$ and $\mathbf{w} - \mathbf{u} = \sum_{i=1}^s \mathbf{v}_i$. We call a Markov basis \mathcal{M} of \mathcal{A} *minimal* if no subset of \mathcal{M} is a Markov basis of \mathcal{A} . For a vector $\mathbf{u} \in \mathcal{L}(\mathcal{A})$, we denote by $\mathbf{u}^+, \mathbf{u}^-$ the unique vectors in \mathbb{N}^n such that $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$. According to a classical result by Diaconis and Sturmfels, if \mathcal{M} is a minimal Markov basis of \mathcal{A} , then the set $\{x^{\mathbf{u}^+} - x^{\mathbf{u}^-} : \mathbf{u} \in \mathcal{M}\}$ is a minimal generating set of $I_{\mathcal{A}}$ (see [9, Theorem 3.1]). The union of all minimal Markov bases of \mathcal{A} is called the *universal Markov basis* of \mathcal{A} and is denoted by $\mathcal{M}(\mathcal{A})$ (see [12, Definition 3.1]). In a Markov basis or in the universal Markov basis we identify elements that differ by a sign, that means if \mathbf{u} belongs in one basis then we consider also that $-\mathbf{u}$ belongs to the same basis but we write only one of them.

The *indispensable subset* of the universal Markov basis $\mathcal{M}(\mathcal{A})$, which is denoted by $\mathcal{S}(\mathcal{A})$, is the intersection of all minimal Markov bases of \mathcal{A} via the same identification. The *Graver basis* of \mathcal{A} , $\mathcal{G}(\mathcal{A})$, is the subset of $\mathcal{L}(\mathcal{A})$ whose elements have no *proper conformal decomposition*; namely, an element $\mathbf{u} \in \mathcal{L}(\mathcal{A})$ belongs to the Graver basis $\mathcal{G}(\mathcal{A})$ if whenever \mathbf{u} can be written in the form $\mathbf{v} +_c \mathbf{w}$, where $\mathbf{v}, \mathbf{w} \in \mathcal{L}(\mathcal{A})$ with $\mathbf{u}^+ = \mathbf{v}^+ + \mathbf{w}^+$ and $\mathbf{u}^- = \mathbf{v}^- + \mathbf{w}^-$, then we conclude that either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ (see [17, Section 4]). The *Graver basis* of \mathcal{A} is always a finite set and contains the universal Markov basis of \mathcal{A} , see [17, Section 7]. Therefore, we have the following inclusions

$$\mathcal{S}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{A}).$$

The notion of a semiconformal decomposition was introduced in [12, Definition 3.9]. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{L}(\mathcal{A})$. We say that $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$ is a **semiconformal decomposition** of \mathbf{u} if a) $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and b) $\mathbf{v}(i) > 0$ implies that $\mathbf{w}(i) \geq 0$, and $\mathbf{w}(i) < 0$ implies that $\mathbf{v}(i) \leq 0$, for $1 \leq i \leq n$. Here $\mathbf{v}(i)$ denotes the i^{th} coordinate of the vector \mathbf{v} . We call the decomposition **proper** if both \mathbf{v}, \mathbf{w} are nonzero. For two integer vectors \mathbf{a}, \mathbf{b} , the relation $\mathbf{a} \geq \mathbf{b}$ means that every coordinate of $\mathbf{a} - \mathbf{b}$ is positive or zero, while the relation $\mathbf{a} > \mathbf{b}$ means $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. It is easy to see that $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$ if and only if $\mathbf{u}^+ \geq \mathbf{v}^+$ and $\mathbf{u}^- \geq \mathbf{w}^-$. We remark that $\mathbf{0}$ cannot be written as the semiconformal sum of two nonzero vectors, since $\mathcal{L}(\mathcal{A}) \cap \mathbb{N}^n = \{\mathbf{0}\}$. For example, given the matrix $A = (1, 3, 6, 8)$, the elements $(1, -1, -1, 1)$, $(4, 0, -2, 1)$, $(5, -1, -3, 2)$, $(3, 1, -1, 0)$ belong to $\text{Ker}_{\mathbb{Z}}(A)$ and $(5, -1, -3, 2) = (1, -1, -1, 1) +_c (4, 0, -2, 1)$, while $(4, 0, -2, 1) = (1, -1, -1, 1) +_{sc} (3, 1, -1, 0)$.

The lack of a proper semiconformal decomposition is not only a sufficient condition for an element to be in $\mathcal{S}(\mathcal{A})$, as was shown in [12, Lemma 3.10], but it is also a necessary condition by [5, Proposition 1.1].

Proposition 2.1. *The set of indispensable elements $\mathcal{S}(\mathcal{A})$ of \mathcal{A} consists of all nonzero vectors in $\mathcal{L}(\mathcal{A})$ with no proper semiconformal decomposition.*

Let $\mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_l \in \mathcal{L}(\mathcal{A})$, $l \geq 2$. We say that $\mathbf{u} =_{ssc} \mathbf{u}_1 + \dots + \mathbf{u}_l$ is a **strongly semiconformal decomposition** if $\mathbf{u} = \mathbf{u}_1 + \dots + \mathbf{u}_l$ and the following conditions are satisfied:

$$\mathbf{u}^+ > \mathbf{u}_1^+ \quad \text{and} \quad \mathbf{u}^+ > \left(\sum_{j=1}^{i-1} \mathbf{u}_j \right) + \mathbf{u}_i^+ \quad \text{for all } i = 2, \dots, l.$$

When $l = 2$, we simply write $\mathbf{u} = \mathbf{u}_1 +_{ssc} \mathbf{u}_2$. Note that $\mathbf{u} = \mathbf{u}_1 +_{ssc} \mathbf{u}_2$ implies that $\mathbf{u}^+ > \mathbf{u}_1^+$ and $\mathbf{u}^- > \mathbf{u}_2^-$. We say that the decomposition is **proper** if all $\mathbf{u}_1, \dots, \mathbf{u}_l$ are nonzero. We remark that if $\mathbf{u} =_{ssc} \mathbf{u}_1 + \dots + \mathbf{u}_l$ is proper, then $\mathbf{u}^+, \mathbf{u}^+ - \mathbf{u}_1, \dots, \mathbf{u}^+ - \sum_{i=1}^l \mathbf{u}_i = \mathbf{u}^- \in \mathbb{N}^n$ and are, therefore, distinct elements of $\mathcal{F}_{\mathbf{u}}$. For example, given the matrix $A = (1, 3, 6, 8)$, the elements $(4, 0, -2, 1)$, $(-5, -1, 0, 1)$, $(3, 1, -1, 0)$, $(6, 0, -1, 0)$ belong to $\text{Ker}_{\mathbb{Z}}(A)$ and $(4, 0, -2, 1) =_{ssc} (-5, -1, 0, 1) + (3, 1, -1, 0) + (6, 0, -1, 0)$.

We also have the following characterisation of the elements of the universal Markov basis according to [5].

Proposition 2.2. *The universal Markov basis $\mathcal{M}(A)$ of A consists of all nonzero vectors in $\mathcal{L}(A)$ with no proper strongly semiconformal decomposition.*

In fact, as shown in [5], we have the following relationship between these decompositions

$$\text{proper conformal} \Rightarrow \text{proper strongly semiconformal} \Rightarrow \text{proper semiconformal} .$$

Let $\mathbf{u} \in \mathcal{L}(A)$. The fiber $\mathcal{F}_{\mathbf{u}}$ is the set $\{\mathbf{t} \in \mathbb{N}^n : \mathbf{u}^+ - \mathbf{t} \in \mathcal{L}(A)\}$. We have that the fibre $\mathcal{F}_{\mathbf{u}}$ is a finite set, since $\mathcal{L}(A) \cap \mathbb{N}^n = \{\mathbf{0}\}$.

Proposition 2.3. *Let $\mathbf{u} \in \mathcal{L}(A)$. There is a bijection between the elements of the fiber $\mathcal{F}_{\mathbf{u}}$ and the ways that \mathbf{u} can be written as semiconformal decomposition.*

Proof. Let $\mathbf{t} \in \mathbb{N}^n$ be an element in the fiber $\mathcal{F}_{\mathbf{u}}$. Then, $\mathbf{u}^+ - \mathbf{t} \in \mathcal{L}(A)$, as well as $\mathbf{t} - \mathbf{u}^- \in \mathcal{L}(A)$, since both $\mathbf{u}^+, \mathbf{u}^-$ belong to $\mathcal{F}_{\mathbf{u}}$. Set $\mathbf{v} = \mathbf{u}^+ - \mathbf{t}$ and $\mathbf{w} = \mathbf{t} - \mathbf{u}^-$. Then, $\mathbf{u} = \mathbf{v} + \mathbf{w}$, and also, $\mathbf{u}^+ \geq \mathbf{v}^+$ and $\mathbf{u}^- \geq \mathbf{w}^-$, since $\mathbf{t} \in \mathbb{N}^n$. This implies that $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$.

For the converse, suppose we have a semiconformal decomposition $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$, where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{L}(A)$. Then $\mathbf{u}^+ \geq \mathbf{v}^+$ and $\mathbf{u}^- \geq \mathbf{w}^-$, which implies that $\mathbf{u}^+ - \mathbf{v} = \mathbf{u}^- + \mathbf{w} \in \mathbb{N}^n$. Note that $\mathbf{u}^+ - (\mathbf{u}^+ - \mathbf{v}) = \mathbf{v} \in \mathcal{L}(A)$. This implies that $\mathbf{u}^+ - \mathbf{v} = \mathbf{u}^- + \mathbf{w}$ is an element in the fiber $\mathcal{F}_{\mathbf{u}}$. \square

Proposition 4.13 in [17] states that certain bases of a toric ideal behave well with respect to elimination. Let $\mathcal{B} \subset \mathcal{A}$, then for the Graver bases we have $\mathcal{G}(\mathcal{B}) = \mathcal{G}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$. The corresponding statement is true also for the universal Gröbner bases and for the circuits.

However, the corresponding statement is not true in general for the Markov bases or the universal Markov bases, i.e.

$$\mathcal{M}(\mathcal{B}) \neq \mathcal{M}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) .$$

For example, generic toric ideals [15] are toric ideals generated by binomials with full support. Moreover, all elements in a minimal Markov basis are indispensable, which means that the universal Markov basis is a minimal Markov basis. However, generic toric ideals are generated by binomials with full support, therefore, it follows that $\mathcal{M}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$ if \mathcal{B} is a proper subset of \mathcal{A} . This shows that $\mathcal{M}(\mathcal{B}) \neq \mathcal{M}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$ whenever the ideal $I_{\mathcal{B}}$ is not zero, for a generic toric ideal $I_{\mathcal{A}}$. On the contrary, Markov bases of Lawrence liftings behave well with respect to certain eliminations, which is the content of Theorem 2.4.

For $A \in \mathcal{M}_{m \times n}(\mathbb{N})$ as above and $r \geq 2$, the r -th Lawrence lifting of A is denoted by $A^{(r)}$ and is the $(rm + n) \times rn$ matrix

$$A^{(r)} = \begin{pmatrix} \overbrace{\begin{matrix} A & 0 & & 0 \\ 0 & A & & 0 \\ & & \ddots & \\ 0 & 0 & & A \end{matrix}}^{r\text{-times}} \\ I_n & I_n & \cdots & I_n \end{pmatrix},$$

see [16]. We write $\mathcal{L}(\mathcal{A}^{(r)})$ for $\text{Ker}_{\mathbb{Z}}(A^{(r)})$ and identify an element of $\mathcal{L}(\mathcal{A}^{(r)})$ with an $r \times n$ matrix: each row of this matrix corresponds to an element of $\mathcal{L}(\mathcal{A})$ and the sum of its rows is zero. The *type* of an element of $\mathcal{L}(\mathcal{A}^{(r)})$ is the number of nonzero rows of this matrix. The *Markov complexity*, $m(\mathcal{A})$, is the largest type of any vector in the universal Markov basis of $\mathcal{A}^{(r)}$ as r varies. According to [6, Theorem 3.3], since $\mathcal{L}(\mathcal{A}) \cap \mathbb{N}^n = \{\mathbf{0}\}$, all minimal Markov bases of $\mathcal{A}^{(r)}$ have the same complexity for $r \geq 2$.

Let $\mathcal{B} \subset \mathcal{A} = \{a_1, a_2, \dots, a_n\}$ and, after renumeration, let $\mathcal{B} = \{a_1, a_2, \dots, a_s\}$. For $\mathbf{u} \in \mathcal{L}(\mathcal{B}^{(r)})$, we denote by $\sigma(\mathbf{u})$ an element of $\mathcal{L}(\mathcal{A}^{(r)})$ written as an $r \times n$ matrix, such that the first s columns are the columns of \mathbf{u} and the last $n - s$ columns are zero columns. Let $\mathbf{v} \in \mathcal{L}(\mathcal{A}^{(r)})$, then we denote by $\pi(\mathbf{v})$ an $r \times s$ matrix with columns the first s columns of \mathbf{v} . In general, $\pi(\mathbf{v}) \notin \mathcal{L}(\mathcal{B}^{(r)})$, but if the last $n - s$ columns of \mathbf{v} are zero, then $\pi(\mathbf{v}) \in \mathcal{L}(\mathcal{B}^{(r)})$.

For simplicity, we will denote $\sigma(\mathcal{M}(\mathcal{B}^{(r)}))$ by $\mathcal{M}(\mathcal{B}^{(r)})$ and $\sigma(\mathcal{L}(\mathcal{B}^{(r)}))$ by $\mathcal{L}(\mathcal{B}^{(r)})$.

Theorem 2.4. *For the universal Markov bases $\mathcal{M}(\mathcal{A}^{(r)})$ and $\mathcal{M}(\mathcal{B}^{(r)})$ of $\mathcal{A}^{(r)}$ and $\mathcal{B}^{(r)}$ respectively, it holds that $\mathcal{M}(\mathcal{B}^{(r)}) = \mathcal{M}(\mathcal{A}^{(r)}) \cap \mathcal{L}(\mathcal{B}^{(r)})$.*

Proof. We will first show that $\mathcal{M}(\mathcal{B}^{(r)}) \subseteq \mathcal{M}(\mathcal{A}^{(r)}) \cap \mathcal{L}(\mathcal{B}^{(r)})$. Let \mathbf{u} be an element of the universal Markov basis $\mathcal{M}(\mathcal{B}^{(r)})$, then $\mathbf{u} \in \mathcal{L}(\mathcal{B}^{(r)})$.

Suppose that $\sigma(\mathbf{u}) \notin \mathcal{M}(\mathcal{A}^{(r)})$, then by Proposition 2.2, there is a proper strongly semiconformal decomposition of $\sigma(\mathbf{u})$

$$\sigma(\mathbf{u}) =_{ssc} \mathbf{u}_1 + \cdots + \mathbf{u}_l,$$

where each \mathbf{u}_i is an element of the lattice $\mathcal{L}(\mathcal{A}^{(r)})$. From the way the element $\sigma(\mathbf{u})$ is defined, the last $n - s$ columns of the matrix $\sigma(\mathbf{u})$ are zero. We claim that all \mathbf{u}_j also have the last $n - s$ columns equal to zero. Let us consider one element on the j -th column of the last $n - s$ columns of \mathbf{u}_1 that is non-zero. Since the sum of the entries of each column of the matrix \mathbf{u}_1 is zero, there exists at least one element on the j -th column of \mathbf{u}_1 which is positive. Suppose this element is the element $(\mathbf{u}_1)_{ij}$ which lies on the i -th row and j -th column. But then $\mathbf{u}^+ > \mathbf{u}_1^+$ and $\mathbf{u}_{ij} = 0 < (\mathbf{u}_1)_{ij}$, which is a contradiction. Therefore, the whole column j would be zero and, subsequently, each of the $n - s$ last columns of \mathbf{u}_1 would be zero.

Suppose that for some t , $1 < t \leq l$, the last $n - s$ columns of the elements $\mathbf{u}_1, \dots, \mathbf{u}_{t-1}$ are zero. Let us consider one element on the j -th column of \mathbf{u}_t that is non-zero, where $s + 1 \leq j \leq n$. Since the sum of the entries of each column of the matrix \mathbf{u}_t is zero, there exists at least one element on the j -th column of \mathbf{u}_t which is positive. Suppose this element is the element $(\mathbf{u}_t)_{ij}$ which lies on the i -th row and j -th column. But then $\mathbf{u}^+ > (\sum_{j=1}^{t-1} \mathbf{u}_j) + \mathbf{u}_t^+$ and $\mathbf{u}_{ij} = 0 < (\mathbf{u}_t)_{ij}$, which is a contradiction.

By induction, all \mathbf{u}_j have the last $n - s$ columns equal to zero, which means that $\pi(\mathbf{u}_j) \in \mathcal{M}(\mathcal{B}^{(r)})$ for $j = 1, \dots, l$. Thus, $\mathbf{u} =_{ssc} \pi(\mathbf{u}_1) + \cdots + \pi(\mathbf{u}_l)$. According to Proposition 2.2,

this is a contradiction, since $\mathbf{u} \in \mathcal{M}(\mathcal{B}^{(r)})$ and as such should have no proper strongly semiconformal decomposition.

To prove the direction $\mathcal{M}(\mathcal{B}^{(r)}) \supset \mathcal{M}(\mathcal{A}^{(r)}) \cap \mathcal{L}(\mathcal{B}^{(r)})$, let $\mathbf{v} \in \mathcal{M}(\mathcal{A}^{(r)}) \cap \mathcal{L}(\mathcal{B}^{(r)})$. If we assume that $\pi(\mathbf{v}) \notin \mathcal{M}(\mathcal{B}^{(r)})$, then there exists a proper strongly semiconformal decomposition $\pi(\mathbf{v}) =_{ssc} \mathbf{v}_1 + \cdots + \mathbf{v}_l$ with each $\mathbf{v}_i \in \mathcal{L}(\mathcal{B}^{(r)})$. But then $\mathbf{v} =_{ssc} \sigma(\mathbf{v}_1) + \cdots + \sigma(\mathbf{v}_l)$ with each $\sigma(\mathbf{v}_i) \in \mathcal{L}(\mathcal{A}^{(r)})$. According to Proposition 2.2, this is a contradiction, since $\mathbf{v} \in \mathcal{M}(\mathcal{A}^{(r)})$. \square

As an application, we show that if the Markov complexity $m(\mathcal{A})$ of a monomial curve \mathcal{A} is equal to 2, then for any subset of three elements $\mathcal{B} \subset \mathcal{A}$ the corresponding toric ideal $I_{\mathcal{B}}$ is complete intersection.

Corollary 2.5. *If a monomial curve $\mathcal{A} = (l_1, l_2, \dots, l_m)$ in \mathbb{A}^m has Markov complexity 2, then for any i, j, k in $\{1, 2, \dots, m\}$ the monomial curve $\mathcal{B} = (l_i, l_j, l_k)$ in \mathbb{A}^3 is complete intersection.*

Proof. Suppose that there exist i, j, k in $\{1, 2, \dots, m\}$ such that the monomial curve $\mathcal{B} = (l_i, l_j, l_k)$ in \mathbb{A}^3 is not complete intersection.

Then by Theorem 2.6 in [5] we know that $m(\mathcal{B}) = 3$. This means that for any r -th Lawrence lifting $r \geq 3$, $\mathcal{B}^{(r)}$ has type 3 elements inside the universal Markov basis $\mathcal{M}(\mathcal{B}^{(r)})$. By Theorem 2.4, there is a type 3 element inside $\mathcal{M}(\mathcal{A}^{(r)})$ as well. This means that the Markov complexity is $m(\mathcal{A}) \geq 3$, which is a contradiction. \square

Remark 2.6. We note that the converse of Corollary 2.5 is not true. In the next sections, we will give examples of monomial curves $\mathcal{A} = (l_1, l_2, \dots, l_m)$ in \mathbb{A}^m with arbitrary large Markov complexity, such that for any i, j, k in $\{1, 2, \dots, m\}$ the monomial curve $\mathcal{B} = (l_i, l_j, l_k)$ in \mathbb{A}^3 is complete intersection.

3. THE FAMILY OF MONOMIAL CURVES $\mathcal{A}_n = (1, n, n^2 - n, n^2 - 1)$

In this section we give the guiding example of the paper; a family of monomial curves \mathcal{A}_n in \mathbb{A}^4 which has the special structure that all members in the family are complete intersections, but also every member of the family satisfies the conclusion of Corollary 2.5 i.e. for any i, j, k in $\{1, 2, 3, 4\}$ the monomial curve $\mathcal{B} = (l_i, l_j, l_k)$ in \mathbb{A}^3 is complete intersection, where $l_1 = 1, l_2 = n, l_3 = n^2 - n, l_4 = n^2 - 1$. We will also present here some properties governing some semiconformal sums associated to these monomial curves.

Let us consider the example of the monomial curve $\mathcal{A}_n = (1, n, n^2 - n, n^2 - 1)$. For this curve, there is always the following element of type n in $\mathcal{L}(\mathcal{A}_n^{(n)})$:

$$\mathbf{u} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & \ddots & & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix},$$

since every row is in $\mathcal{L}(\mathcal{A}_n)$ and the sum of each column is zero. Note that the first $n-2$ rows are of the form $(1, -1, -1, 1)$, while the last two are $(0, 0, n+1, -n)$ and $(n-2, 2-n, -3, 2)$. The following Lemmas study the ways that two of the above elements of $\mathcal{L}(\mathcal{A}_n)$ can be written semiconformally under some special conditions. Note that for big n , there are thousands of elements in the fibers of the above elements, which according to

Proposition 2.3, means that there are thousand of different ways of writing these elements as semiconformal sums.

Lemma 3.1. *Consider the element $u = (1, -1, -1, 1) \in \mathcal{L}(\mathcal{A}_n)$. Then u can be written as a semiconformal decomposition $u = v +_{sc} w$ with the first element of the first term $v_1 = 1$ in exactly the following two ways*

$$(1) \quad (1, -1, -1, 1) = (1, -1, -1, 1) +_{sc} (0, 0, 0, 0)$$

$$(2) \quad (1, -1, -1, 1) = (1, -n, 0, 1) +_{sc} (0, n-1, -1, 0).$$

Proof. Suppose that $u = v +_{sc} w$ for some vectors $v, w \in \mathcal{L}(\mathcal{A}_n)$. Then Proposition 2.3 implies that $u^+ - v = u^- + w = (\alpha, \beta, \gamma, \delta) \in \mathbb{N}^4$. Therefore, the semiconformal sum $u = v +_{sc} w$ is alternatively written as

$$\begin{aligned} u &= (u^+ - (\alpha, \beta, \gamma, \delta)) +_{sc} ((\alpha, \beta, \gamma, \delta) - u^-) \\ &= (1 - \alpha, -\beta, -\gamma, 1 - \delta) +_{sc} (\alpha, \beta - 1, \gamma - 1, \delta) \end{aligned}$$

Since the element $(\alpha, \beta, \gamma, \delta) \in \mathcal{F}_u$, we have that $u^+ - (\alpha, \beta, \gamma, \delta) \in \mathcal{L}(\mathcal{A}_n)$. This implies that

$$\alpha + \beta n + \gamma(n^2 - n) + \delta(n^2 - 1) = n^2.$$

We are interested in establishing what happens when $\alpha = 0$, since we are in the case that $v_1 = 1$. In this case $\beta n + \gamma(n^2 - n) + \delta(n^2 - 1) = n^2$ where $\beta, \gamma, \delta \in \mathbb{N}$. Then δ can take the values of 1 and 0. Suppose that $\delta = 1$ then $\beta n + \gamma(n^2 - n) = 1$ which is a contradiction, since $n \geq 2$ and n divides 1. Therefore $\delta = 0$. Then $\beta n + \gamma(n^2 - n) = n^2$, which has only two solutions: $(\beta, \gamma) = (1, 1)$ or $(\beta, \gamma) = (n, 0)$. Therefore $(\alpha, \beta, \gamma, \delta) = (0, 1, 1, 0)$ or $(\alpha, \beta, \gamma, \delta) = (0, n, 0, 0)$ and this gives us only two cases for the semiconformal decomposition $u = (1, -1, -1, 1) +_{sc} (0, 0, 0, 0)$ or $u = (1, -n, 0, 1) +_{sc} (0, n-1, -1, 0)$. \square

Lemma 3.2. *Consider the element $u = (1, -1, -1, 1) \in \mathcal{L}(\mathcal{A}_n)$. Then u can be written as a semiconformal decomposition $u = v +_{sc} w$ with the second element of the second term $w_2 = -1$ in exactly the following three ways*

$$(3) \quad (1, -1, -1, 1) = (0, 0, 0, 0) +_{sc} (1, -1, -1, 1)$$

$$(4) \quad (1, -1, -1, 1) = (1 - n, 0, -1, 1) +_{sc} (n, -1, 0, 0)$$

$$(5) \quad (1, -1, -1, 1) = (1 - n^2, 0, 0, 1) +_{sc} (n^2, -1, -1, 0).$$

Proof. Suppose that $u = v +_{sc} w$ for some vectors $v, w \in \mathcal{L}(\mathcal{A}_n)$. Then by Proposition 2.3 the semiconformal sum $u = v +_{sc} w$, is alternatively written as

$$(6) \quad u = (u^+ - (\alpha, \beta, \gamma, \delta)) +_{sc} ((\alpha, \beta, \gamma, \delta) - u^-)$$

$$(7) \quad = (1 - \alpha, -\beta, -\gamma, 1 - \delta) +_{sc} (\alpha, \beta - 1, \gamma - 1, \delta)$$

where the element $(\alpha, \beta, \gamma, \delta) \in \mathcal{F}_u$. Therefore $u^+ - (\alpha, \beta, \gamma, \delta) \in \mathcal{L}(\mathcal{A}_n)$. This means that

$$\alpha + \beta n + \gamma(n^2 - n) + \delta(n^2 - 1) = n^2.$$

We are interested in establishing what happens when $\beta = 0$, since we are in the case that $w_2 = -1$. In this case $\alpha + \gamma(n^2 - n) + \delta(n^2 - 1) = n^2$ where $\alpha, \gamma, \delta \in \mathbb{N}^n$. Then δ can take the values of 1 and 0. Indeed, if $\delta \geq 2$ then $\alpha + \gamma(n^2 - n) + \delta(n^2 - 1) > n^2$ which is contradiction. If $\delta = 1$ then $\alpha = 1$ and $\gamma = 0$, therefore $(\alpha, \beta, \gamma, \delta) = (1, 0, 0, 1)$. In the case that $\delta = 0$, we get $\alpha + \gamma(n^2 - n) = n^2$, which has only two solutions, namely $(\alpha, \beta, \gamma, \delta) = (n, 0, 1, 0)$ and $(\alpha, \beta, \gamma, \delta) = (n^2, 0, 0, 0)$.

Therefore, by equation (7) we only have the following three cases for the semiconformal decomposition $u = (0, 0, 0, 0) +_{sc} (1, -1, -1, 1)$ or $u = (1 - n, 0, -1, 1) +_{sc} (n, -1, 0, 0)$ or $u = (1 - n^2, 0, 0, 1) +_{sc} (n^2, -1, -1, 0)$. \square

Lemma 3.3. *Consider the element $u = (2 - n, n - 2, -3, 2) \in \mathcal{L}(\mathcal{A}_n)$. If u can be written as a semiconformal decomposition $u = v +_{sc} w$ with the first entries v_1, w_1 non-positive and the second entries v_2, w_2 non-negative, then $v = \mathbf{0}$ or $w = \mathbf{0}$.*

Proof. Suppose that $u = v +_{sc} w$ for some nonzero vectors $v, w \in \mathcal{L}(\mathcal{A}_n)$. The semiconformal sum $u = v +_{sc} w$, is alternatively written as

$$\begin{aligned} (8) \quad u &= (u^+ - (\alpha, \beta, \gamma, \delta)) +_{sc} ((\alpha, \beta, \gamma, \delta) - u^-) \\ (9) \quad &= (-\alpha, (n - 2) - \beta, -\gamma, 2 - \delta) +_{sc} (\alpha - (n - 2), \beta, \gamma - 3, \delta) \end{aligned}$$

Since $(\alpha, \beta, \gamma, \delta)$ belongs to the fiber \mathcal{F}_u , we have that $u^+ - (\alpha, \beta, \gamma, \delta) \in \mathcal{L}(\mathcal{A}_n)$. This means that

$$(10) \quad \alpha + \beta n + \gamma(n^2 - n) + \delta(n^2 - 1) = 3n^2 - 2n - 2$$

which also gives $\alpha - \delta \equiv -2 \pmod{n}$. The initial conditions about the entries v_1, w_1 and the entries v_2, w_2 imply that $0 \leq \alpha \leq n - 2$ and $0 \leq \beta \leq n - 2$.

Noting that $0 \leq \delta \leq 2$, we distinguish three cases for the value of δ . In the case that $\delta = 2$, we have that $\alpha \equiv 0 \pmod{n}$ which together with $0 \leq \alpha \leq n - 2$ imply that $\alpha = 0$. Then equation (10) gives $\beta n + \gamma(n^2 - n) = n^2 - 2n$, which in turn implies that $\gamma = 0$ and $\beta = n - 2$. Therefore $(\alpha, \beta, \gamma, \delta) = (0, n - 2, 0, 2)$ obtaining the semiconformal decomposition $u = (0, 0, 0, 0) +_{sc} (2 - n, n - 2, -3, 2)$.

Now if $\delta = 1$, we get that $\alpha \equiv -1 \pmod{n}$ and together with $0 \leq \alpha \leq n - 2$ gives a contradiction.

Finally, if $\delta = 0$, then $\alpha \equiv -2 \pmod{n}$ which together with $0 \leq \alpha \leq n - 2$ imply that $\alpha = n - 2$. Then equation (10) becomes $n - 2 + \beta n + \gamma(n^2 - n) = 3n^2 - 2n - 2$ which in turn gives $\beta + \gamma(n - 1) = 3n - 3$. This means that β is a multiple of $n - 1$ and since $0 \leq \beta \leq n - 2$ the only option is for $\beta = 0$ and $\gamma = 3$. Therefore $(\alpha, \beta, \gamma, \delta) = (n - 2, 0, 3, 0)$ gives us the semiconformal decomposition $u = (2 - n, n - 2, -3, 2) +_{sc} (0, 0, 0, 0)$. \square

4. MARKOV COMPLEXITY OF MONOMIAL CURVES

In this section, we prove the main result of this paper regarding the unboundedness of the Markov complexity of monomial curves in $\mathbb{A}^m, m \geq 4$. We use the properties of semiconformal decompositions for the special monomial curve $\mathcal{A}_n = (1, n, n^2 - n, n^2 - 1)$ shown in Section 3, as well as Theorem 2.4 regarding the good behaviour of Markov bases of higher Lawrence liftings with respect to elimination.

Theorem 4.1. *Monomial curves in \mathbb{A}^4 may have arbitrary large Markov complexity.*

Proof. We will show that the type n element

$$\mathbf{u} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n + 1 & -n \\ 2 - n & n - 2 & -3 & 2 \end{pmatrix}$$

belongs to every Markov basis of $\mathcal{A}_n^{(n)}$. Which means we wish to show that the element \mathbf{u} is indispensable, namely that it belongs to $S(\mathcal{A}^{(n)})$, the intersection of all the minimal Markov bases. Let us assume on the contrary that the element \mathbf{u} is not indispensable. Proposition 1.1 in [5], implies that \mathbf{u} admits a proper semiconformal decomposition $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$, where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{L}(\mathcal{A}_n^{(n)})$ such that

$$\mathbf{v}_{ij} > 0 \Rightarrow \mathbf{w}_{ij} \geq 0 \text{ and } \mathbf{w}_{ij} < 0 \Rightarrow \mathbf{v}_{ij} \leq 0,$$

for any $1 \leq i \leq n, 1 \leq j \leq 4$. In terms of signs, for each row of the vector \mathbf{u} we have

$$\begin{aligned} (1, -1, -1, 1) &= (*, \ominus, \ominus, *) +_{sc} (\oplus, *, *, \oplus) \\ (0, 0, n+1, -n) &= (\ominus, \ominus, *, \ominus) +_{sc} (\oplus, \oplus, \oplus, *) \\ (n-2, 2-n, -3, 2) &= (\ominus, *, \ominus, *) +_{sc} (*, \oplus, *, \oplus). \end{aligned}$$

The symbol \ominus means that the corresponding integer is non positive, the symbol \oplus non negative and the symbol $*$ means that it can take any value.

Let $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$ be a semiconformal decomposition of \mathbf{u} , then the sign pattern of the elements \mathbf{v}, \mathbf{w} is:

$$\mathbf{u} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix} = \begin{pmatrix} * & \ominus & \ominus & * \\ * & \ominus & \ominus & * \\ & & \ddots & \\ * & \ominus & \ominus & * \\ \ominus & \ominus & * & \ominus \\ \ominus & * & \ominus & * \end{pmatrix} +_{sc} \begin{pmatrix} \oplus & * & * & \oplus \\ \oplus & * & * & \oplus \\ & & \ddots & \\ \oplus & * & * & \oplus \\ \oplus & \oplus & \oplus & * \\ * & \oplus & * & \oplus \end{pmatrix}.$$

Considering that the sum of every column should be zero, we conclude that the last element of the second column of \mathbf{v} , $\mathbf{v}_{n,2}$, is non-negative and the last element of the first column of \mathbf{w} , $\mathbf{w}_{n,1}$, is non-positive. This means that in the n^{th} row the elements highlighted in grey above are; $\mathbf{v}_{n,1}, \mathbf{w}_{n,1}$ which are non-positive and the elements $\mathbf{v}_{n,2}, \mathbf{w}_{n,2}$ which are non-negative. From Lemma 3.3, we distinguish two cases for the last row: first case that the last row of \mathbf{w} is zero or second case that the last row of \mathbf{v} is zero.

In the first case, the decomposition of \mathbf{u} becomes

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix} = \begin{pmatrix} * & \ominus & \ominus & * \\ * & \ominus & \ominus & * \\ & & \ddots & \\ * & \ominus & \ominus & * \\ \ominus & \ominus & * & \ominus \\ 2-n & n-2 & -3 & 2 \end{pmatrix} +_{sc} \begin{pmatrix} \oplus & * & * & \oplus \\ \oplus & * & * & \oplus \\ & & \ddots & \\ \oplus & * & * & \oplus \\ \oplus & \oplus & \oplus & * \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The first column of \mathbf{w} , highlighted in grey above, is non negative and adds to zero, thus, all the column is zero. So,

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & \ominus & \ominus & * \\ 1 & \ominus & \ominus & * \\ & & \ddots & \\ 1 & \ominus & \ominus & * \\ 0 & \ominus & * & \ominus \\ 2-n & n-2 & -3 & 2 \end{pmatrix} +_{sc} \begin{pmatrix} 0 & * & * & \oplus \\ 0 & * & * & \oplus \\ & & \ddots & \\ 0 & * & * & \oplus \\ 0 & \oplus & \oplus & * \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By Lemma 3.1, we see that if the first element of the first vector in the semiconformal decomposition of $(1, -1, -1, 1)$ is 1, then there are exactly two ways of decomposing semiconformally the element $(1, -1, -1, 1)$; namely

$$(1, -1, -1, 1) +_{sc} (0, 0, 0, 0) \text{ or } (1, -n, 0, 1) +_{sc} (0, n-1, -1, 0).$$

This means that each one of the first $n-2$ rows of \mathbf{v} should be either $(1, -1, -1, 1)$ or $(1, -n, 0, 1)$. By looking at the second column of \mathbf{v} , highlighted in gray above, we see that the first $n-2$ elements are either -1 or $-n$, the $(n-1)^{th}$ element is non-positive and the last is $n-2$. Since they add to zero, this forces all of the first $n-2$ elements to be -1 and the $(n-1)^{th}$ element to be zero. So -1 in the second position means that we are in the first decomposition $(1, -1, -1, 1) = (1, -1, -1, 1) +_{sc} (0, 0, 0, 0)$. Therefore, we get

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & * & \ominus \\ 2-n & n-2 & -3 & 2 \end{pmatrix} +_{sc} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \oplus & * \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Looking at \mathbf{w} , in particular the entries highlighted in gray above, and considering that each column adds to zero, we have that

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix} +_{sc} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the second case, the decomposition of \mathbf{u} becomes

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix} = \begin{pmatrix} * & \ominus & \ominus & * \\ * & \ominus & \ominus & * \\ & & \ddots & \\ * & \ominus & \ominus & * \\ \ominus & \ominus & * & \ominus \\ 0 & 0 & 0 & 0 \end{pmatrix} +_{sc} \begin{pmatrix} \oplus & * & * & \oplus \\ \oplus & * & * & \oplus \\ & & \ddots & \\ \oplus & * & * & \oplus \\ \oplus & \oplus & \oplus & * \\ 2-n & n-2 & -3 & 2 \end{pmatrix}.$$

The second column of \mathbf{v} , highlighted in gray above, is non positive and adds to zero, thus, all the column is zero. Then

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix} = \begin{pmatrix} * & 0 & \ominus & * \\ * & 0 & \ominus & * \\ & & \ddots & \\ * & 0 & \ominus & * \\ \ominus & 0 & * & \ominus \\ 0 & 0 & 0 & 0 \end{pmatrix} +_{sc} \begin{pmatrix} \oplus & -1 & * & \oplus \\ \oplus & -1 & * & \oplus \\ & & \ddots & \\ \oplus & -1 & * & \oplus \\ \oplus & 0 & \oplus & * \\ 2-n & n-2 & -3 & 2 \end{pmatrix}.$$

By Lemma 3.2, we have that if the second element of the second vector in the semiconformal decomposition of $(1, -1, -1, 1)$ is -1 , then there are exactly three ways of

decomposing semiconformally the $(1, -1, -1, 1)$; namely $(0, 0, 0, 0) +_{sc} (1, -1, -1, 1)$ or $(1 - n, 0, -1, 1) +_{sc} (n, -1, 0, 0)$ or $(1 - n^2, 0, 0, 1) +_{sc} (n^2, -1, -1, 0)$.

This means that each one of the first $n - 2$ rows of \mathbf{w} should be either $(1, -1, -1, 1)$ or $(n, -1, 0, 0)$ or $(n^2, -1, -1, 0)$. By looking at the first column of \mathbf{w} , highlighted in gray above, we see that the first $n - 2$ elements are either 1 or n or n^2 , the $(n - 1)^{th}$ element is non-negative and the last is $2 - n$. They all add to zero, so that forces all of the first $n - 2$ elements to be 1 and the $(n - 1)^{th}$ to be zero. Having 1 in the first position means that we are in the first decomposition $(1, -1, -1, 1) = (0, 0, 0, 0) +_{sc} (1, -1, -1, 1)$. Therefore,

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & \ominus \\ 0 & 0 & 0 & 0 \end{pmatrix} +_{sc} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & \oplus & * \\ 2-n & n-2 & -3 & 2 \end{pmatrix}.$$

Looking at \mathbf{v} , particularly the entries highlighted in gray above, and considering that each column adds to zero, we have that

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +_{sc} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix}.$$

Therefore, the decomposition $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$ can never be proper. Thus, we conclude that \mathbf{u} is indispensable, therefore, it belongs to all Markov bases. \square

Remark 4.2. We do not claim that the Markov complexity $m(\mathcal{A}_n)$ of \mathcal{A}_n is n , but at least n . Indeed, consider the example of the monomial curve $\mathcal{A}_5 = (1, 5, 20, 24) \in \mathbb{A}^4$. Then using the computational package 4ti2 (see [1]), we compute a Markov basis of $\mathcal{A}_n^{(r)}$ for $r \leq 6$. Table 1 includes the number of elements of the Markov basis of $\mathcal{A}_n^{(r)}$ as well as the largest type of any vector in the universal Markov basis of $\mathcal{A}_n^{(r)}$, for each $r = 1, \dots, 6$.

r th Lawrence lifting	# elements of Markov basis	Type
2	46	2
3	174	3
4	528	4
5	1520	5
6	4110	6
7	10206	6

TABLE 1. The monomial curve $\mathcal{A}_5 = (1, 5, 20, 24)$ in \mathbb{A}^4

Therefore, this implies that the Markov complexity is at least 6. In fact, the elements of type 6 in the sixth Lawrence lifting are

$$\begin{pmatrix} 0 & 0 & -6 & 5 \\ -2 & 2 & -4 & 3 \\ -2 & 2 & -4 & 3 \\ -2 & 2 & -4 & 3 \\ 3 & -3 & 3 & -2 \\ 3 & -3 & 3 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -6 & 5 \\ 0 & 0 & -6 & 5 \\ -1 & 1 & -5 & 4 \\ -1 & 1 & -5 & 4 \\ -1 & 1 & -5 & 4 \\ 3 & -3 & 3 & -2 \end{pmatrix},$$

as well as all elements coming from permutation of the rows of the above matrices.

Note that the monomial curve $\mathcal{A}_n = (1, n, n^2 - n, n^2 - 1)$ is complete intersection, therefore, the following Corollary directly follows from Theorem 4.1.

Corollary 4.3. *Complete intersection monomial curves in \mathbb{A}^4 may have arbitrary large Markov complexity.*

Corollary 4.4. *Monomial curves in \mathbb{A}^m , $m \geq 4$, may have arbitrary large Markov complexity.*

Proof. The proof for general $m \geq 5$ follows from Theorem 2.4 and Theorem 4.1.

Suppose there is a $d \in \mathbb{N}$ such that $m(\mathcal{A}) \leq d$ for all monomial curves \mathcal{A} in \mathbb{A}^m . If we consider the monomial curve $\mathcal{A} = (1, n, n^2 - n, n^2 - 1, a_{m-4}, \dots, a_m)$ in \mathbb{A}^m , where $n \geq d + 1$ and a_{m-4}, \dots, a_m are any natural numbers, then this means that the largest type of any vector in the universal Markov basis of $\mathcal{A}^{(r)}$ as r varies would be at most d .

We know by Theorem 4.1, that $m(\mathcal{B}) \geq n$ for $\mathcal{B} = \{1, n, n^2 - n, n^2 - 1\} \subset \mathcal{A}$. This means that for any r -th Lawrence lifting with $r \geq n$, $\mathcal{B}^{(r)}$ has an element of type at least n inside the universal Markov basis $\mathcal{M}(\mathcal{B}^{(r)})$. By Theorem 2.4, there is an element of type at least n inside $\mathcal{M}(\mathcal{A}^{(r)})$ as well. This means that the Markov complexity is $m(\mathcal{A}) \geq n$. Since $n \geq d + 1$, we immediately reach a contradiction. \square

REFERENCES

- [1] 4ti2 team, 4ti2 - a software package for algebraic, geometric and combinatorial problems on linear spaces, available at www.4ti2.de (2007).
- [2] S. Aoki, A. Takemura, Minimal basis for connected Markov chain over $3 \times 3 \times K$ contingency tables with fixed two dimensional marginals, *Aust. N. Z. J. Stat.* **45**, 229–249 (2003).
- [3] Y. Berstein, S. Onn, The Graver complexity of integer programming, *Ann. Comb.* 13 (2009), no. 3, 289–296.
- [4] H. Charalambous, A. Katsabekis, A. Thoma, Minimal systems of binomial generators and the indispensable complex of a toric ideal, *Proc. Amer. Math. Soc.* **135**, 3443–3451 (2007).
- [5] H. Charalambous, A. Thoma, M. Vladoiu, Markov complexity of monomial curves, *Journal of Algebra* **417**, 391–411 (2014).
- [6] H. Charalambous, A. Thoma, M. Vladoiu, Markov bases and generalized Lawrence liftings, *Annals of Combinatorics* **19**, 661–669 (2015).
- [7] J. A. De Loera, R. Hemmecke, S. Onn, R. Weismantel, N-fold integer programming, *Discrete Optimization* **5** (2008) 231–241.
- [8] J. A. De Loera, S. Onn, Markov bases of three-way tables are arbitrarily complicated, *Journal of Symbolic Computation* **41** (2006) 173–181.
- [9] P. Diaconis, B. Sturmfels, Algebraic algorithms for sampling from conditional distributions, *Ann. Statist.* **26**, 363–397 (1998).
- [10] M. Drton, B. Sturmfels, S. Sullivant, *Lectures on algebraic statistics*, Oberwolfach Seminars, 39. Birkhäuser Verlag, Basel, viii+171 pp (2009).
- [11] E. Finhold, R. Hemmecke, Lower bounds on the Graver complexity of M-fold matrices. *Ann. Comb.* 20 (2016), no. 1, 73–85.
- [12] S. Hoşten, S. Sullivant, A finiteness theorem for Markov bases of hierarchical models, *J. Combin. Theory Ser. A* **114**, 311–321 (2007).

- [13] T. Kudo, A. Takemura, A lower bound for the Graver complexity of the incidence matrix of a complete bipartite graph, *J. Comb.* 3 (2012), no. 4, 695-708.
- [14] S. Onn, *Nonlinear Discrete Optimization: An Algorithmic Theory*, EMS Publishing House, (2010).
- [15] I. Peeva, B. Sturmfels, Generic lattice ideals, *J. Amer. Math. Soc.*, Vol. **11 (2)**, 363–373 (1998).
- [16] F. Santos, B. Sturmfels, Higher Lawrence configurations, *J. Combin. Theory Ser. A* **103**, 151–164 (2003).
- [17] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, University Lecture Series, No 8, MAS, R.I. (1995).

DIMITRA KOSTA, SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GLASGOW, UNIVERSITY PLACE, GLASGOW, G12 8SQ UNITED KINGDOM
E-mail address: `Dimitra.Kosta@glasgow.ac.uk`

APOSTOLOS THOMA, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, PANEPISTEMIOUPOLE, IOANNINA 45110, GREECE
E-mail address: `athoma@uoi.gr`