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Rigid body dynamics using equimomental systems of point-masses

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Abstract The inertia matrix of any rigid body is the same as the inertia matrix of some system of four point-masses. In this work, the possible disposition of these point-masses is investigated. It is found that every system of possible point-masses with the same inertia matrix can be parameterised by the elements of the orthogonal group in four-dimensional modulo-permutation of the points. It is shown that given a fixed inertia matrix, it is possible to find a system of point-masses with the same inertia matrix but where one of the points is located at some arbitrary point. It is also possible to place two point-masses on an arbitrary line or three of the points on an arbitrary plane. The possibility of placing some of the point-masses at infinity is also investigated. Applications of these ideas to rigid body dynamics are considered. The equation of motion for a rigid body is derived in terms of a system of four point-masses. These turn out to be very simple when written in a 6-vector notation.

1 Introduction

Two systems of rigidly connected point-masses are said to be equimomental if their inertia about any line in space is equal. This implies that the inertia matrices of the two systems are equal. The study of these systems has a long but rather opaque history. Most articles refer to the treatise by Routh, "Dynamics of Rigid Bodies" [9]. It was shown here that every rigid body is equimomental to a system of four point-masses rigidly connected together. The original idea seems to be due to Reye in a paper published in 1865 [10]. In these works, several other theorems about the disposition of systems of point-masses equimomental to a body are given, in particular various ellipsoids that the points can lie on are described.

In 1929 Franklin [3], refers to the book by Routh but comments, "While some of these results are in the literature, they are relatively so little known that their interest justifies a connected account from an elementary standpoint." Franklin's approach is to look at pairs of point-masses, in particular when such systems can be equimomental. By sliding points into coincidence, the number of point-masses can always be reduced to four without changing the inertia properties of the system. A little later Sommerville [15] takes a different approach but also refers to Routh as a source for the result that four point-masses are required in general. Some 20 years later Talbot writes "The standard textbooks on Statics or Mechanics say very little about equimomental systems, and what is said usually refers to principal axes and moments, or is confined to laminae." [16]. Talbot shows, for example, that if two systems of point-masses have the same total mass, the same centroid and have the

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same moments of inertia about six general lines then the systems are equimomental, that is they have the same moments of inertia about any line in space. The approach is to set up and solve systems of linear equations. A proof along these lines is given for the theorem that, in general, four point-masses are required to form an equimomental system to any rigid body.

In 1977 Konstantinov [8], used the ideas of systems of equimomental points to study the dynamics of a particular robot. The paper refers to earlier work from 1959 by Konstantinov et al., titled "Système de masses concentrees, equivalentes à un corps solide", but this work is not readily available. The 1977 paper also refers to the paper by Reye and is one of the very few that were found to do so.

Huang looked again at equimomental systems of four point-masses in the 1990s [6]. The focus here was to find explicit expressions for the points. In the early 2000s, Chaudhary looked again at the problem for his PhD thesis. Surprisingly, he couldn't find a proof of the theorem that the minimum number of points required was four, so gave another proof. However, he preferred to work with systems of seven point-masses and used these to address the problem of balancing mechanisms, see [2] and several other works by this author. In 2014, three Physicists rediscovered the problem yet again. They claimed, "This important property seems to have been forgotten, as we have not found any proper demonstration at all." [4].

In the present work, the space of all possible four point-mass systems equimomental to a given body is identified as the quotient of the Lie group $O(4)$ by the symmetric group of permutations on four letters. This result is believed to be novel, however, given the size and age of the literature on the subject it is difficult to be certain of this.

Next, some theorems about where the masses can be placed are proved. In particular, the possibility of placing some points at non-physical locations is investigated. That is, some of the point-masses may be located at "points at infinity". Again, the results here are believed to be novel.

These ideas are then applied to the dynamics of a rigid body. Specifically, the equations of motion for the body are written in terms of the position and accelerations of the points in an equimomental system. When three of the point-masses are located at infinity, the equations of motion take a particularly simple form. This six-component vector form of the equations of motion is also believed to be novel and may have applications to the design and analysis of mechanisms and robots.

1.1 Background

For rigid body dynamics, it is often useful to use a six-dimensional formalism to describe the position and orientation of the body, see for example [11]. In such a formalism, the inertia of the body is represented as a 6×6 matrix with the general form,

$$N = \begin{pmatrix} \mathbb{I} & m\,C \\ m\,C^{\mathrm{T}} & m\,I_3 \end{pmatrix}, \tag{1}$$

where m is the mass of the body; $\mathbb{I}$ is usual 3×3 inertia matrix of the body; I_3 is the 3×3 identity matrix and C is an antisymmetric matrix corresponding to the position vector of the centre of mass. Suppose $\mathbf{c} = (c_x,\ c_y,\ c_z)^{\mathrm{T}}$ is the position vector of the centre of mass then C is defined by requiring $C\,\mathbf{p} = \mathbf{c} \times \mathbf{p}$ for arbitrary position vectors $\mathbf{p}$.

In this work, it will be useful to adopt a slightly different formalism. The independent entries in the inertia matrix N can be arranged as a symmetric 4×4 matrix,

$$\widetilde{\Xi} = \begin{pmatrix} \frac{1}{2}(-I_{xx} + I_{yy} + I_{zz}) & -I_{xy} & -I_{xz} & mc_x \\ -I_{xy} & \frac{1}{2}(I_{xx} - I_{yy} + I_{zz}) & -I_{yz} & mc_y \\ -I_{xz} & -I_{yz} & \frac{1}{2}(I_{xx} + I_{yy} - I_{zz}) & mc_z \\ mc_x & mc_y & mc_z & m \end{pmatrix},$$

this is usually referred to as the pseudo-inertia matrix.[1] The entries I_{ij} refer to the corresponding entries of the 3×3 inertia matrix, so for example

$$I_{xy} = -\int_{\text{body}} xy\,\rho\,\mathrm{dvol}$$

[1] This name seems to date back to [1], in [13] this was referred to as the "homogeneous plane-distance inertia matrix".

where ρ is the density of the body. Notice that the corresponding element of the pseudo-inertia will be positive.

The above can be thought of as specifying a linear mapping from the 6×6 inertia matrices to the space of pseudo-inertia matrices. Moreover, it is clear that the map is invertible. Hence, we may assume that two rigid bodies are equimomental if and only if their pseudo-inertia matrices are the same.

The pseudo-inertia matrix of a point-mass located at position vector $\mathbf{p}$ and with mass m can be written as the product,

$$\widetilde{\Xi} = m\,\tilde{\mathbf{p}}\,\tilde{\mathbf{p}}^{\mathrm{T}}. \tag{2}$$

In this expression, $\tilde{\mathbf{p}}$ will be referred to as an extended position vector and has the form $\tilde{\mathbf{p}} = (p_x,\ p_y,\ p_z,\ 1)^{\mathrm{T}}$. The expression above is a simple consequence of the definitions of the inertia matrix. Notice that this also tells us how the pseudo-inertia matrix transforms under a rigid body displacement, the extended vectors clearly transform according to the standard homogeneous representation of SE(3) and hence the inertia matrix will transform according to

$$\widetilde{\Xi}' = G\,\widetilde{\Xi}\,G^{\mathrm{T}}$$

where

$$G = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}$$

with R the 3×3 rotation matrix of the displacement and $\mathbf{t}$ its translation vector.

It is a classical theorem that for any rigid body there is a translation which will position the centre of mass at the origin and a rotation that will align the principal directions of inertia with the coordinate axes, see for example [7, Chap. 9]. Such a rigid displacement will hence diagonalise the 6×6 inertia matrix. It is clear that the same rigid displacement will also diagonalise the pseudo-inertia matrix. If the principal moments of inertia are $m\,k_1^2$, $m\,k_2^2$, and $m\,k_3^2$ then the diagonal entries of the pseudo-inertia matrix will be

$$m\,a^2 = \frac{m}{2}\left(-k_1^2 + k_2^2 + k_3^2\right), \quad m\,b^2 = \frac{m}{2}\left(k_1^2 - k_2^2 + k_3^2\right), \quad m\,c^2 = \frac{m}{2}\left(k_1^2 + k_2^2 - k_3^2\right) \quad \text{and}\, m.$$

The fact that the principal moments of inertia satisfy the triangle inequality ensures that these quantities are all positive.

Using this representation, we can think of the space of all possible inertia matrices as points in a projective space $\mathbb{P}^9$, with homogeneous coordinates given by the 10 independent entries of $\widetilde{\Xi}$. Note that the 10 independent entries can be thought of as 6 specifying the position and orientation of the body, 3 for the radii of gyration, and one for the mass of the body. This introduces some unphysical points, in particular the hyperplane determined by $\tilde{\xi}_{44} = 0$. It should also be kept in mind that no distinction between positive definite and non-positive definite matrices has been made.

The space of all point-masses can be viewed as the image of the quadratic Veronese map from $\mathbb{P}^3$ to $\mathbb{P}^9$. The image is a three-dimensional variety of degree $2^3 = 8$ [5]. It is determined by a number of quadratic equations; these are the equations which express the fact that $\widetilde{\Xi}$ has rank one, see Eq. (2).

1.2 Four point-masses

The result mentioned above, that every rigid body is equimomental to a system of four point-masses, will be revisited below. The presentation here roughly follows [15].

Consider four point-masses with equal mass, arranged at the vertices of a regular tetrahedron. The vertices of a regular tetrahedron can be viewed as four of the vertices of a regular cube, see Fig. 1. The tetrahedron's vertices are located at the points

$$\mathbf{q}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{q}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{q}_4 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

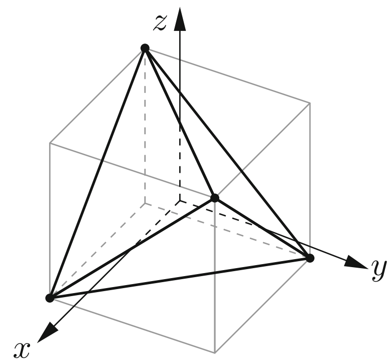

Fig. 1 A regular tetrahedron determined by a regular cube

This tetrahedron can be embedded in $\mathbb{R}^4$ using extended position vectors,

$$\tilde{\mathbf{q}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{q}}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{q}}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{q}}_4 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}. \tag{3}$$

Notice that these extended vectors satisfy the relations, $\tilde{\mathbf{q}}_i^{\mathrm{T}}\,\tilde{\mathbf{q}}_j = 0$ when $i \neq j$, and $\tilde{\mathbf{q}}_i^{\mathrm{T}}\,\tilde{\mathbf{q}}_i = 4$ for $i = 1, \ldots, 4$. Moreover, the extended vectors satisfy,

$$\sum_{i=1}^{4} \tilde{\mathbf{q}}_i\,\tilde{\mathbf{q}}_i^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$
$$+ \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = 4\,I_4.$$

Above we saw that for an arbitrary rigid body there is always a rigid change of coordinates that will make the pseudo-inertia matrix diagonal; $\widetilde{\Xi} = m\,\mathrm{diag}(a^2,\ b^2,\ c^2,\ 1)$. The points can be moved using a non-rigid transformation $\widetilde{D} = \mathrm{diag}(a,\ b,\ c,\ 1)$ so that the extended position vectors of the points become $\tilde{\mathbf{p}}_i = \widetilde{D}\,\tilde{\mathbf{q}}_i$,

$$\tilde{\mathbf{p}}_1 = \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_2 = \begin{pmatrix} -a \\ -b \\ c \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_3 = \begin{pmatrix} a \\ -b \\ -c \\ 1 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{p}}_4 = \begin{pmatrix} -a \\ b \\ -c \\ 1 \end{pmatrix}. \tag{4}$$

Placing four equal masses $m/4$ at these points produces a system with the required inertia matrix,

$$\frac{m}{4}\sum_{i=1}^{4} \tilde{\mathbf{p}}_i\,\tilde{\mathbf{p}}_i^{\mathrm{T}} = \frac{m}{4}\sum_{i=1}^{4} \widetilde{D}\,\tilde{\mathbf{q}}_i\,\tilde{\mathbf{q}}_i^{\mathrm{T}}\,\widetilde{D}^{\mathrm{T}} = m\,\widetilde{D}\,I_4\,\widetilde{D}^{\mathrm{T}} = \widetilde{\Xi}. \tag{5}$$

2 Solutions with unequal masses

Are there other sets of four points equimomental to the original body? Suppose that in Eq. (5) we had used $\widetilde{D}\,U$ rather than just $\widetilde{D}$, where $U \in O(4)$ is an orthogonal 4×4 matrix. The equation would then give

$$\frac{m}{4}\sum_{i=1}^{4} \widetilde{D}\,U\,\tilde{\mathbf{q}}_i\,\tilde{\mathbf{q}}_i^{\mathrm{T}}\,U^{\mathrm{T}}\,\widetilde{D}^{\mathrm{T}} = m\,\widetilde{D}\,U\,I_4\,U^{\mathrm{T}}\,\widetilde{D}^{\mathrm{T}} = \widetilde{\Xi},$$

the same as before since $U\,U^{\mathrm{T}} = I_4$. However, the extended vectors of the points would be given by $\tilde{\mathbf{p}}_i' = \widetilde{D}\,U\,\tilde{\mathbf{q}}_i$. Since $O(4)$ is a six-dimensional Lie group, this gives us a six parameter family of solutions. This family is not isomorphic to $O(4)$ since the points are unordered, any permutation of the four points will give the same system, so in fact the family of solutions will be isomorphic to the quotient $O(4)/S_4$ where S_4 denotes the symmetric group on four letters.

Suppose the $O(4)$ matrix has the form

$$U = \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix}$$

where $V \in O(3)$ is a 3×3 orthogonal matrix, then it is clear that the values of the masses are unchanged but their locations are moved. For example, suppose we choose

$$V = \begin{pmatrix} -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{6} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \end{pmatrix}.$$

Then the points on the regular tetrahedron would be rotated to $\mathbf{q}_i' = V\,\mathbf{q}_i$, as extended vectors this gives

$$\tilde{\mathbf{q}}_1' = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{q}}_2' = \begin{pmatrix} \frac{2}{3}\sqrt{6} \\ 0 \\ -\frac{1}{3}\sqrt{3} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{q}}_3' = \begin{pmatrix} -\frac{1}{3}\sqrt{6} \\ \sqrt{2} \\ -\frac{1}{3}\sqrt{3} \\ 1 \end{pmatrix}, \quad \text{and} \quad \tilde{\mathbf{q}}_4' = \begin{pmatrix} -\frac{1}{3}\sqrt{6} \\ -\sqrt{2} \\ -\frac{1}{3}\sqrt{3} \\ 1 \end{pmatrix}. \tag{6}$$

This is again a regular tetrahedron but with one vertex positioned on the z axis.

2.1 Rotation about the xy-plane

In four dimensions, the axis of a rotation is a two-dimensional linear subspace. The 3D rotations discussed in the previous Section can be thought of as 4D rotations where the 4D rotation axis is given by the span of the usual axis in 3D and a vector in the direction of the 4th coordinate, the w-direction say.

A rotation about the xy-plane would be given by the orthogonal matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{7}$$

Using this to transform the point-masses gives

$$U\,\tilde{\mathbf{q}}_1 = \begin{pmatrix} 1 \\ 1 \\ \cos\theta - \sin\theta \\ \cos\theta + \sin\theta \end{pmatrix}, \quad U\,\tilde{\mathbf{q}}_2 = \begin{pmatrix} -1 \\ -1 \\ \cos\theta - \sin\theta \\ \cos\theta + \sin\theta \end{pmatrix},$$

$$U\,\tilde{\mathbf{q}}_3 = \begin{pmatrix} 1 \\ -1 \\ -\cos\theta - \sin\theta \\ \cos\theta - \sin\theta \end{pmatrix} \quad \text{and} \quad U\,\tilde{\mathbf{q}}_4 = \begin{pmatrix} -1 \\ 1 \\ -\cos\theta - \sin\theta \\ \cos\theta - \sin\theta \end{pmatrix}.$$

To view these as points in $\mathbb{R}^3$, the fourth coordinate must be 1. This implies that the masses of the first two points change from $\frac{m}{4}$ to $\frac{m}{4}\ (\cos\theta + \sin\theta)^2 = \frac{m}{2}\cos^2(\frac{\pi}{4} - \theta)$. The masses of the second pair of points must also change, to $\frac{m}{4}\ (\cos\theta - \sin\theta)^2 = \frac{m}{2}\cos^2(\frac{\pi}{4} + \theta)$. So, we see that equimomental systems of points with unequal masses are also given by this construction. Notice that these masses are never negative.

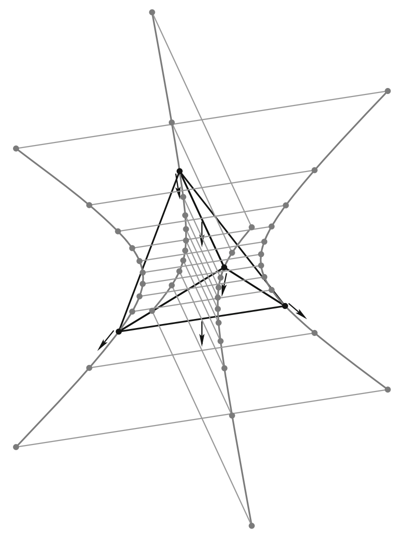

Fig. 2 The effect of a 4D rotation on a tetrahedron of point-masses

This also tells us about the position of the transformed points in $\mathbb{R}^3$. Dividing by the fourth components and applying simple trigonometric relations gives

$$(\cos\theta+\sin\theta)\ U\,\tilde{\mathbf{q}}_1=\begin{pmatrix}\frac{\sqrt{2}}{2}\sec\left(\frac{\pi}{4}-\theta\right)\\ \frac{\sqrt{2}}{2}\sec\left(\frac{\pi}{4}-\theta\right)\\ \tan\left(\frac{\pi}{4}-\theta\right)\\ 1\end{pmatrix},\quad (\cos\theta+\sin\theta)\ U\,\tilde{\mathbf{q}}_2=\begin{pmatrix}-\frac{\sqrt{2}}{2}\sec\left(\frac{\pi}{4}-\theta\right)\\ -\frac{\sqrt{2}}{2}\sec\left(\frac{\pi}{4}-\theta\right)\\ \tan\left(\frac{\pi}{4}-\theta\right)\\ 1\end{pmatrix},$$

$$(\cos\theta-\sin\theta)\ U\,\tilde{\mathbf{q}}_3=\begin{pmatrix}\frac{\sqrt{2}}{2}\sec\left(\frac{\pi}{4}+\theta\right)\\ -\frac{\sqrt{2}}{2}\sec\left(\frac{\pi}{4}+\theta\right)\\ -\tan\left(\frac{\pi}{4}+\theta\right)\\ 1\end{pmatrix}\ \text{and}\ (\cos\theta-\sin\theta)\ U\,\tilde{\mathbf{q}}_4=\begin{pmatrix}-\frac{\sqrt{2}}{2}\sec\left(\frac{\pi}{4}+\theta\right)\\ \frac{\sqrt{2}}{2}\sec\left(\frac{\pi}{4}+\theta\right)\\ -\tan\left(\frac{\pi}{4}+\theta\right)\\ 1\end{pmatrix}.$$

The paths of the points as θ increases are illustrated in Fig. 2. Notice that all four points move downwards. The two lower vertices go to infinity and then reappear at the top of the figure but with their positions swapped. After π rad. the points return to their original positions but with left and right points exchanged. After 2π rad. each vertex returns to its starting position.

2.2 4D rotation from a different viewpoint

Suppose we apply the 4D rotation U, given in (7), to the vertices of the tetrahedron given in (6).

The first point will move according to

$$U\tilde{\mathbf{q}}_1'=\begin{pmatrix}0\\0\\ \sqrt{3}\cos\theta-\sin\theta\\ \cos\theta+\sqrt{3}\sin\theta\end{pmatrix}=2\sin\left(\theta+\frac{\pi}{6}\right)\begin{pmatrix}0\\0\\ \cot\left(\theta+\frac{\pi}{6}\right)\\ 1\end{pmatrix}.$$

The mass of this point will change from $m/4$ in the original configuration to $m\sin^2(\theta+\frac{\pi}{6})$.

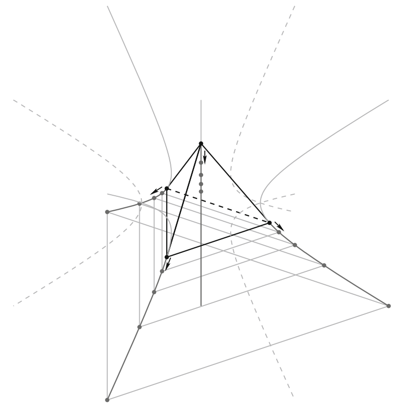

Fig. 3 The effect of a 4D rotation on a tetrahedron of point-masses

The other three points will move to

$$U\tilde{\mathbf{q}}_2' = \frac{2\sqrt{3}}{2}\cos\left(\theta + \frac{\pi}{6}\right)\begin{pmatrix} \sqrt{2}\sec\left(\theta + \frac{\pi}{6}\right) \\ 0 \\ -\tan\left(\theta + \frac{\pi}{6}\right) \\ 1 \end{pmatrix}, \qquad U\tilde{\mathbf{q}}_3' = \frac{\sqrt{3}}{2}\cos\left(\theta + \frac{\pi}{6}\right)\begin{pmatrix} -(1/\sqrt{2})\sec\left(\theta + \frac{\pi}{6}\right) \\ \sqrt{3/2}\sec\left(\theta + \frac{\pi}{6}\right) \\ -\tan\left(\theta + \frac{\pi}{6}\right) \\ 1 \end{pmatrix},$$

and

$$U\tilde{\mathbf{q}}_4' = \frac{\sqrt{3}}{2}\cos\left(\theta + \frac{\pi}{6}\right)\begin{pmatrix} -(1/\sqrt{2})\sec\left(\theta + \frac{\pi}{6}\right) \\ -\sqrt{3/2}\sec\left(\theta + \frac{\pi}{6}\right) \\ -\tan\left(\theta + \frac{\pi}{6}\right) \\ 1 \end{pmatrix}.$$

The mass of each of the last three points will change to $\frac{m}{3}\cos^2(\theta + \frac{\pi}{6})$. Notice that, as θ increases, the first point moves towards the centre of mass, reaching it when $\theta = \pi/3$ rad. As θ increases further, the point moves along the same line and becomes an ideal point or point at infinity when $\theta = 5\pi/6$ rad. The three other points travel on a plane which moves away from the centre of mass as θ increases but stays normal to the z-direction. These points reach infinity when the first point reaches the centre of mass, see Fig. 3.

3 Placement of the point-masses

The question remains: Are all systems of four points equimomental to a given inertia matrix given by the construction given in the previous Section? This can be answered in the affirmative by the following.

Theorem 1 *Let $\widetilde{\Xi}$ be the inertia matrix of a general rigid body. Every system of four point-masses equimomental to $\widetilde{\Xi}$ is given by some element $U \in O(4)$.*

Proof From the above, it is clear that we only need to consider diagonal pseudo-inertia matrices. Suppose some diagonal inertia matrix $\widetilde{\Xi}$ is equimomental to a system of four point-masses m_1, m_2, m_3, and m_4, located at $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$, respectively. We can always transform the points using the diagonal matrix $\widetilde{D}^{-1} = \mathrm{diag}(1/a,\ 1/b,\ 1/c,\ 1)$ to get four extended vectors, $\tilde{\mathbf{q}}_i = \widetilde{D}^{-1}\tilde{\mathbf{p}}_i$, $i = 1, \ldots, 4$. The extended vectors then satisfy

$$\sum_{i=1}^{4} m_i\, \tilde{\mathbf{q}}_i\, \tilde{\mathbf{q}}_i^{\mathrm{T}} = m\, I_4.$$

If we further scale each of these extended vectors by $\sqrt{m_i/m}$ to produce vectors, $\tilde{\mathbf{q}}'_i = \sqrt{m_i/m}\,\tilde{\mathbf{q}}_i$, these vectors will satisfy

$$\sum_{i=1}^{4} \tilde{\mathbf{q}}'_i\, \tilde{\mathbf{q}}'^{\mathrm{T}}_i = I_4.$$

Now, let $\tilde{\mathbf{n}}_1$ be a unit vector perpendicular to the three linearly independent vectors $\tilde{\mathbf{q}}'_2$, $\tilde{\mathbf{q}}'_3$, and $\tilde{\mathbf{q}}'_4$. Multiplying this on the right of the above equation gives $\tilde{\mathbf{q}}'_1\left(\tilde{\mathbf{q}}'^{\mathrm{T}}_1\,\tilde{\mathbf{n}}_1\right) = \tilde{\mathbf{n}}_1$. This shows that $\tilde{\mathbf{q}}'_1$ and $\tilde{\mathbf{n}}_1$ are parallel, and hence $\tilde{\mathbf{q}}'_1$ is perpendicular to $\tilde{\mathbf{q}}'_2$, $\tilde{\mathbf{q}}'_3$, and $\tilde{\mathbf{q}}'_4$. Multiplying by $\tilde{\mathbf{q}}'_1$ on the right now shows that $\tilde{\mathbf{q}}'_1$ is a unit vector. This can be repeated for the other three extended vectors showing that the set of all four vectors $\mathbf{q}'_1$, $\tilde{\mathbf{q}}'_2$, $\tilde{\mathbf{q}}'_3$, and $\tilde{\mathbf{q}}'_4$ comprises a set of mutually orthogonal unit vectors, sometimes called an orthonormal frame. Finally, since the group $O(4)$ acts transitively on orthonormal frames, this set of four point-masses can be transformed into any other orthonormal frame by some matrix $U \in O(4)$. In particular, they may be transformed into the vertices of a regular tetrahedron given in the previous Section. □

Notice that the above theorem holds for all possible rigid bodies, even those with two or three equal principal moments of inertia.

This leads to the following characterisation of quadruples of points that can be equimomental to a rigid body:

Theorem 2 *A system of four points $\tilde{p}_1, \ldots, \tilde{p}_4$ can be equimomental to a rigid body with pseudo-inertia matrix $\widetilde{\Xi}$ if and only if*

$$\tilde{\mathbf{p}}_i^{\mathrm{T}}\, \widetilde{\Xi}^{-1}\, \tilde{\mathbf{p}}_j = \begin{cases} 4, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Proof If the four points $\tilde{\mathbf{p}}_i$ form an equimomental system for a body with pseudo-inertia matrix $\widetilde{\Xi}$ then it is clear from the above that we can change coordinates so that the pseudo-inertia matrix is diagonal, with diagonal entries ma^2, mb^2, mc^2, and m. In this coordinate system, the points will have components as given in Eq. (4). The relation of the theorem is now easily verified.

Conversely, suppose we have a set of four points $\tilde{\mathbf{p}}_i$, satisfying the relation in the theorem. The inverse of the pseudo-inertia matrix can be diagonalised, let G be the rigid body transformation which sends $\widetilde{\Xi}^{-1}$ to $G^{\mathrm{T}}\widetilde{\Xi}^{-1}G = (1/m)\,\mathrm{diag}(a^{-2},\, b^{-2},\, c^{-2},\, 1)$. Using $\widetilde{D}^{-1} = \mathrm{diag}(a^{-1},\, b^{-1},\, c^{-1},\, 1)$, the points $\widetilde{D}^{-1}G^{-1}\tilde{\mathbf{p}}_i$ are then mutually orthogonal. These points can then be transformed to the points $\tilde{\mathbf{q}}_i$ given in Eq. (3) by a suitable $U \in O(4)$. Hence the original points $\tilde{\mathbf{p}}_i$ can be equimomental to the rigid body by Theorem 1. □

3.1 Placement at arbitrary points, lines or planes

Next, we look at the possible placement of the point-masses. The following results show that:

Theorem 3 *For a general rigid body, an equimomental system of four point-masses can always be chosen with one mass at an arbitrary point.*

Proof Let $\widetilde{\Xi} = \frac{m}{4}\sum_{i=1}^{4} \tilde{\mathbf{p}}_i\, \tilde{\mathbf{p}}_i^{\mathrm{T}}$ be the pseudo-inertia of the body. We want to find an equimomental system of point-masses so that one of the point-masses is located at $\tilde{\mathbf{r}}$.

First we can perform a rigid change of coordinates so that the pseudo-inertia matrix is in diagonal form, $G^{-1}\widetilde{\Xi}G^{-T} = \mathrm{diag}(ma^2,\, mb^2,\, mc^2,\, m)$. This transformation must also be applied to the target point, $\tilde{\mathbf{r}}' = G^{-1}\tilde{\mathbf{r}}$.

Next we scale the pseudo-inertia matrix and target point so that the pseudo-inertia matrix is a multiple of the 4×4 identity matrix,

$$\widetilde{D}^{-1}G^{-1}\widetilde{\Xi}G^{-T}\widetilde{D}^{-T} = mI_4,$$

this means the points will be transformed to $\tilde{\mathbf{q}}_i = \widetilde{D}^{-1}G^{-1}\tilde{\mathbf{p}}_i$ and also $\tilde{\mathbf{r}}'' = \widetilde{D}^{-1}G^{-1}\tilde{\mathbf{r}}$.

Then apply a 3D rotation to bring one of the points, $\mathbf{q}_1$ say, into coincidence with the position vector of $\mathbf{r}''$. Call this rotation U_1. Now perform a 4D rotation with axis spanned by the two directions perpendicular to $\tilde{\mathbf{r}}''$. This will move the point $U_1\tilde{\mathbf{q}}_1$ along the vector, and so we can bring it into coincidence with $\tilde{\mathbf{r}}''$, see

Sect. 2.2. Call this rotation U_2. Note that this move will change the values of the masses, but the three masses at $U_1\tilde{\mathbf{q}}_2,\ U_1\tilde{\mathbf{q}}_3$, and $U_1\tilde{\mathbf{q}}_4$ will be equal.

Finally, we put things back where we found them by rescaling and transforming back to the original coordinates. The overall transformation of the point will be

$$G\widetilde{D}U_2U_1\widetilde{D}^{-1}G^{-1}\tilde{\mathbf{p}}_1 = \tilde{\mathbf{r}}.$$

□

Notice that the other points, in the above procedure, will be moved by the transformations applied to the first point. Also, the transformation given here is not unique. The transformation U_2 could be composed with a 3D rotation about the position vector $\tilde{\mathbf{r}}''$. This vector is perpendicular to the plane determined by the other three points, and hence this transformation will move these three points in their plane. In fact, the three points will lie on a single circle as the rotation angle changes, hence after rescaling and transforming back to the original coordinates the three points lie on an ellipse. This observation seems to be in the same vein as Routh's equimomental ellipsoid as discussed in [6].

Next, we have similar theorems for pairs of points and lines in space.

Theorem 4 *For a general rigid body, an equimomental system of four point-masses can always be chosen so that two of the four point-masses lie on an arbitrary line not through the body's centre of mass.*

Proof Let the target line be ℓ and suppose that $\mathbf{r}$ is the point at the foot of the perpendicular from ℓ to the body's centre of mass. As in the previous theorem, we can use a rigid body transformation G^{-1} and a scaling transform $\widetilde{D}^{-1}$ to make the pseudo-inertia matrix proportional to the 4×4 identity matrix. Now assume that we wish to place the points $\mathbf{p}_1$ and $\mathbf{p}_2$ on the line ℓ. Here we concentrate on the point $\mathbf{p}_{12} = (1/2)(\mathbf{p}_1 + \mathbf{p}_2)$, the mid-point of the edge of the tetrahedron. As in the previous theorem, we rotate this point in 3D so that it lies along the direction of $\mathbf{r}''$. Then we rotate in 4D so that $\mathbf{p}_{12}$ and $\mathbf{r}''$ coincide. Again, this is the transformation that will change the masses of the points. A final 3D rotation about the axis of $\mathbf{r}''$ will bring the two points onto the transformed line through $\mathbf{r}''$. Lastly, we transform using $\widetilde{D}$ and G to regain the original coordinates. □

Suppose that a line is given in terms of its Plücker coordinates,

$$\ell = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}$$

where $\boldsymbol{\omega} = \mathbf{a} - \mathbf{b}$ and $\mathbf{v} = \mathbf{b} \times \mathbf{a}$ for any two distinct points $\mathbf{a},\ \mathbf{b}$ on the line. Now, the action of a rigid body displacement on the line is given by the adjoint representation of the group SE(3). The action of the scaling transformation $\widetilde{D}$ on the line is given by the 6×6 matrix

$$\begin{pmatrix} D & 0 \\ 0 & \mathrm{Adj}(D) \end{pmatrix},$$

that is the scaled line $\mathcal{L}$ is given in terms of the original line ℓ by

$$\mathcal{L} = \begin{pmatrix} D & 0 \\ 0 & \mathrm{Adj}(D) \end{pmatrix} \ell.$$

Here D is the diagonal 3×3 matrix $D = \mathrm{diag}(a,\ b,\ c)$ and $\mathrm{Adj}(D)$ denotes its adjugate, $\mathrm{Adj}(D) = abcD^{-1} = \mathrm{diag}(bc,\ ac,\ ab)$. In terms of representations, if we consider the matrix $\widetilde{D}$ as an element of the standard representation of the group $GL(4)$ then the matrix acting on the lines is the corresponding element of the antisymmetric square of the standard representation. The matrix can also be found by considering the action of the group on points and then computing the Plücker coordinates of the line joining the points.

Finally here, there is also a similar result for planes in space:

Theorem 5 *For a general rigid body, an equimomental system of four point-masses can always be chosen so that three of the four point-masses lie on an arbitrary plane not containing the body's centre of mass.*

The proof of this is much the same as for the previous two theorems.

Proof Let the target plane be $\boldsymbol{\pi}$ and suppose that $\mathbf{r}$ is the intersection of the plane with the line normal to the plane through the centre of mass of the body. This time we transform so that the pseudo-inertia is proportional to the identity matrix, move the centroid of three points defining a face of the tetrahedron into coincidence with $\mathbf{r}''$ and then transform back to the original coordinates. □

Again, this transformation is not unique. As above we can rotate the points about the normal to the plane through the centre of mass before transforming back to the original coordinates. The three points on the given plane will lie on an ellipse.

3.2 Choosing the masses

In the previous Subsection, the possible placement of the system of equimomental points was considered. It is possible to specify the masses of the points to some extent. This will be explored here.

Theorem 6 *For a general rigid body, an equimomental system of four point-masses can always be chosen so that one mass has a given, arbitrary value smaller than the body's total mass.*

Proof Consider the 4D rotation studied in Sect. 2.2. The mass of the first point-mass is given by $m \sin^2(\theta + \frac{\pi}{6})$, where m is the mass of the rigid body. Since the sine function varies between -1 and 1, the mass of the point will vary between 0 and 1. No transformation can increase the mass of a single point-mass beyond the total mass of the body since the sum of the four masses must sum to the total mass of the body and the masses are all positive. □

Considering two of the point-masses we have:

Theorem 7 *For a general rigid body, an equimomental system of four point-masses can always be chosen so that two of the masses have arbitrary, but equal, value less than half of the body's total mass.*

Proof If an equimomental system of point-masses has two equal masses then any 4D rotation that preserves this equality must have a (2D) axis that is parallel to the line joining the two points. See Sect. 2.1, where the axis of the rotation is the xy-plane. The masses of the first two points, in this example, are both given by $\frac{m}{2}\cos^2(\frac{\pi}{4} - \theta)$. Hence, choosing different values for the parameter θ, the equal masses can be chosen to be any value between 0 and $\frac{m}{2}$. □

Finally here, if we require three equal masses then,

Theorem 8 *For a general rigid body, an equimomental system of four point-masses can always be chosen with three arbitrary, but equal masses less than one third of the body's total mass.*

Proof To see this look again at Sect. 2.2. This time the three equal masses are given by $\frac{m}{3}\cos^2(\theta + \frac{\pi}{6})$, which lies between 0 and $\frac{m}{3}$ as θ varies. □

3.3 The distance between points and planes

To find the distance between a single point and the plane determined by the other three points in a system of four point-masses equimomental to a fixed inertia matrix $\widetilde{\Xi}$, we first need the following result.

Theorem 9 *For any system of four point-masses that are equimomental to a fixed pseudo-inertia $\widetilde{\Xi}$, the quantity*

$$\kappa = (m_1 m_2 m_3 m_4) \det \left(\tilde{\mathbf{p}}_1 \,\middle|\, \tilde{\mathbf{p}}_2 \,\middle|\, \tilde{\mathbf{p}}_3 \,\middle|\, \tilde{\mathbf{p}}_4 \right)^2$$

is constant and equal to $\kappa = \det(\widetilde{\Xi})$. Here the point-masses are located at the positions $\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_4$, and their masses are $m_1, \ldots, m_4$, respectively.

Proof Define the 4×4 matrix

$$P = \left(\sqrt{m_1}\tilde{\mathbf{p}}_1 \,\middle|\, \sqrt{m_2}\tilde{\mathbf{p}}_2 \,\middle|\, \sqrt{m_3}\tilde{\mathbf{p}}_3 \,\middle|\, \sqrt{m_4}\tilde{\mathbf{p}}_4\right),$$

that is, the columns of the matrix are the extended position vectors of the points weighted by the square root of the mass of the point. If this system of point-masses is equimomental to $\widetilde{\Xi}$, then

$$\widetilde{\Xi} = P P^{\mathrm{T}}.$$

So, $\det(\widetilde{\Xi}) = \det(P)^2$. Finally, we can see that

$$\det(P) = \sqrt{m_1 m_2 m_3 m_4}\, \det\left(\tilde{\mathbf{p}}_1 \,\middle|\, \tilde{\mathbf{p}}_2 \,\middle|\, \tilde{\mathbf{p}}_3 \,\middle|\, \tilde{\mathbf{p}}_4\right).$$

Squaring this settles the theorem. □

A plane can be represented by a 4-component vector,

$$\tilde{\boldsymbol{\pi}} = \begin{pmatrix} \boldsymbol{\pi} \\ -d \end{pmatrix},$$

where $\boldsymbol{\pi}$ is a 3-component unit vector normal to the plane, and d is the perpendicular distance between the plane and the origin of coordinates. In particular, if $\mathbf{p}$ is a point on the plane then it satisfies the linear equation

$$\tilde{\boldsymbol{\pi}}^{\mathrm{T}}\tilde{\mathbf{p}} = \boldsymbol{\pi} \cdot \mathbf{p} - d = 0.$$

If $\mathbf{q}$ is any point, not necessarily on the plane, then the perpendicular distance in the direction of $\tilde{\boldsymbol{\pi}}$ from the point to the plane is given by $\tilde{\boldsymbol{\pi}}^{\mathrm{T}}\tilde{\mathbf{q}}$, see [13]. This can be used to show the following.

Theorem 10 *The perpendicular distance δ, between a single point $\mathbf{p}_1$ with mass m_1, and the plane determined by the other three points $\mathbf{p}_i = (x_i,\ y_i,\ z_i)^{\mathrm{T}}$ each with mass m_i, in a system of four point-masses equimomental to a fixed inertia matrix $\widetilde{\Xi}$, is given by*

$$\delta = \frac{1}{k\sqrt{m_1 m_2 m_3 m_4}} \det(\widetilde{\Xi})^{\frac{1}{2}}$$

where

$$k^2 = \det\begin{pmatrix} y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \\ 1 & 1 & 1 \end{pmatrix}^2 + \det\begin{pmatrix} x_2 & x_3 & x_4 \\ z_2 & z_3 & z_4 \\ 1 & 1 & 1 \end{pmatrix}^2 + \det\begin{pmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ 1 & 1 & 1 \end{pmatrix}^2.$$

Proof The plane through the points $\tilde{\mathbf{p}}_2,\ \tilde{\mathbf{p}}_3$, and $\tilde{\mathbf{p}}_4$ is proportional to the exterior product,

$$k\tilde{\boldsymbol{\pi}} = \tilde{\mathbf{p}}_2 \wedge \tilde{\mathbf{p}}_3 \wedge \tilde{\mathbf{p}}_4,$$

see [17] for example. The constant here is necessary because the first three components of the exterior product do not necessarily form a unit 3-vector. The components of the 4-vector $\tilde{\mathbf{p}}_2 \wedge \tilde{\mathbf{p}}_3 \wedge \tilde{\mathbf{p}}_4$ are given by

$$\det\begin{pmatrix} y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \\ 1 & 1 & 1 \end{pmatrix}, \quad -\det\begin{pmatrix} x_2 & x_3 & x_4 \\ z_2 & z_3 & z_4 \\ 1 & 1 & 1 \end{pmatrix}, \quad \det\begin{pmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } -\det\begin{pmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{pmatrix}.$$

Hence, the constant satisfies

$$k^2 = \det\begin{pmatrix} y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \\ 1 & 1 & 1 \end{pmatrix}^2 + \det\begin{pmatrix} x_2 & x_3 & x_4 \\ z_2 & z_3 & z_4 \\ 1 & 1 & 1 \end{pmatrix}^2 + \det\begin{pmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ 1 & 1 & 1 \end{pmatrix}^2.$$

The distance between $\tilde{\mathbf{p}}_1$ and the plane through the other three points is

$$\delta = \frac{1}{k}\tilde{\boldsymbol{\pi}}^{\mathrm{T}}\tilde{\mathbf{p}}_1 = \frac{1}{k}\tilde{\mathbf{p}}_1 \wedge \tilde{\mathbf{p}}_2 \wedge \tilde{\mathbf{p}}_3 \wedge \tilde{\mathbf{p}}_4 = \frac{1}{k}\det\left(\tilde{\mathbf{p}}_1 \,\middle|\, \tilde{\mathbf{p}}_2 \,\middle|\, \tilde{\mathbf{p}}_3 \,\middle|\, \tilde{\mathbf{p}}_4\right).$$

The result then follows by substituting for the determinant of the extended position vectors of the points using the result of Theorem 9. □

Clearly, we may permute the indices of the points to obtain the distances from the other points in the system to their corresponding planes.

4 Ideal points

By representing points in space as 4-component vectors, non-physical ideal points are introduced. These ideal points are often referred to as points at infinity. It is clear that when choosing four points equimomental to a given rigid body we are not limited to the physical points of three-dimensional space.

The problem of finding four rank one 4×4 symmetric matrices, which sum to the given full rank matrix $\widetilde{\Xi}$, is a standard problem in linear algebra. The standard solution would be to find the eigenvalues, λ_i, and eigenvectors, $\tilde{\mathbf{e}}_i$, of the matrix. For the diagonal matrix $\widetilde{\Xi} = m \operatorname{diag}\left(a^2, b^2, c^2, 1\right)$, we would get, $\widetilde{\Xi} = \lambda_1 \tilde{\mathbf{e}}_1 \tilde{\mathbf{e}}_1^{\mathrm{T}} + \lambda_2 \tilde{\mathbf{e}}_2 \tilde{\mathbf{e}}_2^{\mathrm{T}} + \lambda_3 \tilde{\mathbf{e}}_3 \tilde{\mathbf{e}}_3^{\mathrm{T}} + \lambda_4 \tilde{\mathbf{e}}_4 \tilde{\mathbf{e}}_4^{\mathrm{T}}$, where $\lambda_1 = ma^2$, $\lambda_2 = mb^2$, $\lambda_3 = mc^2$ and $\lambda_4 = m$. The eigenvectors are given by

$$\tilde{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{e}}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{8}$$

Notice that the first three points are non-physical, they lie on the plane at infinity, the other point is located at the body's centre of mass. However, the extended vectors $\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_4$ form an orthonormal frame for $\mathbb{R}^4$, as do the extended vectors $\tilde{\mathbf{q}}_1, \ldots, \tilde{\mathbf{q}}_4$ given in Eq. (3) (after scaling by $1/4$) . As mentioned above, the group $O(4)$ acts transitively on the set of orthonormal frames, and hence this standard solution by eigenvectors (suitably scaled) lies in the family of solutions parameterised by $O(4)/S_4$. In fact, it is easy to see that the orthogonal matrix mapping the $\tilde{\mathbf{e}}_i$ to the (scaled) $\tilde{\mathbf{q}}_i$ vectors is just

$$V = \frac{1}{4}\left(\tilde{\mathbf{q}}_1 \,\middle|\, \tilde{\mathbf{q}}_2 \,\middle|\, \tilde{\mathbf{q}}_3 \,\middle|\, \tilde{\mathbf{q}}_4\right) = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Putting this another way we have that $\tilde{\mathbf{e}}_i = \frac{1}{4} V^{\mathrm{T}} \tilde{\mathbf{q}}_i$.

The above discussion can be summarised in the following theorem.

Theorem 11 *For a general rigid body, an equimomental system of four point-masses can be chosen so that one point is located at the body's centre of mass and the other three are located on the plane at infinity.*

Notice that this gives a practical way of finding equimomental systems of point-masses, all that is required is to compute the eigenvalues and eigenvectors of the pseudo-inertia matrix and then select a suitable $O(4)$ rotation if physical points are required.

These ideas can be extended to consider systems with only two or one point on the plane at infinity.

Theorem 12 *For a general rigid body, an equimomental system of four point-masses can be chosen so that two are located on the plane at infinity and the line joining the finite points passes through the centre of mass of the body.*

Proof Suppose that the pseudo-inertia matrix of the body has the eigenvector expansion as given in (8). That is,

$$\widetilde{\Xi} = \lambda_1 \tilde{\mathbf{e}}_1 \tilde{\mathbf{e}}_1^{\mathrm{T}} + \lambda_2 \tilde{\mathbf{e}}_2 \tilde{\mathbf{e}}_2^{\mathrm{T}} + \lambda_3 \tilde{\mathbf{e}}_3 \tilde{\mathbf{e}}_3^{\mathrm{T}} + \lambda_4 \tilde{\mathbf{e}}_4 \tilde{\mathbf{e}}_4^{\mathrm{T}} = m\widetilde{D} I_4 \widetilde{D}^{\mathrm{T}},$$

where as in previous Sections $\widetilde{D} = \operatorname{diag}(a, b, c, 1)$. Now, apply the 4D rotation U, given in (7), to the points $\tilde{\mathbf{e}}_i$. Under this rotation, the first two ideal points do not move: $U\tilde{\mathbf{e}}_1 = \tilde{\mathbf{e}}_1$ and $U\tilde{\mathbf{e}}_2 = \tilde{\mathbf{e}}_2$. The other two points become

$$U\tilde{\mathbf{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} = \sin\theta \begin{pmatrix} 0 \\ 0 \\ \cot\theta \\ 1 \end{pmatrix} \quad \text{and} \quad U\tilde{\mathbf{e}}_4 = \begin{pmatrix} 0 \\ 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} = \cos\theta \begin{pmatrix} 0 \\ 0 \\ -\tan\theta \\ 1 \end{pmatrix}.$$

For all values of θ, except $\theta = 0,\ \pi$, and $\theta = \pm\frac{\pi}{2}$, these two points lie in $\mathbb{R}^3$. That is, they are finite points. Let us write

$$\tilde{\mathbf{e}}'_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \tilde{\mathbf{e}}'_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \tilde{\mathbf{e}}'_3 = \begin{pmatrix}0\\0\\\cot\theta\\1\end{pmatrix},\quad \tilde{\mathbf{e}}'_4 = \begin{pmatrix}0\\0\\-\tan\theta\\1\end{pmatrix}.$$

So, the finite points $\mathbf{e}'_3$ and $\mathbf{e}'_4$ will have masses $\lambda_3\sin^2\theta$ and $\lambda_4\cos^2\theta$, respectively. The pseudo-inertia matrix of the body is then given by

$$\widetilde{\Xi} = \lambda_1\,\tilde{\mathbf{e}}'_1\,\tilde{\mathbf{e}}'^{\mathrm{T}}_1 + \lambda_2\,\tilde{\mathbf{e}}'_2\tilde{\mathbf{e}}'^{\mathrm{T}}_2 + \lambda_3\sin^2\theta\,\tilde{\mathbf{e}}'_3\,\tilde{\mathbf{e}}'^{\mathrm{T}}_3 + \lambda_4\cos^2\theta\,\tilde{\mathbf{e}}'_4\,\tilde{\mathbf{e}}'^{\mathrm{T}}_4.$$

It is straightforward to see that the centre of mass, located at $\mathbf{c} = (0,\ 0,\ 0)^{\mathrm{T}}$ in these coordinates, lies on the line joining $\mathbf{e}'_3$ and $\mathbf{e}'_4$; specifically $\mathbf{c} = \tan\theta\,\mathbf{e}'_3 + \cot\theta\,\mathbf{e}'_4$. □

In a similar vein we have

Theorem 13 *For a general rigid body, an equimomental system of four point-masses can be chosen so that one point is located on the plane at infinity and the plane determined by the finite points contains the centre of mass of the body.*

Proof Here we can proceed as in the previous theorem to produce the points, $\tilde{\mathbf{e}}'_1,\ \tilde{\mathbf{e}}'_1,\ \tilde{\mathbf{e}}'_3$, and $\tilde{\mathbf{e}}'_4$. But now we subject these points to another 4D rotation, say,

$$V = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\phi & 0 & -\sin\phi\\ 0 & 0 & 1 & 0\\ 0 & \sin\phi & 0 & \cos\phi\end{pmatrix}.$$

This will give the points

$$\tilde{\mathbf{e}}''_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \tilde{\mathbf{e}}''_2 = \begin{pmatrix}0\\\cot\phi\\0\\1\end{pmatrix},\quad \tilde{\mathbf{e}}''_3 = \cos\phi\begin{pmatrix}0\\-\tan\phi\\\cot\theta\sec\phi\\1\end{pmatrix},\quad \tilde{\mathbf{e}}''_4 = \cos\phi\begin{pmatrix}0\\-\tan\phi\\-\tan\theta\sec\phi\\1\end{pmatrix}.$$

The pseudo-inertia matrix for the body can then be written

$$\widetilde{\Xi} = \lambda_1\,\tilde{\mathbf{e}}''_1\,\tilde{\mathbf{e}}''^{\mathrm{T}}_1 + \lambda_2\sin^2\phi\,\tilde{\mathbf{e}}''_2\tilde{\mathbf{e}}''^{\mathrm{T}}_2 + \lambda_3\cos^2\phi\sin^2\theta\,\tilde{\mathbf{e}}''_3\,\tilde{\mathbf{e}}''^{\mathrm{T}}_3 + \lambda_4\cos^2\phi\cos^2\theta\,\tilde{\mathbf{e}}''_4\,\tilde{\mathbf{e}}''^{\mathrm{T}}_4.$$

The fourth column of $\widetilde{\Xi}$ gives the body's centre of mass, but now it will be a linear combination of $\tilde{\mathbf{e}}''_2$, $\tilde{\mathbf{e}}''_3$, and $\tilde{\mathbf{e}}''_4$. So, we see that the centre of mass of the body will lie on the plane determined by the three finite point-masses. □

5 Dynamics

Here the equations of motion for a rigid body using the pseudo-inertia matrix are found. In this way, it will be straightforward to replace the pseudo-inertia matrix for a rigid body by the equimomental system of point-masses.

The angular and linear momenta of a rigid body are given by

$$\begin{aligned}\mathbf{j} &= \mathbb{I}\,\boldsymbol{\omega} + m\,\mathbf{c}\times\mathbf{v},\\ \mathbf{l} &= m\,\boldsymbol{\omega}\times\mathbf{c} + m\,\mathbf{v}\end{aligned}$$

where $\mathbf{j}$ and $\mathbf{l}$ are the angular and linear momentum vectors of the body, respectively, and $\boldsymbol{\omega}$ and $\mathbf{v}$ are the angular and linear parts of the body's velocity twist. As above, m is the mass of the body, $\mathbf{c}$ is the position of its centre of mass, and $\mathbb{I}$ is the 3×3 inertia matrix of the body.

5.1 Momentum

To proceed we need the following, easily shown, result concerning 3×3 antisymmetric matrices: Let A and B be 3×3 antisymmetric matrices corresponding to the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. That is, $A\,\mathbf{p} = \mathbf{a} \times \mathbf{p}$ for any vector $\mathbf{p}$ and similarly for B. Now it is simple to see that

$$A\,B - B\,A = \mathbf{b}\,\mathbf{a}^{\mathrm{T}} - \mathbf{a}\,\mathbf{b}^{\mathrm{T}} = E$$

where the vector corresponding the antisymmetric matrix E is $\mathbf{e} = \mathbf{a} \times \mathbf{b}$. So, the commutator of two 3×3 antisymmetric matrices corresponds to the vector product of the corresponding vectors.

Now the momentum co-twist of the body can be written as a 4×4 antisymmetric matrix $\widetilde{M}$,

$$\widetilde{M} = S\,\widetilde{\Xi} - \widetilde{\Xi}\,S^{\mathrm{T}} = \begin{pmatrix} \Omega & \mathbf{v} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Xi & m\,\mathbf{c} \\ m\,\mathbf{c}^{\mathrm{T}} & m \end{pmatrix} - \begin{pmatrix} \Xi & m\,\mathbf{c} \\ m\,\mathbf{c}^{\mathrm{T}} & m \end{pmatrix} \begin{pmatrix} -\Omega & 0 \\ \mathbf{v}^{\mathrm{T}} & 0 \end{pmatrix} = \begin{pmatrix} J & \mathbf{l} \\ -\mathbf{l}^{\mathrm{T}} & 0 \end{pmatrix}.$$

The 3×3 antisymmetric matrix J here corresponds to the angular momentum vector $\mathbf{j}$ since

$$J = \Omega\,\Xi + \Xi\,\Omega + m\,\left(\mathbf{v}\,\mathbf{c}^{\mathrm{T}} - \mathbf{c}\,\mathbf{v}^{\mathrm{T}}\right).$$

To see this recall that

$$\Xi = \int_{\text{body}} \mathbf{p}\,\mathbf{p}^{\mathrm{T}}\,\rho\,\mathrm{dvol},$$

so

$$\Omega\,\Xi + \Xi\,\Omega = \int_{\text{body}} \left((\boldsymbol{\omega} \times \mathbf{p})\,\mathbf{p}^{\mathrm{T}} - \mathbf{p}\,\left(\boldsymbol{\omega} \times \mathbf{p}\right)^{\mathrm{T}}\right)\,\rho\,\mathrm{dvol} = \int_{\text{body}} E\,\rho\,\mathrm{dvol}$$

where E is the 3×3 antisymmetric matrix corresponding to the vector $\mathbf{e} = \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p})$, using the relation for the commutator above. Integrating this vector would give

$$\int_{\text{body}} \mathbf{e}\,\rho\,\mathrm{dvol} = \int_{\text{body}} \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p})\,\,\rho\,\mathrm{dvol} = \mathbb{I}\,\boldsymbol{\omega}.$$

Using the relation for the commutator above again, we see that the vector corresponding to the 3×3 antisymmetric matrix $J = \Omega\,\Xi + \Xi\,\Omega + m\,\left(\mathbf{v}\,\mathbf{c}^{\mathrm{T}} - \mathbf{c}\,\mathbf{v}^{\mathrm{T}}\right)$ is just the angular momentum,

$$\mathbf{j} = \mathbb{I}\,\boldsymbol{\omega} + m\,\mathbf{c} \times \mathbf{v}.$$

Under a rigid body displacement, the momentum co-twist transforms according to the representation

$$\widetilde{M}' = G\,\widetilde{M}\,G^{\mathrm{T}}.$$

This can be seen by looking at the transformation properties of the pseudo-inertia matrix and the 4×4 twist matrices,

$$S' = G\,S\,G^{-1} \quad \text{and} \quad \Xi' = G\,\Xi\,G^{\mathrm{T}}.$$

5.2 Equations of motion

The equations of motion of a rigid body can be found by computing the time derivative of the momentum co-twist $\widetilde{M}$,

$$\widetilde{M} = S\,\widetilde{\varXi} - \widetilde{\varXi}\,S^{\mathrm{T}} = S\,G\,(t)\;\widetilde{\varXi}_0\,G^{\mathrm{T}}\,(t) - G\,(t)\;\widetilde{\varXi}_0\,G^{\mathrm{T}}\,(t)\;S^{\mathrm{T}}.$$

So,

$$\frac{d}{\mathrm{d}t}\widetilde{M} = \dot{S}\,\widetilde{\varXi} - \widetilde{\varXi}\,\dot{S}^{\mathrm{T}} + S^2\,\widetilde{\varXi} - \widetilde{\varXi}\,S^{2T},$$

since $\frac{d}{\mathrm{d}t}G\,(t) = S\,G\,(t)$.

The time derivative of the momentum co-twist is the wrench applied to the body. This must be written in a form compatible with the representation above, so the force and moment vector must be combined into a 4×4 antisymmetric matrix,

$$\widetilde{W} = \begin{pmatrix} \varGamma & \mathbf{F} \\ -\mathbf{F}^{\mathrm{T}} & 0 \end{pmatrix},$$

where $\varGamma$ is the 3×3 antisymmetric matrix corresponding to the torque $\boldsymbol{\tau}$ applied to the body and $\mathbf{F}$ is the force applied to the body. So, the equation of motion of a rigid body can be written in this 4×4 matrix formalism as

$$\dot{S}\,\widetilde{\varXi} - \widetilde{\varXi}\,\dot{S}^{\mathrm{T}} + S^2\,\widetilde{\varXi} - \widetilde{\varXi}\,S^{2T} = \widetilde{W}. \tag{9}$$

Now suppose we replace the pseudo-inertia matrix $\widetilde{\varXi}$ with an equimomental system of four point-masses. Let us assume that

$$\widetilde{\varXi} = m_1\,\tilde{\mathbf{p}}_1\,\tilde{\mathbf{p}}_1^{\mathrm{T}} + m_2\,\tilde{\mathbf{p}}_2\,\tilde{\mathbf{p}}_2^{\mathrm{T}} + m_3\,\tilde{\mathbf{p}}_3\,\tilde{\mathbf{p}}_3^{\mathrm{T}} + m_4\,\tilde{\mathbf{p}}_4\,\tilde{\mathbf{p}}_4^{\mathrm{T}}.$$

Substituting this into the equation of motion (9) for the rigid body will produce terms of the form

$$m_i\,(\dot{S} + S^2)\tilde{\mathbf{p}}_i\,\tilde{\mathbf{p}}_i^{\mathrm{T}} = m_i\left(\begin{pmatrix} \dot{\varOmega} & \dot{\mathbf{v}} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \varOmega^2 & \varOmega\,\mathbf{v} \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} \mathbf{p}_i \\ 1 \end{pmatrix}\begin{pmatrix} \mathbf{p}_i^{\mathrm{T}} & 1 \end{pmatrix}$$

and their transpose. Comparing the above equation with the expression for the acceleration of a point we see that the above can be written

$$m_i\,(\dot{S} + S^2)\,\tilde{\mathbf{p}}_i\,\tilde{\mathbf{p}}_i^{\mathrm{T}} = m_i\begin{pmatrix} \ddot{\mathbf{p}}_i \\ 0 \end{pmatrix}\begin{pmatrix} \mathbf{p}_i^{\mathrm{T}} & 1 \end{pmatrix}.$$

Then subtracting the transpose of this gives

$$m_i\,\left((\dot{S} + S^2)\,\tilde{\mathbf{p}}_i\,\tilde{\mathbf{p}}_i^{\mathrm{T}} - \tilde{\mathbf{p}}_i\,\tilde{\mathbf{p}}_i^{\mathrm{T}}\,(\dot{S} + S^2)^{\mathrm{T}}\right) = m_i\begin{pmatrix} \ddot{\mathbf{p}}_i\,\mathbf{p}_i^{\mathrm{T}} - \mathbf{p}_i\,\ddot{\mathbf{p}}_i^{\mathrm{T}} & \ddot{\mathbf{p}}_i \\ -\ddot{\mathbf{p}}_i^{\mathrm{T}} & 0 \end{pmatrix}.$$

The 3×3 antisymmetric matrix $\ddot{\mathbf{p}}_i\,\mathbf{p}_i^{\mathrm{T}} - \mathbf{p}_i\,\ddot{\mathbf{p}}_i^{\mathrm{T}}$ corresponds to the 3-vector $\mathbf{p}_i \times \ddot{\mathbf{p}}_i$. The equation of motion for the rigid body (9) can therefore be written in terms of 6-component vectors as

$$m_1\begin{pmatrix} \mathbf{p}_1 \times \ddot{\mathbf{p}}_1 \\ \ddot{\mathbf{p}}_1 \end{pmatrix} + m_2\begin{pmatrix} \mathbf{p}_2 \times \ddot{\mathbf{p}}_2 \\ \ddot{\mathbf{p}}_2 \end{pmatrix} + m_3\begin{pmatrix} \mathbf{p}_3 \times \ddot{\mathbf{p}}_3 \\ \ddot{\mathbf{p}}_3 \end{pmatrix} + m_4\begin{pmatrix} \mathbf{p}_4 \times \ddot{\mathbf{p}}_4 \\ \ddot{\mathbf{p}}_4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{F} \end{pmatrix}. \tag{10}$$

These 6-component vectors are, of course, wrenches; $\boldsymbol{\tau}$ is the total moment acting on the body—corresponding to the 3×3 antisymmetric matrix $\varGamma$.

Notice that for each point-mass we have a wrench, $m_i\begin{pmatrix} \mathbf{p}_i \times \ddot{\mathbf{p}}_1 \\ \ddot{\mathbf{p}}_i \end{pmatrix}$. This could be thought of as a line through the point $\mathbf{p}_i$ in the direction of the acceleration of the point. The set of all such lines is known to form a tetrahedral line complex, see [12].

Suppose that as a set of four equimomental point-masses we choose to use the centre of mass and three ideal point as found in Sect. 4. Let us write these as $\tilde{\mathbf{e}}_1,\ \tilde{\mathbf{e}}_2,\ \tilde{\mathbf{e}}_3$ for the ideal points and $\tilde{\mathbf{c}}$ for the centre of mass. Then since the fourth component of any ideal point is zero, it is not difficult to see that the equations of motion become

$$\lambda_1\begin{pmatrix} \mathbf{e}_1 \times \ddot{\mathbf{e}}_1 \\ \mathbf{0} \end{pmatrix} + \lambda_2\begin{pmatrix} \mathbf{e}_2 \times \ddot{\mathbf{e}}_2 \\ \mathbf{0} \end{pmatrix} + \lambda_3\begin{pmatrix} \mathbf{e}_3 \times \ddot{\mathbf{e}}_3 \\ \mathbf{0} \end{pmatrix} + m\begin{pmatrix} \mathbf{c} \times \ddot{\mathbf{c}} \\ \ddot{\mathbf{c}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{F} \end{pmatrix}. \tag{11}$$

Here $m = m_1 + m_2 + m_3 + m_4$ is the total mass of the body, and each λ_i is m times the radius of gyration about the axis determined by $\mathbf{e}_i$.

If the origin of the coordinate system used is located at the body's centre of mass then clearly $\mathbf{c} = \mathbf{0}$ though $\dot{\mathbf{c}} \neq \mathbf{0}$. This means that the equations of motion (11) decouple, and hence the problems involving the equations of motion of a rigid body can be significantly simplified.

6 Conclusions

The main idea behind this work is to explore the concept of equimomental systems of point-masses. Although some of the ideas here are rather old and well studied, for example the dynamics of a rigid body, it is hoped that a new look at these problems with the benefit of modern methods may lead to new insights.

There are several applications of these ideas which could be pursued. The problem of mechanism balancing is a key example. This seems to have been extensively studied by Chaudhary and co-workers, see for example [2]. However, as mentioned above, that work seems to concentrate on systems of seven point-masses. Another possibility might be to look at the design of robot links in order to simplify the dynamics computations. Some progress in this direction was made in [14]. Finally, another possible application might be found in the problem of identifying the inertia parameters of a body from experimental data.

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