ADAM MICKIEWICZ UNIVERSITY IN POZNAŃ

DOCTORAL THESIS

On the role of entanglement in the formation and stability of composite bosons

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"The best possible knowledge of a whole does not necessarily include the best possible knowledge of its parts."

Erwin Schrödinger

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List of Publications

[1] Zakarya Lasmar, Dagomir Kaszlikowski, and Paweł Kurzyński, Phys. Rev. A **96**, 032325 (2017).

[2] Zakarya Lasmar, Adam S. Sajna, Su-Yong Lee, and Paweł Kurzyński. Phys. Rev. A **98**, 062105 (2018).

[3] Zakarya Lasmar, P. Alexander Bouvrie, Adam S. Sajna, Malte C. Tichy, and Paweł Kurzyński. arXiv:1902.08157

[4] Marcin Karczewski, Su-Yong Lee, Junghee Ryu, Zakarya Lasmar, Dagomir Kaszlikowski, Paweł Kurzynski. arXiv:1902.08159

The present dissertation consists of two parts which are mainly based on the papers [1–3]

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Abstract

On the role of entanglement in the formation and stability of composite bosons

by mgr. Zakarya Lasmar

Composite bosons are many-body systems made of many elementary bosons, or an even number of elementary fermions. Recently, it was suggested that quantum entanglement can be understood as the origin of the bosonic character of the simplest systems of this type (bipartite composite bosons). This idea motivated a new direction of research: quantum information oriented study of composite particles. In the present work, the role of entanglement in the formation and dynamics of composite bosons is studied. We show that, in some special situations, two entangled fermions can exhibit bosonic behaviour while being specially separated. We propose a nonlocal scheme that leads two pairs of entangled fermions to form an analogue of a two-partite bosonic Fock state. Also, we show that in some situations entanglement can provide stability for a composite particle. In this case, the interaction-free dynamics of an entangled bipartite system appear to mimic the behaviour of two interacting particles. In addition, we discuss the possible reasons that can limit the influence of entanglement on the behaviour of composite particles. The no-signalling condition seems to play a crucial role, which suggests that interactions in some situations become necessary. Regarding the formation of composite bosons, I present an entanglement-based method to study the bosonic quality of fermionic multipartite systems. Using this method, I examine the bosonic quality of the ground state of the extended one-dimensional Hubbard model while tuning the strength of interactions.

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Streszczenie

Rola splątania w procosie formowania i stabilności bozonów złożonych

autorstwa mgra Zakaryi Lasmara

Bozony złożone to układy wielociałowe powstałe z wielu podstawowych bozonów lub z parzystej liczby podstawowych fermionów. W ostatnich czasach zasugerowano, że wielkością odpowiedzialną za bozonową własność najprostrzych układów tego typu (dwuciałowych bozonów złożonych) jest splątanie kwantowe. Ten pomysł zapoczątkował nowy kierunek badań: analizę własności cząstek złożonych w języku teorii informacji kwantowej. W tej rozprawie omówiona zostanie rola splątania w procesie powstawania i dynamiki bozonów złożonych. Pokażemy, że w szczególnych przypadkach, dwa splątane fermiony mogą wykazywać bozonowe właściwości nawet jeśli znajdują się w dwóch odległych miejscach. Zaproponujemy metodę pozwalającą na przekształcenie stanu dwóch par splątanych fermionów w dwucząstkowy bozonowy stan Foka. Następnie pokażemy, że w niektórych sytuacjach splątanie może zapewnić stabilność cząstki złożonej. W tym przypadku dynamika nieodziałującej pary splątanych cząstek przypomina dynamikę dwóch oddziałujących cząstek. Ponadto, zbadamy możliwe przyczyny ograniczeń wpływu splątania na zachowanie się cząstek złożonych. Zasada braku sygnalizowania wydaje się odgrywać tutaj kluczową rolę, co sugeruje, że oddziaływanie między cząstkami może być czasem niezbędne. Odnośnie problemu powstawania bozonów złożonych, zaproponuję metodę badania bozonowej własności wielocząstkowych układów fermionowych, która jest oparta na splątaniu wielociałowym. Ta metoda zostanie użyta do zbadania bozonowości stanu podstawowego rozszerzonego modelu Hubbarda w sytuacji kiedy siła oddziaływań pomiędzy cząstkami rośnie.

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Contents

Introduction

In nature, elementary particles can be classified into two categories, bosons or fermions. For example, we all know that electrons are particles which behave in a fermionic way, while photons exhibit a the typical bosonic behaviour. The difference between these two types of particles can be manifested via several effects and phenomena. Due to the Pauli exclusion principle, the same quantum state cannot be occupied by more than one fermion. On the other hand, bosons should not experience any restrictions on their occupation numbers. Another simple scenario in which fermions and bosons exhibit drastically different behaviours is when two indistinguishable particles meet at a symmetric beam splitter. If these particles are indistinguishable bosons, they will always appear at the same output, i.e. they bunch. However, if they are indistinguishable fermions, they will always appear at different outputs, i.e. they anti-bunch. This phenomenon is called the Hong-Ou-Mandel effect [5].

Most of systems studied in laboratories consist of composite particles, e.g. molecules, atoms or even neutrons and protons. Basically, most of bosonic particles are composed of several elementary bosons or an even number of elementary fermions or some combination of both. Hence, the term *composite bosons* is used by the community studying such systems [6]. Inquisitively, we can say that a composite boson emerges when its constituents are strongly attracted to each other. This will bound the constituents together, and hence it can be treated as single particle. As long as the external forces are weaker than the interactions between the constituents, the composite particle can be effectively described by its centre of mass in addition to its total momentum.

Remarkably, bound states of composite particles are usually strongly entangled. For instance, bipartite composite particles such as Hydrogen atoms, Cooper pairs or excitons can exist in pure states while the states of each of theirs constituents are highly mixed. It can be intuitive to imagine a Hydrogen atom in harmonic trap. The coulomb interaction will keep the electron in the vicinity of the proton, while both constituents get delocalized within the well. Ergo, if the size of the trap is much larger than the radius of the Hydrogen atom, the electron and the proton become highly entangled. This idea was further developed, and it was suggested that the bosonic behaviour of a bipartite composite particle is proportional to the amount of quantum correlations between the constituents [7, 8]. This has stimulated many works by authors from the quantum information community [9–28]. However, some ambiguity regarding the role of entanglement behind the behaviour of composite bosons remained undressed. This is because in all the previous works, the authors presented studies of entanglement properties of stationary quantum states describing bipartite composite particles. Here, I will consider the dynamics of such systems. In order to single out the effects of entanglement, I will focus on interaction-free dynamics. For instance, a scenario in which the entangled constituents are spatially separated, so they do not interact while remaining correlated.

The present work consists of two parts. In the first three chapters, I will study the behaviour of composite bosons with a special focus on the role of quantum correlations in interaction-free dynamics of bipartite composite bosons. In the fourth chapter, I will study the formation of composite bosons made of more than two constituents and discuss their bosonic quality and entanglement properties. Bellow, I will briefly describe the structure of this thesis.

In the first chapter, I will first introduce few concepts from the theory of composite bosons. Then, I will present a study of non-local bosonic behaviour of two elementary fermions. I will consider two bipartite composite bosons with spatially separated constituents. My aim will be to show that these two composite bosons can bunch solely via local operations. However, studying the change of the entanglement properties of system through its evolution suggests that interaction between constituents of the same type is necessary.

In the second chapter, I will first state some definitions regarding what can be considered as stable composite particle, or a decayed one. Considering these definitions, I will study the interaction-free spread of a bipartite composite particle in a one-dimensional space. My aim will be to examine the role of quantum correlations in preserving the stability of the system. Then, I will study the effects of non-zero temperatures over the dynamics of the same system.

In the third chapter, I will first recall few aspects of the standard Mach-Zhender Interferometer (MZI). Next, I will present an idea of a MZI-like setup, within the one-dimensional Hubbard model. Considering a bipartite composite particle spreading over a discrete lattice, my aim will be to take advantage of the stability due to entanglement (discussed in the second chapter) for observing the collective de Broglie wavelength of the considered system. I will show that if the constituents are highly entangled, proper post-selective measurements can lead to the observation of a collective behaviour the composite particle.

In the fourth chapter, I will first discuss the formation of composite bosons made of two fermions. Then, I will extend these results to multipartite composite bosons. I will consider the one-dimensional Hubbard model with two types of interactions, i.e. long-range and short-range interactions. Mainly, I will examine the ground state of a system made of many pairs of fermions and discuss its corresponding entanglement properties and bosonic quality. My aim is to observe the behaviour of the total system as a function of the strength of interactions. I will show that the control of the strength of interactions is crucial for the formation and control of the composite structures.

Chapter 1

Nonlocal bunching of composite bosons

The results included in this chapter were published as a regular article in Physical Review A [1].

1.1 Introduction

When is it reasonable to expect two fermions to behave like a single boson? This question has been actively investigated for many years, because the problem applies to a wide range of topics, like superconducting Cooper pairs [29], Bose-Einstein Condensates [30, 31] and Excitons [6, 31, 32]. In 2005, C.K. Law suggested a hypothesis to address the aforementioned question:

> " *...quantum entanglement provides an understanding of the origin of composite behaviour ...This implies an interesting picture that constituent particles are somehow bounded by quantum entanglement. Mechanical binding forces are not essential, they serve only as physical means to enforce quantum correlations.*" [7]

In other words, the bosonic behaviour is solely due to entanglement, and that the role of interactions is to provide a mechanism to create quantum correlations. He supported that by positively testing his hypothesis on a specific class of quantum states. Also, he suggested extending this class of states to a more general one as an open problem. In 2010, W. K. Wootters and his team published an argument proving that Law's claim is in fact more general than Law himself expected. They showed that it is valid for all bipartite quantum states [8]. Remarkably, this idea has stimulated the community working on composite bosons, and it was further developed in a number of works [9–28].

However, since entanglement stands as the origin of all bosonic behaviour, Law also argued in his paper that two spatially separated fermions can behave like a single boson [7]. In fact, Wootters and his team have come to a similar conclusion, and their comment on that can be summarized in the following quote

> *"Taking this idea to its logical conclusion, Law notes that two particles can be highly entangled even if they are far apart. Could we treat such a pair of fermions as a composite boson? The above analysis suggests that we can do so. However, we would have to regard the pair as a very fragile boson in the absence of an interaction that would preserve the pair's entanglement in the face of external disturbances. On this view, the role of interaction in creating a composite boson is not fundamentally to keep the two particles close to each other, but to keep them entangled."* [8]

The analysis upon which they arrived to this conclusion was by considering a system made of N pairs of fermions, and studying the ladder structure of its states — a detailed discussion of their argument will be presented later in this chapter. Also, they considered the change of the structure after adding or subtracting a pair to/from the total system. In other words, their analysis was limited to quantum systems in stationary states. Therefore it is natural to ask: What about the dynamics of such systems? This is the main question which I will consider in this chapter.

First, I will consider a pair of fermions undergoing a beam splitting operation. My primary goal will be to show that entanglement is not enough by itself to keep the constituents close to each other, i.e. to keep the composite particle stable. I will show that interaction has a fundamental role in this scenario, namely entanglement production. Then, I will consider two identical pairs of fermions, and discuss the necessary conditions for their bunching, via the Hang-Ou-Mandel effect. In the last part of this chapter, for two pairs of fermions, I will propose a nonlocal bunching scenario. In this case, I will show that interaction is required only between constituents of the same type. In addition, I will show that the probabilities for the success of such a scenario depends directly on the amount of entanglement between the constituents, which is in agreement with the previous findings.

1.2 Preliminaries

1.2.1 Coboson made of two fermions

In this section, I will present the general argument which suggests that two elementary fermions can behave like a single boson when they are highly entangled. First, Let us consider an arbitrary state of a bipartite system

$$|\psi\rangle = \sum_{m,n=1}^{\infty} \gamma_{m,n} a_m^\dagger b_n^\dagger |0\rangle. \tag{1.1}$$

such that $a_m^\dagger$ ($b_n^\dagger$) stands for the creation operator of a fermion of type A (B) in mode m (n). The coefficients $\gamma_{m,n}$ are complex numbers, such that $\gamma_{m,n}\gamma_{m,n}^* = |\gamma_{m,n}|^2$ is the probability of finding particle A in mode m and particle B in mode n. However, using the Schmidt decomposition [33], one can rewrite the state (1.1) in the form

$$|\psi\rangle = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\tilde{a}_i^\dagger \tilde{b}_i^\dagger |0\rangle. \tag{1.2}$$

This can be achieved by finding the proper rotation of the basis.

$$\begin{aligned} \tilde{a}_i^\dagger &= \sum_k \alpha_{i,k} a_k^\dagger, \\ \tilde{b}_i^\dagger &= \sum_k \beta_{i,k} b_k^\dagger. \end{aligned} \tag{1.3}$$

For simplicity, I will drop the ˜. The Schmidt coefficients are probabilities, therefore always positive real numbers

$$0 \leq \lambda_i \leq 1. \tag{1.4}$$

As a matter of fact, the Schmidt rank is the number of non-zero Schmidt coefficients. Let us denote it by d. We can say that the state $|\psi\rangle$ is separable if and only if $d = 1$, and entangled otherwise. In addition to that, the set $\{\lambda_i\}_{i=1}^d$ needs to satisfy the normalization condition.

$$\sum_{i=1}^{d} \lambda_i = 1. \tag{1.5}$$

In order to quantify the amount of entanglement, one can compute the purity of the reduced state of particle A or B. Note that the purity takes values between 0 and 1. 1 corresponds to a separable state while 0 corresponds to an infinitely entangled state. In fact, since we have a bipartite systems, there is only one way to bipartition it. Ergo, for pure states, both purities computed from both reduced states will always be equal to each other.

$$P = \mathrm{Tr}\{\rho_a^2\} = \mathrm{Tr}\{\rho_b^2\}, \tag{1.6}$$

where ρ_a and ρ_b are the reduced density matrices of particle A and B, respectively. These can be computed from the state $|\psi\rangle$ by using the method discussed in [34]. For instance

$$\rho_{a_{n,m}} = \langle\psi| a_m^\dagger a_n |\psi\rangle. \tag{1.7}$$

Which leads to

$$\rho_A = \sum_{i=1}^{d} \lambda_i a_i^\dagger |0\rangle\langle 0| a_i, \tag{1.8}$$

$$\rho_B = \sum_{i=1}^{d} \lambda_i b_i^\dagger |0\rangle\langle 0| b_i. \tag{1.9}$$

Therefore, the purity takes the general definition

$$P = \sum_{i=1}^{d} \lambda_i^2. \tag{1.10}$$

Considering this form of the purity in addition to the normalization condition expressed in Eq. (1.5), we can say that the value of P is bounded

$$\frac{1}{d} \leq P \leq 1. \tag{1.11}$$

Clearly, the upper bound corresponds to a separable state ($d = 1$). On the other hand, the lower bound corresponds to the maximally entangled state for which all the non-zero Schmidt coefficients are equal to each other. In the limit of an infinitely large Schmidt rank, $d \to \infty$, the lower bound will drop to zero.

Now, let us define, in the Schmidt basis, the creation operator of a composite particle made of two entangled constituents, A and B.

$$c^\dagger = \sum_i \sqrt{\lambda_i} a_i^\dagger b_i^\dagger. \tag{1.12}$$

The action of this operator on vacuum will engender a single composite particle state of the form of $|\psi\rangle$ in Eq. (1.2). This composite particle has an internal structure described by the sum over the index i.

$$c^\dagger|0\rangle = |\psi\rangle \equiv |1\rangle. \tag{1.13}$$

Since A and B are fermions, one might ask if the pair is a composite boson. Using the results from [13] we can write

$$[c, c^\dagger] = 1 - \Delta, \tag{1.14}$$

such that $[c, c^\dagger] = cc^\dagger - c^\dagger c$ is the commutation relation of the composite particle.

$$\Delta = \sum_{i=1}^{d} \lambda_i (a_i^\dagger a_i + b_i^\dagger b_i). \tag{1.15}$$

It is clear from Eq. (1.14) that the operator $c^\dagger$ is not perfectly bosonic. This can be explained by the fact that this composite particle has an internal structure and that it has fermionic constituents which have to obey the Pauli exclusion principle. The operator Δ describes the deviation of the behaviour of $c^\dagger$ from the perfect bosonic one. Evidently, the expectation value of Δ should vanish for perfect bosons [13].

Another mathematical aspect of bosonic operators, is the construction of Fock states. For example, by using the definition in Eq. (1.12) we can write

$$(c^\dagger)^N|0\rangle = \sqrt{\chi_N N!}|N\rangle. \tag{1.16}$$

Here, $|N\rangle$ is an N-partite Fock state. Because the operator $c^\dagger$ is not perfectly bosonic, the normalization factor χ_N is required so the Fock state is normalized. In fact, it can be shown that χ_N has the form [6–8, 13]

$$\chi_N = \frac{1}{N!}\langle 0|c^N(c^\dagger)^N|0\rangle = \frac{1}{N!}\sum_{\substack{k_1\cdots k_N\\ \text{all different}}} \lambda_{k_1}\cdots\lambda_{k_N}. \tag{1.17}$$

If we add to an N-partite Fock state another composite boson, we get

$$\begin{aligned} c^\dagger|N\rangle &= \frac{1}{\sqrt{\chi_N N!}}c^\dagger(c^\dagger)^N|0\rangle. \\ &= \sqrt{\frac{\chi_{N+1}}{\chi_N}}\frac{\sqrt{N+1}}{\sqrt{\chi_{N+1}(N+1)!}}(c^\dagger)^{N+1}|0\rangle. \\ &= \alpha_{N+1}\sqrt{N+1}|N+1\rangle. \end{aligned} \tag{1.18}$$

Such that

$$\alpha_{N+1} = \frac{\chi_{N+1}}{\chi_N}. \tag{1.19}$$

However, when we subtract a composite boson from $|N\rangle$, the system can lose the ladder structure. From Eq. (1.18), we can write

$$\langle N-1|c|N\rangle = (\langle N|\alpha_N\sqrt{N})|N\rangle. \tag{1.20}$$

However, if we apply the operator c on $|N\rangle$, the value of $\langle N-1|c|N\rangle$ should remain the same. Therefore, we write

$$\langle N-1|c|N\rangle = \langle N-1|(\alpha_N\sqrt{N}|N-1\rangle + |\varepsilon_N\rangle). \tag{1.21}$$

Hence,

$$c|N\rangle = \alpha_N\sqrt{N}|N-1\rangle + |\varepsilon_N\rangle. \tag{1.22}$$

The state $|\varepsilon_N\rangle$ is orthogonal to $|N-1\rangle$, and it corresponds to a pure state of an N-partite system describing all the permutations lacking the proper bosonic ladder structure. Obviously, in the case of perfect bosons, we should have

$$\langle\varepsilon_N|\varepsilon_N\rangle = 0. \tag{1.23}$$

In order to examine the equation above, let us compute the general form of $\langle \varepsilon_N | \varepsilon_N \rangle$ for an arbitrary state describing a composite particle. For that, we need to evaluate the expectation value of the commutation relation given in Eq. (1.14)

$$\begin{aligned}\langle N|[c, c^\dagger]|N\rangle &= \langle N|cc^\dagger|N\rangle - \langle N|c^\dagger c|N\rangle. \\ &= \langle N+1|\alpha_{N+1}^2(N+1)|N+1\rangle - \langle N-1|\alpha_N^2(N)|N-1\rangle \\ &\quad - \langle \varepsilon_N|\varepsilon_N\rangle. \\ &= \alpha_{N+1}^2(N+1) - \alpha_N^2(N) - \langle \varepsilon_N|\varepsilon_N\rangle \end{aligned} \tag{1.24}$$

The expectation value of the right hand side of Eq. (1.14) takes the form

$$\begin{aligned}\langle N|1-\Delta|N\rangle &= \langle N|N\rangle - \langle N|\sum_{i=1}^{d}\lambda_i(a_i^\dagger a_i + b_i^\dagger b_i)|N\rangle. \\ &= 1 - 2(1-\alpha_{N+1}^2) = 2\alpha_{N+1}^2 - 1. \end{aligned} \tag{1.25}$$

In fact, it has been shown in [6] that the value of $\langle N|\Delta|N\rangle$ has to be equal to $2(1-\alpha_{N+1}^2)$. On the other hand, considering Eq. (1.14), we know that the right-hand sides of Eqs. (1.24) and (1.25) should be equal to each other.

$$2\alpha_{N+1}^2 - 1 = \alpha_{N+1}^2(N+1) - \alpha_N^2(N) - \langle \varepsilon_N|\varepsilon_N\rangle, \tag{1.26}$$

which can be rewritten as

$$\langle \varepsilon_N|\varepsilon_N\rangle = 1 - \alpha_{N+1}^2 + (\alpha_{N+1}^2 - \alpha_N^2)N. \tag{1.27}$$

It is clear from the equation above that $\langle \varepsilon_N|\varepsilon_N\rangle \to 0$ when $\frac{\chi_N}{\chi_{N-1}} \to 1$. Starting from this idea, C. K. Law considered a double Gaussian state for each pair, and showed that the total system made of N pairs can behave like a collection of bosons.

$$\psi(x_A, x_B) = \mathcal{N} e^{-(x_A+x_B)^2/\sigma_c^2} e^{-(x_A-x_B)^2/\sigma_r^2}. \tag{1.28}$$

$\mathcal{N}$ is a normalization constant. x_A (x_B) corresponds to the position of the fermion A (B). σ_c and σ_r stands for the width along the direction of the centre of mass and the relative distance, respectively. For such a special state, the normalization factors can be computed exactly. Interestingly, the expressions of χ_N and $\frac{\chi_{N+1}}{\chi_N}$ depend on the degree of entanglement between each pair [7].

$$\frac{\chi_{N+1}^{(D.G.)}}{\chi_N^{(D.G.)}} \approx 1 - NP, \tag{1.29}$$

where the upper script $(D.G.)$ stands for *Double Gaussian*, and P for the purity of each pair. The value NP indicates a deviation from the ideal bosonic behaviour. In simple words, if each pair is sufficiently entangled, for a given N, the total system will exhibit an ideal bosonic behaviour. However, even if this has been proven for double Gaussians, C. K. Law speculated that there might be other classes of quantum states for which the same conclusions will hold.

In fact, the hypothesis of C. K. Law was suggested for composite bosons made of two elementary fermions or two elementary bosons [7]. However, in this work I will only consider the fermionic case.

In 2010, W. K. Wootters and his team published a proof extending Law's conclusions to be valid for any bipartite state [8]. Basically, using equations (1.5, 1.10 and 1.17), they showed that the normalization ratio is in fact bounded like

$$1 - NP \leq \frac{\chi_{N+1}}{\chi_N} \leq 1 - P. \tag{1.30}$$

Clearly, when each pair is strongly entangled, $P \to 0$, both bounds will converge to 1, which implies an ideal bosonic behaviour.

In 2012, M. C. Tichy, P. A. Bouvrie and K. Mølmer published another proof for bounds which are tighter than the ones mentioned above [19]. The same team has also considered the general case of composite bosons made of two elementary bosons [20]. This results marked the beginning of a new direction of research, namely a quantum information based studies on composite bosons. Since then this topic has been attracting more and more attention and so far many works has been publish [9–28].

1.2.2 Maximally entangled cobosons made of two fermions

The creation operator of a bipartite composite boson $c^\dagger$ has the most general form defined in the Eq. (1.12). For such a definition, the entanglement between the constituents depend on the choice of the Schmidt distribution $\{\lambda_i\}_{i=1}^d$, see Eq. (1.10). In the limit of maximal entanglement the Schmidt distribution need to be uniform $\{\lambda_i = \frac{1}{d}\}_{i=1}^d$. In this case, the purity depends only on the Schmidt rank $P = \frac{1}{d}$ and the operator $c^\dagger$ takes the form

$$c^\dagger = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} a_i^\dagger b_i^\dagger, \tag{1.31}$$

which leads to

$$\chi_N = \frac{d!}{d^N(d-N)!}, \quad (1.32)$$

$$\alpha_N = \sqrt{\frac{d-N+1}{d}}, \quad (1.33)$$

$$\langle \varepsilon_N | \varepsilon_N \rangle = 0, \quad (1.34)$$

$$1 - \frac{N}{d} \leq \frac{\chi_{N+1}}{\chi_N} \leq 1 - \frac{1}{d}. \quad (1.35)$$

Clearly, when the Schmidt rank is much larger than the number of composite bosons $N \ll d$ the system exhibits an ideal bosonic behaviour $\frac{\chi_{N+1}}{\chi_N} \to 1$.

1.3 Entanglement and stability under beam splitting transformation

1.3.1 Single coboson

The most general form of the Hamiltonian for a single particle Beam Splitter (BS) can be written as

$$\mathcal{H}_{BS} = a_L^\dagger a_R + a_R^\dagger a_L. \quad (1.36)$$

One can easily check that

$$\mathcal{H}_{BS}|+_{BS}\rangle = |+_{BS}\rangle = \frac{1}{\sqrt{2}}(a_R^\dagger + a_L^\dagger)|0\rangle, \quad (1.37)$$

$$\mathcal{H}_{BS}|-_{BS}\rangle = -|-_{BS}\rangle = -\frac{1}{\sqrt{2}}(a_R^\dagger - a_L^\dagger)|0\rangle, \quad (1.38)$$

where, $|+_{BS}\rangle$ and $|-_{BS}\rangle$ are the eigenvectors of $\mathcal{H}_{BS}$, and the corresponding eigenvalues are $+1$ and -1, respectively. Hence, the evaluation of the time evolution generated by $\mathcal{H}_{BS}$ is straightforward. Assuming natural units, we can write

$$a_R^\dagger|0\rangle \to \frac{1}{\sqrt{2}}(e^{-it}|+_{BS}\rangle + e^{+it}|-_{BS}\rangle), \quad (1.39)$$

$$a_L^\dagger|0\rangle \to \frac{1}{\sqrt{2}}(e^{-it}|+_{BS}\rangle - e^{+it}|-_{BS}\rangle), \quad (1.40)$$

which can be rewritten as

$$a_R^\dagger \to \cos(t) a_R^\dagger - i\sin(t) a_L^\dagger, \quad (1.41)$$

$$a_L^\dagger \to \cos(t) a_L^\dagger - i\sin(t) a_R^\dagger. \quad (1.42)$$

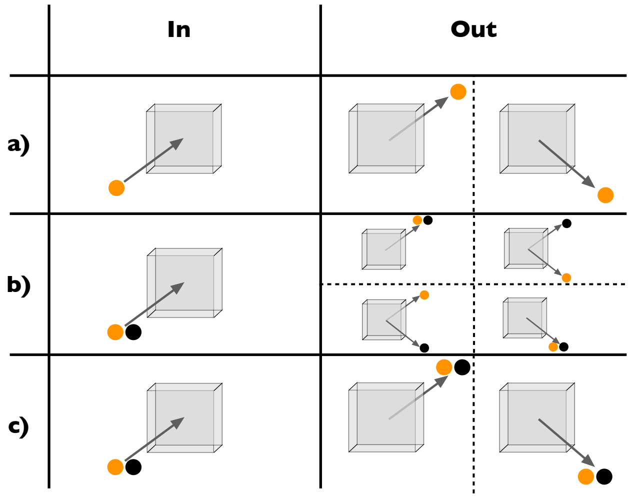

Figure 1.1: Three scenarios for beam-splitting of elementary and composite particles, with and without interactions. a) An elementary particle can appear at one of the two outputs. b) An interaction-free transformation of two particles - Each particle evolves independently of the other one. Consequently, among the four possible outcomes two correspond to the decay of the composite particle. c) Transformation of two interacting particles - They exhibit a collective behaviour and always remain together. This is similar to the single particle behaviour. [1]

Hence, if the time of evolution is $t = \frac{\pi}{4}$, we get a symmetric BS.

$$a_R^\dagger \to \frac{1}{\sqrt{2}}(a_R^\dagger - i a_L^\dagger), \tag{1.43}$$

$$a_L^\dagger \to \frac{1}{\sqrt{2}}(a_L^\dagger - i a_R^\dagger). \tag{1.44}$$

In this case, a single particle has equal chances for passing through or getting reflected, see Fig. 1.1 a).

Now, let us consider a bipartite system described by Eq. (1.2). In this case, each constituent will have two degrees of freedom, $a_{i,X}^\dagger$ and $b_{i,Y}^\dagger$. Namely, one corresponding to the internal structure and the other one to the in/output of the BS, $i = 1, \cdots, d$ and $X, Y = L, R$, respectively. If we assume a standard BS operation, i.e. interaction-free, the evolution of each particle will be independent of the other. For this operation the Hamiltonian take the form

$$\mathcal{H}_{\text{ind}} = \mathcal{H}_A + \mathcal{H}_B, \tag{1.45}$$

such that

$$\mathcal{H}_A = \sum_{i=1}^{d}(a_{i,L}^{\dagger}a_{i,R} + a_{i,R}^{\dagger}a_{i,L}), \tag{1.46}$$

$$\mathcal{H}_B = \sum_{i=1}^{d}(b_{i,L}^{\dagger}b_{i,R} + b_{i,R}^{\dagger}b_{i,L}). \tag{1.47}$$

By using the results found for the single particle case (1.43), one can easily show that $\mathcal{H}_{\text{ind}}$ will lead to the transformation

$$c_L^{\dagger}|0\rangle \equiv \frac{1}{\sqrt{d}}\sum_{i=1}^{d} a_{i,L}^{\dagger}b_{i,L}^{\dagger}|0\rangle \rightarrow \tag{1.48}$$
$$\frac{1}{2\sqrt{d}}\sum_{i=1}^{d}(a_{i,L}^{\dagger}b_{i,L}^{\dagger} - ia_{i,R}^{\dagger}b_{i,L}^{\dagger} - ia_{i,L}^{\dagger}b_{i,R}^{\dagger} - a_{i,R}^{\dagger}b_{i,R}^{\dagger})|0\rangle.$$

Clearly, the constituents have 50% of chance to go out through the same output, and 50% of chance to go out through different ones. The later can be interpreted as a decay of the composite particle (see Fig. 1.1 b).

In order to prevent the composite particle from decaying, one can consider a BS operation for which the constituents can interact. In fact, this was already studied in many previous works [23–25, 28]. In this discussion, I will consider a model similar to the one proposed in [24]. In this model, the Hamiltonian will be similar to $\mathcal{H}_{\text{ind}}$, but with one extra term:

$$\mathcal{H}_{\text{dep}} = \mathcal{H}_A + \mathcal{H}_B + \mathcal{H}_{\text{int}}, \tag{1.49}$$

such that

$$\mathcal{H}_{\text{int}} = -\gamma \sum_{X=R,L}\sum_{i=1}^{d} a_{i,X}^{\dagger}a_{i,X}b_{i,X}^{\dagger}b_{i,X}. \tag{1.50}$$

This term describes an interaction between particles A and B, parametrized by the constant γ. When this interaction is strongly attractive $\gamma \gg 1$ the evolution of the composite particle can be approximated like

$$c_L^{\dagger}|0\rangle \equiv \frac{1}{\sqrt{d}}\sum_{i=1}^{d} a_{i,L}^{\dagger}b_{i,L}^{\dagger}|0\rangle \rightarrow \tag{1.51}$$
$$\frac{1}{\sqrt{2d}}\sum_{i=1}^{d}(a_{i,L}^{\dagger}b_{i,L}^{\dagger} - a_{i,R}^{\dagger}b_{i,R}^{\dagger})|0\rangle \equiv \frac{1}{\sqrt{2}}(c_L^{\dagger} - c_R^{\dagger})|0\rangle.$$

In this transformation, the two components will always exit through the same output, which can be seen as a collective behaviour. This is because no information regarding the internal structure of the system was exposed, see Fig. 1.1 c).

At this stage of the discussion, one might wonder if it is possible to avoid interaction while preserving the stability of the composite particle. In fact, in order to understand the role of interaction it is important to note that the BS model discussed so far has a particular feature. Namely, the evolution does not effect the internal structure, and acts solely on the $X = R, L$ degree of freedom. Consequently, we can simplify the system by considering a pair of fermions in a separable state. Regardless of the details of the transformation, we would require the following evolution to occur.

$$a_L^\dagger b_L^\dagger \to \frac{1}{\sqrt{2}}(a_L^\dagger b_L^\dagger + e^{i\varphi} a_R^\dagger b_R^\dagger)|0\rangle, \tag{1.52}$$

where the phase φ is arbitrary. Note that this transformation is similar to the one for the single particle case, see (1.43). But, from transformation (1.52) one can easily show that it is entangling since the single particle state evolves from a pure to a mixed one.

$$\rho_A = a_L^\dagger|0\rangle\langle 0|a_L \to \frac{1}{2}(a_L^\dagger|0\rangle\langle 0|a_L + a_R^\dagger|0\rangle\langle 0|a_R). \tag{1.53}$$

We can also show that for a composite boson described by (1.31) the single fermion purity decreases through the transformation with interaction (1.51) from $\frac{1}{d}$ to $\frac{1}{2d}$. While in the transformation without interaction (1.48) the purity remains unchanged.

This suggests that the stability of a composite particle undergoing a BS transformation requires entanglement production, which implies that interaction has an important role and cannot be avoided. While this argument is sufficient to prove that the role of interaction is fundamental and not limited to the protection of the system against external forces, the same conclusion can be drown if we consider the no-signalling principle. This will be discussed in details in the third Chapter.

1.3.2 The Hong-Ou-Mandel effect

One of the most fundamental effects that distinguish bosonic from fermionic behaviour is the so-called Hong-Ou-Mandel effect [5]. We all know that in nature elementary particles can be either fermions, like electrons and protons, or bosons, like photons. In fact, two particles of the same type that enter different parts of a $50/50$ beam splitter will either go out through the same output or through different outputs, depending on the nature of these particles. Using (1.43) we can write the following transformation

$$f_R^\dagger f_L^\dagger \rightarrow \frac{1}{2}(f_R^\dagger - i f_L^\dagger)(f_L^\dagger - i f_R^\dagger), \tag{1.54}$$

$$h_R^\dagger h_L^\dagger \rightarrow \frac{1}{2}(h_R^\dagger - i h_L^\dagger)(h_L^\dagger - i h_R^\dagger), \tag{1.55}$$

where $f^\dagger$ and $h^\dagger$ stands for the creation operator of an elementary fermion and boson, respectively. The transformations above can be rewritten as

$$f_R^\dagger f_L^\dagger \rightarrow \frac{1}{2}(f_R^\dagger f_L^\dagger - i f_L^\dagger f_L^\dagger - i f_R^\dagger f_R^\dagger - f_L^\dagger f_R^\dagger), \tag{1.56}$$

$$h_R^\dagger h_L^\dagger \rightarrow \frac{1}{2}(h_R^\dagger h_L^\dagger - i h_L^\dagger h_L^\dagger - i h_R^\dagger h_R^\dagger - h_L^\dagger h_R^\dagger). \tag{1.57}$$

We know that two identical fermions cannot occupy the same state, thus we can write $f_L^\dagger f_L^\dagger|0\rangle = f_R^\dagger f_R^\dagger|0\rangle = 0$. On the other hand, we know that states of identical bosons are symmetric under particle permutations, which means $h_R^\dagger h_L^\dagger - h_L^\dagger h_R^\dagger = 0$. Consequently, we can simplify the transformations above to

$$f_R^\dagger f_L^\dagger \rightarrow \frac{1}{2}(f_R^\dagger f_L^\dagger - f_L^\dagger f_R^\dagger) = f_R^\dagger f_L^\dagger, \tag{1.58}$$

$$h_R^\dagger h_L^\dagger \rightarrow \frac{-i}{2}(h_L^\dagger h_L^\dagger + h_R^\dagger h_R^\dagger). \tag{1.59}$$

Clearly, identical fermions always go out via different outputs, which is called anti-bunching. On the other hand, elementary bosons always go out from the same output, which corresponds to bunching. In this work, I will consider identical pairs of fermions and ask if they can behave like two elementary bosons, i.e. if it possible to make them bunch.

1.3.3 Two cobosons

In the previous subsection, we saw that for a two-port beam splitter, the Hang-Ou-Mandel effect suggests that two indistinguishable bosons, prepared at input as $a_L^\dagger a_R^\dagger|0\rangle$, will bunch, i.e. will be transformed to $\frac{1}{\sqrt{2}}(a_L^{\dagger 2} + a_R^{\dagger 2})|0\rangle$. Since it is a fundamental manifestation of bosonic behaviour, naturally, one may ask if composite bosons can also bunch. In order to answer this question, let us consider the ideal bunching transformation of maximally entangled composite bosons (1.31)

$$|\psi_i\rangle = c_L^\dagger c_R^\dagger|0\rangle \equiv \frac{1}{d}\sum_{i,j=1}^{d} a_{i,L}^\dagger b_{i,L}^\dagger a_{j,R}^\dagger b_{j,R}^\dagger|0\rangle \rightarrow \tag{1.60}$$
$$\frac{1}{2d\sqrt{\chi_2}}\sum_{i,j=1}^{d}(a_{i,L}^\dagger b_{i,L}^\dagger a_{j,L}^\dagger b_{j,L}^\dagger + a_{i,R}^\dagger b_{i,R}^\dagger a_{j,R}^\dagger b_{j,R}^\dagger)|0\rangle,$$
$$\equiv \frac{(c_L^{\dagger 2} + c_R^{\dagger 2})}{2\sqrt{\chi_2}}|0\rangle = |\psi_f\rangle,$$

where

$$\chi_2 = 1 - \frac{1}{d} = 1 - P. \tag{1.61}$$

Now that we have defined the desired transformation, let us examine the entanglement properties of both initial and final states. It is crucial to note that, in this particular case, the total system is made of indistinguishable subsystems. Of the total of 4 particles, two are of type A and two are of type B. This requires the use of the approach presented in [35]. However, considering the kind of information we are interested in, it is sufficient to examine the purity of some subsystems.

Since the system is made of two components for each type of particles, the trace of each single particle reduced density matrix is equal to 2 [34]. Ergo, renormalization is required. Using the definition in (1.7), considering particle of type A, we can show that its single particle state does not change through the transformation (1.60).

$$\rho_{A_i} = \rho_{A_f} = \frac{1}{2d}\sum_{X=L,R}\sum_{i=1}^{d} a_{i,X}^\dagger|0\rangle\langle 0|a_{i,X}. \tag{1.62}$$

Thus, the value of the single-particle purity does not change too, $P = \frac{1}{2d}$. Of course, since the system is symmetric with regards to the type of particles, the single particle state for type B is analogue to the one for type A.

At this point, we need to consider two-partite states. More precisely, here we are interested in reduced states of particles of the same type. By definition, the matrix elements can be computed using the formula

$$\rho_{A_{kl,nm}} = \langle\psi|a_m a_n a_k^\dagger a_l^\dagger|\psi\rangle. \tag{1.63}$$

We get

$$\rho_{A_i} = \frac{1}{d^2}\sum_{i,j=1}^{d} a^\dagger_{i,L}a^\dagger_{j,R}|0\rangle\langle 0|a_{j,R}a_{i,L}, \tag{1.64}$$

$$\rho_{A_f} = \frac{1}{d^2\chi_2}\sum_{\substack{i,j=1\\ i>j}}^{d} \Big(a^\dagger_{i,L}a^\dagger_{j,L}|0\rangle\langle 0|a_{j,L}a_{i,L} \tag{1.65}$$
$$+ \; a^\dagger_{i,R}a^\dagger_{j,R}|0\rangle\langle 0|a_{j,R}a_{i,R}\Big) .$$

Clearly, the final state is different from the initial one. In fact, the two-partite purity changes from $P_i^{(2)} = \frac{1}{d^2}$ to $P_f^{(2)} = \frac{1}{d(d-1)}$.

Therefore, we can say that the transformation (1.60) affects the entanglement at the level of two particles. We observe that at the two particle level, for particles of the same type, the entanglement decreases through this transformation. Hence, interaction may not be required between particles of different types. Nevertheless, some kind of interaction between particles of the same type may lead to the necessary decrease of entanglement, $P_i^{(2)} < P_f^{(2)}$.

1.4 Nonlocal bunching

Now, our goal is to confirm the conjecture suggested in the previous section. I fact, this will also answer to the question: can a pair of spatially separated fermions exhibit some kind of single boson behaviour? This requires particles A and B to be spatially separated, so they cannot interact while remaining entangled.

Let us say that Alice and Bob are conducting two spatially separated experiments. They share two composite bosons, which are split into two elementary fermions. Particles of type A are in Alice's laboratory, while particles of type B are with Bob. In order to show that the transformation (1.60) can be realised using only local operations, we can take advantage of a Bell-like setup [36]. Usually, this kind of setup is considered for disproving local realistic description of measurements with spatially separated constituents. However, in this study we are not interested in that. Our goal is to achieve the transformation (1.60). The entanglement between particles A and B does not depend on the distance between them, ergo each composite boson can be described by the state (1.13). In addition to that, let us consider that the constituents belonging to each composite boson are occupying a different mode, i.e. modes labelled $X = R, L$. Therefore, we can write the initial state

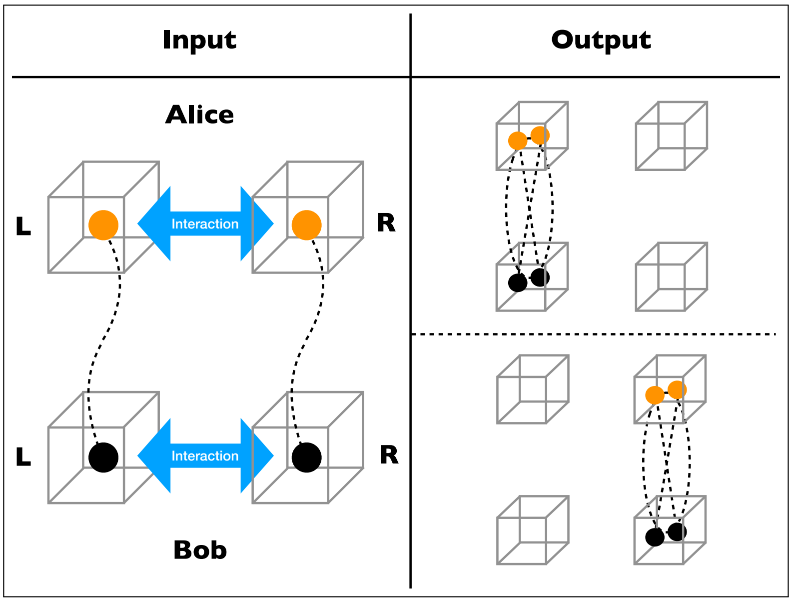

Figure 1.2: Sketch of the nonlocal bunching scenario of two composite bosons. The possible outcomes correspond to particles occupying the same mode. In this case, one cannot distinguish which particle belongs to which composite boson. [1]

$$|\psi_i\rangle = c_L^\dagger c_R^\dagger|0\rangle \equiv \frac{1}{d}\sum_{i,j=1}^{d} a_{i,L}^\dagger b_{i,L}^\dagger a_{j,R}^\dagger b_{j,R}^\dagger|0\rangle. \tag{1.66}$$

As a matter of fact, if particles of the same type are allowed to interact via the Hamiltonian (see Fig. 1.2),

$$\begin{aligned}\mathcal{H}_A = \sum_{\substack{i,j=1\\ i>j}}^{d} & (a_{i,L}^\dagger a_{j,L}^\dagger a_{j,R}a_{i,L} + a_{i,R}^\dagger a_{j,R}^\dagger a_{i,R}a_{j,L} \\ & + a_{i,L}^\dagger a_{j,R}^\dagger a_{j,L}a_{i,L} + a_{j,L}^\dagger a_{i,R}^\dagger a_{j,R}a_{i,R}),\end{aligned} \tag{1.67}$$

and

$$\begin{aligned}\mathcal{H}_B = \sum_{\substack{i,j=1\\ i>j}}^{d} & (b_{i,L}^\dagger b_{j,L}^\dagger b_{j,R}b_{i,L} + b_{i,R}^\dagger b_{j,R}^\dagger b_{i,R}b_{j,L} \\ & + b_{i,L}^\dagger b_{j,R}^\dagger b_{j,L}b_{i,L} + b_{j,L}^\dagger b_{i,R}^\dagger b_{j,R}b_{i,R}),\end{aligned} \tag{1.68}$$

we will arrive at a transformation that is very similar to (1.60). Moreover, this generated transformation depends on the Schmidt rank, i.e. the amount of entanglement between the constituents. In fact, for highly entangled composite bosons, the Hamiltonians (1.67) and (1.68) will generate a transformation that is identical to (1.60).

1.4.1 Evolution generated by $\mathcal{H}_A$ and $\mathcal{H}_B$

First, let us consider the Hamiltonian ($\mathcal{H}_A$) for particles of type A. The eigenvectors of this Hamiltonian are

$$|1\rangle = \frac{1}{\sqrt{2}}(a^{\dagger}_{i,L}a^{\dagger}_{j,L} + a^{\dagger}_{i,L}a^{\dagger}_{j,R}), \tag{1.69}$$

$$|2\rangle = \frac{1}{\sqrt{2}}(a^{\dagger}_{i,L}a^{\dagger}_{j,L} - a^{\dagger}_{i,L}a^{\dagger}_{j,R}), \tag{1.70}$$

$$|3\rangle = \frac{1}{\sqrt{2}}(a^{\dagger}_{i,R}a^{\dagger}_{j,L} - a^{\dagger}_{i,R}a^{\dagger}_{j,R}), \tag{1.71}$$

$$|4\rangle = \frac{1}{\sqrt{2}}(a^{\dagger}_{i,R}a^{\dagger}_{j,L} + a^{\dagger}_{i,R}a^{\dagger}_{j,R}), \tag{1.72}$$

$$|5\rangle = a^{\dagger}_{i,L}a^{\dagger}_{i,R}, \tag{1.73}$$

and the corresponding eigenvalues are: $+1$ for (1.69, 1.72), -1 for (1.70, 1.71) and 0 for (1.73).

If we consider an initial state of the form

$$\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle). \tag{1.74}$$

It will evolve and become at time t

$$\frac{1}{\sqrt{2}}(e^{-it}|1\rangle - e^{it}|2\rangle). \tag{1.75}$$

Actually, the state (1.75) can be rewritten as

$$(-i\sin(t)a^{\dagger}_{i,L}a^{\dagger}_{j,L} + cos(t)a^{\dagger}_{i,L}a^{\dagger}_{j,R})|0\rangle. \tag{1.76}$$

Using a similar approach, the state

$$\frac{1}{\sqrt{2}}(|3\rangle + |4\rangle), \tag{1.77}$$

will evolve to

$$\frac{1}{\sqrt{2}}(e^{-it}|4\rangle + e^{it}|3\rangle), \tag{1.78}$$

which can be rewritten as

$$(-i\sin(t)a^{\dagger}_{i,R}a^{\dagger}_{j,R} + \cos(t)a^{\dagger}_{i,R}a^{\dagger}_{j,L})|0\rangle. \tag{1.79}$$

Consequently, for $t = \frac{\pi}{2}$, $\mathcal{H}_A$ will generate the following evolutions

$$a^\dagger_{i,L} a^\dagger_{j,R} \rightarrow -i a^\dagger_{i,L} a^\dagger_{j,L} \quad \text{for } i > j, \tag{1.80}$$

$$a^\dagger_{j,L} a^\dagger_{i,R} \rightarrow -i a^\dagger_{i,R} a^\dagger_{j,R} \quad \text{for } i > j, \tag{1.81}$$

$$a^\dagger_{i,L} a^\dagger_{i,R} \rightarrow a^\dagger_{i,L} a^\dagger_{i,R}. \tag{1.82}$$

By following similar steps, it can be easily shown that the same holds for particles of type B with the Hamiltonian $\mathcal{H}_B$ (1.68).

Ergo, the state (1.66) will evolve to

$$\begin{aligned} &\frac{1}{d}\Big(\sum_{i>j=1}^{d} - a^\dagger_{i,\,L} b^\dagger_{i,\,L} a^\dagger_{j,\,L} b^\dagger_{j,\,L} - a^\dagger_{i,\,R} b^\dagger_{i,\,R} a^\dagger_{j,\,R} b^\dagger_{j,\,R} \\ &\quad + \sum_{k=1}^{d} a^\dagger_{k,\,L} b^\dagger_{k,\,L} a^\dagger_{k,\,R} b^\dagger_{k,\,R} \Big) |0\rangle \\ &\qquad = \Big(- \frac{(c^\dagger_L)^2 + (c^\dagger_R)^2}{2} + \frac{1}{d} \sum_{k=1}^{d} a^\dagger_{k,\,L} b^\dagger_{k,\,L} a^\dagger_{k,\,R} b^\dagger_{k,\,R} \Big) |0\rangle, \\ &\qquad = -\sqrt{1-P}|\psi_f\rangle + \sqrt{P}|\gamma\rangle, \end{aligned} \tag{1.83}$$

such that

$$|\psi_f\rangle = \frac{(c^\dagger_L)^2 + (c^\dagger_R)^2}{2\sqrt{\chi_2}} |0\rangle, \tag{1.84}$$

$$|\gamma\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} \left(a^\dagger_{k,L} b^\dagger_{k,L} a^\dagger_{k,R} b^\dagger_{k,R} \right) |0\rangle. \tag{1.85}$$

Here, we have a local evolution, because particles of the same type are in the same spatial location. Clearly, this evolution can lead to the state $|\psi_f\rangle$. The probability of this case to happen $1-P$ depends solely on the amount of entanglement inside each composite boson. Which means that the constituents need to be strongly correlated ($d \gg 1$ and $P \rightarrow 0$) in order to recover the transformation (1.60). Note that this result is in agreement with previous claims regarding the bosonic quality and its relation to the degree of entanglement.

The scenario discussed above is an atypical bunching scenario, because the bosons considered here are not elementary, but composite particles. This makes interaction crucial for the stability of the system when going through a beam splitter. However, one need to be careful when considering interacting constituents. For instance, previously suggested scenarios like [24] may seem like the standard one (interaction-free bunching scenario using elementary bosons). This is because particles A and B are allowed to interact only to provide stability to the composite boson. However, since all the constituents of both composite bosons are in the same mode, one cannot distinguish which particle of type A belongs to the first or the second composite boson. The same holds also for particles of type B. Consequently, the interaction will bind all the constituents into a four-partite system, which can be interpreted as a two-particle Fock state.

1.5 Summary

In this chapter, we saw that an entangled pair of spatially separated fermions can exhibit a single particle bosonic behaviour. Using two copies of such a pair, it is possible to locally evolve the system and arrive at a state which can be interpreted as an analogue of a two-partite bosonic Fock state. Also, we saw that it is impossible for a pair of entangled fermions to undergo an interaction free beam splitting operation, in an intact manner. Consequently, we can say that such pairs cannot be considered as single bosons in an unambiguous way. In addition, it is important to mention that these composite bosons are very fragile. Even if entangled pairs of spatially separated fermions can exhibit single boson behaviour, they would be extremely sensitive to external disturbance [8].

Chapter 2

On dynamical stability of composite particles

The results included in this chapter were published as a regular article in Physical Review A [2].

2.1 Introduction

A composite particle is a divisible system, made of at least two indivisible constituents. This divisible nature implies that the system has an internal structure. In other words, it has information hidden inside of it. Understanding the reasons and conditions which make the constituents exhibit a collective behaviour, as if they make a single entity to the outside observer, is one of the key problems of the field of many body physics, and complex quantum systems, for recent reviews see [6, 37].

Previous works suggested that the information embedded within the internal structure of composite bosons has a crucial role behind some collective behaviour [7–9, 38–42]. In the previous chapter, we arrived at a similar conclusion. However, while considering the stability of a composite particle going through a beam splitting operation, we also saw that interactions are necessary, namely for entanglement production or consumption (depending on the case). This lead us to ask: is it possible to have a situation where only entanglement is responsible for a collective behaviour? This is the question I will consider in this chapter.

In the quantum regime, we know that objects can exhibit particle-like behaviour, while in other situations the same objects can behave in a wave-like manner. This is called wave-particle duality, and this is going to be of crucial importance in the following discussions. In fact, in the previous chapter, our approach for studying composite particles exposed their particle-like nature. In this case, we saw that interactions are required in order to keep them stable. However, when a composite particle is behaving like a wave, thing will be different regarding its stability. In this chapter, I will consider a system made of two constituents, and I will be interested in the way the system will spread via an interaction-free evolution.

In fact, the concept of a composite particle is often ambiguous. This is why I will start this chapter by stating few definitions regarding what can be considered as stable composite particle, or a decayed one. Considering these definitions, I will discuss an interaction-free spread of a bipartite system, in a double Gaussian state, over a one-dimensional space. More precisely, I will be interested in the role of the quantum correlations in preserving the stability of the system. In the last part of this chapter, I will examine the effects of thermalization over the spread of the same system.

2.2 Preliminaries

2.2.1 Single vs. composite particles

The most obvious property of a particle is the fact that it is a localized object. Unlike waves, a particle can be associated with a single position. Therefore, when a particle is confined within a region of space, and a number of detectors is distributed over this region, at most one detector will click every time a measurement is performed. Ergo, one can say that a single particle can be associated with a single detector click. In addition to the localization, one can enumerate many other properties (like spin, charge ... etc) which are fundamental to what one can consider as a particle. However, in this discussion, only the position and momentum, or velocity, will bring a significant impact. This is why only these two variables will be brought to focus.

First, let us consider a system made of many elementary constituents. If after performing the same measurement described above, only one detector will click, such a system can be viewed as a composite particle. In other words, a composite particle should mimic the localization property of a single particle. After an evolution of time t, this composite particle is considered to be stable as long as its constituents remain close enough to each other. Otherwise, one can say that the composite particle has decayed. Also, by following this line of thoughts, one can say that a composite particle requires few parameters to be fully described. However, if this composite particle will fall apart, one will need more parameters to describe it. This is because when a composite particle falls apart, some correlations are lost. For instance, since a composite particle has all its constituents close to each other, then the knowledge of the position of one of the constituents can imply some information regarding the position of the other constituents. On the other hand, in order to express the state of a decayed composite particle one needs to know about the position and momentum of each constituent.

As matter of fact, the stability of a composite particle can be maintained if its constituents are properly correlated, or if they are attractively interacting. When the constituents are initially close to each other, and all their velocities are identical, the entire system will move toward the same direction with the same speed. This will keep the intra-constituents distances constant as long as they do not experience any external forces. In such a scenario, the composite particle will remain intact while being in a perpetual motion. Here, the source of the composite particle's stability is the correlations between the velocities and positions of its constituents. However, if external forces are exerted, stability might become temporary. If all the constituents experience the same forces, the change of their velocities will be the same. Consequently, the composite particle will remain stable, as in the previous example. However, if these forces act differently on each constituent, then their velocities will start to diverge, and so will be their intra-constituents distances. In this case, the composite particle will inevitably fall apart. By the time this constituents will be far enough from each other to make two detectors click, the system can no longer be considered as a composite particle. When the constituents are attractively interacting, even if the external forces do not act in the same way on all the constituents, the composite particle will remain stable, provided that the intra-constituents interactions are strong enough (see Fig 2.1).

2.2.2 Quantum particles

Within the domain of quantum theory, one can no longer think about particles following the behaviour mentioned in the previous section. Due to the particle-wave duality, an initially localized quantum object will evolve in time and spontaneously disperse like a wave. This object will be in a superposition of being in many different positions at the same time, i.e. it will be delocalized. In such a case, performing a measurement of the position of this object as described in the previous section will lead to the collapse of the wave function. In other words, the performed operation of measurement will bring the object back to a localized state. This means that only a single detector, in an indeterministic way, will click. At the moment of the measurement, the particle will be within the region associated with the click, and all the other superpositions will be destroyed. Similarly to the classical ones, a quantum particle can also be associated with a single click.

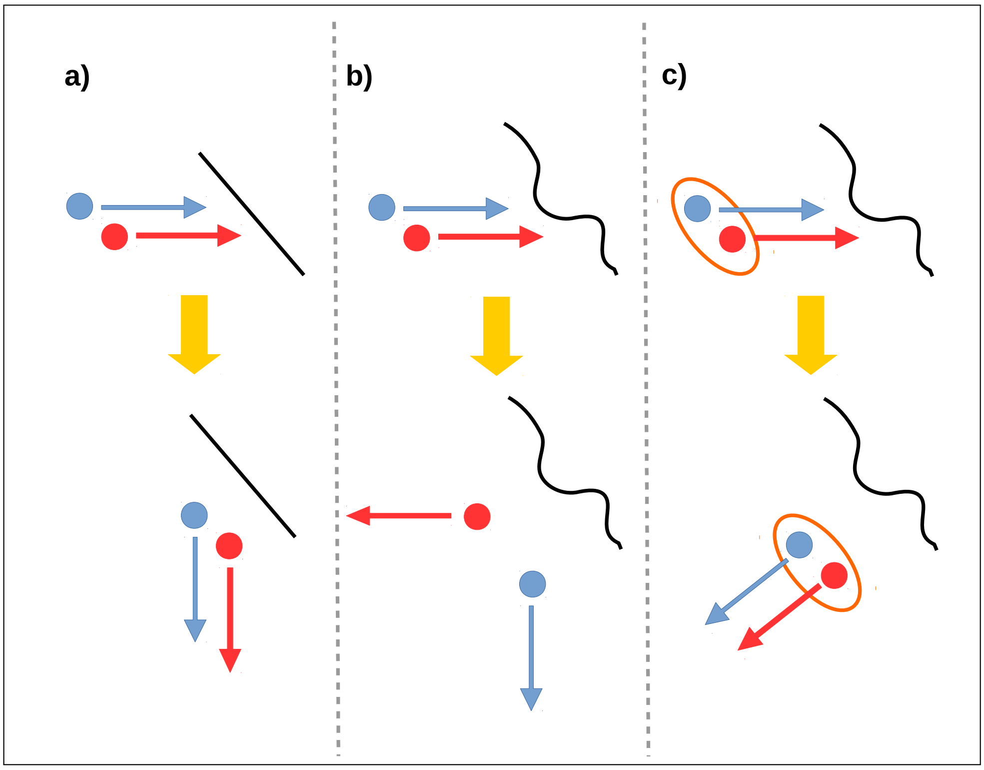

Figure 2.1: Sketch of three scenarios describing the dynamics of a classical composite particle. a) The two constituents are not interacting but their momenta are correlated. After reflecting of a flat wall both momenta will change in similar way. Consequently, both constituents will remain close to each other. b) Two non-interacting constituents with correlated momenta scattered by an irregular wall away from each other. The correlation between their momenta gets disturbed because each particle is reflected by the wall at a different angle. c) Two attractively interacting constituents scattered by an irregular wall remain together. Even if they get reflect at different angles, the interaction keeps them close to each other. Hence, their momenta remain correlated. [2]

On one hand, the concept of compositeness, within a classical framework, is closely related to the correlations between the positions and momenta of the constituents. However, the physics of a quantum composite particle have to obey the Heisenberg uncertainty principle. In other words, one is not allowed to know simultaneously both position and momentum of each constituent. For instance, if a bipartite system is initially localized in a single position, then both constituents are in the same position. Nevertheless, in such a case, one is unable to know any thing about their velocities. Therefore, each one will spread independently, which will lead to the decay of the composite particle as a whole. This is because the knowledge of the state of one particle will say noting about the other one. Thus, it will be highly unlikely for the constituents to remain close to each other.

Assuming a bipartite system spreading via a free evolution. If the constituents remain close enough to each other, performing a measurement will lead to a single detector click. In this case, the total quantum system mimics a classical composite particle. Considering the conclusions drawn by C. K. Law and C. Chudzicki et al. [7, 8], it is natural to think about the role of entanglement behind such a composite behaviour. In other words, we can ask: *is it possible to keep these two constituents close to each other without interactions and using only entanglement as a resource for the stability of the total system?* In this chapter, I will consider this question and discuss the free dynamics of a bipartite system in a double Gaussian state. My goal is to identify a single particle behaviour of the total system and to examine how it depends on entanglement.

2.3 Spread of a composite particle

2.3.1 A composite particle in a double Gaussian state

Gaussian wave packets are particularly suited for studying the considered problem. This is because they are mathematically simple while having interesting properties. However, the conclusions which will be drown later can also be applicable for more general class of states. First, let us recall some elementary definitions. A standard single-particle Gaussian packet can be expressed by

$$\psi(x, t = 0) = \mathcal{N} \exp(-x^2/2\sigma^2). \tag{2.1}$$

Here, the initial position is centred around $x = 0$ and the momentum is also centred around the same value $p = 0$. $\mathcal{N}$ is a normalization constant and $\sigma/\sqrt{2}$ is the so-called standard deviation. For an interaction-free evolution the standard deviation will grow with time as

$$\Delta x(t) = \frac{1}{\sqrt{2}}\sqrt{\sigma^2 + \frac{\hbar^2 t^2}{m^2\sigma^2}}, \tag{2.2}$$

where m is the mass of the particle. Note that at $t = \frac{m}{\hbar}\sigma^2$, the standard deviation would have grown by the factor of $\sqrt{2}$. In fact, it is possible to rewrite the equation (2.2) as

$$\Delta x(t) = \frac{1}{\sqrt{2}}\frac{\hbar}{m\sigma}\sqrt{\left(\frac{m\sigma^2}{\hbar}\right)^2 + t^2}. \tag{2.3}$$

Clearly, for $t \gg \frac{m}{\hbar}\sigma^2$, the time-dependent standard deviation can be approximated as

$$\Delta x(t) \approx \frac{\hbar}{\sqrt{2}m\sigma} t. \tag{2.4}$$

This means that for a greater initial variance, the wave packet spread is slower.

Now, let us consider two one-dimensional particles in a double Gaussian state.

$$\psi(x_1, x_2, t = 0) = \mathcal{N} e^{-\frac{(x_1 - x_2)^2}{4\sigma^2}} e^{-\frac{(x_1 + x_2)^2}{4\Sigma^2}}, \tag{2.5}$$

such that $\sigma/\sqrt{2}$ and $\Sigma/\sqrt{2}$ are the standard deviations along the directions of the relative distance $\frac{x_1 - x_2}{\sqrt{2}}$ and the centre of mass $\frac{x_1 + x_2}{\sqrt{2}}$, respectively. In the same way used in chapter 1, let us consider the purity of a single particle density matrix as a measurement of the degree of entanglement between the particles. We already know that its general definition is given by

$$P = \text{Tr}\{\rho_1(x_1, x_1')^2\} = \text{Tr}\{\rho_2(x_2, x_2')^2\}. \tag{2.6}$$

The purity was calculated in [7] and it has the form

$$P = \frac{1 - \mathcal{Z}}{1 + \mathcal{Z}}, \tag{2.7}$$

where

$$\mathcal{Z} = \left(\frac{\sigma - \Sigma}{\sigma + \Sigma}\right)^2. \tag{2.8}$$

Thus, it can be rewritten as

$$P = \frac{2\sigma\Sigma}{\sigma^2 + \Sigma^2}. \tag{2.9}$$

Note that only when $\sigma = \Sigma$ the purity is equal to 1, which means that the state is separable. In the limit of $\sigma \ll \Sigma$ or $\sigma \gg \Sigma$ the particles are strongly correlated. Actually, the entanglement properties of the double Gaussian quantum states was already studied in [7]. Now, let us consider the dynamics.

2.3.2 Free spread of a composite particle

First, let us examine an interaction-free evolution of the state (2.5) via the free particle Hamiltonian defined by

$$\mathcal{H}_{\text{free}} = \frac{p_1^2 + p_2^2}{2m}, \tag{2.10}$$

assuming that both particles have the same mass. In fact, one can rewrite $\mathcal{H}_{\text{free}}$ as

$$\mathcal{H}_{\text{free}} = \frac{(p_1 + p_2)^2}{4m} + \frac{(p_1 - p_2)^2}{4m}, \tag{2.11}$$

or

$$\mathcal{H}_{\text{free}} = \mathcal{H}_{+} + \mathcal{H}_{-}, \tag{2.12}$$

where $\mathcal{H}_{\pm} = \frac{(p_1 \pm p_2)^2}{4m}$. This suggests that each Gaussian in (2.5) will evolve like a single particle free Gaussian wave-packet, independently of the other one. Consequently, the standard deviations of each Gaussian will grow similarly to (2.2).

When the system is in a separable state, i.e. $\sigma = \Sigma$, the state (2.5) becomes

$$\mathcal{N} e^{-x_1^2/2\sigma^2} e^{-x_2^2/2\sigma^2}. \tag{2.13}$$

Thus, each particle will exhibit an independent free evolution, and consequently the system as whole will not be able to fulfil the aforementioned criteria for composite particles.

Remarkably, when the constituents are highly entangled, the system will mimic an evolution of a composite particle. Note that in this case the factorization (2.13) is no longer possible. Also, when the system evolves for long times, the standard deviations corresponding to the centre of mass and the relative position will grow as t/Σ and t/σ, respectively. Which means that for $\Sigma \ll \sigma$, the centre of mass will get delocalized, while the distance between the particles will grow slowly (see Fig. 2.2).

Now, let us define criteria for the decay of a composite particle. For that, let $\delta \equiv \sigma/\sqrt{2}$ be the initial size of the composite particle and $\tau \equiv \frac{m}{\hbar}\sigma^2$ be its lifetime. As long as the size of the system is less than $\sqrt{2}\delta$, we say that the composite particle is still stable. Otherwise, we say that it has decayed. In fact, for $t > \tau$, the size of the composite particle will be large enough to consider that a decay has occurred. Using (2.2) and (2.9), we can easily show that the centre of mass spreads over the distance

$$\Delta_{cm}(t) = \frac{1}{\sqrt{2}} \sqrt{\Sigma^2 + \frac{\hbar^2 t^2}{m \Sigma^2}}. \tag{2.14}$$

For $t = \tau$, we can write

$$\Delta_{cm}(\tau) = \frac{1}{\sqrt{2}} \sqrt{\frac{\Sigma^4 + \sigma^4}{\Sigma^2}} = \frac{\delta}{P}\sqrt{4 - 2P^2}. \tag{2.15}$$

Clearly, in this case the distance $\Delta_{cm}(\tau)$ depends only on the initial size of the composite particle δ and the degree of entanglement P.

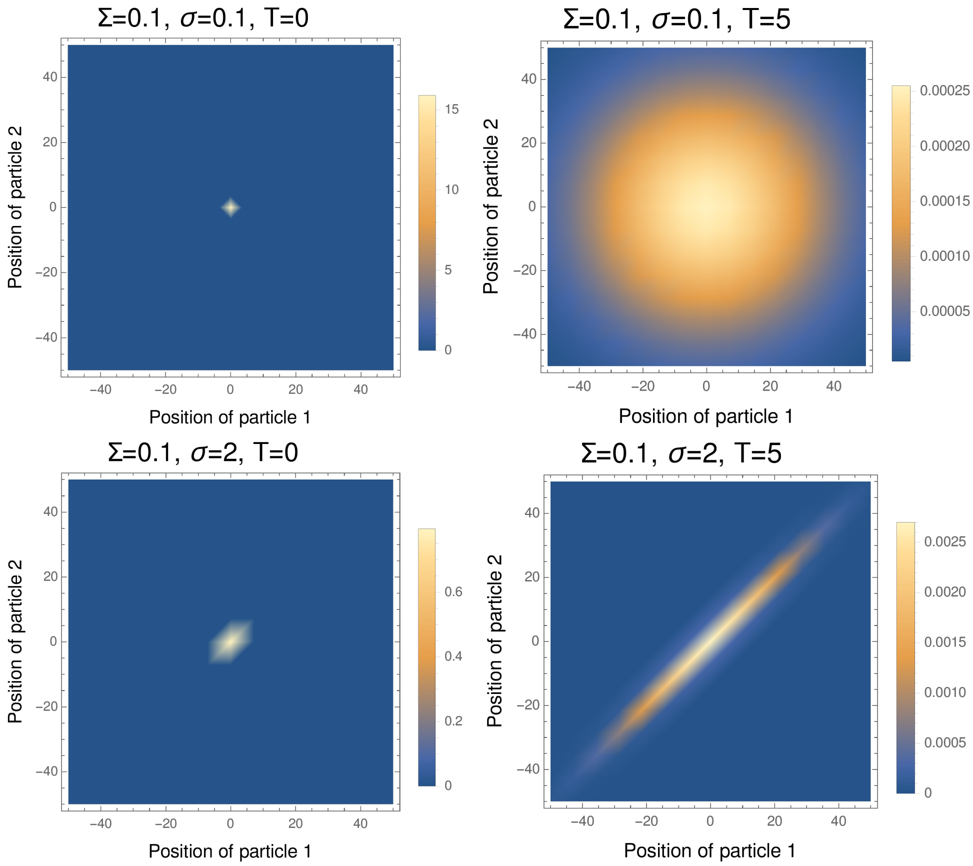

Figure 2.2: Interaction-free evolution of two particles in a double Gaussian state for a separable case (top) and an entangled one (bottom). Natural units are considered ($\hbar = 1$ and $m = 1$). [2]

2.3.3 Effect of temperature on the spread of a composite particle

Now, let us consider the same scenario as above, but with non-zero temperatures. Using the model presented in [43], we can redefine the initial state as

$$\begin{aligned} \psi_{k_1,k_2}(x_1,x_2,t=0) &= \mathcal{N} e^{-\frac{(x_1-x_2)^2}{4\sigma^2}+i\frac{(k_1-k_2)(x_1-x_2)}{2}} \\ &\times \ e^{-\frac{(x_1+x_2)^2}{4\Sigma^2}+i\frac{(k_1+k_2)(x_1+x_2)}{2}}, \end{aligned} \tag{2.16}$$

where k_1 (k_2) stands for the momentum associated with the first (second) particle. In case of a non-zero temperature, these momenta need to be random. Hence, we have to associate them with some probability distributions $\mu_1(k_1)$ and $\mu_2(k_2)$.

For convenience, we can assume k_1 and k_2 to be discrete, independent and identically distributed according to Maxwell distribution.

$$\mu(k,T) = \frac{1}{Z} e^{-\frac{\hbar^2 k^2}{2k_B T}}, \tag{2.17}$$

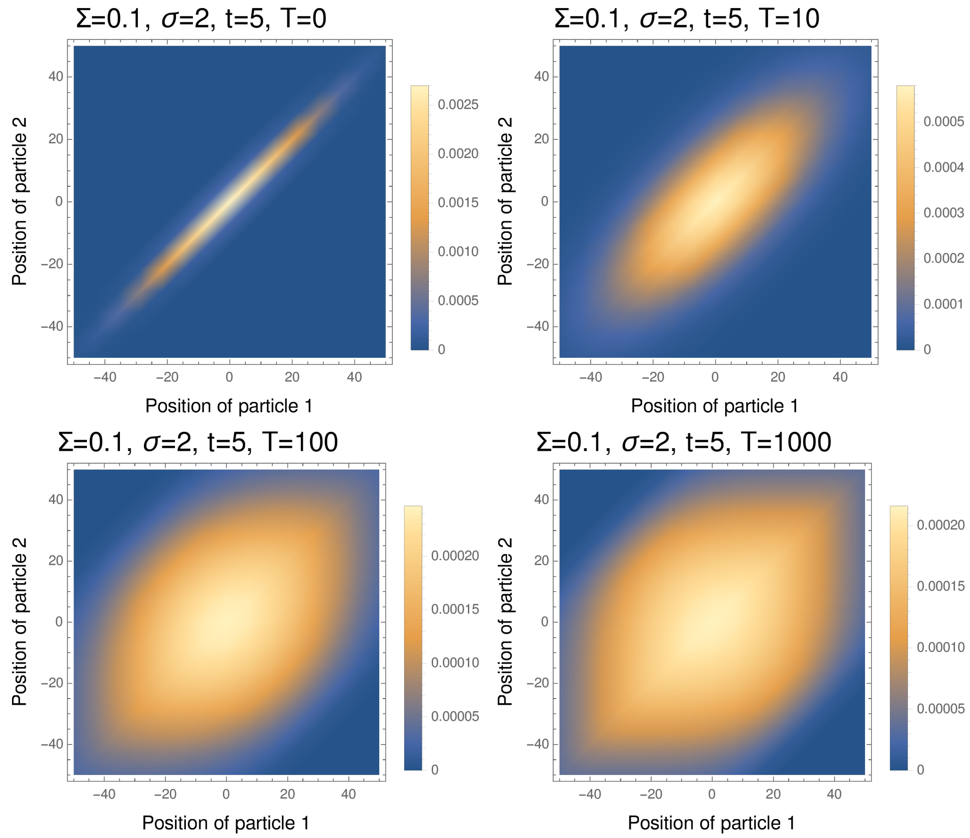

Figure 2.3: Effects of thermalization on the dynamics of two particles in a double Gaussian state for few temperatures. Natural units are considered $k_B = 1$ and $\hbar = 1$. [2]

where Z stands for the partition function, $Z = \sum_k e^{-\frac{\hbar^2 k^2}{2k_B T}}$, T is the temperature, and k_B is the Boltzmann constant. The effect of the thermalization on the spread of the considered composite particle is shown in Fig. 2.3. Here, each momentum can take values $\pm\frac{\hbar n\pi}{5}$, where $n = 0, 1, \ldots, 10$. Clearly, we can see that the distance between the constituents grows as temperatures get higher. Therefore, we can say that, as expected, the increase of temperature will lead to a faster decay of our composite particle.

Remarkably, the effect of thermalization is mainly focused on the relative distance between the constituents. To explain why, let us define $\Delta_T(k)$ as the thermal spread of the momentum. For a particle of mass m, after time t, the thermal fluctuations give rise to a spread which will cover the distance $\frac{\Delta_T(k)}{m}t$. Note that $\Delta_T(k) \leq 4\hbar\pi$. In addition, we saw earlier in this chapter that, in the limit of long times of evolution, the spread will scale as $\approx \frac{\hbar}{\sqrt{2}m\sigma}t$. Consequently, the effect of thermalization will dominate the natural spreading of our composite particle if $\frac{\hbar}{\sqrt{2}\sigma} \ll \Delta_T(k)$. In this discussion, the initial state was always prepared such that $\Sigma \ll \sigma$. As temperatures get higher, the thermal fluctuations will dominate first the spreading of the Gaussian associated with the relative distance. In the limit $\frac{\hbar}{\sqrt{2}\sigma} \ll \frac{\hbar}{\sqrt{2}\Sigma}$, the temperatures will need to get even higher for the thermal fluctuations to dominate the spreading of both Gaussians. Therefore, we can argue that if the temperature is low enough, the effects of thermalization are limited to one degree of freedom, namely the relative distance.

From [43], we know that the standard deviation of the relative distance, assuming thermalization, changes as

$$\Delta_T(x_1 - x_2)(t) = \frac{1}{\sqrt{2}}\sqrt{\sigma^2 + \left(\frac{\hbar^2}{m^2\sigma^2} + \frac{k_B T}{m}\right) t^2}. \tag{2.18}$$

Clearly, for high temperatures, the composite particle will decay after time $\tau_T \approx \sqrt{\frac{m}{k_B T}}$. Note that in this case, the initial entanglement does not have any role.

From the discussion above, one can easily show that the thermal fluctuations dominate the spread of our composite particle when

$$T > \frac{\hbar^2}{m k_B \sigma^2}. \tag{2.19}$$

For a numerical example, let is consider a system made of two electrons at one hundred nanometres apart (10^{-7} m). From the inequality above we get $T > 0.088K$. Note that the considered distance between the electrons is within the range known for Cooper pairs [44]. Remarkably, the temperatures $T > 0.088K$ are also within the range of temperatures at which the formation of Cooper pairs are observed [45]. However, one need to keep in mind that in this work we are considering an over simplified toy model. Therefore, in realistic situations other effects should be also taken into account.

2.3.4 Entanglement as resource for the stability of composite particles

We saw in the previous chapter that the stability of a composite particle going through a splitting operation requires the generation of entanglement. We considered a transformation of the form

$$c^\dagger_{x_0} \to \sum_x \alpha_x c^\dagger_x, \tag{2.20}$$

such that $\sum_x |\alpha_x|^2 = 1$ and

$$c^\dagger_{x_0} = \sum_{i=1}^{d} \sqrt{\lambda_i} a^\dagger_{x_0,i} b^\dagger_{x_0,i}. \tag{2.21}$$

$c^\dagger_{x_0}$ is the creation operator of a composite particle in mode x_0. The Schmidt rank d with the Schmidt distribution $\{\lambda_i\}_{i=1}^{d}$ indicate the degree of entanglement stored inside each composite particle. In the case of $d = 1$, we can write $c^\dagger_{x_0} = a^\dagger_{x_0} b^\dagger_{x_0}$, which means that the constituents a and b are disentangled. Therefore, the transformation (2.20) takes the form

$$a^\dagger_{x_0} b^\dagger_{x_0} \to \sum_x \alpha_x a^\dagger_x b^\dagger_x. \tag{2.22}$$

Clearly, the transformation above creates entanglement, because its right hand side is very similar to the definition in Eq. (2.21), see the discussion on the purity and the Schmidt decomposition of bipartite states in chapter 1. The fact that entanglement production is needed implies that the transformation above is not possible without interaction.

However, in this chapter we saw that a bipartite non-interacting system can get delocalized while its constituents remain close to each other. If fact, this phenomena does not contradict the conclusion stated above, because the actual transformation in this case is of the form

$$c^\dagger_{x_0} \to \sum_x \alpha_x d^\dagger_x, \tag{2.23}$$

such that

$$c^\dagger_{x_0} = \sum_y \beta_y a^\dagger_{x_0,y} b^\dagger_{x_0,y}, \tag{2.24}$$

$$d^\dagger_x = \sum_y \gamma_{x,y} a^\dagger_{x,y} b^\dagger_{x,y}. \tag{2.25}$$

Note that the entanglement in this case is stored in two degrees of freedom. One corresponds to the internal structure of the composite particle, encoded over y in $\sum_y \gamma_{x,y} a^\dagger_{x,y} b^\dagger_{x,y}$ and the second corresponds to the spatial position of the composite particle encoded over x in $\sum_x \alpha_x d^\dagger_x$. The idea of the study presented in this chapter consists of the following: the total entanglement should remain the same through the evolution. So, if the entanglement encoded over the spatial degree of freedom increases, the entanglement stored within the internal structure of the system should decrease. In other words, considering the component a, its initial state is given by

$$\rho_a(0) = \sum_y |\beta_y|^2 a^\dagger_{x_0,y}|0\rangle\langle 0|a_{x_0,y}, \tag{2.26}$$

and it final state takes the form

$$\rho_a(t) = \sum_{x,y} |\alpha_x|^2 |\gamma_{x,y}|^2 a^\dagger_{x,y}|0\rangle\langle 0|a_{x,y}. \tag{2.27}$$

The idea is to have the same purities for both of the states above equal to each other.

To conclude, a composite particle remains stable during a free evolution because the entanglement stored within its internal structure is consumed. Once all entanglement is converted to its spatial degree of freedom, i.e. its internal structure runs out of quantum correlations, $d^\dagger_x = a^\dagger_x b^\dagger_x$, the decay of the system become inevitable. Note that this interpretation explains why the more entangled is the initial state, the further the system will spread before it falls apart, see Eq. (2.15).

2.4 Summary

In this chapter, the problem of stability of quantum systems was studied. Here, the aim was to investigate the role of entanglement behind the stability of composite particles. I considered a free evolution of a bipartite quantum system in a double Gaussian state. We saw that even if the constituents are not attracted to each other, they will remain close to each other, provided that their initial state is properly entangled. Namely, if the Gaussian corresponding to the centre of mass is much narrower than the one corresponding to the relative distance, the first will spread much faster than the second. This means that the centre of mass will get delocalized much faster than the growth of the relative distance. Also, the expression of the distance over which the centre of mass will get delocalized before the composite particle will decay was shown. The explicit form of this expression suggests that it describes a distance which depends solely on the initial size and the amount of entanglement of the system. We saw that during the spread of the system the internal entanglement stored inside the composite particle is consumed and converted into spatial entanglement. This allows the composite particle to get delocalized without falling apart. In the last part of this chapter, I examined the effects of thermalization on the dynamics of the same system. As expected, we saw that increasing temperatures lead to a faster decay. In addition, I discussed the thermal contributions specific to each degree of freedom. For our particular double Gaussian states, if the temperatures are low enough, the thermal contribution will only effect the relative distances.

Chapter 3

On the de Broglie wavelength of composite particles

The results included in this chapter were published as a regular article in Physical Review A [2].

3.1 Introduction

One of the key episodes that has lead to the rise of quantum physics to the theory which we know today is the discovery of the concept of wave-particle duality. Back in 1905, A. Einstein published an argument explaining the photoelectric effect [46]. Basically, he showed that this effect is due to a particle-like behaviour of light. At that time, this was a very counter-intuitive concept, since it was unanimously accepted that light was of an electromagnetic wave. In 1924, L. de Broglie argued in his Ph.D. thesis that the idea of Einstein can be applied the other way around for massive particles [47]. The idea of de Broglie was experimentally confirmed in 1927 by C. Davisson and L. Germer using electrons (they observed diffraction patterns after scattering of a crystal of nickel metal) [48]. To make the theory consistent with itself, since massive particles can behave in a wave-like manner, de Broglie introduced the concept of an effective wavelength to be associated to this massive particles. Nowadays, we call it the *de Broglie wavelength*. Let us denote it by λ_{dB}. Also, he gave a mathematical definition for this wavelength

$$\lambda_{dB} = \frac{h}{p}, \tag{3.1}$$

such that h stands for the Planck constant, and p for the momentum of the particle. Clearly, this wavelength depends on the mass of the particle in question, since we can write

$$p = mv, \tag{3.2}$$

where v and m stand for the velocity and the mass of the particle, receptively.

In the third volume of *The Feynman Lectures on Physics* published in 1965, R. P. Feynman outlined a thought experiment of single electrons passing through a double-slit. By that time, experiments using beams of electrons in double-slit experiment was already realized, e.g. in 1961 by C. Jönsson [49]. However, the work of C. Jönsson has only illustrated a collective wave-like behaviour of a beam of electrons, while the thought experiment of Feynman aimed to show that individual electrons can behave like waves. The first experimental realization of single-electron diffractions was done by G. Pozzi and his team in 1974 [50]. However, in this experiment the authors mimicked the double-slits by using a prism to diffract individual electrons, and they observed the build-up of the diffracted electrons into a the fringe pattern. It was until 2008, that the same team reported the first experiment using an actual double-slit [51], and in 2012 they reported the first experiment where they measured the arrival of each electron individually [52].

In a single-electron double-slit experiment, each electron will be in a superposition and will pass through both slits. When it will arrive at the screen behind the slits, it will interfere with itself. In this scenario, for the electron to arrive to a given point on the screen, it can take one of two paths (one path corresponding to each slit). Because these two paths can have different lengths, the two superpositions will arrive to the screen with a relative phase shift. This is why the interference of each electron with itself will lead to the well known fringe pattern. Interestingly, this fringe pattern will have a period which depends on the de Broglie wavelength of the particle in question, λ_{dB}. In other words, the double-slit scheme allow for experimental observation of the de Broglie wavelength.

In 1891, L. Zehnder proposed the idea of an interferometer using two mirrors and two half-reflecting mirrors [53]. A beam of light can be split into two beams and later recombined in order to observe the interferences. In a follow-up article published in 1892, L. Mach suggested that by tuning the position of the mirrors, we can change the length of each path [54]. In this case, the difference between the length of the two paths will lead to a phase shit between the two light beams. This made it possible to observe the interferences of the two beams as a function of the relative offset between their phases. Nowadays, this scheme is called the Mach-Zehnder interferometer (MZI). In Figure 3.1, a sketch of Mach-Zehnder interferometer is illustrated. We can see that when the light beam will arrive to first Beam Splitter (BS_1) it will be split between two paths. After reflecting from the mirrors, the beam will either go directly to the second Beam Splitter (BS_2), or pass through phase shifting device (φ) before

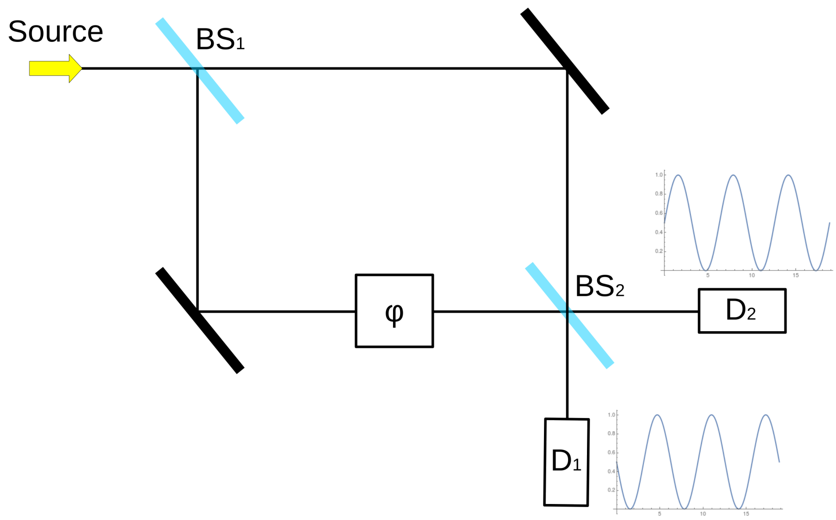

Figure 3.1: Schematic representation of Mach-Zehnder interferometer. [2]

arriving at BS_2. This device can change the length of the corresponding path, which will lead to relative phase shift between the phases of the two beams.

As a matter of fact, the MZI is considered as a simpler, yet more controllable, version of the double-slit experiment. This is because in the quantum regime the concept of both of them is based on a particle in a superposition taking two paths at the same time and interfering with itself at the output. In 1995, Y. Yamamoto and his team first suggested to use a MZI for observing the collective de Broglie of composite particles [38]. This has stimulated many works investigating different methods for observing and manipulating this collective behaviour [39–42]. In addition, the concept of collective de Broglie wavelengths has been proven to be applicable for quantum metrology [55]. Since it can allow for imaging and sensing with precisions beyond the classical limit.

In this work, my goal will be to show that the collective behaviour due to entanglement is conditioned on the performed measurement. Here, my study of compositeness is related to previous works within quantum optics [38–42] and quantum information theory [1, 7–28, 56]. I will first discuss in details the standard MZI. Later, I will propose a MZI-like setup, within the one-dimensional Hubbard model, and then discuss its similarities with the standard one. I will consider a bipartite system in a double Gaussian state spreading over a discrete lattice. Then, I will examine conditions under which a MZI-like setup can lead to the observation of the collective de Broglie wavelength of the considered system.

3.2 Preliminaries

3.2.1 Standard Mach-Zehnder Interferometer

Now, let us go through the mathematics of what happens to a particle through a MZI. Initially, let a single particle be at the first input of BS_1. As we saw in the first chapter, the corresponding operation will lead to the transformation (1.43)

$$a_1^\dagger \to \frac{1}{\sqrt{2}}(a_1^\dagger - ia_2^\dagger). \tag{3.3}$$

Then, the particle in the second path, $a_2^\dagger$, will undergo a phase shift and acquire an extra phase factor $e^{-i\varphi} = e^{-i\frac{2\pi\Delta x}{\lambda_{dB}}}$

$$\frac{1}{\sqrt{2}}(a_1^\dagger - ia_2^\dagger) \to \frac{1}{\sqrt{2}}(a_1^\dagger - ie^{-i\varphi}a_2^\dagger). \tag{3.4}$$

Where Δx stands for the difference between of the lengths of the two paths, and λ_{dB} is the de Broglie wavelength of this single particle. After, the particle will undergo a second beam splitting operation

$$\begin{aligned}\frac{1}{\sqrt{2}}(a_1^\dagger - ie^{-i\varphi}a_2^\dagger) &\to \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(a_1^\dagger - ia_2^\dagger) - ie^{-i\varphi}\frac{1}{\sqrt{2}}(a_2^\dagger - ia_1^\dagger)),\\ &\to \frac{1}{2}((1-e^{-i\varphi})a_1^\dagger - i(1+e^{-i\varphi})a_2^\dagger).\end{aligned} \tag{3.5}$$

Therefore the probabilities of having the particle at detector 1 or 2 will take the form

$$\langle a_1^\dagger a_1\rangle = \frac{1}{2}(1-\cos(\frac{2\pi\Delta x}{\lambda_{dB}^{(1)}})), \tag{3.6}$$

$$\langle a_2^\dagger a_2\rangle = \frac{1}{2}(1+\cos(\frac{2\pi\Delta x}{\lambda_{dB}^{(1)}})). \tag{3.7}$$

Clearly, the two probabilities are periodic as a function of the extra length on the second path, Δx. The period of these oscillations corresponds to an extra length that is exactly equal to the de Broglie wavelength of a single constituent, $\langle a_1^\dagger a_1\rangle|_{\Delta x=0}= \langle a_1^\dagger a_1\rangle|_{\Delta x=\lambda_{dB}^{(1)}}$.

Now, let us consider the case of two elementary particles. For simplicity, we can assume that both particles have the same mass and velocity. Hence, we can say $\lambda_{dB}^{(1)}$ correspond to each one of these single particles. In addition, the collective de Broglie ($\lambda_{dB}^{(2)}$) corresponding to the total system can be written as

$$\lambda_{dB}^{(2)} = \frac{\lambda_{dB}^{(1)}}{2}. \tag{3.8}$$

First, let us assume that both particle are in input one of BS_1, and follow the same steps taken above.

$$a_1^\dagger b_1^\dagger \to \frac{1}{2}(a_1^\dagger - i a_2^\dagger)(b_1^\dagger - i b_2^\dagger). \tag{3.9}$$

The operators $a^\dagger$ and $b^\dagger$ stand for the creation operators of the first and second particle, respectively. Then, the particles on the second path will acquire an extra phase factor $e^{-i\varphi}$

$$\frac{1}{2}(a_1^\dagger - i a_2^\dagger)(b_1^\dagger - i b_2^\dagger) \to \frac{1}{2}(a_1^\dagger - i e^{-i\varphi} a_2^\dagger)(b_2^\dagger - i e^{-i\varphi} b_2^\dagger). \tag{3.10}$$

Before the detection, a second beam splitting operation will be performed at BS_2.

$$\frac{1}{2}(a_1^\dagger - i e^{-i\varphi} a_2^\dagger)(b_2^\dagger - i e^{-i\varphi} b_2^\dagger) \to \tag{3.11}$$
$$\to \frac{1}{4}((1 - e^{-i\varphi})a_1^\dagger - i(1 + e^{-i\varphi})a_2^\dagger)((1 - e^{-i\varphi})b_1^\dagger - i(1 + e^{-i\varphi})b_2^\dagger).$$

Hence, the probabilities of having both particle at the same detector take the form

$$\langle a_1^\dagger a_1 b_1^\dagger b_1\rangle_{\text{ind}} = |\frac{1}{4}(1 - 2e^{-i\varphi} + e^{-i2\varphi})|^2, \tag{3.12}$$
$$\langle a_2^\dagger a_2 b_2^\dagger b_2\rangle_{\text{ind}} = |\frac{-1}{4}(1 + 2e^{-i\varphi} + e^{-i2\varphi})|^2, \tag{3.13}$$

such that $|z|$ correspond to the modulus of the complex number z. The subscript $_{\text{ind}}$ stands for independent particles. Here, we see that we have oscillations as function of φ in addition to double oscillations as function of 2φ. This indicates an ambiguous behaviour of total system which we cannot interpret as a single particle behaviour.

On the other hand, using the transformation (3.11), the probabilities of having each particle at a different detector can be written as

$$\langle a_1^\dagger a_1 b_2^\dagger b_2\rangle_{\text{ind}} = |\frac{-i}{4}(1 - e^{-i2\varphi})|^2, \tag{3.14}$$
$$\langle a_2^\dagger a_2 b_1^\dagger b_1\rangle_{\text{ind}} = |\frac{-i}{4}(1 - e^{-i2\varphi})|^2. \tag{3.15}$$

Of course, using the equations above one can easily show that the sum of the probabilities corresponding to all the possible outcomes is equal to 1.

$$\langle a_1^\dagger a_1 b_1^\dagger b_1\rangle_{\text{ind}} + \langle a_2^\dagger a_2 b_2^\dagger b_2\rangle_{\text{ind}} + \langle a_1^\dagger a_1 b_2^\dagger b_2\rangle_{\text{ind}} + \langle a_2^\dagger a_2 b_1^\dagger b_1\rangle_{\text{ind}} = 1. \tag{3.16}$$

In this discussion, we focus on the situations where a composite particle exhibits single partite dynamics. The probabilities of having each particle at a different detector, $\langle a_1^\dagger a_1 b_2^\dagger b_2\rangle_{\text{ind}}$ and $\langle a_2^\dagger a_2 b_1^\dagger b_1\rangle_{\text{ind}}$, correspond to outcomes which fail to fulfil our definition of single particle behaviour stated in the previous chapter. Hence, we are only interested in the cases for which the constituents stay together, i.e. outcomes corresponding to $\langle a_1^\dagger a_1 b_1^\dagger b_1\rangle_{\text{ind}}$ and $\langle a_2^\dagger a_2 b_2^\dagger b_2\rangle_{\text{ind}}$.

Now, let us consider the case of interacting particles, i.e. the beam splitting operations are performed via the Hamiltonian described by Eq. (1.49). In this case, both particles will always go out from the same output

$$a_1^\dagger b_1^\dagger \rightarrow \frac{1}{\sqrt{2}}(a_1^\dagger b_1^\dagger - i a_2^\dagger b_2^\dagger). \tag{3.17}$$

Consequently, both particles will either undergo the phase shift together or avoid it together.

$$\frac{1}{\sqrt{2}}(a_1^\dagger b_1^\dagger - i a_2^\dagger b_2^\dagger) \rightarrow \frac{1}{\sqrt{2}}(a_1^\dagger b_1^\dagger - i e^{-i\varphi} a_2^\dagger e^{-i\varphi} b_2^\dagger). \tag{3.18}$$

Also, they will go out of BS$_2$ through the same output

$$\frac{1}{\sqrt{2}}(a_1^\dagger b_1^\dagger - i e^{-i\varphi} a_2^\dagger e^{-i\varphi} b_2^\dagger) \rightarrow \frac{1}{2}((1-e^{-i2\varphi})a_1^\dagger b_1^\dagger - i(1+e^{-i2\varphi})a_2^\dagger b_2^\dagger). \tag{3.19}$$

This time, the probabilities of having both particle at the same detector take the form

$$\begin{aligned}\langle a_1^\dagger a_1 b_1^\dagger b_1\rangle_{\text{int}} &= \frac{1}{2}(1-\cos(\frac{2\pi\Delta x}{\lambda_{dB}^{(1)}/2})) = \frac{1}{2}(1-\cos(\frac{2\pi\Delta x}{\lambda_{dB}^{(2)}})), &(3.20)\\ \langle a_2^\dagger a_2 b_2^\dagger b_2\rangle_{\text{int}} &= \frac{1}{2}(1+\cos(\frac{2\pi\Delta x}{\lambda_{dB}^{(1)}/2})) = \frac{1}{2}(1+\cos(\frac{2\pi\Delta x}{\lambda_{dB}^{(2)}})). &(3.21)\end{aligned}$$

Here, the subscript $_{\text{int}}$ stands for interacting particles. Visibly, the period of these oscillations correspond to the collective de Broglie wavelength of the composite particle,

$$\langle a_1^\dagger a_1 b_1^\dagger b_1\rangle_{\text{int}}|_{\Delta x=0} = \langle a_1^\dagger a_1 b_1^\dagger b_1\rangle_{\text{int}}|_{\Delta x=\lambda_{dB}^{(2)}}. \tag{3.22}$$

Therefore, we can consider this as a single particle behaviour of the total composite system. Note that, later in this chapter, the interferences will be plotted as a function of φ. In this case, one will need to keep in mind that $\varphi|_{\Delta x=\lambda_{dB}^{(2)}} = \pi$ and $\varphi|_{\Delta x=\lambda_{dB}^{(1)}} = 2\pi$. In other words, a period equal to 2π implies individual behaviour of the constituents, while a period equal to π may correspond to a collective behaviour of the total system.

We saw in this section that keeping the constituents together while going through the interferometer is crucial for observing its collective de Broglie wavelength. As we saw in the first chapter, keeping two particles together through a beam splitting operation requires the generation of entanglement. Hence, interaction (or some kind of post selection) is inevitable for observing a collective behaviour of the total system. However, we saw in the previous chapter that entangled non-interacting particles can stay close to each other while their centre of mass spreads over the space. Ergo, one can expect that such a phenomenon can lead to the observation of the collective de Broglie wavelength in a generalized MZI scheme. Later in this chapter, this hypothesis will be investigated while considering our definition of composite particles (from last chapter) and without revealing the internal structure of system.

3.2.2 Discrete double Gaussian state

Similarly to the previous chapter, let us consider the evolutions of double Gaussian states. Here, these sates are defined in a discrete and finite space as follows

$$|\psi\rangle = \mathcal{N} \sum_{x_1,x_2=1}^{d} e^{\frac{(x_1+x_2)^2}{4\sigma^2}} e^{\frac{(x_1-x_2)^2}{4\Sigma^2}} a^\dagger_{x_1} b^\dagger_{x_2} |0\rangle, \tag{3.23}$$

where $a^\dagger_{x_1}$ ($b^\dagger_{x_2}$) stands for a creation operator of the first (second) constituent at position x_1 (x_2). Also, let us assume periodic boundary conditions, i.e., $x_j + d \equiv x_j$ for $j = 1, 2$. This system will evolve via the Hamiltonian $\mathcal{H} = \mathcal{H}_{\text{free}} + V$, such that

$$\mathcal{H}_{\text{free}} = -\sum_{x=1}^{d} \left(a^\dagger_{x+1} a_x + a^\dagger_x a_{x+1} + b^\dagger_{x+1} b_x + b^\dagger_x b_{x+1} \right), \tag{3.24}$$

will give rise to the free spread of our composite particle. The term V stands for a potential which will be exerted on the system.

3.2.3 Measurements

Now, let us recall few statements from the previous chapter. We assumed that a bipartite system can be considered as a single entity if its constituents are close to each other. So, we associated the concept of a composite particle with a single detector click. However, for this discussion, we will need to concretely define when two objects are close and when they are spatially separated.

By definition, in a double Gaussian state the particles do not necessarily share the same position. However, the relative distance between the constituents has a standard deviation $\delta = \sigma/\sqrt{2}$, which we can consider as the size of the composite particle. In fact, when we treat a composite system as single entity we need to discard any information regarding its internal structure. For instance, we can limit our focus to the centre of mass of the system, provided that the relative distance is bellow some threshold Δ ($\Delta \geq \delta$). Practically, we can implement this threshold by coarse graining the space.

From the definition of the state in Eq. (3.23), we can see that the two constituents of the system are assumed to be distinguishable. However, we would like to avoid revealing the internal structure of our composite particle. Thus, we should neither be able to distinguish the constituents, nor to address them individually. This can be achieved by appropriately choosing the operation for performing the measurement. Our space is made of d different positions, with periodic boundary conditions. For the coarse graining, let us say we would like to divide it to m unit cells. Therefore, each cell will need to have a size of $\Delta = \frac{d}{m}$ positions. We associate to each cell a detector which can be described by the operator

$$D_j = \sum_{x_1,x_2=j\Delta+1}^{(j+1)\Delta} a^{\dagger}_{x_1} a_{x_1} b^{\dagger}_{x_2} b_{x_2}. \tag{3.25}$$

The index j stands for the label associated to each coarse grained unit cell. The operator D_j has an eigenvalue equal to 1 only if both particles are located within the j^{th} cell, $j\Delta + 1 \leq x_1, x_2 \leq (j+1)\Delta$. Otherwise, it has an eigenvalue equal to 0.

All things considered, in this measurement scenario we assume that the space is coarse grained into m unit cells. Each cell is associated with a detector. At most, one detector will click, register an outcome 1, if both constituents are within its corresponding cell. In case of a decay of our composite particle, which we interpret as having each constituent in a different cell, all the detectors will register an outcome 0 and will not click. However, one need to keep in mind that coarse graining in this protocol might give rise to some imperfections. In some cases, the relative distance can be shorter than Δ, while the constituents are in different unit cells. For example, for $x_1 = \Delta - 1$ and $x_2 = \Delta + 1$, neither of the detectors D_0 or D_1 will click, even if $|x_2 - x_1| = 2$. However, if Δ is large enough, the probabilities for such cases to happen will be relatively low.

It might be tempting to think of different detections operation, such as

$$D^{(a)}_j + D^{(b)}_j = \sum_{x_1=j\Delta+1}^{(j+1)\Delta} a^{\dagger}_{x_1} a_{x_1} + \sum_{x_2=j\Delta+1}^{(j+1)\Delta} b^{\dagger}_{x_2} b_{x_2}. \tag{3.26}$$

This operation will lead to two clicks every time a measurement is performed (one for each constituent). Of course, such an operation can be used for checking if the composite particle has decayed or not. Nevertheless, the aim of this work is to investigate the role of entanglement. Since the average value of this operation takes the form

$$\langle D_j^{(a)} + D_j^{(b)} \rangle = \mathrm{Tr}\{D_j^{(a)}\rho_a\} + \mathrm{Tr}\{D_j^{(b)}\rho_b\}, \tag{3.27}$$

we can say that no information is revealed regarding the quantum correlations between the constituents. In order to access such information, one will need to consider the variance or higher moments.

3.3 Mach-Zehnder-like setup

The main idea of this scheme is as follows. We consider a discrete lattice of d positions with periodic boundary conditions. We coarse grain this space into 4 unit cells, $\Delta = d/4$, labelled by $j = 0, 1, 2, 3$. The initial state should be prepared at the centre of one of these unit cells. Then, we let the wave-packet spread to the two neighbouring cells and apply a phase shift only in one of them. After that, we let the system evolve again and let it recombine and interfere with itself at the forth cell where the measurements will be performed. To be more specific, let us go through the details step by step.

First - State preparation: We prepare the initial state at the centre of the cell labelled by $j = 1$, i.e the double Gaussian should be centred around the position $3d/8$

$$|\psi\rangle = \mathcal{N} \sum_{x_1,x_2=1}^{d} e^{\frac{(x_1+x_2-3d/4)^2}{4\sigma^2}} e^{\frac{(x_1-x_2)^2}{4\Sigma^2}} a^\dagger_{x_1} b^\dagger_{x_2} |0\rangle. \tag{3.28}$$

In order to be consistent with our definition of a composite particle, we choose the parameters σ and Σ such that the entire wave-packet is located only within one unit cell, $j = 1$. Then, we let it spread to the two neighbouring cells, $j = 0$ and $j = 2$, which will play the role of the two paths in the standard MZI.

Second - Splitting of the wave-packet: We let the system evolve via the Hamiltonian $\mathcal{H}_{\text{free}}$ for a time period we call $T/2$ (see Eq. (3.24)). Here, we are trying to realize a $50/50$ beam splitting operation to the cells $j = 0$ and $j = 2$. However, via $\mathcal{H}_{\text{free}}$, it is impossible to realize that with perfect efficiency. This is due to the fact that the ideal $50/50$ splitting is a perfectly periodic operation. However, since the ratio of the eigenvalues of $\mathcal{H}_{\text{free}}$ is in general irrational, the operator $e^{-i\mathcal{H}_{\text{free}}t}$ is quasi-periodic.

$$E(k_1, k_2) = -2\cos\left(\frac{2\pi}{d}k_1\right) - 2\cos\left(\frac{2\pi}{d}k_2\right), \tag{3.29}$$

where $E(k_1, k_2)$, and $k_1, k_2 = 0, 1, \ldots, d-1$ stands for the spectrum of $\mathcal{H}_{\text{free}}$ and the momentum of the first and the second particle. Nevertheless, it is possible to approximately find value for $T/2$ such that $e^{-i\mathcal{H}_{\text{free}}2T} \approx \mathbb{1}$. In other words, our wave-packet starts to spread from the cell $j = 1$ to the rest of the space. Since we are assuming that our space is finite and with periodic boundary conditions, the wave-packet will eventually continue spreading and at some point it will recombine around its initial position, $3d/8$. This state does not have to match perfectly the initial wave-packet. Here we are interested in the time where its fidelity with the initial wave-packet is at its maximum, we chose to call that time $2T$. At half that time, T, the wave-packet should be at the opposite cell $j = 3$, and at the quarter of that time, $T/2$, the wave-packet should be in a superposition at the two neighbouring cells, $j = 0$ and $j = 2$ (see Fig. 3.2, first row).

Third - Phase shift: Now that our wave-packet is split between cells $j = 0$ and $j = 2$, a phase shift is applied at the cell $j = 2$ via the operator $U_\varphi = e^{-iV\varphi}$, such that

$$V = \sum_{x_1, x_2 = d/2+1}^{3d/4} (a^\dagger_{x_1} a_{x_1} + b^\dagger_{x_2} b_{x_2}). \tag{3.30}$$

and φ correspond to the time period over which the potential is exerted.

Fourth - Recombination of the wave-packet: Now, we let the system evolve again via the operator $U = e^{-i\mathcal{H}_{\text{free}}T/2}$. As discussed above, this should lead to the recombination of the wave-packet in the opposite unit cell, $j = 3$.

Fifth - Measurement: At this point, we apply the operator D_3 in order to verify if both constituents are within the corresponding cell (see the definition of D_3 in Eq. (3.25)).

The evolution of the initial wave-packet defined in Eq. (3.28) was numerically simulated for three case, entangled without interactions ($\Sigma = 0.01$ and $\sigma = 2$), separable without interactions ($\Sigma = 0.01$ and $\sigma = 0.01$), and the same separable state with interactions. In the later case, the dynamics were generated via $\mathcal{H}_{\text{free}} + \mathcal{H}_{\text{int}}$, such that $\mathcal{H}_{\text{int}}$

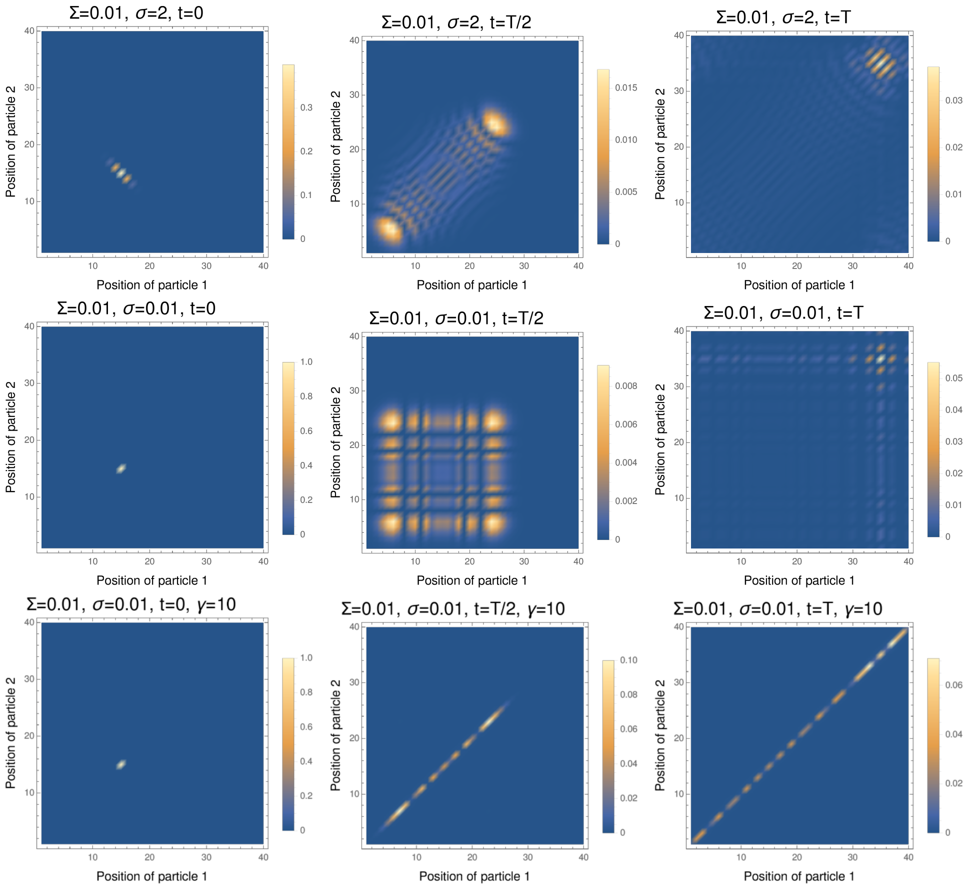

Figure 3.2: Two-particle probability density plots showing the evolution in the MZI-like setup for $\varphi = 0$. The first row corresponds to entangled initial conditions, the second one to separable initial conditions, and the third one to the evolution with an interaction ($\gamma = 10$) between the particles. $T = 11$ for the first two cases and $T = 50$ for the last one. [2]

correspond to a point-like interaction term.

$$\mathcal{H}_{\text{int}} = -\gamma \sum_{x=1}^{d} a_x^\dagger a_x b_x^\dagger b_x, \tag{3.31}$$

γ stands for an interaction parameter (attractive interactions correspond to a positive γ and repulsive otherwise). In this simulations, the parameters were chosen as $d = 40$, $T = 11$, and $\gamma = 10$. In order to illustrate the splitting and recombination of the wave-packet, the evolution of the system without the phase shift $\varphi = 0$ is presented in Fig. 3.2.

On the other hand, in the case with phase shifts, I considered three different sizes for the detector, D_3. This will allow us to examine the effects of revealing the internal structure of our composite particle on the observed interferences. First, I assumed that the detector can cover the entire cell $j = 3$, which correspond to a size $\Delta = 10$ (the detector clicks if $31 \leq x_1, x_2 \leq 40$). For the second case, I assumed a

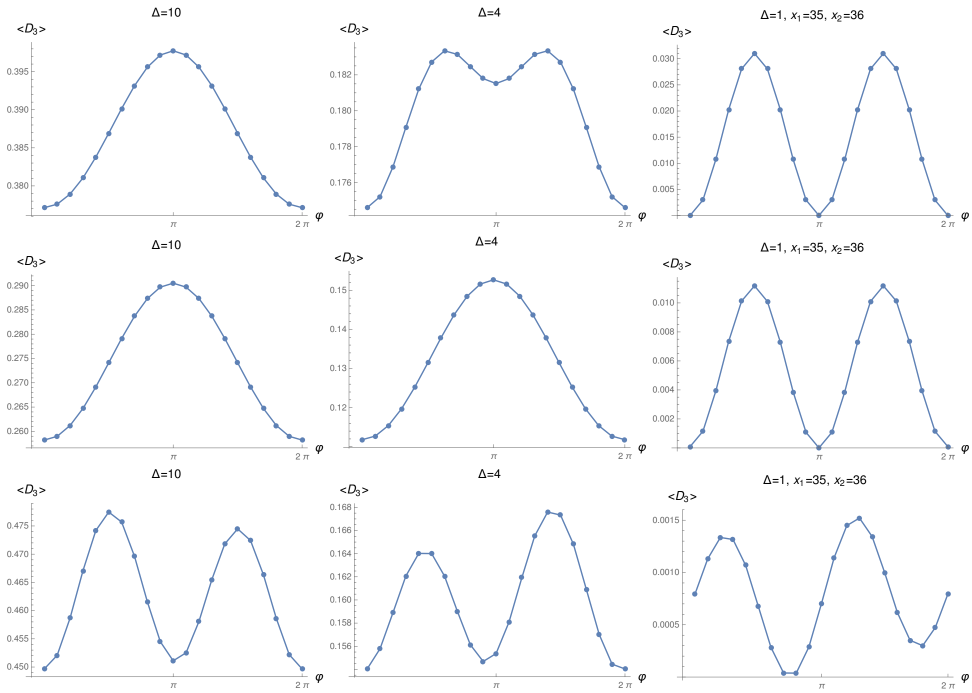

Figure 3.3: Average values $\langle D_3 \rangle$ as functions of φ. The first row corresponds to entangled initial conditions, the second one to separable initial conditions, and the third one to the evolution with an interaction ($\gamma = 10$) between the particles. The graphs in each row represent different coarse graining of space which corresponds to different resolutions of the detector D_3. The coarse graining effect is especially important in the first two cases with no interaction. [2]

smaller size $\Delta = 4$ to be covered by the detector, which means that we get a click if both particles are within $34 \leq x_1, x_2 \leq 37$ (a small part at the centre of the cell $j = 3$). For the last case, I considered the smallest size possible, $\Delta = 1$, which corresponds to a click if the first and second constituent is at position $x_1 = 35$ and $x_2 = 36$, respectively. For these three cases, the expectation value $\langle D_3 \rangle$ is plotted as a function of the applied phase shift φ in Fig. 3.3.

As mentioned above, when the oscillations have a period of 2π it corresponds to a single particle de Broglie wavelength, $\lambda_{dB}^{(1)}$. On the other hand, a period equal to π may imply a collective behaviour of the composite particle, $\lambda_{dB}^{(2)}$. Clearly, when the detector covers the entire cell, $j = 3$, the collective behaviour is only visible when the constituents are attractively interacting. However, when $\Delta = 4$, we can see that the entangled state starts to lead to single particle behaviour of the total system. This can be attributed to the fact that in this case Δ and the composite particle are of comparable sizes. In other words, we can say that the detector starts to reveal the internal structure. For $\Delta = 1$, we can see that the period of the oscillations is equal to π, even when the initial state is separable. In the previous chapter, we saw that separable states cannot lead to a single particle behaviour without interaction. Ergo, this double oscillations is a consequence of the detection strategy, and neither due to the entanglement nor to a collective behaviour. For a separable state, we can write $\langle a^\dagger_{x_1} a_{x_1} b^\dagger_{x_2} b_{x_2}\rangle = \langle a^\dagger_{x_1} a_{x_1}\rangle\langle b^\dagger_{x_2} b_{x_2}\rangle$. Assuming that both $\langle a^\dagger_{35} a_{35}\rangle$ and $\langle b^\dagger_{36} b_{36}\rangle$ are periodic functions with a period equal to 2π, it is possible for the product $\langle a^\dagger_{35} a_{35}\rangle\langle b^\dagger_{36} b_{36}\rangle$ to have a period equal to π. Note that a similar case was already discussed when we considered the standard MZI (see Eqs. (3.14) and (3.15)).

3.4 Why interaction is so important?

In this chapter, we saw that properly entangled states (3.23) of a bipartite non-interacting system can manifest "some" composite behaviour. In this case, the success of observing composite features depends on the measurement strategy. But, in general, interaction is required for observing a true manifestation of compositeness, that does not depend on the performed measurement. In this section I will discuss why non-interacting systems cannot exhibit a fully composite behaviour, and why interaction is so important for this behaviour.

First, let us reconsider the standard MZI. In the case of two particles with no interactions, from Eq. (3.11) we can write the probability of having particle A at detector 1 as

$$\langle a^\dagger_1 a_1\rangle_{\text{ind}} = \frac{1}{2}(1-\cos(\varphi)). \tag{3.32}$$

Note that in this case we have an individual behaviour of particle A. On the other hand, when particles are allowed to interact, using Eq. (3.20) we get

$$\langle a^\dagger_1 a_1\rangle_{\text{int}} = \frac{1}{2}(1-\cos(2\varphi)). \tag{3.33}$$

In this case, we have a composite behaviour which corresponds to the collective de Broglie wavelength, $\lambda^{(2)}_{dB}$. Note that in Eq. (3.32), we have an identical probability to the one we got for a single particle in the standard MZI, see Eq. (3.6). This can be explained by the fact that the free-evolution of each particle is independent of the behaviour of the other one. In other words, during the evolution, if particles do not interact, they do not know about each other. For instance, by looking at the evolution of particle A, we cannot say anything about the evolution of particle B. In fact, we can not know if particle B is in the interferometer or not. However, when interaction is included, things get different. In this case, particles can communicate, and share information. Such communication is not possible without interaction. In fact, this is a variant of the *no-signalling condition*

For example, let us consider $\varphi = \pi$. We see that in the independent case, the particle A has zero chance of being at D_1, $\langle a_1^\dagger a_1 \rangle_{\text{ind}}|_{\varphi=\pi} = 0$. On the other hand, when interaction is included, the probability corresponding to the same outcome is equal to one, $\langle a_1^\dagger a_1 \rangle_{\text{int}}|_{\varphi=\pi} = 1$. Thus, by allowing interaction and just looking for particle A, we can say if particle B is in the interferometer or not. Ergo, we can say that a composite particle requires interaction whenever it is involved in a scenario that allows for signalling between its constituents. Note that post-selective measurements can also simulate an effective interaction, and consequently can lead to the observation of some composite features.

3.5 Summary

In this chapter I considered a Mach-Zehnder-like setup and my aim was to observe the collective de Broglie wavelength of a bipartite system. I considered a discrete space, which was divided into 4 coarse grained unit cells. We saw that when the system is prepared in a double Gaussian wave-packet, and properly entangled, the entire system can exhibit single particle behaviour and spread over the space without falling apart. We saw that by considering periodic boundary conditions, the dynamics of the system can mimic the dynamics of a particle in a standard Mach-Zehnder interferometer, and thus allow for the observation of a composite feature of the system, namely the collective de Broglie wavelength. For the measurements, I considered three different strategies, $\Delta = 10$, $\Delta = 4$, and $\Delta = 1$. The interference patterns were computed numerically for the considered MZI. We saw that only when the measurement precision is comparable to the size of the system, $\Delta = 4$, that entanglement can lead to the observation of some composite behaviour. However, one need to keep in mind that the true compositeness requires the constituents to stay correlated, and therefore interactions are fundamentally required.

Chapter 4

On the formation of multipartite composite bosons

The results included in this chapter were published as a preprint version of a regular article [3]. Currently, it is under review in Physical Review A.

4.1 Introduction

The concept of composite bosons has attracted the attention of many authors from the quantum information community [7–28]. However, all the previous works published so far were considering composite bosons made two constituents. It is only natural to ask if this concept is also applicable for composite particles made of many constituents. In this chapter, I will present a generalization to composite bosons made of $2N$ fermionic constituents.

In fact, a system made of N indistinguishable pairs of fermions has many possible ways to form composite bosons. For instance, it can be a single composite boson made $2N$ constituents. Also, the system can split and form two composite bosons, each one made of N elementary fermions (assuming N is even). It is also possible to have N bipartite composite bosons. Thus, one might ask: what are the conditions necessary for the formation of these possible assemblies? and what is their respective bosonic quality? Here, I will consider these questions. I will study the ground state of the one-dimensional Hubbard model while tuning the strength of the inter-particle interactions. Specifically, I will discuss the behaviour of N identical pairs of elementary fermions on a lattice, and examine their corresponding bosonic quality.

Considering a large number of constituents implies many *possible* structures. This leads to the question: how can we control the structure of composite particles in laboratory? The relevance of this question stems from the fact that some structures seems to appear spontaneously. In some situations, this spontaneous behaviour can be disadvantageous. Especially, if it does not lead to the structure of interest. For instance, the Bose-Einstein condensation of atomic Hydrogen [57]. The main challenge here was to keep the Hydrogen atoms from forming H_2 molecules, e.g. by using high magnetic fields to achieve spin polarization [58]. In this chapter, I will show that the control of the strength of interactions has a crucial role behind the formation and control of composite structures.

4.2 Composite boson made of a single A-B pair

In this section, I consider a system made of a single A-B pair within the Hubbard model. My aim is to find the ground state and to examine its bosonic properties depending on the strength of interactions. Here, I will show that this system can exhibit a good bosonic behaviour in the limit of strong point-like interactions. This is a quick warm up, since the behaviour of such a system was already discussed in a previous work [24].

4.2.1 A bipartite composite boson

In the first chapter, we discussed the idea of composite bosons in a maximally entangled state

$$c^\dagger|0\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} a_k^\dagger b_k^\dagger|0\rangle. \tag{4.1}$$

For this state, the Schmidt distribution is uniform, i.e. all the non-zero Schmidt coefficients are equal to each other, $\{\lambda_i = 1/d\}_{i=1}^{d}$. Hence, the name "uniform state" was allocated to (4.1) in [19]. The internal structure for this state depends on a single variable, d. This makes the evaluation of its bosonic quality relatively easy, see Eqs. (1.32 – 1.35). However, for this discussion, it is more convenient to consider a more general description of a composite boson made of two distinguishable fermions.

$$c_{s,r}^\dagger|0\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} e^{i\frac{2\pi}{d}kr} a_k^\dagger b_{k+s}^\dagger|0\rangle. \tag{4.2}$$

such that, $r, s = 0, 1, \ldots, d-1$. Note that regardless of the indices r and s the quality of the composite boson created by $c_{s,r}^\dagger$ depends only on the Schmidt rank, d. One can easily show that the purity for a given d corresponds to its smallest possible value, $P = 1/d$. Ergo, we can say that the operator $c_{s,r}^\dagger$ creates a composite boson with the best possible quality.

As matter of fact, expression (4.2) describes a set of d^2 orthonormal states

$$\langle 0|c_{s,r}c^{\dagger}_{s',r'}|0\rangle = \delta_{s,s'}\delta_{r,r'}, \tag{4.3}$$

such that $\delta_{x,y}$ stands for the Kronecker delta. However, as reported in [6], this orthogonality is not preserved for states corresponding to more than one composite boson.

$$|\langle 0|c^{\dagger N}_{s,r}|c^{\dagger N}_{s',r'}|0\rangle| \geq 0 \neq \delta_{s,s'}\delta_{r,r'}. \tag{4.4}$$

Interestingly, when $N = d$, the Pauli exclusion principle implies that every mode is occupied, which correspond to a unique state,

$$c^{\dagger d}_{s,r}|0\rangle \equiv c^{\dagger d}_{s',r'}|0\rangle, \tag{4.5}$$

regardless of the values of s, s', r and r'.

Now, let us consider that our system is described by the operator $c^{\dagger}_{s,r}$ but in a superposition for different values of the indices s and r.

$$\sum_{s,r=0}^{d-1} \alpha_{s,r}c^{\dagger}_{s,r}|0\rangle. \tag{4.6}$$

In this case, many terms might cancel out. In fact, it is possible to remain only with a single term, i.e. $a^{\dagger}_k b^{\dagger}_{k'}$. This corresponds to a separable state, therefore the system cannot exhibit any bosonic quality. On the other hand, we can say that the more terms will remain, the more ideal the bosonic behaviour will be.

4.2.2 Bosonic quality of the Hubbard ground state

In 1968, E. H. Lieb and F. Y. Wu showed that the one dimensional Hubbard model have an exact solution with Bethe ansatz [59]. However, in this section, we are considering two distinguishable particles (an A-B pair) within the same model. I assume that attractive interaction occurs when both particles are in the same site. When the energy corresponding to this interaction is much stronger than the kinetic energy of the particles, the ground state of the system will be a superposition corresponding to both particles always occupying the same site. In this case, the A-B pair can be approximated by a hardcore boson. Inspired by the results reported in [59], I will show that the bipartite ground state is simply a sum of maximally entangled terms of the form (4.2).

First, let us consider that the Hubbard Hamiltonian contains two terms

$$\mathcal{H} = J\mathcal{H}_0 + U\mathcal{H}_p, \tag{4.7}$$

such that J and U are positive parameters. $\mathcal{H}_0$ and $\mathcal{H}_p$ stand for the hoping and point-like interaction terms, respectively.

$$\mathcal{H}_0 = -\sum_{i=0}^{d-1} \left(a_i^\dagger a_{i+1} + b_i^\dagger b_{i+1} + h.c. \right). \tag{4.8}$$

$$\mathcal{H}_p = -\sum_{i=0}^{d-1} a_i^\dagger a_i b_i^\dagger b_i. \tag{4.9}$$

In addition, I consider the periodic boundary condition $d \equiv 0$.

Now, let us discuss the action of $\mathcal{H}$ on $|s,r\rangle \equiv c_{s,r}^\dagger |0\rangle$. One can easily show that

$$\begin{aligned} \mathcal{H}|s,r\rangle = & - & J(1+e^{i\frac{2\pi}{d}r})|s+1,r\rangle \\ & - & J(1+e^{-i\frac{2\pi}{d}r})|s-1,r\rangle \\ & - & U\delta_{s,0}|s,r\rangle. \end{aligned} \tag{4.10}$$

Clearly, $\mathcal{H}$ does not affect the degree of freedom described by r. Consequently, every value of r will lead to an independent set of equations. The hoping contribution is described by $-J(1+e^{\pm i\frac{2\pi}{d}r})$. Hence, its smallest possible value corresponds to $-2J$ for $r=0$.

For a moment, let us consider an infinite lattice, $-\infty \leq s \leq \infty$, which means $d \to \infty$. In this case, the states (4.2) correspond to a purity equal to zero

$$P = \lim_{d\to\infty} \frac{1}{d} = 0. \tag{4.11}$$

This implies an ideal bosonic behaviour of the states (4.2), regardless of the number of composite bosons, N.

Now, let us consider

$$|\psi_0\rangle = \sum_{s=-\infty}^{\infty} \alpha_s |s,0\rangle, \tag{4.12}$$

as a possible ground state for $\mathcal{H}$. We can write

$$\mathcal{H}|\psi_0\rangle = \varepsilon|\psi_0\rangle, \tag{4.13}$$

such that ε stands for the ground state energy. In order to determine the values α_s and ε we need to solve a set of recurrence equations of the form

$$-\frac{\varepsilon}{2J}\alpha_s = \alpha_{s+1} + \alpha_{s-1}, \tag{4.14}$$

for $s \neq 0$, and an atypical equation

$$-\frac{(U+\varepsilon)}{2J}\alpha_0 = \alpha_1 + \alpha_{-1}. \tag{4.15}$$

First, let us focus on the typical equations (4.14). In general, this kind of recurrence equations have solutions of the form

$$\alpha_s = A r_0^s + B r_0^{-s}, \tag{4.16}$$

such that A and B are constants, while r_0 and r_0^{-1} are roots of

$$r^2 + \frac{\varepsilon}{2J} r + 1 = 0. \tag{4.17}$$

In order to keep the ground state normalized, we need to assume the condition

$$\lim_{s\to\pm\infty} \alpha_s = 0. \tag{4.18}$$

Obviously, assuming $|r_0| \leq 1$ means $|\frac{1}{r_0}| \geq 1$. Thus, considering the condition (4.18) implies

$$\alpha_s = A r_0^s \quad (s > 0), \tag{4.19}$$
$$\alpha_s = B r_0^{-s} \quad (s < 0), \tag{4.20}$$

In addition, by substituting the general form (4.16) in our recurrence equations (4.12), we can get

$$\varepsilon = -2J(r_0 + r_0^{-1}). \tag{4.21}$$

Now, let us plug the above in

$$-\frac{\varepsilon}{2J}\alpha_1 = \alpha_2 + \alpha_0, \tag{4.22}$$
$$-\frac{\varepsilon}{2J}\alpha_{-1} = \alpha_0 + \alpha_{-2}, \tag{4.23}$$

which we can rewrite as

$$A(r_0^2 + 1) = A r_0^2 + \alpha_0, \tag{4.24}$$
$$B(r_0^2 + 1) = B r_0^2 + \alpha_0. \tag{4.25}$$

Ergo, we get $A = B = \alpha_0$.

Now, let us substitute ε and α_s in the atypical equation (4.14). After simplifications, we can write

$$-\bar{U} + r_0 + r_0^{-1} = 2r_0, \tag{4.26}$$

such that $\bar{U} = U/2J$. Now, let us multiply the equation above by r_0 and write

$$r_0^2 + \bar{U} r_0 - 1 = 0. \tag{4.27}$$

Clearly, the solutions to the equation above have the form

$$r_0 = -\frac{1}{2}(\bar{U} \pm \sqrt{\bar{U}^2 + 4}). \tag{4.28}$$

In order to have $|r_0| \leq 1$, we choose the solution with the minus sign.

From the results above, we can write the final form of the solution as

$$\alpha_s = \mathcal{A} \left(\frac{\sqrt{U^2 + 16J} - U}{4J} \right)^{|s|}, \tag{4.29}$$

such that $\mathcal{A}$ is a normalization constant. In addition, the corresponding energy is

$$\varepsilon = -\sqrt{U^2 + 16J^2}. \tag{4.30}$$

From the discussion above we can say that the ground state (4.12) does not necessarily correspond to a perfect composite boson. Basically, it is a sum of terms that might cancel each other and decrease the total amount of entanglement between the particles. This might even lead to separable state, and thus the lack of bosonic quality. The results above imply that in the strong interaction regime, $U \gg J$, things are less ambiguous. In this regime, we have $\alpha_0 \to 1$ while $\alpha_s \to 0$ for $s \neq 0$. Hence, the ground state of $\mathcal{H}$ is simply $c^\dagger_{0,0}|0\rangle$, which describes a composite boson with the best possible quality. Note that, in the same limit, the ground state energy tends to take a negative value of the interaction parameter, $\varepsilon \to -U$.

In order to examine the origin of the bosonic behaviour observed above, we need to consider the same system without the hoping term (4.8). In this case, assuming $J = 0$ will lead to a degenerate ground state which we can write as

$$\eta^\dagger_k|0\rangle \equiv a^\dagger_k b^\dagger_k|0\rangle, \tag{4.31}$$

such that

$$\eta^{\dagger 2}_k = 0, \quad \eta^\dagger_k \eta^\dagger_{k'} = \eta^\dagger_{k'} \eta^\dagger_k. \tag{4.32}$$

Which means that any state of both particles occupying the same site is a ground state. In fact, the relations (4.32) are not bosonic. However, they are proper to the so-called hardcore bosons. Clearly, these relations stand for a special case, $d = 1$, of the commutation relations for composite bosons discussed in first chapter, Eq. (1.14). When $d = 1$, the system is in a separable state and the contribution of Δ, i.e. the deviation from the perfect bosonic behaviour becomes

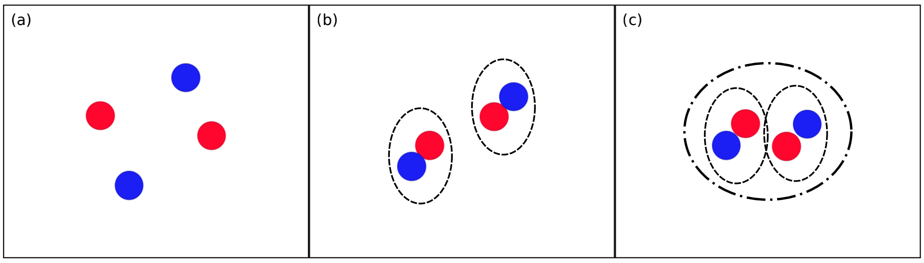

Figure 4.1: Possible assemblies with two fermions of type A and two fermions of type B. (a) No interactions – All the constituents are free. (b) Strong point-like interactions – Formation of two independent A-B pairs. (c) Strong point-like interactions and nearest-neighbour interactions – Formation of a single compound made of two A-B pairs. [3]

important. Therefore, we can say that the hoping term is required in order to lift this degeneracy, and give rise to the entanglement necessary for a good bosonic behaviour.

4.3 Composite boson made of a two A-B pairs

In the last section, we saw that, in the regime of strong point-like interactions and an infinite lattice, the Hubbard Hamiltonian leads to an ideal bosonic behaviour of a single A-B pair. Now, I will consider a system made of two A-B pairs within the same model. My aim is to examine the bosonic properties of the ground states depending on the strength of interactions. For this system, we can have composite bosons similar to the one observed in the previous section, i.e. two bipartite composite bosons as shown in the Fig. 4.1 (b). The second possible combination would be a four-partite composite boson made of all the constituents, see Fig. 4.1 (c). Clearly, in order to observe the later possibility, we will need to make all the constituents interact with each other. However, the point-like interaction leads to the formation of hardcore boson, $\eta_k^\dagger \equiv a_k^\dagger b_k^\dagger$, which have to obey the Pauli exclusion. Ergo, the on-site interaction, per se, is not enough to make this hardcore bosons interact with each other. Intuitively, we can say that the introduction of some long-rang interaction, e.g. between first-nearest neighbours, is required for the formation of the larger composite boson.

In this discussion, I will first present the states describing two bipartite composite bosons and a four-partite one. I will examine the differences between these states and analyse the ground state of the system. I will investigate the transition of the system from the first type of composite bosons to the other one as a function of the strength of the nearest-neighbour interaction.

4.3.1 Two bipartite composite bosons

Considering a single composite boson, while $U \gg J$, the ground state of $\mathcal{H}$ takes the form

$$c^{\dagger}_{0,0}|0\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \eta^{\dagger}_{k}|0\rangle. \tag{4.33}$$

The operator $c^{\dagger}_{0,0}$ corresponds to the creation operator of our composite boson. Therefore, we can write the two-partite Fock state as

$$\frac{c^{\dagger 2}_{0,0}}{\sqrt{2\chi_2}}|0\rangle = \frac{1}{d\sqrt{2\chi_2}} \sum_{k,k'=0}^{d-1} \eta^{\dagger}_{k}\eta^{\dagger}_{k'}|0\rangle, \tag{4.34}$$

such that $\chi_2 = 1 - P = (d-1)/d$ is the normalization factor, see its definition in chapter 1. Bellow, I will show that the state (4.34) can correspond to the ground state of the system, assuming the extended one-dimensional Hubbard model.

4.3.2 Four-partite entangled states

Now, let us consider the general case of a system made of two hardcore bosons. Similarly to the definition (4.2), the states describing such a system can be written in terms of the operators $\eta^{\dagger}_{k}$ as

$$q^{\dagger}_{s,r}|0\rangle \equiv \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{i\frac{2\pi}{d}kr}\eta^{\dagger}_{k}\eta^{\dagger}_{k+s}|0\rangle, \tag{4.35}$$

such that $r = 0, 1, \ldots, d-1$. However, since $\eta^{\dagger}_{k}\eta^{\dagger}_{k+s} = \eta^{\dagger}_{k+s}\eta^{\dagger}_{k}$ and assuming that d is even, we can consider that $s = 1, \ldots, d/2$ without loosing any generality.

Clearly, the above states span an orthonormal basis,

$$\langle 0|q_{s',r'}q^{\dagger}_{s,r}|0\rangle = \delta_{s,s'}\delta_{r,r'}. \tag{4.36}$$

In other words, every possible state of two hardcore bosons can be described by some linear combination of the states above. Hence, the two-partite Fock state (4.34) can be written in terms of $q^{\dagger}_{s,r}$ operators as

$$\frac{c^{\dagger 2}_{0,0}}{\sqrt{2\chi_2}}|0\rangle = \sqrt{\frac{2}{d}} \sum_{s=1}^{d/2} q^{\dagger}_{s,0}|0\rangle. \tag{4.37}$$

Now, let us discuss the differences between the states (4.34) and (4.35). While the state (4.35) counts d different terms, the state (4.34) contains quadratically more terms. This implies that the information encoded within each one of them is somehow different. First, we need to keep in mind that both of them describe two hardcore bosons, which indicates bipartite correlations between the constituents of each hardcore boson. If fermion A is at position k, we can say that a fermion of type B is also at the same position. However, in the state (4.34) every term has the form $\eta_k^\dagger \eta_{k'}^\dagger$, such that the indices k and k' are independent. Therefore, knowing the position k of one hardcore boson does not imply any information about the position k' of the second one, apart from the knowledge that it is not at the same place as the first one $k \neq k'$. On the other hand, the state (4.35) counts terms of the form $\eta_k^\dagger \eta_{k+s}^\dagger$, such that s is a constant. This means that knowing the position one constituent implies information about the positions of the remaining three particles. This can be considered as a genuine four-partite correlations.

4.3.3 Bosonic properties of the four-partite entangled state

Let us examine the bosonic behaviour of a four-partite system described by the state $q_{1,0}^\dagger|0\rangle$. As discussed in the first chapter, I will evaluate and analyse the normalization constant $\chi_N^{(2)}$ associated to the operator $q_{1,0}^\dagger$.

$$\langle 0|q_{1,0}^N q_{1,0}^{\dagger N}|0\rangle = \chi_N^{(2)} N!\,. \tag{4.38}$$

It is clear that the operator $q_{1,0}^\dagger$ can give rise to a ladder structure similar to Eqs. (1.18) and (1.22).

In this case, the normalization constant takes the form (for a proof see the section 4.4.2)

$$\chi_N^{(2)} = \frac{N!}{d^N}\binom{d-N}{d-2N}. \tag{4.39}$$

Thus, the normalization ratio can be written as

$$\frac{\chi_{N+1}^{(2)}}{\chi_N^{(2)}} = \left(1 - \frac{N+1}{d}\right)\Pi_{i=1}^N\left(1 - \frac{2}{d+i-2N}\right) \leq 1. \tag{4.40}$$

The value of the normalization ratio is upper bounded by 1. Remarkably, it is also lower bounded by

$$\frac{\chi_{N+1}^{(2)}}{\chi_N^{(2)}} \geq \left(1 - \frac{N+1}{d}\right)\left(1 - \frac{2}{d+1-2N}\right)^N. \tag{4.41}$$

The relations above implies that $\lim_{d\gg N} \frac{\chi^{(2)}_{N+1}}{\chi^{(2)}_{N}} \to 1$, because

$$\lim_{d\gg N}\left(1-\frac{N+1}{d}\right)\left(1-\frac{2}{d+1-2N}\right)^{N} \to 1. \tag{4.42}$$

This means that the operator $q^{\dagger}_{1,0}$ acts as an ideal bosonic one in the limit $d \gg N$.

4.3.4 Extended Hubbard model

Now, let us consider the Hamiltonian

$$\mathcal{H} = J\mathcal{H}_0 + U\mathcal{H}_p + \gamma\mathcal{H}_{nn}. \tag{4.43}$$

which is basically the one considered before (4.7) in addition to the term

$$\mathcal{H}_{nn} = -\sum_{k=0}^{d-1} a^{\dagger}_{k} a_k b^{\dagger}_{k+1} b_{k+1}, \tag{4.44}$$

that describes an interaction between first nearest-neighbours. Here, I assume that all the parameters in the Hamiltonian (4.43) are positive, $J, U, \gamma \geq 0$.

It might seem that the model above considers only nearest-neighbour interactions between particles of different types, i.e. A and B, while omitting the same kind of interactions between particles of the same type, i.e. (A and A) or (B and B). However, the point-like interactions described by the term $\mathcal{H}_p$ will give rise to an indirect interaction between particles of the same type when occupying first-neighbouring positions. For instance, if particle A is at site k, since we are considering hardcore bosons, it will interact with the particle B which is in the same position via $\mathcal{H}_p$. On the other hand, if the other hardcore boson is at position $k \pm 1$, the particle A will also interact with the particle B, belonging to the second pair, via $\mathcal{H}_{nn}$. Thus, we can say that both particles B, each one belonging to a different hardcore boson, will experience an indirect nearest-neighbour interaction mediated by the particle A. Of course, the same also holds for an indirect interaction between A and A mediated by B.

Intuitively, we can say that in the limit of strong nearest-neighbour interactions both hardcore bosons will tend to stick next to each other. However, the aim of this discussion is to examine the entanglement and the bosonic properties of the ground state as the strength of the nearest neighbour interactions changes, i.e. as a function of γ. In order to have a system that is susceptible to this interactions, we need to consider relatively small values for the variable γ.

Here, we are interested in deriving an effective Hamiltonian of the form

$$\mathcal{H}^{(1)} = J\mathcal{H}_0 + U\mathcal{H}_p, \tag{4.45}$$

which can approximate the original one (4.43) in the limit $U \gg J \gg \gamma$. First, let us define the projector on the ground state of $\mathcal{H}_p$ as

$$P_g = \sum_{j_1<j_2<\cdots<j_N} \left(\bigotimes_{n=1}^{N} \eta^\dagger_{j_n} \left|0\right\rangle \left\langle 0\right| \eta_{j_n} \right). \tag{4.46}$$

The unitary evolution operator for $\mathcal{H}_p$ can be expressed as

$$\begin{aligned} \mathcal{U}_g &= e^{-it\mathcal{H}_p/\hbar} = P_g e^{-itE_g/\hbar} + P_e e^{-itE_e/\hbar}, \\ &= P_g e^{-itNU/\hbar} + P_e e^{-it(N-1)U/\hbar}, \end{aligned} \tag{4.47}$$

such that P_e stands for the projector on the first excited state which corresponds to a state of an independent A-B pair in addition to a compound made of $N-1$ pairs with energy $(N-1)U$. It has been shown that at first order of perturbation [60]

$$\begin{aligned} P_g\mathcal{U}_I^{(1)}P_g &= -\frac{i}{\hbar}\int_0^t P_g\mathcal{H}_I(t_1)P_g dt_1, \\ &= -\frac{i}{\hbar}\int_0^t P_g\mathcal{U}_g(t_1)^\dagger\mathcal{H}_0\mathcal{U}_g(t_1)P_g dt_1 = 0, \end{aligned} \tag{4.48}$$

because $P_g\mathcal{H}_0P_g = 0$. At the second order [60], we have

$$\begin{aligned} P_g\mathcal{U}_I^{(2)}P_g &= -\frac{i}{\hbar}\int_0^t P_g\mathcal{H}_I(t_1)dt_1\int_0^{t_1}\mathcal{H}_I(t_2)P_g dt_2, \\ &= -\frac{i}{\hbar}\int_0^t\int_0^{t_1} dt_1 dt_2 P_g\left(\mathcal{U}_g(t_1)^\dagger\mathcal{H}_0\mathcal{U}_g(t_1)\right) \\ &\quad \times \left(\mathcal{U}_g(t_2)^\dagger\mathcal{H}_0\mathcal{U}_g(t_2)\right)P_g, \\ &= -\frac{i}{\hbar}\int_0^t\int_0^{t_1} dt_1 dt_2 P_g\mathcal{H}_0^2 P_g e^{i(t_2-t_1)U/\hbar}, \\ &= -\left(\frac{i}{\hbar}\right)^2\frac{\hbar}{iU}\left(t + \frac{\hbar}{iU}e^{-itU/\hbar} - 1\right)P_g\mathcal{H}_0^2P_g, \\ &\overset{U\gg J}{\approx} \frac{i}{U\hbar}P_g\mathcal{H}_0^2P_g, \end{aligned} \tag{4.49}$$

such that

$$\begin{aligned} P_g\mathcal{H}_0^2P_g &= -N4J^2\sum_{k=0}^{d-1}\eta_k^\dagger\eta_k - 2J^2\sum_{k=0}^{d-1}(\eta_k^\dagger\eta_{k+1} + h.c.) \\ &\quad + 4J^2\sum_{k=0}^{d-1}\eta_k^\dagger\eta_k\eta_{k+1}^\dagger\eta_{k+1}. \end{aligned} \tag{4.50}$$

In the limit $U \gg J$, the effective Hamiltonian $\mathcal{H}^{(1)}$ takes the form

$$\mathcal{H}^{(1)}_{\text{eff}} = - N\left(U+\frac{4J^2}{U}\right)\sum_{k=0}^{d-1}\eta_k^\dagger\eta_k - \frac{2J^2}{U}\sum_{k=0}^{d-1}(\eta_k^\dagger\eta_{k+1}+h.c.) + \frac{4J^2}{U}\sum_{k=0}^{d-1}\eta_k^\dagger\eta_k\eta_{k+1}^\dagger\eta_{k+1}. \tag{4.51}$$

assuming that $P_g\mathcal{H}_pP_g = -NU$.

Now, let us consider the full Hamiltonian $\mathcal{H}$. At first order we get,

$$P_g\mathcal{H}_{nn}P_g = -2\gamma\sum_{k=0}^{d-1}\eta_k^\dagger\eta_k\eta_{k+1}^\dagger\eta_{k+1}. \tag{4.52}$$

Note that the second order contributions are negligible since they depend on $\gamma^2/U \ll J^2/U$. Thus, in the limit $U \gg J \gg \gamma$, the effective Hamiltonian takes the form

$$\mathcal{H}_{\text{eff}} = - N\left(U+\frac{4J^2}{U}\right)\sum_{k=0}^{d-1}\eta_k^\dagger\eta_k - \frac{2J^2}{U}\sum_{k=0}^{d-1}(\eta_k^\dagger\eta_{k+1}+h.c.) - \left(2\gamma-\frac{4J^2}{U}\right)\sum_{k=0}^{d-1}\eta_k^\dagger\eta_k\eta_{k+1}^\dagger\eta_{k+1}. \tag{4.53}$$

In the expression above, the first term is constant when the number of particles is fixed. Ergo, we can drop it and write

$$\begin{aligned}\mathcal{H}_{\text{eff}} = &- \frac{2J^2}{U}\sum_{k=0}^{d-1}(\eta_k^\dagger\eta_{k+1}+h.c.) \\ &- \left(2\gamma-\frac{4J^2}{U}\right)\sum_{k=0}^{d-1}\eta_k^\dagger\eta_k\eta_{k+1}^\dagger\eta_{k+1}. \\ = &- \bar{J}\sum_{k=0}^{d-1}(\eta_k^\dagger\eta_{k+1}+h.c.) - \bar{\gamma}\sum_{k=0}^{d-1}\eta_k^\dagger\eta_k\eta_{k+1}^\dagger\eta_{k+1}.\end{aligned} \tag{4.54}$$

For simplicity, let us define $\bar{J} = 2\frac{J^2}{U}$ and $\bar{\gamma} = 2(\gamma - \bar{J})$. Remarkably, this approximation reduced our system from a four-partite to a two-partite problem.

As discussed in the previous section, let us investigate the form of the ground state by analysing the action of $\mathcal{H}_{\text{eff}}$ on $q_{s,r}^\dagger|0\rangle \equiv |s,r\rangle_q$. We get

$$\begin{aligned}\mathcal{H}_{\text{eff}}|s,r\rangle_q = & - \bar{J}(1+e^{i\frac{2\pi}{d}r})|s+1,r\rangle_q \\ & - \bar{J}(1+e^{-i\frac{2\pi}{d}r})|s-1,r\rangle_q \\ & - \bar{\gamma}\delta_{s,1}|s,r\rangle_q. \end{aligned} \tag{4.55}$$

Clearly, $\mathcal{H}_{\text{eff}}$ does not affect the parameter r which implies that the hoping contribution takes its smallest values, $-2\bar{J}$, for $r = 0$. Also, s has no upper bound when we consider an infinite discrete lattice, $d \to \infty$.

Now, let us consider the general form of the ground state

$$|\psi_0\rangle = \sum_{s=1}^{\infty} \beta_s |s,0\rangle_q, \tag{4.56}$$

such that

$$\mathcal{H}_{\text{eff}}|\psi_0\rangle = \bar{\varepsilon}|\psi_0\rangle. \tag{4.57}$$

As before, we get a set of typical recurrence equations

$$-\frac{\bar{\varepsilon}}{2\bar{J}}\beta_s = \beta_{s+1} + \beta_{s-1}, \tag{4.58}$$

for $s > 1$, in addition to an atypical equation

$$-\frac{(\bar{\gamma}+\bar{\varepsilon})}{2\bar{J}}\beta_1 = \beta_2, \tag{4.59}$$

for $s = 0$.

The solution to the above recurrence equations can be found with a similar approach to the one considered for Eqs. (4.14) and (4.15). A general form of the solution of Eq. (4.58) can be expressed as

$$\beta_s = Br_0^s + Ar_0^{-s}, \tag{4.60}$$

such that A and B are constants and

$$\bar{\varepsilon} = -2\bar{J}(r_0 + r_0^{-1}). \tag{4.61}$$

Assuming that $r_0 \leq 1$ and because we have $\lim_{s\to\infty} \beta_s = 0$, we can say that $A = 0$. Ergo,

$$\beta_s = Br_0^s. \tag{4.62}$$

Now, we substitute the above in Eq. (4.59). After simplification, we write

$$(r_0 + r_0^{-1} - \frac{\bar{\gamma}}{2\bar{J}})r_0 = r_0^2. \tag{4.63}$$

Ergo,

$$r_0 = \frac{2\bar{J}}{\bar{\gamma}} = \frac{\bar{J}}{\gamma - \bar{J}} = \frac{1}{\frac{\gamma U}{2J^2} - 1}. \tag{4.64}$$

Finally, one can show that the above leads to

$$\beta_s = \mathcal{B}\left(\frac{\bar{J}}{\gamma - \bar{J}}\right)^s, \tag{4.65}$$

$$\bar{\varepsilon} = \frac{4\gamma\bar{J} - 4\bar{J}^2 - 2\gamma^2}{\gamma - \bar{J}}, \tag{4.66}$$

such that $\mathcal{B}$ stands for a normalization constant. In fact, normalization is a crucial assumption for our solution. Since we have the limit $\lim_{s\to\infty}\beta_s \neq 0$ for $\gamma \leq 2\bar{J}$, we can say that the solution above works only in the regime $U \gg \gamma > 2\bar{J}$.

In the limit of strong nearest neighbour interactions we get

$$\lim_{\gamma\gg 2\bar{J}} \beta_1 \gg \lim_{\gamma\gg 2\bar{J}} \beta_k \qquad \text{for } k \geq 2. \tag{4.67}$$

Namely, the coefficient β_1 become much larger than all the other coefficients. Thus, in this limit, we can say that $q^\dagger_{1,0}|0\rangle$ dominates the ground state of the system.

Interestingly, when γ approaches the value $2\bar{J}$ we get

$$\lim_{\gamma\to 2\bar{J}} \beta_s \to \mathcal{B}. \tag{4.68}$$

In this case, all the coefficients are equal to each other. This corresponds to the two-partite Fock state previously presented in Eq. (4.34) or (4.37), i.e. two independent bipartite composite bosons.

The idea of two composite bosons requiring attractive interaction to be probabilistically independent from each other might be surprising. However, it can be explained as follows. In general the ground state of the system contains all the possible configurations of two hardcore bosons, $\eta^\dagger_k\eta^\dagger_{k'}$. However each configuration might contribute with a different amount of energy. In the state (4.34), the probabilities corresponding to the different configurations are all the same. Hence, the composite bosons are independent. If the ground state of the system is to be identical to the state (4.34), the energy contributions will need to be the same for all the different configurations. In other words, the ground state should have the form

$$\mathcal{N}\sum_{k<k'} \eta^\dagger_k\eta^\dagger_{k'}|0\rangle. \tag{4.69}$$

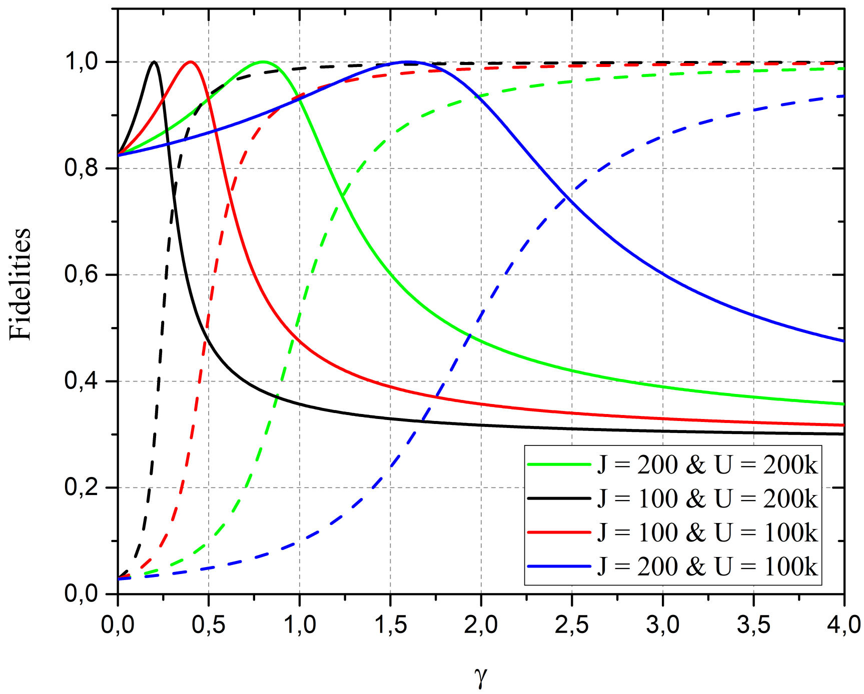

Figure 4.2: Plots of the fidelities $|\langle\psi(\gamma)|q_{1,0}^{\dagger}|0\rangle|^2$ (dashed) and $|\langle\psi(\gamma)|\frac{c_{0,0}^{\dagger 2}}{\sqrt{2\chi_2}}|0\rangle|^2$ (solid) as functions of the strength of the nearest-neighbour interactions, γ. The parameters J and U are fixed at different values while $d = 8$. [3]

such that $\mathcal{N}$ stands for a normalization constant. The action of $\mathcal{H}_{\text{eff}}$ on the state above leads to

$$\mathcal{N} \sum_{k<k'} \varepsilon_{k,k'} \eta_k^{\dagger} \eta_{k'}^{\dagger} |0\rangle. \tag{4.70}$$

Here, $\varepsilon_{k,k'}$ stands for different energy contributions. When $\gamma = 0$, only the hoping contributes to $\varepsilon_{k,k'}$. In this case, we can write $\varepsilon_{k,k+1} = -2\bar{J}$ and $\varepsilon_{k,k'} = -4\bar{J}$ for $k' \geq k+2$. This difference is due to the fact that the operators $\eta_k^{\dagger}$ and $\eta_{k'}^{\dagger}$ correspond to hardcore bosons which cannot occupy the same site, see (4.32). This is why nearest-neighbour interactions are required to be with strength $-\gamma = -2\bar{J}$ when $k' = k+1$ in order to compensate for the difference.

4.3.5 Numerical simulations

Assuming periodic boundary conditions, the ground state $|\psi(\gamma)\rangle$ of the system was evaluated numerically for $d = 8$. These numerical simulations were performed in the limit $U \gg J \gg \gamma$. More precisely, while the parameter of the nearest-neighbour interactions, γ, takes values between 0 and 4, the parameter of the point-link interactions was fixed at two values $U = 1 \times 10^5$ or 2×10^5, and the hoping parameter was fixed as $J = 1 \times 10^2$ or 2×10^2. In Fig. 4.2, the fidelities $|\langle\psi(\gamma)|q_{1,0}^{\dagger}|0\rangle|^2$ and $|\langle\psi(\gamma)|\frac{c_{0,0}^{\dagger 2}}{\sqrt{2\chi_2}}|0\rangle|^2$ are plotted as functions of the parameter γ. Visibly, when the fidelity of the ground state with the two-partite Fock state approaches 1, the fidelity with the four-partite composite boson rises to $1/4$. This is a consequence of the finite nature of the considered lattice. Using Eqs. (4.37) and (4.35), one can easily show that

$$|\langle 0|q_{1,0}\frac{c_{0,0}^{\dagger 2}}{\sqrt{2\chi_2}}|0\rangle|^2 = d\binom{d}{2}^{-1}. \tag{4.71}$$

This implies that these two states will tend to be orthogonal in the limit of an infinite lattice,

$$\lim_{d\to\infty}|\langle 0|q_{1,0}\frac{c_{0,0}^{\dagger 2}}{\sqrt{2\chi_2}}|0\rangle|^2 \to 0. \tag{4.72}$$

As a matter of fact, these simulations were run twice. First, I considered the original Hamiltonian (4.43). Next, I used the effective one (4.54). The obtained results were identical.

4.4 Multipartite composite bosons

Now, let us consider a system made of N fermionic A-B pairs, which makes a total of $2N$ fermions. For this system, many possible multifermionic assemblies can be achieved, see Fig. 4.3. These assemblies can be expected to appear in the model discussed above, see the effective extended Hamiltonian (4.54). For example, when parameter γ is large enough to compensate for Pauli blocking on the hoping contribution, the ground state of the system will correspond to a state of N independent bipartite composite bosons, each made of an A-B pair. From that point, as nearest-neighbour interactions gets stronger, we can expect larger composite particles to emerge in the ground state. Intuitively, we can say that in the limit of strong nearest-neighbour interactions, the system will form a $2N$-partite composite boson. Here, I will consider the transition of the ground state through the different combinations as the strength of the nearest-neighbour interactions increases.

4.4.1 Multipartite entangled states

Let us define the state of a composite particle made of M fermionic A-B pairs

$$q_{(M)}^\dagger|0\rangle \equiv \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\eta_k^\dagger\eta_{k+1}^\dagger\dots\eta_{k+M-1}^\dagger|0\rangle. \tag{4.73}$$

As before, we can say that in the limit of strong point-like interactions, every A-B pair will tend to make a hardcore boson described by the operator $\eta_k^\dagger$. In addition to that, when the nearest-neighbour

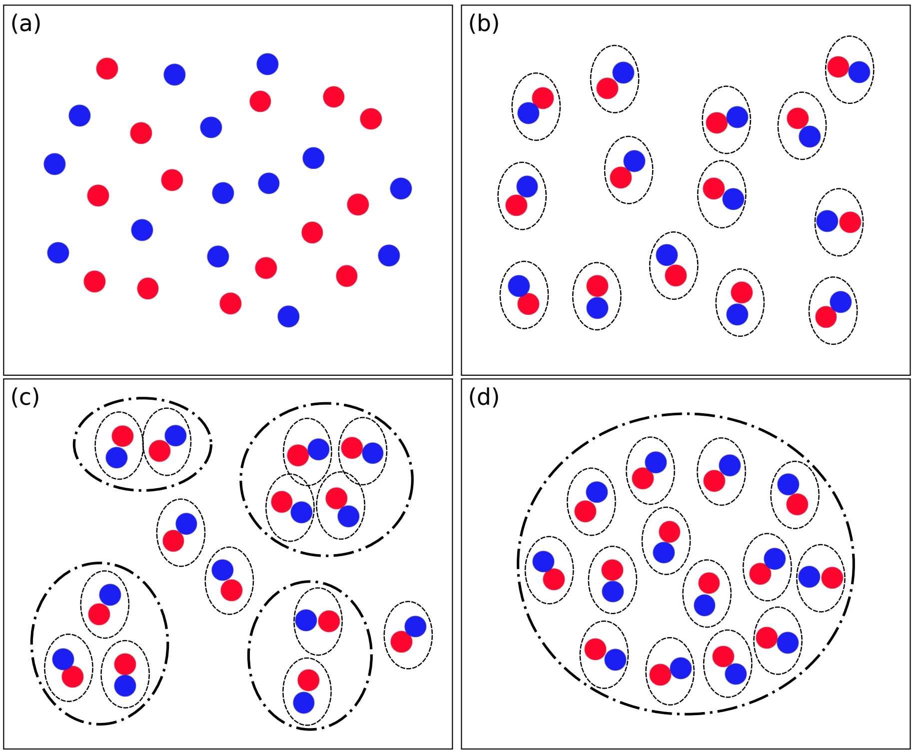

Figure 4.3: Different possible assemblies for many fermions of type A and B. (a) No interactions – All the fermions are free. (b) Strong point-like interactions and weak nearest-neighbour interactions – Formation of bipartite composite bosons, each one made of a single A-B pair. (c) Strong point-like interaction and increasing nearest-neighbour interactions – Emergence of large assemblies of multipartite composite bosons. (d) Strong point-like interaction and strong nearest-neighbour interactions – Formation of a single bosonic compound made of all the fermions. [3]

interactions are also relatively strong, all these hardcore bosons will tend to stick next to each other. Therefore, the ground state of the system in such a limit can be expected to be of the form of the state presented above (4.73). In fact, this state is a general form of previously discussed states. For instance, when $M = 1$ the state above becomes the state of a singe bipartite composite boson $c^\dagger_{0,0}|0\rangle$, see (4.2). Also, when $M = 2$ the state (4.73) corresponds to a single four-partite composite boson $q^\dagger_{1,0}|0\rangle$, see (4.35).

4.4.2 Bosonic properties of multipartite entangled states

For a composite particle made of M fermionic A-B pairs described by $\eta^\dagger_k = a^\dagger_k b^\dagger_k$, we can write an unnormalized state of N such composite particles

$$q^{\dagger N}_{(M)}|0\rangle, \tag{4.74}$$

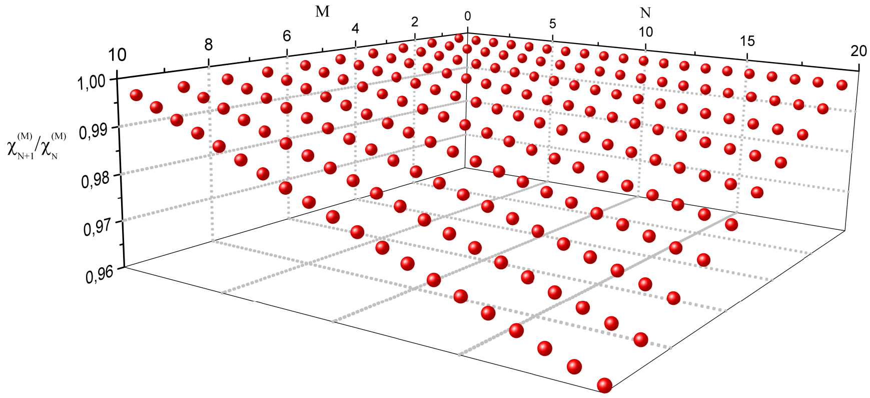

Figure 4.4: The lower bound for the normalization ratio $\frac{\chi^{(M)}_{N+1}}{\chi^{(M)}_{N}}$ as a function of the size (M) and the number (N) of the composite bosons, for $d = 10000$.

such that

$$q^{\dagger}_{(M)} = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \eta^{\dagger}_{k}\eta^{\dagger}_{k+1} \dots \eta^{\dagger}_{k+M-1}. \tag{4.75}$$

Its corresponding normalization factor can be computed using the formula

$$\chi^{(M)}_{N} = \frac{1}{N!} \langle 0 | q^{N}_{(M)} q^{\dagger N}_{(M)} | 0 \rangle. \tag{4.76}$$

However, for convenience purposes, we can define the operator

$$\Gamma^{\dagger}_{k} = \eta^{\dagger}_{k}\eta^{\dagger}_{k+1} \dots \eta^{\dagger}_{k+M-1}, \tag{4.77}$$

which obeys

$$\Gamma^{\dagger}_{k}\Gamma^{\dagger}_{k'} = \Gamma^{\dagger}_{k'}\Gamma^{\dagger}_{k}, \tag{4.78}$$

$$\Gamma^{\dagger}_{k}\Gamma^{\dagger}_{k'} = 0 \text{ if } |k - k'| < M. \tag{4.79}$$

Hence, we can write

$$\begin{aligned}
q^{\dagger N}_{(M)} |0\rangle &= \frac{1}{d^{N/2}} \sum_{k_1,\dots,k_N} \Gamma^{\dagger}_{k_1} \dots \Gamma^{\dagger}_{k_N} |0\rangle, \\
&= \frac{N!}{d^{N/2}} \sum^{*}_{k_1 < \dots < k_N} \Gamma^{\dagger}_{k_1} \dots \Gamma^{\dagger}_{k_N} |0\rangle, \\
&\equiv \frac{N!}{d^{N/2}} \sum^{*}_{k_1 < \dots < k_N} |k_1, \dots, k_N\rangle,
\end{aligned}$$

such that $*$ means that within the sum we consider the relations (4.79). Consequently, we get

$$\chi_N^{(M)} = \frac{N!}{d^N} \sum_{k_1 < \ldots < k_N}^{*} 1 = \frac{N!}{d^N} \binom{N + d - NM}{d - NM}. \tag{4.80}$$

The binomial coefficient stands for the combination with repetitions. For example, let us consider a bowl containing y items, each one marked with a different number from 1 to y. From this bowl, we pick randomly a single item, note its number and put it back in the bowl. If we repeat this process of picking, noting and putting back x times, we will end up with a string of numbers. This sequence of numbers correspond to one possible combination, and obviously there are many other combinations. In fact, the total number of possible combinations is given by $\binom{x+y-1}{x}$. In the formula (4.80), we considered N compounds, each composed of M A-B pairs. Since each compound occupies M positions, after placing N compounds, $d - NM$ positions will remain unoccupied. Here, we consider the N compound and distribute the unoccupied positions around them. For instance, we can put the first empty position before the 1^{st} compound, between the 1^{st} and the 2^{nd}, $\ldots$, after the N^{th}. Then we repeat the same for all the other unoccupied positions. In simpler words, we choose one position from $N + 1$ possibilities, and repeat $d - NM$ times.

Now, let us rewrite the formula (4.80) as

$$\chi_N^{(M)} = \frac{\Pi_{i=1}^{N}(d - NM + i)}{d^N}. \tag{4.81}$$

Hence, the ratio $\frac{\chi_{N+1}^{(M)}}{\chi_N^{(M)}}$ takes the form

$$\begin{aligned} \frac{\chi_{N+1}^{(M)}}{\chi_N^{(M)}} &= \frac{\Pi_{i=1}^{N+1}(d - (N+1)M + i)}{d\Pi_{i=1}^{N}(d - NM + i)}. \\ &= \left(1 - \frac{(N+1)(M-1)}{d}\right) \Pi_{i=1}^{N} \left(1 - \frac{M}{d + i - NM}\right). \end{aligned} \tag{4.82}$$

Note that above value is upper-bounded by 1 and lower-bounded by

$$\left(1 - \frac{(N+1)(M-1)}{d}\right) \left(1 - \frac{M}{d + 1 - NM}\right)^N. \tag{4.83}$$

Clearly, this lower-bound approaches 1 in the limit $d \gg NM$. In Fig. 4.4, the value of the lower bound is plotted for $d = 10000$.

The operator $q^\dagger_{(M)}$ associated with a single compound made of M A-B pairs will exhibit an ideal bosonic behaviour when $d \gg M$.

Therefore, in this limit the state

$$\frac{q_{(M)}^{\dagger N}}{\sqrt{\chi_N^{(M)} N!}}|0\rangle, \tag{4.84}$$

corresponds to a Fock state made of N components.

In fact, the bosonic behaviour observed above is due to multipartite entanglement between the constituents. But, a multipartite entanglement is not enough. Bellow, I will show that any possible separability of the state $q_{(M)}^{\dagger}|0\rangle$ will imply that $q_{(M)}^{\dagger}$ cannot recover the ladder structure, which suggests that entanglement need to be genuinely multipartite.

4.4.3 Why genuine multipartite entanglement is important?

Obviously, the state $q_{\text{sep}}^{\dagger}|0\rangle$ given by

$$q_{\text{sep}}^{\dagger}|0\rangle \equiv \eta_{k_1}^{\dagger} \dots \eta_{k_M}^{\dagger}|0\rangle, \tag{4.85}$$

leads to $q_{\text{sep}}^{\dagger 2}|0\rangle = 0$. This state is separable according to a definition in [61]. Ergo, the system cannot be considered as a composite boson.

If we consider a system made of many A-B pairs, and we assume that at least one hardcore boson is at a well defined mode, we get

$$\left(\eta_{k_1}^{\dagger} \sum_{k_2,\dots,k_M} \alpha_{k_2,\dots,k_M} \eta_{k_2}^{\dagger} \dots \eta_{k_M}^{\dagger}\right)^2 |0\rangle = 0. \tag{4.86}$$

In this case also, the system cannot be considered as a composite boson. This means that all the constituents should be entangled to have $q_{(M)}^{\dagger 2}|0\rangle \neq 0$.

Now, let us consider an arbitrary number of hardcore bosons associated to states which are not genuinely multipartite entangled, i.e. states which are separable with respect to some partitions. Separability in this case implies that we can write

$$\begin{aligned} c^{\dagger}|0\rangle \quad &\equiv \quad \left(\sum_{i_1,i_2,\dots} w_{i_1,i_2,\dots} \eta_{i_1}^{\dagger} \eta_{i_2}^{\dagger} \dots\right) \\ &\quad \times \left(\sum_{j_1,j_2,\dots} v_{j_1,j_2,\dots} \eta_{j_1}^{\dagger} \eta_{j_2}^{\dagger} \dots\right)|0\rangle, \end{aligned} \tag{4.87}$$

such that $w_{i_1,i_2,\ldots}$ and $v_{j_1,j_2,\ldots}$ are normalized symmetric coefficients associated to two partitions

$$\langle 0|cc^{\dagger}|0\rangle = \sum_{\substack{i_1,i_2,\ldots\\ j_1,j_2,\ldots}} |w_{i_1,i_2,\ldots}|^2 |v_{j_1,j_2,\ldots}|^2 = 1. \tag{4.88}$$

In the first chapter, we discussed in details the creation and annihilation operators of composite bosons and how they should obey the ladder structure (1.18). We know that if $c^{\dagger}|0\rangle$ corresponds to a single composite boson, $c^{\dagger 2}|0\rangle$ should correspond to two composite bosons while its norm ≈ 2. Considering the definition (4.87), we can write

$$\begin{aligned} c^{\dagger 2}|0\rangle &= \sum_{\substack{i_1,i_2,\ldots\\ i'_1,i'_2,\ldots}} \left(w_{i_1,i_2,\ldots}\eta^{\dagger}_{i_1}\eta^{\dagger}_{i_2}\ldots\right)\left(w_{i'_1,i'_2,\ldots}\eta^{\dagger}_{i'_1}\eta^{\dagger}_{i'_2}\ldots\right) \\ &\times \sum_{\substack{j_1,j_2,\ldots\\ j'_1,j'_2,\ldots}} \left(v_{j_1,j_2,\ldots}\eta^{\dagger}_{j_1}\eta^{\dagger}_{j_2}\ldots\right)\left(v_{j'_1,j'_2,\ldots}\eta^{\dagger}_{j'_1}\eta^{\dagger}_{j'_2}\ldots\right)|0\rangle. \end{aligned}$$

Assuming non-overlapping terms, i.e. all indices $i_1, i_2, \ldots, i'_1, i'_2, \ldots$ are different, we have

$$\begin{aligned} &\left(w_{i_1,i_2,\ldots}\eta^{\dagger}_{i_1}\eta^{\dagger}_{i_2}\ldots\right)\left(w_{i'_1,i'_2,\ldots}\eta^{\dagger}_{i'_1}\eta^{\dagger}_{i'_2}\ldots\right) = \\ &\quad \left(w_{i'_1,i'_2,\ldots}\eta^{\dagger}_{i'_1}\eta^{\dagger}_{i'_2}\ldots\right)\left(w_{i_1,i_2,\ldots}\eta^{\dagger}_{i_1}\eta^{\dagger}_{i_2}\ldots\right). \end{aligned} \tag{4.89}$$

Obviously, the same also holds for the second partition. Therefore, we can use above formula and write

$$\begin{aligned} c^{\dagger 2}|0\rangle &= 4\sum^{*}_{\substack{i_1,i_2,\ldots\\ i'_1,i'_2,\ldots\\ i_1<i'_1}} \left(w_{i_1,i_2,\ldots}w_{i'_1,i'_2,\ldots}\eta^{\dagger}_{i_1}\eta^{\dagger}_{i_2}\ldots\eta^{\dagger}_{i'_1}\eta^{\dagger}_{i'_2}\ldots\right) \\ &\times \sum^{*}_{\substack{j_1,j_2,\ldots\\ j'_1,j'_2,\ldots\\ j_1<j'_1}} \left(v_{j_1,j_2,\ldots}v_{j'_1,j'_2,\ldots}\eta^{\dagger}_{j_1}\eta^{\dagger}_{j_2}\ldots\eta^{\dagger}_{j'_1}\eta^{\dagger}_{j'_2}\ldots\right)|0\rangle. \end{aligned} \tag{4.90}$$

The sums with $*$ means that only non-overlapping terms are counted. Assuming the inequalities $i_1 < i'_1$ and $j_1 < j'_1$ leads to counting every possible non-overlapping term only once. Clearly, from (4.89) we can say that every non-overlapping term will appear twice. Since we have two partitions in this case, we need to include the factor $2^2 = 4$. Hence, in the case of s partitions, the factor should be 2^s. Therefore, we can write

$$\begin{aligned}\langle 0|c^2 c^{\dagger 2}|0\rangle &= 16 \sum_{\substack{i_1,i_2,\ldots \\ i'_1,i'_2,\ldots \\ i_1<i'_1}}^{*} \sum_{\substack{j_1,j_2,\ldots \\ j'_1,j'_2,\ldots \\ j_1<j'_1}}^{*} \big(|w_{i_1,i_2,\ldots}|^2 \\ &\quad \times |v_{j_1,j_2,\ldots}|^2 |w_{i'_1,i'_2,\ldots}|^2 |v_{j'_1,j'_2,\ldots}|^2\big) . \end{aligned} \tag{4.91}$$

The expression above can be simplified as

$$4 \sum_{\substack{i_1,i_2,\ldots \\ i'_1,i'_2,\ldots}} \sum_{\substack{j_1,j_2,\ldots \\ j'_1,j'_2,\ldots}} |w_{i_1,i_2,\ldots}|^2 |v_{j_1,j_2,\ldots}|^2 |w_{i'_1,i'_2,\ldots}|^2 |v_{j'_1,j'_2,\ldots}|^2 - 4\omega(*) = 4(1-\omega(*)), \tag{4.92}$$

such that

$$\omega(*) = \sum^{\ddagger} |w_{i_1,i_2,\ldots}|^2 |v_{j_1,j_2,\ldots}|^2 |w_{i'_1,i'_2,\ldots}|^2 |v_{j'_1,j'_2,\ldots}|^2 . \tag{4.93}$$

Here, ‡ means that the sum counts only the terms with overlapping indices. When $\omega(*) = 1/2$, the norm of $c^{\dagger 2}|0\rangle$ will be approximately equal to 2. However, in this case the number of overlapping terms needs to be large. This can occur only for weakly entangled states. However, strong entanglement is required for $c^{\dagger N}$ to recover the ladder structure. Consequently, we can say that when $\omega(*) = 1/2$, the norm $\langle 0|c^N c^{\dagger N}|0\rangle \neq N!$ for $N > 2$. In fact, Pauli exclusion implies that states $c^{\dagger N}|0\rangle$ do not have any overlapping terms. Therefore, if the state $c^{\dagger}|0\rangle$ is weakly entangled, we get $\langle 0|c^N c^{\dagger N}|0\rangle \to 0$.

In the limit of large number of modes $d \gg N$ and assuming that the states corresponding to each partition are highly entangled, we can say that the term $\omega(*)$ is relatively small. This implies that the norm of $c^{\dagger 2}|0\rangle$ is $\approx 4 \neq 2$. In case of s partitions, the norm will be $\approx 2^s$. Clearly, a proper bosonic behaviour emerges only if $s = 1$. Hence, genuine multipartite entanglement is necessary for a good bosonic behaviour to take place.

4.4.4 Composite bosons of various sizes

Now, let us consider a system of N fermionic A-B pairs assembled in k composite bosons, which can be of different sizes. Its corresponding state can be written as

$$|M_1 + \ldots + M_k\rangle \equiv \mathcal{N} q^{\dagger}_{(M_1)} \ldots q^{\dagger}_{(M_k)}|0\rangle, \tag{4.94}$$

Here, I assume that $M_1 \geq \ldots \geq M_k$, i.e. the composite bosons are decreasingly ordered with respect to their sizes. The fact that the system is made of N pairs implies the constrain $M_1 + \ldots + M_k = N$. In addition to that, $\mathcal{N}$ stands for a normalization constant. Due to the indistinguishability of fermionic A-B pairs, and due to the Pauli exclusion principle and that the considered lattice is finite, even if the states $q^\dagger_{(M_1)}|0\rangle, \ldots, q^\dagger_{(M_k)}|0\rangle$ are normalized, this normalization will not hold for the state $q^\dagger_{(M_1)} \ldots q^\dagger_{(M_k)}|0\rangle$. Hence, the necessity of the constant $\mathcal{N}$.

For example, let us consider the case of m composite bosons of the identical sizes, $M_i = \ldots = M_{i+m} = M$. Without $\mathcal{N}$, the state $q^\dagger_{(M)} \ldots q^\dagger_{(M)}|0\rangle$ would have a norm proportional to $\chi^{(M)}_m m!$.

As a second example, the state

$$|3+1\rangle = \mathcal{N}\frac{1}{d}\sum_{k,l=0}^{d-1} \eta^\dagger_k \eta^\dagger_{k+1} \eta^\dagger_{k+2} \eta^\dagger_l |0\rangle, \tag{4.95}$$

implies a normalization constant of the form

$$\mathcal{N}^2 = \frac{d^2}{d^2 - 4d}. \tag{4.96}$$

Clearly, in the limit of an infinite lattice, the value of the normalization constant approaches one, $\lim\limits_{d\to\infty} \mathcal{N} \to 1$.

4.4.5 Numerical simulations for N=3 and N=4

Considering a system made of 3 and 4 fermionic A-B pairs, the ground state of the system, $|\psi(\gamma)\rangle$, was numerically evaluated for the effective Hamiltonian (4.54) while assuming the limit $\gamma \ll J \ll U$. In Fig. 4.5, the fidelities $|\langle\psi(\gamma)|M_1 + \ldots + M_k\rangle|^2$ are plotted as functions of $\gamma U/J^2 = \gamma\bar{J}/2$, for $d = 10$. As in the case of 2 A-B pairs, the ground state was evaluated for a variable parameter γ, while the parameters U and J were fixed. In Fig. 4.5, the values of γ were converted to the corresponding values of the ratio $\gamma U/J^2$ in order to illustrate the rescaling of the plots for different values of U and J. As a matter of fact, regardless of the values at which the parameters U and J were fixed, all the rescaled plots were identical. The choice of the lattice size was restricted by the high computational complexity of the problem. Consequently, the corresponding bosonic quality might not be ideal. However, some important qualitative behaviour of the model is still observable via these simulations.

In the case of three pairs, $N = 3$, clearly the ground state of the system is dominated by the state $|1+1+1\rangle$ when the strength of the nearest neighbour interactions is relatively weak. In fact, the ground state matches $|1+1+1\rangle$ when $\gamma U/J^2 = 4$, which corresponds to $\gamma \neq 0$. This

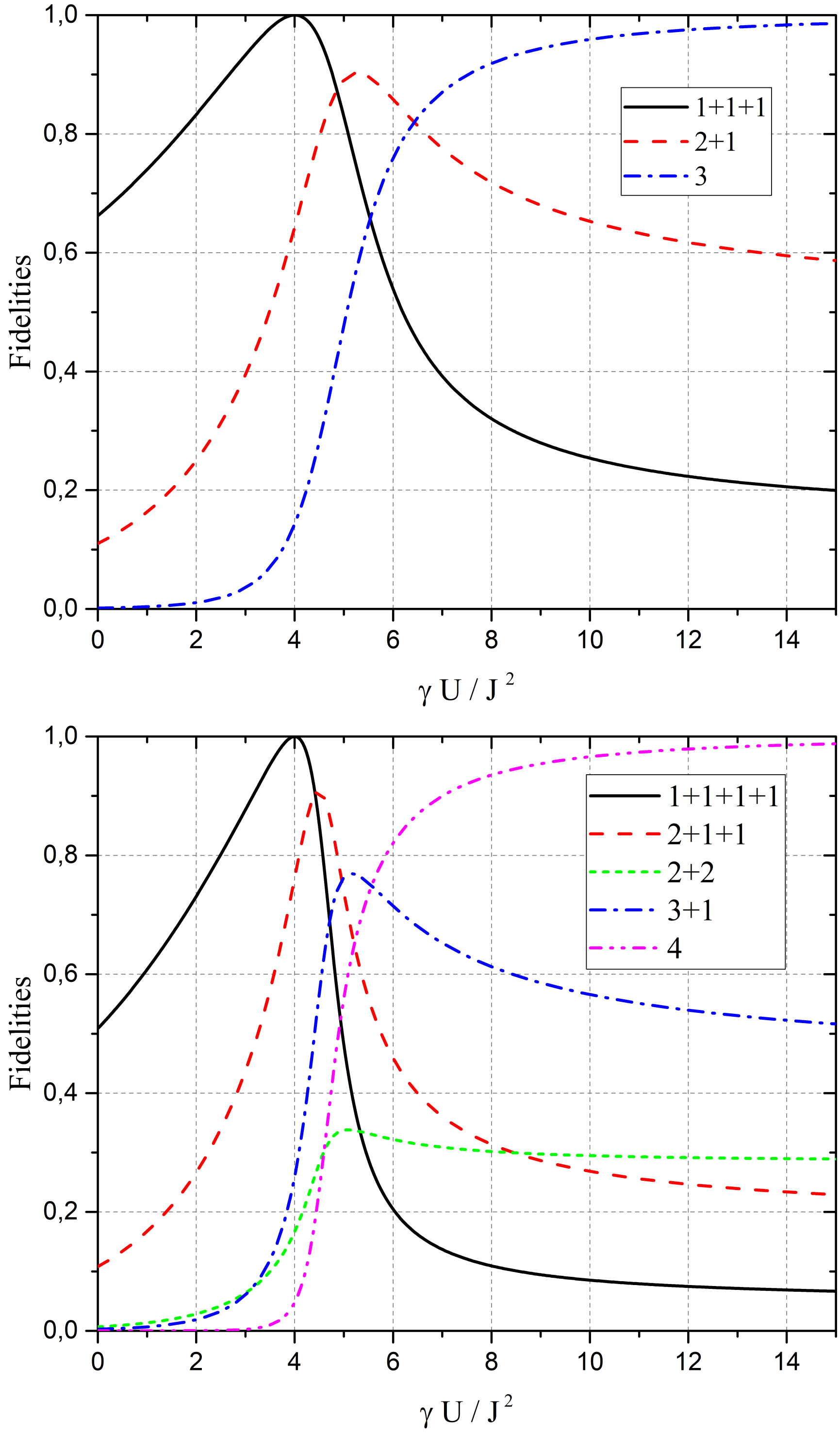

Figure 4.5: The fidelities of the ground state with few possible assemblies of three (top) and four (bottom) A-B pairs as functions of the ratio $\gamma U/J^2$, for $d = 10$. [3]

perfect fidelity can be explained by the same argument discussed in the case $N = 2$. After this maximum, the fidelity $|\langle\psi(\gamma)|1+1+1\rangle|^2$ decreases rapidly, while the ground state becomes briefly dominated by $|2+1\rangle$, but they never reach a perfect match. After that, the ground state becomes dominated by $|3\rangle$. In the limit of strong nearest neighbour interactions, $\gamma U/J^2 \gg 4$, the perfect fidelity is reached, $|\langle\psi(\gamma)|3\rangle|^2 \to 1$.

In the case of four pairs, $N = 4$, we observe that the ground state is dominated by $|1+1+1+1\rangle$ when $\gamma U/J^2 < 4$ and dominated by $|4\rangle$ in the limit $\gamma U/J^2 \gg 4$. The fidelities $|\langle\psi(\gamma)|1+1+1+1\rangle|^2$ and $|\langle\psi(\gamma)|4\rangle|^2$ reach one for $\gamma U/J^2 = 4$ and $\gamma U/J^2 \gg 4$, respectively. This is similar to the behaviour observed in the previous case. However, during the transition from $|1+1+1+1\rangle$ to $|4\rangle$, there is an intermediate region where $|2+1+1\rangle$ and then $|3+1\rangle$ briefly takes over the ground state of the system. As before, the fidelities corresponding to these intermediate states never reach one. The state $|2+2\rangle$ has kept relatively low fidelities with the ground state through the transition, with a maximum around ≈ 0.3. This fidelity $|\langle\psi(\gamma)|2+2\rangle|^2$ is due to the overlap of $|2+2\rangle$ with the other states. Clearly, the value of $|\langle\psi(\gamma)|2+2\rangle|^2$ does not change much after reaching its peak, even in the limit $\gamma U/J^2 \gg 4$. This can be interpreted as a result of the overlap between $|2+2\rangle$ and $|4\rangle$. In this case, the overlap is due to the finiteness of the considered space. As discussed above, in the limit of an infinite lattice, $d \to \infty$, this overlap is expected to vanish.

From the results above, we can suggest a transition of the form

$$|1+1+\ldots\rangle \to |2+1+\ldots\rangle \to |3+1+\ldots\rangle \to \ldots \to |N\rangle, \quad (4.97)$$

for the case of N A-B pairs. Initially, the system is made of N independent bipartite composite bosons. Then, two of these bosonic particles will merge and make a four-partite composite boson, which will absorb the other bipartite composite bosons one after the other. Through this transition, only one large composite boson will emerge, and will grow until it will absorb all the other bipartite ones. Eventually, it will form a single $2N$-partite composite boson in state $|N\rangle$. In the subsection bellow, I will discuss the transition (4.97) in terms of energy. Assuming that the system will always be in its ground state while increasing the strength of the nearest-neighbour interactions, my aim is to show that the transition (4.97) is the most energetically favourable.

4.4.6 Transition from N bipartite composite bosons to a single bosonic particle

Considering the state

$$|M_1+\ldots+M_k\rangle = \mathcal{K} \sum_{j_1,\ldots,j_k} \eta^\dagger_{j_1} \ldots \eta^\dagger_{j_1+M_1-1} \ldots \eta^\dagger_{j_k} \ldots \eta^\dagger_{j_k+M_k-1}|0\rangle, \quad (4.98)$$

such that $\mathcal{K}$ is a normalization constant, and $\eta^\dagger_{j_i}\dots\eta^\dagger_{j_i+M_i-1}$ corresponds to the i'th compound made of M_i A-B pairs. The state above describes a system made of $N=\sum_j M_j$ A-B pairs, which are assembled into k components. The terms in the sum stand for the superposition of all the different arrangements of the compounds in the lattice. However, in the limit $d \gg N$, we can say that the probabilities of having two compounds right next to each other are rather negligible, e.g. $\eta^\dagger_{j_i}\dots\eta^\dagger_{j_i+M_i-1}$ and $\eta^\dagger_{j_{i+1}}\dots\eta^\dagger_{j_{i+1}+M_{i+1}-1}$ such that $j_{i+1}=j_i+M_i$. Consequently, we can write

$$\langle M_1'+\ldots+M_{k'}'|M_1+\ldots+M_k\rangle = 0. \tag{4.99}$$

such that $\{M_1,\dots,M_k\}$ and $\{M_1',\dots,M_{k'}'\}$ are different configurations, i.e. we have $k\neq k'$ or at least there exist two values i for which $M_i\neq M_i'$. Let us call this *the non-adjacency assumption*.

Considering the states $|M_1+\ldots+M_k\rangle$, let us evaluate the expectation value of the effective Hamiltonian (4.54)

$$\langle\mathcal{H}_{\text{eff}}\rangle = \langle\mathcal{H}_k\rangle + \langle\mathcal{H}_p\rangle, \tag{4.100}$$

such that

$$\mathcal{H}_k = -\bar{J}\sum_{k=0}^{d-1}(\eta^\dagger_k\eta_{k+1}+h.c.), \tag{4.101}$$

and

$$\mathcal{H}_p = -\bar{\gamma}\sum_{k=0}^{d-1}\eta^\dagger_k\eta_k\eta^\dagger_{k+1}\eta_{k+1}, \tag{4.102}$$

are the kinetic and potential parts, respectively.

First, let us examine the action of $\mathcal{H}_k$ on a component made of many A-B pairs

$$\begin{aligned}\mathcal{H}_k\eta^\dagger_{j_i}\eta^\dagger_{j_i+1}\dots\eta^\dagger_{j_i+M_i-2}\eta^\dagger_{j_i+M_i-1}|0\rangle = \\ -\bar{J}\eta^\dagger_{j_i-1}\eta^\dagger_{j_i+1}\dots\eta^\dagger_{j_i+M_i-2}\eta^\dagger_{j_i+M_i-1}|0\rangle \\ -\bar{J}\eta^\dagger_{j_i}\eta^\dagger_{j_i+1}\dots\eta^\dagger_{j_i+M_i-2}\eta^\dagger_{j_i+M_i}|0\rangle.\end{aligned} \tag{4.103}$$

Basically, this results in a single A-B pair separated from the rest of the cluster. In general, we can say that the state $|M_1+\ldots+M_k\rangle$ will change to $|M_1'+\ldots+M_{k'}'\rangle$. In the extreme case, when the original state does not describe any single A-B pairs, the resulting state will contain a superposition of states that are all different from $|M_1+\ldots+M_k\rangle$. Ergo, considering the non-adjacency assumption, we can write

$$\langle M_1+\ldots+M_k|\mathcal{H}_k|M_1+\ldots+M_k\rangle = 0 \quad \text{if} \quad M_k>1. \tag{4.104}$$

However, if we consider that $|M_1 + \ldots + M_k\rangle$ describes r single A-B pairs, the action of $\mathcal{H}_k$ will lead to the same original configuration $2r$ times. Considering the non-adjacency of the components, we can assume that each single A-B pair can hop left or right and remain a single pair. If the number of components remains the same, and their sizes as well, the configuration remains unchanged. Hence, we can write

$$\langle M_1 + \ldots + M_k|\mathcal{H}_k|M_1 + \ldots + M_k\rangle = -2r\bar{J}, \\ \text{if} \quad M_{k-r+1} = \ldots = M_k = 1. \tag{4.105}$$

The action of $\mathcal{H}_p$ is much simpler. Clearly, all states $|M_1 + \ldots + M_k\rangle$ are eigenstates of $\mathcal{H}_p$. From the non-adjacency assumption, we can say that A-B pairs are first neighbours only if they are constituents of the same component. Consequently, the eigenvalues depend only on the number of components, regardless of their sizes.

$$\langle M_1 + \ldots + M_k|\mathcal{H}_p|M_1 + \ldots + M_k\rangle = -(N-k)\bar{\gamma}. \tag{4.106}$$

From the two relations above, we can write

$$\langle \mathcal{H}_{\text{eff}} \rangle = -2r\bar{J} - (N-k)\bar{\gamma}. \tag{4.107}$$

Now, let us examine the expectation values $\langle \mathcal{H}_{\text{eff}} \rangle$ for two states describing the same number of components but with different sizes. More precisely, let us assume that they differ in the number of single A-B pairs. From Eq. (4.107), the second term $-(N-k)\bar{\gamma}$ will be the same for both, since it depends only on the number of components, k. On the other hand, the first term $-2r\bar{J}$ depends on the number of single A-B pairs, r. Therefore, the state with the higher number of single pairs will have a lower expectation value, i.e. more energetically favourable. For example, let us consider a system made of five pairs assembled into three components, $k = 3$. Here, we consider two configurations. The first with two single pairs, and the second with only one. Using the formula (4.107), we can write

$$\langle 3+1+1|\mathcal{H}_{\text{eff}}|3+1+1\rangle = -4\bar{J} - 2\bar{\gamma}, \tag{4.108}$$

and

$$\langle 2+2+1|\mathcal{H}_{\text{eff}}|2+2+1\rangle = -2\bar{J} - 2\bar{\gamma}. \tag{4.109}$$

The results above suggests that during the transition from $|1+\ldots+1\rangle$ to $|N\rangle$, the system will pass through states of the form $|M+1+\ldots+1\rangle$. Note that this is in accordance with our previous hypothesis, see Eq. (4.97). In general, we can express the average energy of these

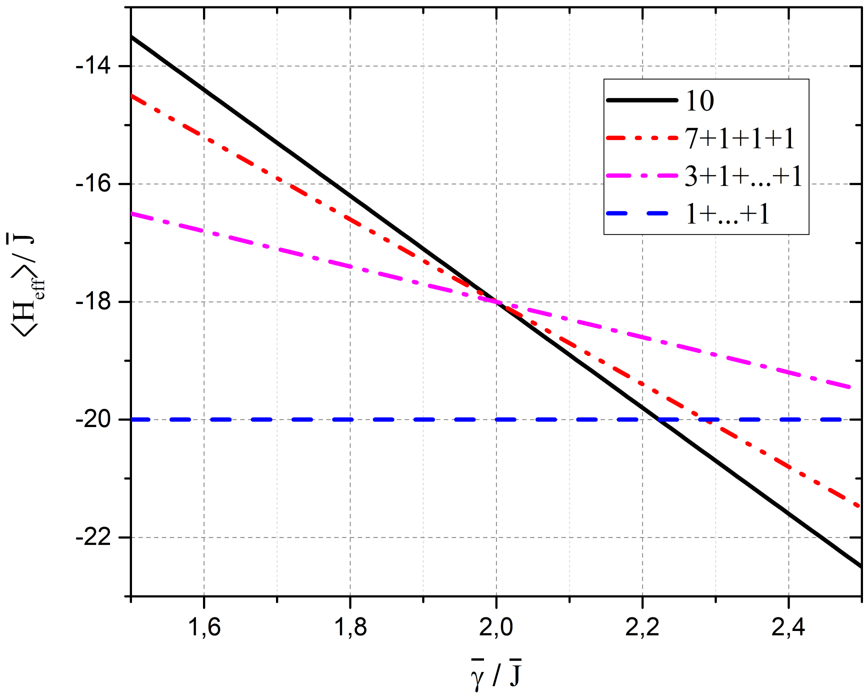

Figure 4.6: The average energy of ten A-B pairs in the states $|10\rangle$, $|7+1+\cdots+1\rangle$, $|3+1+\cdots+1\rangle$, and $|1+1+\ldots+1\rangle$ as functions of the ratio $\bar{\gamma}/\bar{J}$ and under the assumption of no-adjacency, $d \gg N$. [3]

intermediate states as

$$\begin{aligned}&\langle M+1\ldots+1|\mathcal{H}_{\text{eff}}|M+1\ldots+1\rangle = \\ &-(M-1)\bar{\gamma}-2(N-M+\delta_{M,1})\bar{J},\end{aligned} \tag{4.110}$$

such that $\delta_{M,1}$ stands for a Kronecker delta.

In Fig. 4.6, the average energies for some states are plotted as a function of the ratio $\bar{\gamma}/\bar{J}$, considering ten A-B pairs, $N=10$. Clearly, through this transition, only the states $|1+1+\ldots+1\rangle$ and $|10\rangle$ will be associated with lowest value. This suggests that while increasing the strength γ, in the limit $d \gg N$ (no-adjacency assumption), the system will skip the intermediate states and will go from $|1+1+\ldots+1\rangle$ directly to $|N\rangle$. In other words, this transition will consist of a one step process: the assembly sketched in Fig. 4.3 (b) will skip (c) and will get transformed directly to assembly (d). Using Eq. (4.110), we can find the transition point from $|1+\ldots+1\rangle$ to $|N\rangle$

$$\begin{aligned}\langle 1+\ldots+1|\mathcal{H}_{\text{eff}}|1+\ldots+1\rangle &= \langle N|\mathcal{H}_{\text{eff}}|N\rangle, \\ -2N\bar{J} &= -(N-1)\bar{\gamma}.\end{aligned} \tag{4.111}$$

Hence, the transition should take place for

$$\bar{\gamma}/\bar{J} = 2N/(N-1), \tag{4.112}$$

which we can rewrite as

$$\gamma U/J^2 = 2 + 2N/(N-1). \tag{4.113}$$

This is in agreement with our numerical simulations illustrated in Fig. 4.5. In terms of entanglement, the results above suggest that a system containing only bipartite correlations will get transformed directly to a state with only genuine multipartite quantum correlations.

4.5 Summary

In this chapter, I considered the problem of the formation of composite bosons made of several fermionic constituents. We saw that maximally entangled states of a fermionic pair can approximate the ground states of 1-D Hubbard model with point-like interactions. Also, I discussed the cases of 4 fermions and then generalized it to $2N$ fermions. I showed that some multipartite entangled states can correspond to composite bosons. Following the approach of C. K Law [7], I studied their bosonic behaviour. We saw that, for multipartite composite bosons, stronger correlations imply better bosonic quality, similarly to the case of bi-fermions. In addition, I examined qualitatively the correlations between the fermions for different possible arrangements. For instance, we saw that the states of two bipartite composite bosons have only bipartite correlations, while the states of a four-partite composite boson has only genuine four-partite correlations. In the general case, the formation of a composite boson made $2N$ fermions requires genuine $2N$-multipartite fermionic entanglement.

Also, I considered interacting fermions within the one-dimensional Hubbard model. I used an effective Hamiltonian to approximate pairs of fermions as hard-core bosons and I numerically studied the system's ground state, for a finite lattice $d = 8, 10$. We saw that while increasing the strength of the nearest neighbour interaction several possible structures emerged.

Finally, I studied the average energies corresponding to different fermionic arrangements. We saw that the transition is more energetically favourable if it consists of a single step process, in the limit of a large lattice $d \gg 10$. In this case, the increase of the strength of the nearest neighbour interaction will lead to a transformation from many bipartite composite bosons to a single multipartite one, without having any intermediate structures.

Summary

The main thread discussed in this work is the study of many-body systems within the frame of quantum information. First, I presented a non-local bunching scenario of composite bosons. We saw that a single composite boson cannot undergo a beam splitting operation and remain stable without some kind of interactions. In this case, the stability of the composite boson require entanglement generation which suggests the need for interactions or some post-selective measurements. Then, we saw that two spatially separated pairs of highly entangled fermions can behave in a manner proper to two bosons, if local operations are allowed. Using only local operations, two of such pairs can evolve to a state that can be interpreted as a Fock state made of two composite bosons.

Next, I considered a more flexible definition for the stability of a composite particle. I showed that entanglement can lead to the spread of the centre of mass of a bipartite composite particle while the relative distance between its constituents grows relatively slowly. Also, we saw that the distance over which the centre of mass gets delocalized before the decay of the composite particle depends only on its initial size and the amount of entanglement between its constituents. In this case, while the composite particle gets delocalized, the correlations encoded over its internal structure gets transformed into spatial entanglement. This leads to the spread of the composite particle while preventing its decay. Also, I considered the dynamics of the same system under non-zero temperatures. As expected, the thermalization leads to a faster decay of the composite particle. In addition, we saw that the effects of thermalization is not the same for all degrees of freedom. Interestingly, if the temperature is low enough, the thermalization effects only the relative distances.

Then, I presented a Mach-Zehnder-like setup. I considered a discrete lattice with periodic boundary conditions. We saw that the entire system with no interactions can get delocalized over the space without falling apart. Since, we considered periodic boundary conditions, the spread of the composite particle make it interferer with itself which allows for the observation of the collective de Broglie wavelength. We saw that some post-selective measurements were required, along side the strong entanglement of the constituents, in order to observe the collective behaviour of the system. This is due to a variety of the no-signalling principle, since the observation of true composite features requires the constituents to remain correlated.

Finally, I discussed the formation of bipartite and multipartite composite bosons. Also, I considered the transition from the first type to the second one. Assuming a system made of several fermionic components (of two types A and B) within the one-dimensional Hubbard model, we saw that the strength of interactions has a crucial role in determining the dominant structure of the ground state. In order to have only independent bipartite composite bosons, the system should be in the limit of strong short-range interactions and relatively weak long-range interactions. Here, the long-range interactions are needed to compensate for an effective repulsion that occurs due to the Pauli blocking. While increasing the strength of the long-range interactions, the numerical simulations indicated that larger structures of a composite boson made of more than a single pair of fermions emerges. In the limit of strong long-range interactions, we saw that all the fermions merge together and form a single composite boson. In the case of a multipartite composite particle, we saw that its bosonic behaviour requires genuine multipartite entanglement between all the constituents. In addition, in the limit of an infinitely large space, we saw that it is energetically more favourable to have a direct transition from bipartite composite bosons to a single multipartite one made of all the fermionic pairs. This corresponds to a transition from a system only with bipartite correlations directly to a system with only genuine multipartite correlations.

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