Calibration Results for Incomplete Preferences*

Zvi Safra† Uzi Segal‡

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Abstract

in this work we demonstrate that incomplete expected utility preferences are susceptible to criticism similar to that addressed at their complete analogues: even a modest degree of risk aversion in the small is sufficient to imply extreme and unreasonable degree of risk aversion in the large. Our results thus shed doubt on the usefulness of incomplete expected utility preferences for practical and theoretical purposes.

Keywords: Calibration, incomplete preferences, risk aversion

1 Introduction

Rabin [2] showed that what seems to be like a reasonable behavior by an expected utility maximizer with respect to small lotteries necessarily implies very unreasonable reaction to large lotteries. For example, a rejection of a the lottery $(-100, \frac{1}{2}; 110, \frac{1}{2})$ at all wealth levels below 300,000 implies a rejection of the lottery $(-4000, \frac{1}{2}; 6 \cdot 10^7, \frac{1}{2})$. These results were obtained under the

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†Warwick Business School (zvi.safra@wbs.ac.uk) and Emeritus professor, Tel Aviv University.

‡Department of Economics, Boston College (segalu@bc.edu).

assumption that preferences are complete: for every two lotteries X and Y, either $X \succeq Y$ or $Y \succeq X$. Several leading researchers find the assumption that preferences of decision makers are complete to be questionable. Among them are von-Neumann and Morgenstern [7], Aumann [1], and Schmeidler [4, 5, 6]. For example, Schmeidler [6] wrote "out of the seven axioms listed here the completeness of the preferences seems to me the most restrictive and most imposing assumption" and, in one of his earliest works (Schmeidler [4]) he proved the existence of competitive equilibria in markets with (continuum of) agents with incomplete preferences. In another paper [5] he showed the importance of continuity assumptions in dealing with incomplete preferences. Our aim in this paper is to extend Rabin's results to the case of incomplete preferences.

Consider a committee of k decision makers, making decisions by a unanimity rule, where choices between alternatives are made only if all members are in agreement, as otherwise the committee makes no decision. We investigate the plausibility of the assumption that all the members of this committee are expected utility maximizers. Similarly to the case of complete individual preferences we show that even having a minimal degree of group risk aversion or inconclusiveness implies unreasonable behavior on the part of the committee. That is, a uniform rejection of a small gamble implies rejection of extremely attractive (big) gambles. Similarly, a uniform non-acceptance of a small gamble implies non-acceptance of extremely attractive gambles. We also demonstrate that similar results hold if the uniform rejection (non-acceptance) is restricted to an interval.

This demostrates that the limitations of the expected utility model in the case of complete preferences extend to incomplete preferences, and hence this model loses some of its attrctiveness. Whether or not these limitations apply to incomplete non expected utility preferences is beyond the scope of the current project, but we suspect that the more general results of Safra and Segal [3] can be extended to cover incomplete non-EU preferences as well.

2 Incomplete Expected Utility preferences

Let L be a space of finite lotteries, let $\succeq_1, \ldots, \succeq_k$ be a collection of k complete expected utility preferences over it and assume that the preferences $\succeq$ of the decision maker satisfy $X \succeq Y \iff \forall i,\ X \succeq_i Y$. Let u_i be a vNM utility associated with $\succeq_i$. Then

$$X \succeq Y \Longleftrightarrow \forall i, \quad \mathrm{E}[u_i\,(X)] \geqslant \mathrm{E}[u_i\,(Y)] \tag{1}$$

We assume that all preferences are risk averse, hence all u_i are concave. Suppose that for all w, the decision maker prefers δ_w to the lottery $(w-\ell, \frac{1}{2};\, w+g, \frac{1}{2})$. By definition, this happens iff for all i, $\delta_w \succeq_i (w-\ell, \frac{1}{2};\, w+g, \frac{1}{2})$. From Rabin [2] we know that such preferences may lead to implausible rejections of very attractive large lotteries. By definition, so does the decision maker.

The requirement that this rejection occurs at all wealth levels is strong, but as shown by Rabin, similar results can be obtained even if the rejection of the small lotteries is restricted to a bounded domain.

Definition 1 The list $\{w, [a,b], X, \mathbb{X}\}$ is a Rabin-type paradox if a rejection of the lottery X at all wealth levels between a and b implies a rejection of the lottery $\mathbb{X}$ at the wealth level w.

Example 1 The first example is taken from Table 2 in Rabin [2]. The second is from Table 1 in Safra and Segal [3].

- w = 290,000, $a = 0$, b = 300,000, $X = (-100, \frac{1}{2};\, 110, \frac{1}{2})$, $\mathbb{X} = (-1{,}000, \frac{1}{2};\, 718{,}190, \frac{1}{2})$
- $w = a$ = 80,000, b = 120,000, $X = (-100, \frac{1}{2};\, 105, \frac{1}{2})$, $\mathbb{X} = (-5{,}035, \frac{1}{2};\, 10^7, \frac{1}{2})$

Conclusion 1 If $\{w, [a,b], X, \mathbb{X}\}$ is a Rabin-type paradox for $u_1, \ldots, u_k$, then it is also a Rabin-type paradox for a decsion maker with preferences as in eq. (1).

This conclusion connects reasonable rejections of small lotteries with unreasonable rejections of extremely attractive large lotteries. But what happens if a decision maker with incomplete preferences cannot determine whether to accept or reject a certain small lottery? As we show next, such inconclusiveness will lead to inconclusiveness with respect to very attractive large lotteries.

Suppose that we observe that for a certain ℓ and g and for all $w \in I = [a,b]$, the decision maker cannot determine preferences between δ_w and $(w-\ell, \frac{1}{2}; w+g, \frac{1}{2})$. For each such w there must therefore be i and j such that $\delta_w \succ_i (w-\ell,\frac{1}{2};w+g,\frac{1}{2})$ but $(w-\ell,\frac{1}{2};w+g,\frac{1}{2}) \succ_j \delta_w$. Let $I_i = \{w \in I : \delta_w \succ_i (w-\ell,\frac{1}{2};w+g,\frac{1}{2})$ and observe that $I = \cup_i I_i$.

Consider a pair L, G, where G is "large." The decision maker with wealth level a will not accept the lottery $(-L,\frac{1}{2};G,\frac{1}{2})$ iff at least one of the preferences $\succeq_i$ rejects it. In other words, if there exists i such that

$$u_i(a) > \tfrac{1}{2}u_i(a-L) + \tfrac{1}{2}u_i(a+G) \tag{2}$$

Assume wlg that for all i, $u_i(a) = 0$, $u'_i(a_-) := \lim\limits_{x\uparrow a} = 1$. The maximal value of $u(a-L)$ is therefore $-L$. For a given G, inequality (2) is satisfied for some i if $L > \min_i\{u_i(a+G)\}$.

Given $s_i := u_i(b+g)$ and $t_i := u'_i(b+g)$, define the linear (risk averse) continuation $\tilde{u}_i$ of u_i at $b+g$ such that for $x \geqslant b+g$, $\tilde{u}_i(x) = s_i + t_i(x-b-g)$. Inequality (2) is thus satified if

$$L > \min_i\{s_i + t_i(G-b-g)\} \tag{3}$$

Assume for simplicity that $(b+g)-(a-\ell) = kM(\ell+g)$ for some integer M and let $w_m = a+m(\ell+g)$, $m = 0,\ldots,kM-1$. Consider the sets $J = \{w_m\}_{m=0}^{kM-1}$ and $J_i = I_i \cap J$, $i = 1,\ldots,k$. We assume that the incomplete preferences $\succeq$ cannot accept $(w-\ell,\frac{1}{2};w+g,\frac{1}{2})$ at all $w \in [a,b]$, and in particular they cannot accept this lottery at all $w \in J$. For every m there is therefore (at least one) i such that $\delta_{w_m} \succ_i (w_m-\ell,\frac{1}{2};w_m+g,\frac{1}{2})$. As proved by Rabin [2], for this i and m

$$u'_i(w_m+g) \leqslant \tfrac{\ell}{g}u'_i(w_m-\ell)$$

Moreover, if person i has such preferences at r_i points of J, then

$$u'_i(b+g) \leqslant \left(\tfrac{\ell}{g}\right)^{r_i} u'_i(a-\ell)$$

To assure non acceptance, we need to work with the worst-case scenario, in which a combination of the minimum of $u_i(b+g)$ and $u'_i(b+g)$ is as high as possible. A necessary condition for that is that at each point of J only one of the preferences $\succeq_i$ will reject the lottery $(-\ell,\frac{1}{2};g,\frac{1}{2})$. By the definition of the partition of $J_1,\ldots,J_k$ of J,

$$J_i = \{w_m \in J : w_m \succ_i (w_m-\ell,\tfrac{1}{2};w_m+g,\tfrac{1}{2})\}$$

Denote the elements of J_i by $w_i^1 < \ldots < w_i^{m_i}$. To increase the values of s_i and t_i under the assumption that all the functions u_i are concave we'll assume that between w_i^r and w_i^{r+1} the function u_i is linear.

Suppose that at $b+g$, the derivative (from the right) of $\tilde{u}_i$ is greater than that of $\tilde{u}_j$ (in our notation: $t_i > t_j$). Whatever are the values of the corresponding functions at $b+g$ (s_i and s_j), $\tilde{u}_i$ will become larger than $\tilde{u}_j$. To increase the minimum of these values we will thus require that all the slopes of the continuations are the same, that is, $t_1 = \ldots = t_k$. And since at a the derivatives of all the functions u_i are the same and since at each point of J_i the derivative is multiplied by no more than $\frac{\ell}{g}$, we can restrict attention to the case in which $|J_1| = \ldots = |J_k| = M$ and therefore

$$t_1 = \ldots = t_k = \left(\frac{\ell}{g}\right)^M$$

In order to make sure that inequality (2) is satisfied, we therefore need to find a partition of J into k sets of M points such that the minimal value of $\{s_1, \ldots, s_k\}$ will be as high as possible. Denote this value s^*. We now offer several upper bounds for this value.

Given the points $w_i^1, \ldots, w_i^M$, the function u_i is bounded from above by the piecewise linear function v_i, given by

$$v_i(x) = \begin{cases} x - a & x \leqslant w_i^1 \\ v_i(w_i^1) + \frac{\ell}{g}(x - w_i^1) & x \in [w_i^1, w_i^2] \\ \ldots & \\ v_i(w_i^j) + \left(\frac{\ell}{g}\right)^j g(x - w_i^j) & x \in [w_i^j, w_i^{j+1}] \\ \ldots & \\ v_i(w_i^M) + \left(\frac{\ell}{g}\right)^M g(x - w_i^j) & x \in [w_i^M, g+b] \end{cases} \tag{4}$$

The highest possible value of such a function at $b+g$ is when $J_i = \{a + (k-1)M(\ell+g), \ldots, a + (kM-1)(\ell+g) = b\}$ and

$$s_i = u_i(g+b) \leqslant [(k-1)M - 1](g+\ell) + \sum_{m=1}^{M-1} \left(\frac{\ell}{g}\right)^m (g+\ell) + \left(\frac{\ell}{g}\right)^M g$$

As this is the highest possible value any s_i may reach, the highest possible value of s^*, the minimum of $\{s_1, \ldots, s_k\}$, cannot be higher.

We now show that for $k = 2$ a much better upper bound for s^* can be obtained. Let J_1^*, J_2^* be a partition for which s^* is obtained.

Claim 1 Given the partition J_1^*, J_2^*, $|s_1 - s_2| < 2(\ell + g)$.

Proof: Suppose wlg that $s^* = s_1 \leqslant s_2 - 2(\ell + g)$. There are $m \in J_1$ and $m+1 \in J_2$, otherwise, by eq. (4), $s_1 > s_2$. Let $J_1' = J_1 \cup \{m+1\} \setminus \{m\}$ and $J_2' = J_2 \cup \{m\} \setminus \{m+1\}$ with the corresponding s_1' and s_2'. Then $s_1 < s_1' \leqslant s_1 + \ell + g$ and $s_2 > s_2' \geqslant s_2 - (\ell + g)$. But then the partition (J_1', J_2') creates a higher minimal value for $\{s_1, s_2\}$, a contradiction. ■

It is easy to see that the highest possible sum of the values of s_1 and s_2 is obtained when $J_1 = \{0, \ldots, M-1\}$ and $J_2 = \{M, \ldots, 2M-1\}$. In that case, by eq. (4),

$$\bar{s} := s_1 + s_2 \leqslant (\ell + g) \sum_{r=1}^{M-1} \left(\frac{\ell}{g}\right)^r + [M(\ell + g) + g] \left(\frac{\ell}{g}\right)^M + M(\ell + g) + (\ell + g) \sum_{r=1}^{M-1} \left(\frac{\ell}{g}\right)^r + \ell \left(\frac{\ell}{g}\right)^M$$

By claim 1, the higest possible value of the maximum of the s-values of the two agents cannot exceed $\bar{s}/2 + \ell + g$, which is therefore an upper bound for s^*.

3 Calculations

In this section we offer some calculations bassed on the above analysis. These numbers prove that Rabin's argument, namely that reasonable risk aversion in the small leads to extreme level of risk aversion with respect to large lotteries, carries over to incomplete preferences.

Consider the case $k = 2$. As before, let $u_1(a) = u_2(a) = 0$ and $u_1'(a_-) = u_2'(a_-) = 1$. At the point $C := a + (2M-1)(\ell + g) + \ell$, the derivates from the right of both u_1 and u_2 are $\left(\frac{\ell}{g}\right)^M$, and the minimum of the two cannot be higher than

$$B := \tfrac{1}{2}\left[(\ell+g)\sum_{r=1}^{M-1}\left(\tfrac{\ell}{g}\right)^r + [M(\ell+g)+g]\left(\tfrac{\ell}{g}\right)^M + \right.$$

$$\left. M(\ell+g)+(\ell+g)\sum_{r=1}^{M-1}\left(\tfrac{\ell}{g}\right)^r + \ell\left(\tfrac{\ell}{g}\right)^M\right] + \ell + g =$$

$$(\ell+g)\left(\sum_{r=0}^{M-1}\left(\tfrac{\ell}{g}\right)^r + \tfrac{1}{2}\left[(M+1)\left(\tfrac{\ell}{g}\right)^M + M\right]\right)$$

Therefore, at $G \geqslant C$, at least one of the two functions cannot exceed $B + (G-C)\left(\frac{\ell}{g}\right)^M$. For a given L, the decision maker will not accept the lottery $(-L, 1-p; G, p)$ if $-(1-p)L + p\left[B + (G-C)\left(\frac{\ell}{g}\right)^M\right] \leqslant 0$, that is, if

$$G \leqslant C + \left[\left(\tfrac{1-p}{p}\right)L - B\right] \Big/ \left(\tfrac{\ell}{g}\right)^M$$

The following tables show the critical values of G for $\ell = 100$ and some values of g, M, L, and p. For example, in the first table, if the decision maker does not accept the lottery $(-100, \frac{1}{2}; 110, \frac{1}{2})$ at all wealth levels between his curent wealth level w and $w + 8400$, then he will not accept a lottery in which there is a 99.9% chance of losing \$200 and a one-in-a-thoudand chance of winning over 1.3 million dollar. The second table suggests that if the decision maker does not accept the lottery $(-100, \frac{1}{2}; 105, \frac{1}{2})$ at all wealth levels between his curent wealth level w and $w + 41,000$, then he will not accept a lottery in which there is a 99% chance of losing \$1000 and a 1% chance of winning over 11 million dollar.

$M = 20$, $p = 0.001$

$L\backslash g$	\$101	\$105	\$110	\$125
\$100	120,803	258,450	650,804	8,379,847
\$200	242,700	523,515	1,322,881	17,044,790
\$500	608,391	1,318,708	3,339,113	43,039,622
\$1,000	1,217,876	2,644,030	6,699,499	86,364,341
\$5,000	6,093,755	13,246,608	33,582,589	4.3×10^8
\$10,000	12,188,604	26,499,830	67,186,451	8.7×10^8

$M = 100$, $p = 0.01$

$L \backslash g$	\$101	\$105	\$110	\$125
\$200		724,572	96,360,075	3.6×10^{13}
\$500	102,044	4,630,159	5.1×10^{8}	1.8×10^{14}
\$1,000	235,932	11,139,471	1.2×10^{9}	4.3×10^{14}
\$5,000	1,307,039	63,213,969	6.6×10^{9}	2.4×10^{15}
\$10,000	2,645,921	1.3×10^{8}	1.4×10^{10}	4.8×10^{15}

$M = 200$, $p = 0.1$

$L \backslash g$	\$101	\$105	\$110	\$125
\$5,000	114,046	3.5×10^{8}	4.1×10^{12}	5.2×10^{23}
\$10,000	443,267	1.1×10^{9}	1.3×10^{13}	1.6×10^{24}

4 Concluding Remarks

Rabin [2] showed that within the expected utility model, a seemingly reasonable degree of risk aversion with respect to small lotteries implies a rejection of what seems to be extremely attractive large lotteries. Safra and Segal [3] showed that these results hold for all "well-behave" extensions of expected utility, and moreover, they provided much stronger numerical analysis.

When a decision maker has incomplete preferences, his statement "I cannot determine which of the two options is better" indicates that none of the options can be deemed inferior to the other. One would expect therefore that a statement like "I can't tell whether I prefer w or $(w - \ell, \frac{1}{2}; w + g, \frac{1}{2})$" will indicate a lesser degree or risk aversion than an outright preferences for the sure outcome. Yet even this small degree of risk aversion is sufficient to imply non-acceptance of lotteries that should obviously be accepted. Similarly to Rabin's [2] and Safra and Segal's [3] criticism of using one set of complete preferences for the analysis of decision under risk, our results cast doubt on the reality of using incomplete preferences for these purposes.

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