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FIXED POINT RESULTS OF F-RATIONAL CYCLIC CONTRACTIVE MAPPINGS ON 0-COMPLETE PARTIAL METRIC SPACES

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Abstract. Wardowski [19] introduced a new concept of contraction which called F-contraction and proved a fixed point theorem on complete metric space. Following this direction of research, in this paper, we introduce an $F-$rational cyclic contraction on

partial metric spaces and we present new fixed point results for such cyclic contraction in 0-complete partial metric spaces. An example is given to illustrate the main result, also an application to integral equation is given to show the usability of our results.
Keywords: fixed point, F-contractions, (0−complete) partial metric space.

1. Introduction

In 1994, S. G. Matthews [14] introduced the notion of partial metric spaces and obtained various fixed point theorems. In fact, he showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification.

Later on, Romaguera [16] introduced the notions of 0-Cauchy sequences and 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

In 2012, Wardowski [19] introduced a new type of contraction called F-contraction and proved a new fixed point theorem concerning F-contraction. Furthermore, Abbas et al.,[1] generalized the concept of $F-$contraction and proved certain fixed and common fixed point results. Afterwards Secelean [17] proved fixed point theorems consisting of $F-$contractions by iterated function systems. Piri et al.,[15] proved a fixed point result for F-Suzuki contractions for some weaker conditions on the self map of a complete metric space which generalizes the result of Wardowski. Lately, Acar et al.,[5] introduced the concept of generalized multivalued $F-$contraction mappings. Further Altun et al.,[4] extended multivalued mappings with $\delta-$Distance and established fixed point results in complete metric space. Sgroi et al.,[18] established fixed point theorems for multivalued $F-$contractions and obtained the solution of certain functional and integral equations, which was a proper generalization of some multivalued fixed point theorems including Nadler's. Recently Ahmad et al.,[6],[9] recalled the concept of $F-$contraction to obtain some fixed point, and common fixed point results in the context of complete metric spaces.

In this paper, we introduce an $F-$rational cyclic contraction on partial metric spaces and we present new fixed point results for such cyclic contraction in 0-complete partial metric spaces. An example is given to illustrate the main result, also an application to integral equation is given to show the usability of our results.

2. Preliminaries

First we recall some definitions and properties of partial metric spaces.

Definition 2.1. [14] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$($\mathbb{R}^+$ stands for nonnegative reals) such that for all $x, y, z \in X$:

(P1) $x = y \Leftrightarrow p\,(x, x) = p\,(y, y) = p\,(x, y)\,;$

(P2) $p(x, x) \leq p(x, y);$

(P3) $p(x,y) = p(y,x)$;

(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric spaces is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x,y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0. Also, every metric space is a partial metric space, with zero self distance.

Example 2.2. [14] If $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x,y) = \max\{x,y\}$, for all $x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a partial metric space.

For more examples of partial metric spaces, we refer the reader to [8] and the references therein.

Each partial metric p on X generates a T_0 topology $\tau(p)$ on X which has a base topology of open $p-$balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ and $B_p(x,\varepsilon) = \{y \in X : p(x,y) < \varepsilon + p(x,x)\}$.

A mapping $f : X \to X$ is continuous if and only if, whenever a sequence $\{x_n\}$ in X converging with respect to $\tau(p)$ to a point $x \in X$, the sequence $\{fx_n\}$ converges with respect to $\tau(p)$ to $fx \in X$.

Let (X,p) be a partial metric space.

(i) A sequence $\{x_n\}$ in partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n, x)$.

(ii) A sequence $\{x_n\}$ in partial metric space (X,p) is called Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$. The space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

(iii) A sequence $\{x_n\}$ in partial metric space (X,p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges (in $\tau(p)$) to a point $x \in X$ such that $p(x,x) = 0$.

Lemma 2.3. *Let (X,p) be a partial metric space.*

(a) *[2],[12] If $p(x_n, z) \to p(z,z) = 0$ as $n \to \infty$, then $p(x_n, y) \to p(z,y)$ as $n \to \infty$ for each $y \in X$.*

(b) *[16] If (X,p) is complete, then it is $0-$complete.*

It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete. The following example shows that the converse assertion of (b) need not hold.

Example 2.4 ([16])**.** The space $X = [0,+\infty) \cap \mathbb{Q}$ with the partial metric $p(x,y) = \max\{x,y\}$ is 0-complete, but is not complete. Moreover, the sequence $\{x_n\}$ with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X,p), but it is not a 0-Cauchy sequence.

Definition 2.5 ([11])**.** Let (X, d) be a metric space and $f : X \to X$ be a mapping. Then it is said that f satisfies the orbital condition if there exists a constant $k \in (0, 1)$ such that

$$d\left(fx, f^2x\right) \leq k\ d(x, fx), \tag{2.1}$$

for all $x \in X$.

Theorem 2.6 ([3])**.** *Let (X, p) be a $0-$complete partial metric space and $f : X \to X$ be continuous such that*

$$p\left(fx, f^2x\right) \leq k\ p(x, fx) \tag{2.2}$$

holds for all $x \in X$, where $k \in (0, 1)$. Then there exists $z \in X$ such that $p(z, z) = 0$ and $p(fz, z) = p(fz, fz)$.

Definition 2.7 ([11])**.** Let (X, p) be a partial metric space and $f : X \to X$ be a mapping with fixed point set $Fix(f) \neq \phi$. Then f has property (P) if $Fix(f^n) = Fix(f)$, for each $n \in N$.

Lemma 2.8 ([11])**.** *Let (X, p) be a partial metric space, $f : X \to X$ be a self map such that Fix $(f) \neq \phi$. Then f has the property (P) if (2.2) holds for some $k \in (0, 1)$ and either (i) for all $x \in X$, or (ii) for all $x \neq fx$.*

One of the remarkable generalizations of Banach's contraction principle was reported by Kirk et al.,[13] via cyclic contraction.

Theorem 2.9 ([13])**.** *Let $\{A_i\}_{i=1}^m$ be a nonempty closed subset of a complete metric space (X, d) and suppose $f : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ be a mapping satisfying the following conditions:*

(1) *$f(A_i) \subseteq A_{i+1}$ for $1 \leq i \leq m$, where $A_{m+1} = A_1$.*

(2) *$d(fx,\ fy) \leq \psi(d(x, y))$, for all $x \in A_i,\ y \in A_{i+1},\ i \in \{1, 2, \cdots, m\}$,*

where $A_{m+1} = A_1$ and $\psi : [0, 1) \to [0, 1)$ is a function, upper semi-continuous from the right and $0 \leq \psi(t) < t$ for $t > 0$. Then, f has a fixed point $z \in \bigcap_{i=1}^m A_i$.

Wardowski [19] defined the $F-$contraction as follows.

Definition 2.10 ([19])**.** Let (X, d) be a complete metric space. A self mapping $f : X \to X$ is said to be an $F-$contraction if there exists a constant $\lambda > 0$ such that

$$\forall\ x, y \in X,\ d(fx, fy) > 0 \Rightarrow \lambda + F(d(fx, fy)) \leq F(d(x, y)). \tag{1.3}$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, i.e. for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if, $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0, 1)$ such that $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$.

We denote $\mathcal{F}$ the family of all functions F that satisfy the conditions $(F1) - (F3)$.

Example 2.11 ([19])**.** The Family of $\mathcal{F}$ is not empty.

1. $F(x) = \ln(x); x > 0$;
2. $F(x) = x + \ln(x); x > 0$;
3. $F(x) = \ln(x^2 + x); x > 0$;
4. $F(x) = \frac{-1}{\sqrt{x}}; x > 0$.

Theorem 2.12 ([19])**.** *Let (X, d) be a complete metric space and $f : X \to X$ be a $F-$contraction. If f or F is continuous, then we have*

(1) f has a unique fixed point $x^* \in X$.
(2) For all $x \in X$, the sequence $\{T^n x\}$ is convergent to $x^{*.}$

3. Main results and discussion

Let (X, p) be a partial metric space, through out of this paper we mean by Δ_p be the set of all nonempty closed subsets of X.

Definition 3.1. Let (X, p) be a partial metric space, $V_i \in \Delta_p$ for $i = 1, 2, \cdots, m$, $E = \bigcup_{i=1}^{m} V_i$ where $m \in N$. A mapping $f : E \to E$ is called an F-rational cyclic contraction if there exists $F \in \mathcal{F}$ and $\lambda \in \mathbb{R}_+$ such that

1. $f(V_i) \subseteq V_{i+1}, i = 1, 2, ..., m,$ where $V_{m+1} = V_1$,
2. For $x \in V_i$, $y \in V_{i+1}$, $i = 1, 2, ..., m$, with $p(fx, fy) > 0$, we have

$$\lambda + F(p(fx, fy)) \le F(\mathcal{H}_f(x, y)), \tag{3.1}$$

where

$$\mathcal{H}_f(x, y) = ap(x, y) + bp(x, fx) + cp(y, fy) + dp(x, fy) + ep(y, fx) + l\frac{p(x, fx).p(y, fy)}{1 + p(x, y)}, \tag{3.2}$$

and

$$a, b, c, d, e, l \ge 0 \text{ with } a + b + c + d + e + l < 1. \tag{3.3}$$

The main result of this section is the following.

Theorem 3.2. *Let (X,p) be a 0-complete partial metric space, $V_i \in \Delta_p$; $i = 1,2,\cdots,m$ where $m \in \mathbb{N}$ and $E = \bigcup_{i=1}^m V_i$. Suppose that $f : E \to E$ is an F-rational cyclic contraction. Then,*

1. *f has a unique fixed point $z \in E$.*
2. *$p(z,z) = 0$ and $z \in \bigcap_{i=1}^m V_i$.*
3. *for any $x_0 \in E$, the sequence $x_n = f^n x_0$, converges to z in topology $\tau(p)$.*

Proof. Let $x_0 \in E$ be an arbitrary point. Then there exists i_0 such that $x_0 \in V_{i_0}$, so there is $x_1 \in V_{i_0+1}$ where $x_1 = fx_0$. Continue in this process we can construct a sequence $x_n = fx_{n-1} = f^n x_0 \in V_{i_0+n}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of f. From now on assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$ and let $p_n = p(x_n, x_{n+1})$, so $p_n > 0$ for all $n \in \mathbb{N}$. Since $f : E \to E$ is an F-rational cyclic contraction. So, from (3.1) and (3.2) we have that

$$\begin{aligned}
&\lambda + F(p_n) = \lambda + F(p(x_n, x_{n+1})) \\
&= \lambda + F(p(fx_{n-1}, fx_n)) \\
&\leq F\left(\begin{array}{c} ap(x_{n-1}, x_n) + bp(x_{n-1}, x_n) + cp(x_n, x_{n+1}) + dp(x_{n-1}, x_{n+1}) \\ +ep(x_n, x_n) + l\frac{p(x_{n-1},x_n).p(x_n,x_{n+1})}{1+p(x_{n-1},x_n)} \end{array}\right).
\end{aligned}$$

Since $p(x_{n-1}, x_{n+1}) \leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)$, F is strictly increasing and $\frac{p(x_{n-1},x_n).p(x_n,x_{n+1})}{1+p(x_{n-1},x_n)} < p(x_n, x_{n+1})$, then the above inequality becomes

$$\lambda + F(p_n) \leq F((a+b+d)p_{n-1} + (c+d+l)p_n + (e-d)p(x_n, x_n)). \tag{3.4}$$

Since $\lambda > 0$, then

$$F(p_n) \leq \lambda + F(p_n) \leq F((a+b+d)p_{n-1} + (c+d+l)p_n + (e-d)p(x_n, x_n)).$$

But, F is strictly increasing, so we deduce that

$$p_n \leq (a+b+d)p_{n-1} + (c+d+l)p_n + (e-d)p(x_n, x_n)). \tag{3.5}$$

By symmetry of $p(x_{n+1}, x_n) = p(x_n, x_{n+1})$, and using similar argument as above one can deduce that

$$\begin{aligned}
\lambda + F(p(x_{n+1}, x_n)) &= \lambda + F(p(fx_n, fx_{n-1})) \\
&\leq F((a+c+e)p_{n-1} + (b+e+l)p_n + (d-e)p(x_n, x_n)).
\end{aligned}$$

Thus,

$$F(p_n) \leq \lambda + F(p_n) \leq F((a+c+e)p_{n-1} + (b+e+l)p_n + (d-e)p(x_n, x_n))$$

which implies that

$$p_n \leq (a+c+e)p_{n-1} + (b+e+l)p_n + (d-e)p(x_n, x_n). \tag{3.6}$$

Adding up, equations (3.5) and (3.6) we get $p_n \leq \beta p_{n-1}$, where $\beta = \frac{2a+b+c+d+e}{2-b-c-d-e-2l} < 1$, which is a consequence of (3.3). Hence,

$$p_n < p_{n-1}, \text{ for all } n \in \mathbb{N} \tag{3.7}$$

Using property(P2) of partial metric, equations (3.4), (3.7) and the property of strictly increasing of F we get

$$\begin{aligned} \lambda + F(p_n) &\leq F\left((a+b+d)\, p_{n-1} + (c+d+l)\, p_n + (e-d)\, p(x_n, x_n)\right) \\ &\leq F\left((a+b+d)\, p_{n-1} + (c+d+l)\, p_{n-1} + (e-d)\, p_{n-1}\right) \\ &= F\left((a+b+c+d+e+l)\, p_{n-1}\right) \\ &\leq F(p_{n-1}). \end{aligned}$$

Hence, $\lambda + F(p_n) \leq F(p_{n-1})$ for all $n \in \mathbb{N}$. This implies

$$F(p_n) \leq F(p_{n-1}) - \lambda \leq \cdots \leq F(p_0) - n\lambda, \text{ for all } n \in \mathbb{N} \tag{3.8}$$

and so $\lim_{n\to+\infty} F(p_n) = -\infty$. By the property $(F2)$, we get that $p_n \to 0$ as $n \to +\infty$.

Now, by (F3) there exist $k \in (0,1)$ such that $\lim_{n\to+\infty} p_n^k F(p_n) = 0$. By (3.8), the following holds for all $n \in \mathbb{N}$:

$$p_n^k F(p_n) - p_n^k F(p_0) \leq -n\lambda p_n^k \leq 0. \tag{3.9}$$

Letting $n \to +\infty$ in (3.9) we deduce that

$$\lim_{n\to+\infty} n p_n^k = 0. \tag{3.10}$$

By using the continuous function $g(x) = x^{\frac{1}{k}}$; $x \in (0, \infty)$, we get that

$$\lim_{n\to+\infty} n^{\frac{1}{k}} p_n = \lim_{n\to+\infty} g(n p_n^k) = 0. \tag{3.11}$$

Now, by using the limit comparison test with $a_n = p_n$, $b_n = n^{\frac{-1}{k}}$ and equation (3.10) we ensure that the series $\sum_{n=1}^{+\infty} p_n$ is convergent. This implies that $\{x_n\}$ is a $0-$Cauchy sequence. Since E is closed in a 0-complete partial metric (X, p), then E is also 0-complete and there exist $z \in E = \bigcup_{i=1}^{m} V_i$ such that

$$\lim_{n\to\infty} p(x_n, z) = 0 = p(z, z). \tag{3.12}$$

Notice that the iterative sequence $\{x_n\}$ has an infinite number of terms in V_i for each $i = 1, ..., m$. Hence, there is a subsequence of $\{x_n\}$ in each V_i, $i = 1, ..., m$, which converges to z. Using that each V_i, $i = 1, ..., m$, is closed, we conclude that $z \in \bigcap_{i=1}^{m} V_i$.

We shall prove that z is a fixed point of f. Using the triangle inequality (p4) of partial metric space and (3.2) (which is possible since z belongs to each V_i) to obtain

$$\begin{aligned} p(z,fz) &\leq p(z,x_{n+1})+p(x_{n+1},fz)-p(x_{n+1},x_{n+1}) \\ &\leq p(z,x_{n+1})+p(fx_n,fz) \\ &\leq p(z,x_{n+1})+ap(x_n,z)+bp(x_n,x_{n+1})+cp(z,fz)+dp(x_n,fz) \\ &\quad +ep(x_{n+1},z)+l\ \frac{p(x_n,x_{n+1}).p(z,fz)}{1+p(x_n,z)}. \end{aligned} \tag{3.13}$$

Using Lemma 2.3 part (a) and passing to the limit when $n\to\infty$ in (3.13), we obtain that

$$(1-c-d)\,p(z,fz)\leq 0,$$

and hence

$$p(z,fz)=0. \tag{3.14}$$

Now by using triangle inequality (P4), (3.14) and (3.12) we deduce that $p(fz,fz)$ $=0$. Therefore, by (P1) we get $f(z)=z$.

Finally, we will prove the uniqueness, let u be another fixed point of f in E, with $p(u,z)\neq 0$. By the cyclic character of f, we have $u,z\in\bigcap_{i=1}^m V_i$. Since f is an F-rational cyclic contraction and using the property (P2) of partial metric, we have

$$\begin{aligned} \lambda+F(p(u,z)) &= \lambda+F(p(fu,fz)) \\ &\leq F\left(\begin{array}{c} ap(u,z)+bp(u,u)+cp(z,z)+dp(u,z)+ep(u,z) \\ +l\frac{p(u,fu).p(z,fz)}{1+p(u,z)} \end{array}\right) \\ &\leq F((a+b+c+d+e)\,p(u,z)), \end{aligned}$$

which is a contradiction deduced from the strictly increasing property of F and being $a+b+c+d+e<1$, hence $z=u$. Thus z is a unique fixed point of f. □

By taking $F(\alpha)=\alpha+\ln(\alpha)$ in Theorem 3.2 we get the following corollary.

Corollary 3.3. *Let (X,p) be a 0-complete partial metric space, $V_i\in\Delta_p$; $i=1,2,\cdots,m$ where $m\in\mathbb{N}$ and $E=\bigcup_{i=1}^m V_i$. Suppose that $f:E\to E$ and the following conditions are hold:*

1. *$f(V_i)\subseteq V_{i+1}, i=1,2,...,m,$ where $V_{m+1}=V_1$,*
2. *There exist $\lambda>0$ such that for $x\in V_i$, $y\in V_{i+1}$, $i=1,2,...,m$, with $p(fx,fy)>0$, we have*

$$\begin{aligned}\lambda+\ln(p(fx,fy)) &\leq \left(\begin{array}{c} ap\,(x,y)+bp\,(x,fx)+cp\,(y,fy)+dp\,(x,fy) \\ +ep\,(y,fx)+l\frac{p(x,fx).p(y,fy)}{1+p(x,y)} \end{array}\right) \\ &+ \ ln\left(\begin{array}{c} ap\,(x,y)+bp\,(x,fx)+cp\,(y,fy)+dp\,(x,fy) \\ +ep\,(y,fx)+l\frac{p(x,fx).p(y,fy)}{1+p(x,y)} \end{array}\right),\end{aligned}$$

where $a,b,c,d,e,l\geq 0$ *and* $a+b+c+d+e+l<1$. *Then,*

1. f *has a unique fixed point* $z\in E$.
2. $p\,(z,z)=0$ *and* $z\in\bigcap_{i=1}^{m}V_i$.
3. *for any* $x_0\in E$, *the sequence* $x_n=f^nx_0$, *converges to* z *in topology* $\tau(p)$.

By taking $F(\alpha)=\frac{-1}{\sqrt{\alpha}}$ in Theorem 3.2 we get the following corollary.

Corollary 3.4. *Let* (X,p) *be a* 0*-complete partial metric space,* $V_i\in\Delta_p$*;* $i=1,2,\cdots,m$ *where* $m\in\mathbb{N}$ *and* $E=\bigcup_{i=1}^{m}V_i$. *Suppose that* $f:E\rightarrow E$ *and the following conditions are hold:*

1. $f(V_i)\subseteq V_{i+1}, i=1,2,...,m,$ *where* $V_{m+1}=V_1$,
2. *There exist* $\lambda>0$ *such that for* $x\in V_i$, $y\in V_{i+1}$, $i=1,2,...,m$, *with* $p\,(fx,fy)>0$, *we have*

$$\lambda+\frac{-1}{\sqrt{p(fx,fy)}}\leq\frac{-1}{\sqrt{\left(\begin{array}{c} ap\,(x,y)+bp\,(x,fx)+cp\,(y,fy)+dp\,(x,fy)+ep\,(y,fx) \\ +l\frac{p(x,fx).p(y,fy)}{1+p(x,y)} \end{array}\right)}}$$

where $a,b,c,d,e,l\geq 0$ *and* $a+b+c+d+e+l<1$. *Then,*

1. f *has a unique fixed point* $z\in E$.
2. $p\,(z,z)=0$ *and* $z\in\bigcap_{i=1}^{m}V_i$.
3. *for any* $x_0\in E$, *the sequence* $x_n=f^nx_0$, *converges to* z *in topology* $\tau(p)$.

Example 3.5. Let $X=\mathbb{R}$ be equipped with the usual partial metric $p\,(x,y)=\max\{|x|,|y|\}$. Then, clearly (X,p) is $0-$complete. Suppose $V_1=\left[0,\frac{1}{2}\right]$, $V_2=\left[\frac{-1}{6},0\right]$, $V_3=\left[0,\frac{1}{18}\right]$, $V_4=\left[\frac{-1}{54},0\right]$ and $E=\bigcup_{i=1}^{4}V_i$. Define $f:E\rightarrow E$ such that $fx=\frac{-x}{8}$ for all $x\in E$. It is clear that $f(V_i)\subseteq V_{i+1}$.

Take $\lambda = ln(4)$, $a = \frac{1}{2}$ and $b = c = d = e = l = \frac{1}{11}$. Let $x \in V_i$ and $y \in V_{i+1}$ such that either $x \neq 0$ or $y \neq 0$, then

$$\begin{aligned} p(fx, fy) &= \max\{|\frac{-x}{8}|, |\frac{-y}{8}|\} \\ &= \frac{1}{8}\max\{|x|, |y|\} \\ &= \frac{1}{8}p(x, y) \\ &= (\frac{1}{4})(\frac{1}{2})p(x, y). \end{aligned} \tag{3.15}$$

Now take ln for both sides of (3.15) we get

$$\begin{aligned} \ln(p(fx, fy)) &= \ln((\frac{1}{4})(\frac{1}{2})p(x, y)) \\ &= -ln(4) + \ln(\frac{1}{2}p(x, y)) \\ &\leq -ln(4) + \ln\Big(\frac{1}{2}p(x, y) + \frac{1}{11}p(x, fx) + \frac{1}{11}p(y, fy) + \frac{1}{11}p(x, fy) \\ &\quad + \frac{1}{11}p(y, fx) + \frac{1}{11}\frac{p(x, fx)p(y, fy)}{1 + p(x, y)}\Big). \end{aligned}$$

Hence,

$$ln(4) + F(p(x, y)) \leq F(\mathcal{H}_f(x, y)).$$

Therefore, all conditions of Theorem 3.2 are satisfied and we deduce that f has a unique fixed point $z = 0 \in \bigcap_{i=1}^{4} V_i$ and $p(z, z) = 0$ holds true.

Another consequence of Theorem 3.2 is the following.

Theorem 3.6. *Under the assumptions of Theorem 3.2. The function f satisfies the orbital condition (2.2). In particular, there exist $z \in E$ such that $p(z, z) = 0$ and $p(fz, z) = p(fz, fz)$; also, f has the property (P).*

Proof. By Theorem 3.2, the set of fixed points for f is not empty. We will prove that f satisfies condition (2.2) of Theorem 2.6. Let $x \in Y$ be arbitrary. Putting $x = x$ and $y = fx$ in condition (3.1) of Theorem 3.2, we have

$$\begin{aligned} \lambda + F\left(p\left(fx, f^2x\right)\right) &\leq F\left(\mathcal{H}_f(x, fx)\right) \\ &\leq F\left(\begin{array}{c} ap(x, fx) + bp(x, fx) + cp\left(fx, f^2x\right) \\ +dp\left(x, f^2x\right) + ep(fx, fx) + l\frac{p(x,fx).p\left(fx,f^2x\right)}{1+p(x,fx)} \end{array}\right), \end{aligned}$$

By (P4) and repeating the same process as in proof Theorem 3.2, we get that

$$\begin{aligned} &\lambda + F\left(p\left(fx, f^2x\right)\right) \\ &\leq F\left(\begin{array}{c} (a + b + d)p(x, fx) + (c + d + l)p\left(fx, f^2x\right) \\ +(e - d)p(fx, fx) \end{array}\right) \end{aligned} \tag{3.16}$$

by symmetry we have,

$$\begin{aligned} &\lambda + F\left(p\left(fx, f^2x\right)\right) \\ &\leq F\left(\begin{array}{c} (a+c+e)\, p\,(x, fx) + (b+e+l)\, p\left(fx, f^2x\right) \\ + (d-e)\, p\,(fx, fx) \end{array}\right). \end{aligned} \tag{3.17}$$

Using the same argument as in the proof of Theorem 3.2, we deduce that

$$p\left(fx, f^2x\right) \leq \beta p\,(x, fx)\,,$$

where $\beta = \frac{2a+b+c+d+e}{2-(b+c+d+e+2l)} < 1$, which is a consequence of (3.3). Thus, f satisfies the orbital condition. By Theorem 2.6, there exists $z \in E$ such that $p\,(z, z) = 0$ and $p\,(fz, z) = p\,(fz, fz)$. So, by Lemma 2.8, f has the property (P). □

4. Application to integral equations

In this section, we will give an application to some integral equation to show the usability of the main result. Consider the integral equation

$$u(t) = h(u(t)) + \int_0^t H(t,r)\zeta(r, u(r))\, dr, \quad \text{for all } t \in [0,1], \tag{4.1}$$

where, $\zeta : [0,1]\times\mathbb{R} \to \mathbb{R}$, $H : [0,1]\times[0,1] \to \mathbb{R}$ and $h : \mathbb{R} \to [0,\infty)$ are functions.

Let $X = C([0,1])$ be the set of all real continuous functions on $[0,1]$, endowed with the partial metric

$$p(u,v) = \max\{\sup_{t\in[0,1]} |u(t)|, \sup_{t\in[0,1]} |v(t)|\}, \text{ for all } u, v \in X.$$

Clearly, (X, p) is a 0-complete partial metric space.

Let $\kappa, \eta \in X$, $\kappa_0, \eta_0 \in \mathbb{R}$ such that for all $t \in [0,1]$ we have

$$\kappa_0 \leq \kappa(t) \leq \eta(t) \leq \eta_0, \tag{4.2}$$

$$\kappa(t) \leq h(u(t)) + \int_0^t H(t,r)\zeta(r, \eta(r))\, dr, \tag{4.3}$$

and

$$\eta(t) \geq h(u(t)) + \int_0^t H(t,r)\zeta(r, \kappa(r))\, dr. \tag{4.4}$$

Let for all $r \in [0,1]$, $\zeta(r,\cdot)$ and $h(.)$ are decreasing functions, that is,

$$x, y \in \mathbb{R}, x \geq y \text{ implies } \zeta(r,x) \leq \zeta(r,y). \tag{4.5}$$

and

$$h(x) \leq h(y). \tag{4.6}$$

Assume that,

$$\max_{t\in[0,1]} \int_0^1 H(t,s)ds < e^{-\lambda}, \text{for some } \lambda \in (0,\infty). \tag{4.7}$$

and

$$\sup_{r\in[0,1]} |\zeta(r,u(r))| \leq \sup_{r\in[0,1]} |u(r)|. \tag{4.8}$$

Define a mapping $f : X \to X$ by

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))dr; \quad t \in [0,1]. \tag{4.9}$$

Also, suppose that for all $x, y \in \mathbb{R}$ with ($x \leq \eta_0$ and $y \geq \kappa_0$) or ($x \geq \kappa_0$ and $y \leq \eta_0$) we have,

$$|h(u(t))| \leq \tfrac{1}{8} e^{-\lambda} \max\{\sup_{t\in[0,1]} |u(t)|, \sup_{t\in[0,1]} |f(u(t))|\}. \tag{4.10}$$

Theorem 4.1. *Under the assumptions* (4.2)-(4.10)*, the integral equation* (4.1) *has a solution z such that $z \in C([0,1])$ with $\kappa(t) \leq z(t) \leq \eta(t)$ for all $t \in [0,1]$.*

Proof. Define the closed subsets of X, U_1 and U_2 by

$$U_1 = \{u \in X : u \leq \eta\}$$

and

$$U_2 = \{u \in X : u \geq \kappa\}.$$

Also define the mapping $f : U_1 \cup U_2 \to U_1 \cup U_2$ by

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))\, dr, \text{ for all } t \in [0,1].$$

Now we prove that,

$$f(U_1) \subseteq U_2 \text{ and } f(U_2) \subseteq U_1. \tag{4.11}$$

Suppose, $u \in U_1$, that is,

$$u(r) \leq \eta(r), \text{ for all } r \in [0,1].$$

Using condition (4.5) and (4.6) we obtain that

$$\zeta(r,u(r)) \geq \zeta(r,\eta(r)), \text{ for all } r \in [0,1].$$

and

$$h(u(r)) \geq h(\eta(r)), \text{ for all } r \in [0,1].$$

The above inequalities with condition (4.3) imply that

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))\ dr \geq h(\eta(t))$$
$$+ \int_0^t H(t,r)\zeta(r,\eta(r))\ dr = \eta(t) \geq \kappa(t),$$

for all $t \in [0,1]$. Then we have $f(u(t)) \in U_2$. Similarly, let $u \in U_2$, that is,

$$u(r) \geq \kappa(r),\ \text{ for all } r \in [0,1].$$

Using condition (4.5) and (4.6) we obtain that

$$\zeta(r,u(r)) \leq \zeta(r,\kappa(r)),\ \text{ for all } r \in [0,1].$$

and

$$h(u(r)) \leq h(\kappa(r)),\ \text{ for all } r \in [0,1].$$

The above inequalities with condition (4.4) imply that

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))\ dr \leq h(\kappa(t))$$
$$+ \int_0^t H(t,r)\zeta(r,\kappa(r))\ dr = \kappa(t) \leq \eta(t),$$

for all $t \in [0,1]$. Then we have $f(u(t)) \in U_1$. Also, we deduce that (4.11) holds.

Let, $x \in U_1$ and $y \in U_2$. Then from (4.9), for all $t \in [0,1]$, we have

$$\begin{aligned}
|f(x(t))| &= |h(x(t)) + \int_0^t H(t,r)\zeta(r,x(r))\ dr| \\
&\leq |h(x(t))| + |\int_0^t H(t,r)\zeta(r,x(r))\ dr| \\
&\leq |h(x(t))| + \int_0^t |H(t,r)||\zeta(r,x(r))|\ dr \\
&\leq |h(x(t))| + \int_0^t |H(t,r)| \max\{\sup_{r\in[0,1]} |\zeta(r,x(r))|, \sup_{r\in[0,1]} |\zeta(r,y(r))|)\ dr \\
&\leq |h(x(t))| + \max_{t\in[0,1]} \int_0^t H(t,r)p(x,y)dr \\
&\leq |h(x(t))| + \frac{1}{8}e^{-\lambda}p(x,y) \\
&\leq \frac{1}{8}e^{-\lambda}p(x,fx) + \frac{1}{8}e^{-\lambda}p(x,y) \\
&= e^{-\lambda}(\frac{1}{8}p(x,fx) + \frac{1}{8}p(x,y)).
\end{aligned}$$

Thus,

$$\sup_{t\in[0,1]} |f(x(t))| \leq e^{-\lambda}(\frac{1}{8}p(x, fx) + \frac{1}{8}p(x, y)). \tag{4.12}$$

Similarly, we have

$$\sup_{t\in[0,1]} |f(y(t))| \leq e^{-\lambda}(\frac{1}{8}p(y, fy) + \frac{1}{8}p(x, y)). \tag{4.13}$$

Hence, from (4.12) and (4.13) we have

$$\begin{aligned}\max\{\sup_{t\in[0,1]} |f(x(t))|, \sup_{t\in[0,1]} |f(y(t))|\} &\leq e^{-\lambda}(\frac{1}{8}p(x, y) + \frac{1}{8}p(x, fx) + \frac{1}{8}p(y, fy)) \\ &\leq e^{-\lambda}(\frac{1}{8}p(x, y) + \frac{1}{8}p(x, fx) + \frac{1}{8}p(y, fy) \\ &+ \frac{1}{8}p(x, fy) + \frac{1}{8}p(y, fx)).\end{aligned}$$

Therefore,

$$p(fx, fy) \leq e^{-\lambda}(\frac{1}{8}p(x, y) + \frac{1}{8}p(x, fx) + \frac{1}{8}p(y, fy) + \frac{1}{8}p(x, fy) + \frac{1}{8}p(y, fx))$$

and so,

$$\ln(p(fx, fy)) \leq -\lambda + ln\Big(\frac{1}{8}p(x, y) + \frac{1}{8}p(x, fx) + \frac{1}{8}p(y, fy) + \frac{1}{8}p(x, fy) + \frac{1}{8}p(y, fx)\Big),$$

which implies that $\lambda + F(p(fx, fy)) \leq F(\mathcal{H}_f(x, y))$ is satisfied for $F(\alpha) = \ln(\alpha)$ for all $\alpha \in X$ with $a = b = c = d = e = \frac{1}{8}$ and $l = 0$. Hence, all conditions of Theorem 3.2 holds and f has a fixed point z such that $z \in C([0, 1])$ with $\kappa \leq z(t) \leq \eta$ for all $t \in [0, 1]\}$. That is, $z \in U_1 \cap U_2$ is a solution to (4.1). □

Conflict of interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Conclusion

We conclude that every F-rational cyclic contraction mapping $f : \bigcup_{i=1}^m V_i \to \bigcup_{i=1}^m V_i$ defined on a 0-complete partial metric space (X, p) has a unique fixed point $z \in \bigcap_{i=1}^m V_i$ and for any $x_0 \in \bigcup_{i=1}^m V_i$, the sequence $x_n = f^n x_0$ converges to z in topology $\tau(p)$, where V_i is nonempty closed subset of X for each $i = 1, \cdots, m$.

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