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Comment on "Applying a second-kind boundary integral equation for surface tractions in Stokes flow"

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In the article of 2011 by Keaveny and Shelley referred to in the title of this note, formulas (38) and (39) for the surface tractions on a prolate spheroid translating parallel or perpendicular to its axis of symmetry, respectively, contain errors. Those authors, however, did use the correct form of the expressions for the exact solution to compare with their numerical results in figure 1.

We pursued the somewhat lengthy process of computing the exact tractions starting from expressions for the velocity and pressure fields given in Art. 339 of Lamb (1932)'s discussion of the analysis by Oberbeck (1876) (presented also in Happel and Brenner, 1983, section 5-11). Keeping the same notation as in Keaveny and Shelley, we obtained in the case of a prolate spheroid translating parallel to its axis of symmetry

$$f_{\parallel}(\phi) = \frac{4U\eta}{\chi_0 + \alpha_0 c^2}\left[\frac{1}{a^2} - \frac{\epsilon^2}{(1-\epsilon^2)}\frac{\cos^2\phi}{c^2}\right]^{-1/2}, \tag{1}$$

and in the case of a prolate spheroid translating perpendicular to its axis of symmetry

$$f_{\perp}(\phi) = \frac{4U\eta}{\chi_0 + \alpha_1 a^2}\left[\frac{1}{a^2} - \frac{\epsilon^2}{(1-\epsilon^2)}\frac{\cos^2\phi}{c^2}\right]^{-1/2}, \tag{2}$$

with $\epsilon = \sqrt{c^2 - a^2}/c$. Because ϵ must be dimensionless, in Keaveny and Shelley's definition for ϵ, the denominator should be c rather than c^2 as it is written. Apart from the discrepancy in the denominator in the definition of ϵ, the only difference with Keaveny and Shelley's results is that the factor $-1/(1-\epsilon^2)$ in front of the $\cos^2\phi$ is missing in their expressions (38) and (39). Note also that symbol η for the viscosity should be added to the numerator of expressions (44) and (45) of Keaveny and Shelley (2011) for the total drag force acting on the spheroid.

Note that neither Oberbeck nor Lamb presented explicit formulas for the local tractions on the surface of the ellipsoid such as those in Keaveny and Shelley. It is also worth mentioning that both

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Youngren and Acrivos (1975) and Karrila and Kim (1989) validated the results from their numerical boundary integral schemes for the surface tractions on translating spheroids by graphically comparing with the analytical solutions, although they did not include the corresponding expressions. Karrila and Kim, however, listed the work of Brenner (1964) and appendix B of Kim (1986) as the sources. Indeed, when the formulas in Brenner for general ellipsoids are specialized to the case of prolate spheroids, they lead to expressions (1) and (2) of this note.

We thank Prof. E. E. Keaveny for kindly sharing with us his personal erratum on the traction formulas after receiving from us a draft of this note. The expressions in his erratum were used to generate the correct analytical results shown in figure 1 of Keaveny and Shelley (2011) and confirm the validity of our expressions (1) and (2) in this note. We are also grateful to the reviewer who directed us to the article by Karrila and Kim. We acknowledge the support of the Engineering and Physical Sciences Research Council (Grant Nos. EP/N016602/1, EP/P020887/1, and EP/P031684/1) and the Leverhulme Trust (Research Project Grant).

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