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Discrete Weighted Exponential Distribution: Properties and Applications

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Abstract. In this paper, we propose a new lifetime model as a discrete version of the continuous weighted exponential distribution which is called discrete weighted exponential distribution (DWED). This model is a generalization of the discrete exponential distribution which is originally introduced by Chakraborty (2015). We present various statistical indices/properties of this distribution including reliability indices, moment generating function, probability generating function, survival and hazard rate functions, index of dispersion, and stress-strength parameter. We first present a numerical method to compute the maximum likelihood estimations (MLEs) of the models parameters, and then conduct a simulation study to further analyze these estimations. The advantages of the DWED are shown in practice by applying it on two real world applications and compare it with some other well-known lifetime distributions.

1. Introduction

Although, the continuous distributions are more widely used for modelling the lifetime of a component or system, but there would be are situations that discrete lifetime distributions are more plausible to apply due to the nature of the problem. For instance, the discrete distributions would be better choices for modelling the lifetime of an on/off switch or lifetime of a device that is exposed to shocks, etc. The geometric, negative binomial and the multinomial are among more popular discrete models which have been widely used in the reliability analysis and other related applications (Meeker and Escobar, 1998). The common way to construct discrete distributions is based on the discretization of some suitable continuous distributions for example Chakrabortry and Chakravarty (2012, 2014, 2016), Chakrabortry and Gupta (2015), Chakrabortry (2015a, 2015b) and Chakrabortry and Bhati (2016). One of the simplest ways to implement discretization is briefly explained here. We assume a continuous random variable X has the survival function (SF) $S_X(x) = P(X \geq x)$, and a random variable Y is defined as $Y = [X]$. The probability mass function (PMF) of Y is then given by (Kotz et al., 2006)

$$P(Y = y) = P(y \leq X < y+1) = P(X \geq y) - P(X \geq y+1) = S_X(y) - S_X(y+1), \text{for } \ y = 0, 1, 2, \ldots \qquad (1)$$

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The discretization of a continuous distribution using this method retains the same functional form of the survival function, $S_X(.)$. As a result, many reliability characteristics remain unchanged. This method which is widely applied to generate new discrete distributions has received attention in the last four decades. In this regard, Nakagawa and Osaki (1975) obtained the discrete Weibull distribution, Roy (2004) proposed and studied discrete Rayleigh distribution, Kemp (2008) examined the discrete half-normal distribution, Krishna and Singh (2009) obtained the Burr and Pareto discrete distributions, Aghababaei Jazi et al. (2010) introduced and studied properties of the discrete inverse Weibull distribution, and Gomez-Deniz and Calderin (2011) studied the discrete Lindley distribution. Gomez et al. (2014) introduced a weighted exponential distribution (denoted by $EE(\alpha,\beta)$) with the following probability density function (PDF) and SF

$$f_X(x;\alpha,\beta) = \frac{\alpha^2(1+\beta x)e^{-\alpha x}}{\alpha+\beta}, \quad x,\alpha>0\,,\beta\geq 0, \;\; S_X(x) = \left(1+\frac{\alpha\beta x}{\alpha+\beta}\right)e^{-\alpha x}, \tag{2}$$

respectively. Where, α and β are respectively, shape and scale parameters. The PDF of $EE(\alpha,\beta)$ which is a mixture of an exponential density with the inverse scale α (denoted by $E(\alpha)$), and a Gamma distribution with the parameters $(2,\alpha)$ (denoted by $\mathcal{G}(2,\alpha)$), is given by

$$f_X(x;\alpha,\beta) = \frac{\alpha}{\alpha+\beta}\alpha e^{-\alpha x} + \frac{\beta}{\alpha+\beta}\alpha^2 x e^{-\alpha x}. \tag{3}$$

It should be noted that the exponential distribution, $E(\alpha)$ can be derived from $EE(\alpha,\beta)$ by setting $\beta=0$.
The main aim of this paper is to construct a new and novel discrete distribution, so-called *discrete weighted exponential distribution*, from the weighted exponential distribution presented in Equations (2) and (3). We also study important features and properties of DWED and show its usefulness in reliability analysis. It is interesting to note that the DWED enfolds the discrete exponential distribution of Chakraborty (2015) and geometric distribution for some specific values of its parameters. The various generalizations of Geometric distribution (Jain and Consul, 1971; Philippou et al., 1983; Tripathi et al., 1987; Mačutek, 2008; and Gomez-Deniz, 2010) can be also derived similar the DWED. We denote the DWED with its parameters, $(\alpha,\ \beta)$ by $DWE(\alpha,\beta)$, where $\alpha>0$, $\beta\geq 0$. We will show that the DWED exhibits both constant and increasing hazard rates, and it can be applied to a wide range of applications. The advantages of this distribution over other alternative discrete models will be examined and compared through several illustrations.
The paper is organized as follows: Section 2 introduces the $DWE(\alpha,\beta)$ distribution and presents some of its important statistical features and properties including cumulative distribution function (CDF), moment generating function, probability generating function, moments, quantile, modality, survival and hazard rate functions, index of dispersion, entropy and stress-strength parameter. In Section 3, we provide the MLE of the DWEDs parameters. We analyze two real world applications using the DWED in Section 4. Finally, some concluding remarks are given in Section 5.

2. Discrete Weighted Exponential Distribution

2.1. Probability Mass Function, Survival and Hazard Rate Functions

The discrete weighted exponential distribution is formally defined as follows:

Definition 2.1. *A random variable Y has a discrete weighted exponential distribution with parameters $\alpha>0$ and $\beta\geq 0$ denoted by $DWE(\alpha,\beta)$, if*

$$f_Y(y;\alpha,\beta) = P(Y=y) = \frac{\{(\alpha+\beta+\alpha\beta y)(1-e^{-\alpha})-\alpha\beta e^{-\alpha}\}}{\alpha+\beta}e^{-\alpha y}\,;\;\; y\in\{0,1,2,\ldots\}\,. \tag{4}$$

In other words, the PMF of the DWED is derived by substituting the SF of EE distribution given in Equation (2) into Equation (1). The resulting PMF, presented in (4), has an explicit form in terms of (α,β). It is trivial to show that $DWE(\alpha,0)$ distribution coincides with the discrete exponential distribution of Chakraborty

(2015). In addition, the geometric distribution with parameter, $0 < p < 1$ can be derived from $DWE(\alpha, 0)$, by setting $\alpha = \ln(\frac{1}{1-p})$.
The CDF, SF and hazard rate function of $Y \sim DWE(\alpha, \beta)$ are respectively given by

$$F_Y(y) = 1 - \frac{e^{-\alpha(y+1)}\left(\alpha + \beta + \alpha\beta(1 + y)\right)}{\alpha + \beta}, \tag{5}$$

$$S_Y(y) = \left(1 + \frac{\alpha\beta}{\alpha + \beta} y\right) e^{-\alpha y}, \tag{6}$$

and

$$H_Y(y) = \frac{\alpha\beta \left\{y(1 - e^{-\alpha}) - e^{-\alpha}\right\} + (\alpha + \beta)(1 - e^{-\alpha})}{\alpha + \beta + \alpha\beta y}. \tag{7}$$

It can also be illustrated that $H_Y(y)$ is bounded for all y with the following bounds

$$H_Y(0) = 1 - e^{-\alpha}(1 + \frac{\alpha\beta}{\alpha + \beta}), \;\; H_Y(\infty) = 1 - e^{-\alpha},$$
$$H_Y(0) \leq H_Y(y) \leq H_Y(\infty).$$

The ratio of successive probabilities, given in (8) can be used to calculate the probabilities of the $DWE(\alpha, \beta)$ distribution, recursively.

$$r_y = \frac{P(Y = y + 1)}{P(Y = y)} = e^{-\alpha}\left(1 - \frac{\alpha\beta}{((e^{\alpha} - 1)(\alpha + \beta) - \alpha\beta) + \alpha\beta(e^{\alpha} - 1)y}\right), \;\; y \in \{0, 1, 2, 3, \ldots\}. \tag{8}$$

It is trivial to show that for any value of α and β, r_y is a non-increasing function of y. In addition, the DWED is a unimodal distribution as illustrated in the following proposition.

Proposition 2.2. *The discrete weighted exponential distribution is strongly unimodal.*

Proof. It is trivial to show that r_y is a decreasing function. Therefore, we can conclude that the corresponding PMF, $P(Y = y)$ is a log-concave function, and thus strongly unimodal (see also Chakraborty and Chakravarty (2016) for a similar argument). □

An immediate result of Proposition 2.2 is that the DWED has a non-decreasing failure rate. It can be also seen the PMF of $DWE(\alpha, \beta)$ distribution for different shape parameter values and fixed scale parameter ($\beta = 3$) is a unimodal distribution (Figure 1(a)), and the PMF of $DWE(\alpha, \beta)$ distribution with different scale parameter values and the fixed shape parameter ($\alpha = 0.5$) is a deceasing function. The PMF becomes decreasing for all values of α satisfying in the following inequality, $e^{-\alpha}(2 - e^{-\alpha}) \leq 1$.
Figure 2 shows the hazard rate function of the DWED, given in (7), for different values of (α, β). It can be seen that the hazard rate function could be constant, or increasing with respect to y based on the different parameters values.

2.2. *Moments, mean, variance and quantiles*

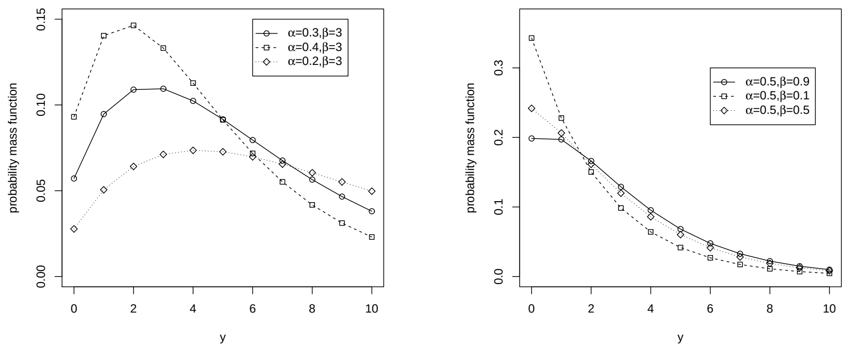
In this subsection, we present the quantiles, moment and probability generating functions, moments and variance of the DWE(α, β) distribution. In the following proposition, we present the general forms of moment and probability generating function of the DWED which can be used to compute moments and variance.

Proposition 2.3. *Let $Y \sim DWE(\alpha, \beta)$. The moment and probability generating functions of Y are*

$$M_Y(t) = \frac{\left((\alpha+\beta)(1-e^{-\alpha}) - \alpha\beta e^{-\alpha}\right) + e^{t-\alpha}\left(\alpha\beta - (\alpha+\beta)(1-e^{-\alpha})\right)}{\left(\alpha+\beta\right)(1-e^{t-\alpha})^2}; \;\; t < \alpha,$$

$$\phi_Y(t) = \frac{\left((\alpha+\beta)(1-e^{-\alpha}) - \alpha\beta e^{-\alpha}\right) + te^{-\alpha}\left(\alpha\beta - (\alpha+\beta)(1-e^{-\alpha})\right)}{\left(\alpha+\beta\right)(1-te^{-\alpha})^2}; \;\; t < e^{\alpha},$$

respectively.



(*a*) (*b*)

Figure 1: Probability mass function of the discrete weighted exponential distribution for (a) different values of α and the fixed $\beta = 3$; and (b) for the fixed $\alpha = 0.5$ and different values of β.

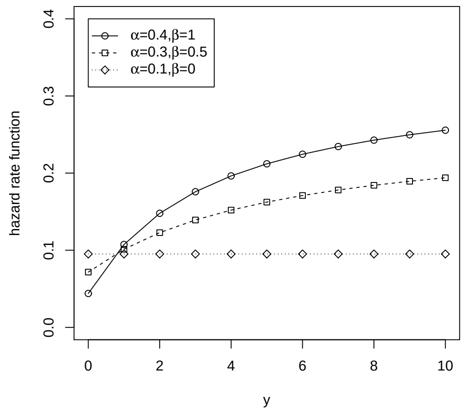


Figure 2: Hazard rate function of the discrete weighted exponential distribution.

The quantile function of the discrete weighted exponential distribution, denoted by $Q(u)$, derived as the root of $F(Q(u)) = u$, is given by

$$\left(\frac{e^{-\alpha}(\alpha+\beta+\alpha\beta)}{\alpha+\beta} + \frac{\alpha\beta e^{-\alpha}}{\alpha+\beta}Q(u)\right)e^{-\alpha\, Q(u)} = 1-u. \tag{9}$$

By taking logarithm from the both sides of Equation (9), we have

$$\log\left(\frac{e^{-\alpha}(\alpha+\beta+\alpha\beta)}{\alpha+\beta} + \frac{\alpha\beta e^{-\alpha}}{\alpha+\beta}Q(u)\right) - \alpha Q(u) = \log(1-u). \tag{10}$$

It can be illustrated that the solution of the following equation

$$\log(A+Bx) + Cx = \log(D)$$

is given by

$$x = \frac{1}{C}W\left(\frac{CD}{B}e^{\frac{AC}{B}}\right) - \frac{A}{B},$$

where $W(.)$ is the Lambert function for the input values $A, B, C, D,$ and $B, C \neq 0$ (Valluri et al., 2000). Therefore, the quantile function, Q(u) for $\beta \neq 0$, can be similarly obtained as follows:

$$Q(u) = \left[\frac{(-1)\left(\alpha\beta + \alpha + \beta + \beta\, W\left(\frac{(\alpha+\beta)(u-1)e^{-\frac{\alpha+\beta}{\beta}}}{\beta}\right)\right)}{\alpha\beta} \right], \tag{11}$$

where $0 < u < 1$, and [.] denotes the floor function (Valluri et al., 2000).

It should be noted that when $\beta = 0$, the DWE distribution will be reduced to discrete exponential distribution (or geometric distribution) and its quantile is the same as their quantiles.

It is thus trivial to show that the median of the DWE distribution, when $\beta \neq 0$, is given by

$$Q(\frac{1}{2}) = \left[\frac{(-1)\left(\alpha\beta + \alpha + \beta + \beta\, W\left(-\frac{(\alpha+\beta)e^{-\frac{\alpha+\beta}{\beta}}}{2\beta}\right)\right)}{\alpha\beta} \right].$$

By using the series expansion of Lambert W function, the quantile function given in Equation (11) can be rewritten as follows:

$$Q(u) = \left[\frac{(-1)\left(\alpha\beta + \alpha + \beta + \beta \sum_{i=1}^{\infty} \frac{(-i)^{i-1}\left(\frac{(\alpha+\beta)(u-1)e^{-\frac{\alpha+\beta}{\beta}}}{\beta}\right)^{i}}{i!}\right)}{\alpha\beta} \right].$$

The non-central moments of the DWE distribution are formally presented in Proposition 2.4.

Proposition 2.4. *Let $Y \sim DWE(\alpha, \beta)$, the r^{th} ($r = 1, 2, \ldots$) moment of Y is given by*

$$\mu_r = E\left(Y^r\right) = \frac{(\alpha+\beta)(1-e^{-\alpha}) - \alpha\beta e^{-\alpha}}{\alpha+\beta}\Phi(e^{-\alpha}, -r, 0) + \frac{\alpha\beta e^{-\alpha}}{\alpha+\beta}\Phi(e^{-\alpha}, -(r+1), 0), \tag{12}$$

where $\Phi(.)$ is the Lerch function (Gradshteyn and Ryzhik, 2000) which is defined as

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}; \quad |z| < 1.$$

The mean and second-order moment about zero of Y can be respectively obtained by replacing $r = 1$ and $r = 2$ in (12) as follows:

$$\begin{aligned} E(Y) &= \tfrac{1}{e^{\alpha}-1} + \tfrac{3\alpha\beta}{(e^{\alpha}-1)^3(\alpha+\beta)}, \\ E(Y^2) &= \tfrac{\left\{1+e^{\alpha}-2e^{2\alpha}+e^{4\alpha}-e^{5\alpha}\right\}}{(e^{\alpha}-1)^3 e^{\alpha}(-1+2e^{\alpha}-e^{2\alpha})} + \tfrac{2\alpha\beta(1-2e^{\alpha}-2e^{2\alpha})}{(e^{\alpha}-1)^3(\alpha+\beta)(-1+2e^{\alpha}-e^{2\alpha})}. \end{aligned} \tag{13}$$

It is then trivial to compute variance of Y, $Var(Y)$, based on the first and second moments. Figure 3 and Table 1 illustrate the means of the $DWE(\alpha, \beta)$ distribution for different α and β. It can be observed that $E(Y)$ increases with β (for the fixed α) and decreases with α (for the fixed β). Figure 3 and Table 1 illustrate the variances of the $DWE(\alpha, \beta)$ distribution for different α and β. For the fixed α, $Var(Y)$ decreases with β, and for the fixed β, $Var(Y)$ decreases with α (Figure 3 and Table 1).

Table 1: Mean and Variance for fixed α and fixed β.

α	β	Mean	α	β	Variance
0.3	0.5	15.99	0.3	0.5	473.63
0.3	2	21.13	0.3	2	432.40
0.3	6	22.87	0.3	6	406.46
0.2	0.7	47.51	0.2	0.7	2262.18
0.6	0.7	2.96	0.6	0.7	26.03
3	0.7	0.05	3	0.7	0.05

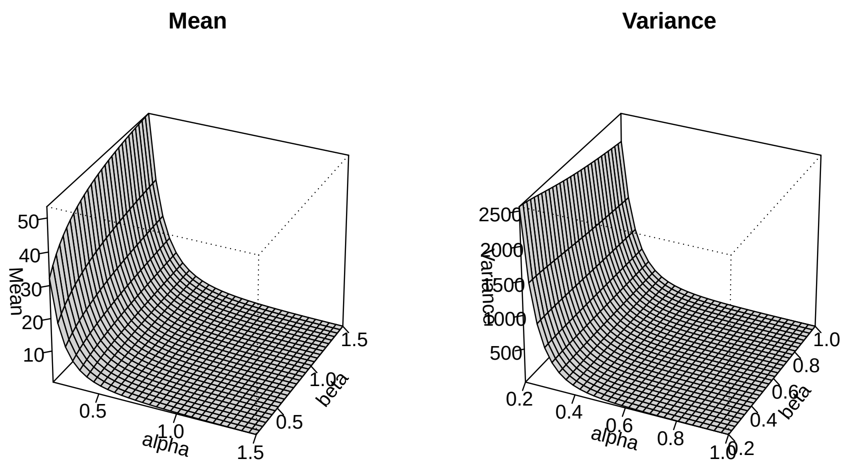


Figure 3: Means and variances of the DWE distribution for the different values of (α, β).

2.3. Index of Dispersion

In probability theory and statistics, the index of dispersion is a normalized measure of the dispersion of a probability distribution (Upton and Cook, 2006). It is a measure used to quantify whether a set of observed occurrences is dispersed compared to a standard statistical model. The index of dispersion (ID) is defined as variance divided by the mean of a distribution. If ID value is greater than one, the corresponding distribution is over-dispersed, and if it is less than one, the distribution is under-dispersed. Figure 4 represents the ID plot of the DWE distribution for different values of α and β. It can be observed that ID is greater than 1 for all values of α and β. Therefore, we can conclude that the DWE distribution is always over-dispersed. The ID decreases as β increases for fixed value of α and also decreases, as α increases for fixed value of β.

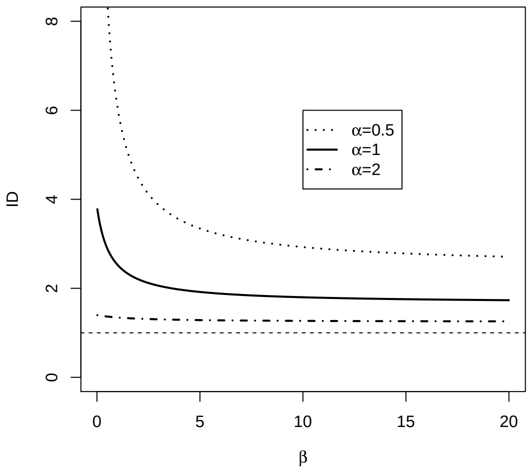


Figure 4: Index of Dispersion Plot of discrete weighted exponential distribution.

2.4. Entropy

The concept of entropy plays a vital role in information theory. Entropy measures provide important tools to indicate variety in distributions at particular moments in time and to analyze evolutionary processes over time. These measures are the variation of the uncertainty in the distribution of a random variable. In this section, we compute the Renyi entropy of $DWE(\alpha, \beta)$ distribution. The Renyi entropy is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \sum_y (P(Y = y))^\gamma \right\}, \tag{14}$$

where $\gamma > 0$ and $\gamma \neq 1$ (Renyi, 1961). For the DWE distribution and when γ is an integer number, we can write

$$\sum_{y=0}^{\infty} (P(Y = y))^\gamma = \sum_{j=0}^{\gamma} \frac{((\alpha+\beta)(1-e^{-\alpha}) - \alpha\beta e^{-\alpha})^j (\alpha\beta e^{-\alpha})^{\gamma-j}}{(\alpha+\beta)^\gamma} \Phi\left(e^{-\alpha\gamma}, j-\gamma, 0\right), \tag{15}$$

where $\Phi(.)$ is the Lerch function. This series will be convergent if $e^{\alpha\gamma} > 1$ by the ratio test. As a result, the equation given in (14) will become

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left(\sum_{j=0}^{\gamma} \frac{((\alpha+\beta)(1-e^{-\alpha}) - \alpha\beta e^{-\alpha})^j (\alpha\beta e^{-\alpha})^{\gamma-j}}{(\alpha+\beta)^\gamma} \Phi\left(e^{-\alpha\gamma}, j-\gamma, 0\right) \right).$$

The cumulative residual entropy (Rao et al., 2004) is also defined by

$$I_C = -\sum_{y=0}^{\infty} P(Y > y)\, \log\left(P(Y > y)\right). \tag{16}$$

By using the Taylor series expansion for $log(1 + x)$ and substituting it in Equation (16), we have

$$\begin{aligned} I_C = &(1 + \tfrac{\alpha\beta}{\alpha+\beta})e^{-\alpha} \sum_{j=1}^{\infty}\sum_{k=0}^{j} \left(\tfrac{-\alpha\beta}{\alpha+\beta}\right)^j \binom{j}{k} \Phi(e^{-\alpha}, -k, 0) \\ &+ e^{-\alpha} \sum_{j=1}^{\infty}\sum_{k=0}^{j} (-1)^j \left(\tfrac{\alpha\beta}{\alpha+\beta}\right)^{j+1} \binom{j}{k} \Phi(e^{-\alpha}, -(k+1), 0) \\ &+ \tfrac{\alpha\left(e^{2\alpha}(\alpha+\beta+\alpha\beta)-e^{\alpha}(\alpha+\beta-\alpha\beta)\right)}{(\alpha+\beta)(e^{\alpha}-1)^3}. \end{aligned}$$

2.5. Stress-strength parameter

In the context of reliability, the stress-strength model describes the life of a component which has a random strength Y subjected to a random stress X. A component fails at the instant when the stress applied to it exceeds the strength, and the component will function satisfactorily if $Y > X$. Therefore, $R = P(X < Y)$ can be considered as a measure for the component reliability. It has many applications, particularly in engineering concepts such as structural reliability, deterioration modeling of rocket motors, assessment of static fatigue of ceramic components, evaluating fatigue failure of aircraft structures, and modeling the ageing of concrete pressure vessels.

Suppose Y and X are independent discrete weighted exponential random variables with parameters (α_1, β_1) and (α_2, β_2), respectively. The $R = P(X < Y)$ can be then expressed as

$$\begin{aligned} R = P(Y > X) &= \sum_{x=0}^{\infty} \left(\tfrac{\alpha_1+\beta_1+\alpha_1\beta_1(x+1)}{\alpha_1+\beta_1}\right) e^{-\alpha_1(x+1)} \left(\tfrac{(\alpha_2+\beta_2+\alpha_2\beta_2 x)(1-e^{-\alpha_2})-\alpha_2\beta_2 e^{-\alpha_2}}{\alpha_2+\beta_2}\right) e^{-\alpha_2 x} \\ &= \frac{\left\{\begin{array}{l} e^{2(\alpha_1+\alpha_2)}\left(-\alpha_1\beta_2 - \beta_1\alpha_2 - \alpha_1\alpha_2(1+\beta_2+\beta_1) - \beta_1\beta_2\left(\alpha_1\alpha_2 + \alpha_2 + \alpha_1 + 1\right)\right) \\ + e^{\alpha_1+2\alpha_2}\left(-2\alpha_1\beta_2 - 2\beta_1\alpha_2 - \alpha_1\alpha_2(2+\beta_1-\beta_2-2\beta_1\beta_2) - \beta_1\beta_2(\alpha_1-\alpha_2)\right) \\ + e^{\alpha_1+\alpha_2}\left(2\alpha_1\beta_2 + 2\beta_1\alpha_2 - \alpha_1\alpha_2(\beta_1\beta_2 - 2 - \beta_2 - \beta_1) - \beta_1\beta_2(-2-\alpha_2-\alpha_1)\right) \\ + \left(\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2\right)\left(e^{\alpha_2}-1\right) \end{array}\right\}}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)(e^{\alpha_1+\alpha_2}-1)^3}. \end{aligned} \tag{17}$$

3. Estimation of Parameters

In this section, we provide the point estimations of the parameters of the discrete weighted exponential distribution using the maximum likelihood method. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample drawn from a DWE(α, β) distribution. The log-likelihood function of this sample is given by

$$\ell = \log L = -\alpha n\bar{y} - n\log(\alpha+\beta) + \sum_{i=1}^{n} \log\left((\alpha+\beta+\alpha\beta y_i)(1-e^{-\alpha}) - \alpha\beta e^{-\alpha}\right). \tag{18}$$

The likelihood equations are then given by

$$\begin{aligned} &\tfrac{d\ell}{d\alpha} = 0 \\ &\quad\Rightarrow\ -n\bar{y} - \tfrac{n}{\alpha+\beta} + \sum_{i=1}^{n} \tfrac{(1+\beta y_i)(1-e^{-\alpha}) + e^{-\alpha}\left(\alpha(1+\beta(y_i+1))\right)}{(\alpha+\beta+\alpha\beta y_i)(1-e^{-\alpha}) - \alpha\beta e^{-\alpha}} = 0, \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\tfrac{d\ell}{d\beta} = 0 \\ &\quad\Rightarrow\ -\tfrac{n}{\alpha+\beta} + \sum_{i=1}^{n} \tfrac{(1+\alpha y_i)(1-e^{-\alpha}) - \alpha e^{-\alpha}}{(\alpha+\beta+\alpha\beta y_i)(1-e^{-\alpha}) - \alpha\beta e^{-\alpha}} = 0. \end{aligned} \tag{20}$$

The solutions of likelihood Equations (19) and (20) provide the maximum likelihood estimators (MLEs) of α and β which can be obtained by a numerical method such as the two dimensional Newton-Raphson type procedure.

The Fisher's information matrix is given by

$$I_y(\alpha,\beta) = \begin{bmatrix} -E\left(\frac{d^2}{d\alpha^2} Log\, L\right) & -E\left(\frac{d^2}{d\alpha d\beta} Log\, L\right) \\ -E\left(\frac{d^2}{d\beta d\alpha} Log\, L\right) & -E\left(\frac{d^2}{d\beta^2} Log\, L\right) \end{bmatrix}. \tag{21}$$

where the second partial derivatives are given below:

$$\begin{aligned}
\frac{d^2\ell}{d\alpha^2} &= \frac{n}{(\alpha+\beta)^2} + \sum_{i=1}^{n} \left(\frac{2e^{-\alpha}\left(1+\beta y_i-\alpha-\beta-\alpha\beta y_i+\beta-\alpha\beta\right)}{(\alpha+\beta+\alpha\beta y_i)(1-e^{-\alpha})-\alpha\beta e^{-\alpha}} \right.\\
&\qquad \left. - \frac{\left((1+\beta y_i)(1-e^{-\alpha})+e^{-\alpha}\left(\alpha+\alpha\beta y_i+\alpha\beta\right)\right)^2}{\left((\alpha+\beta+\alpha\beta y_i)(1-e^{-\alpha})-\alpha\beta e^{-\alpha}\right)^2} \right),\\
\frac{d^2\ell}{d\beta^2} &= \frac{n}{(\alpha+\beta)^2} - \sum_{i=1}^{n} \left(\frac{\left((1+\alpha y_i)(1-e^{-\alpha})-\alpha e^{-\alpha}\right)^2}{\left((\alpha+\beta+\alpha\beta y_i)(1-e^{-\alpha})-\alpha\beta e^{-\alpha}\right)^2} \right),\\
\frac{d^2\ell}{d\beta d\alpha} &= \frac{n}{(\alpha+\beta)^2} + \sum_{i=1}^{n} \left(\frac{y_i(1-e^{-\alpha})+\alpha e^{-\alpha}(y_i+1)}{(\alpha+\beta+\alpha\beta y_i)(1-e^{-\alpha})-\alpha\beta e^{-\alpha}} \right.\\
&\qquad \left. - \frac{\left((1+\alpha y_i)(1-e^{-\alpha})-\alpha e^{-\alpha}\right)\left((1+\beta y_i)(1-e^{-\alpha})+e^{-\alpha}(\alpha+\alpha\beta y_i+\alpha\beta)\right)}{\left((\alpha+\beta+\alpha\beta y_i)(1-e^{-\alpha})-\alpha\beta e^{-\alpha}\right)^2} \right).
\end{aligned}$$

One can show that the DWE distribution satisfies the regularity conditions which are fulfilled for parameters in the interior of the parameter space, but not on the boundary (see, e.g., Ferguson, 1996, p. 121). That is, $I_y^{\frac{1}{2}}(\alpha,\beta)\left((\hat{\alpha},\hat{\beta})^T - (\alpha,\beta)^T\right)$ converges in distribution to bivariate normal with the (vector) mean zero and the identity covariance matrix. The Fisher's information matrix given in Equation (21) can be approximated as follows

$$I_y(\hat{\alpha},\hat{\beta}) \approx \begin{bmatrix} -\frac{d^2}{d\alpha^2} Log\, L\Big|_{(\hat{\alpha},\hat{\beta})} & -\frac{d^2}{d\alpha d\beta} Log\, L\Big|_{(\hat{\alpha},\hat{\beta})} \\ -\frac{d^2}{d\beta d\alpha} Log\, L\Big|_{(\hat{\alpha},\hat{\beta})} & -\frac{d^2}{d\beta^2} Log\, L\Big|_{(\hat{\alpha},\hat{\beta})} \end{bmatrix},$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of α and β, respectively (see also Gomez-Deniz, 2010).

3.1. *Simulation study*

In this section, we study the performance and accuracy of the MLEs of the parameters of the DWED by conducting various simulations for different sample sizes and different parameter values. To generate data from DWE distribution, we can simply use the PDF given in Definition 1 and CDF given in Equation (5). A code written in R is provided in Appendix A.

The simulation study using Monte Carlo method is conducted for $N = 5000$ iteration and for different sample sizes $n = 200, 500, 1000$ and different parameter values $(\alpha,\beta) = (0.3, 0.7)$, $(\alpha,\beta) = (0.7, 0.3)$, $(\alpha,\beta) = (0.9, 0.9)$, $(\alpha,\beta) = (0.2, 0.2)$, $(\alpha,\beta) = (1.1, 0.4)$ and $(\alpha,\beta) = (0.7, 1.4)$. In this simulation, mean bias and mean square error of the MLE of the parameters, defined below, are computed and discussed.

1. Mean bias (*Bias*) of the MLE of the parameter of interest, θ (e.g., α or β) is defined as follows:

$$Bias_\theta(n) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right).$$

2. Mean squared error (MSE) of the MLE of the parameter of interest, θ is defined as follows:

$$MSE_\varepsilon(n) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)^2,$$

Table 2: Monte Carlo Simulation Results: Mean Bias and MSE in parenthesis.

(α, β)	$\hat{\alpha}$	$\hat{\beta}$
$n = 200$		
(0.3,0.7)	0.0026(0.0006)	0.2088(0.5099)
(0.7,0.3)	-0.0036(0.0119)	0.2180(0.2809)
(0.9,0.9)	-0.0004(0.0149)	0.5332(2.9958)
(0.2,0.2)	-0.0003(0.0004)	0.0400(0.0266)
(1.1,0.4)	-0.0004(0.0376)	0.4956(1.4359)
(0.7,1.4)	-0.0001(0.0047)	0.6787(5.1876)
$n = 500$		
(0.3,0.7)	-0.0021(0.0002)	0.0427(0.0810)
(0.7,0.3)	0.0002(0.0057)	0.0781(0.0737)
(0.9,0.9)	0.0078(0.0044)	0.0145(0.3717)
(0.2,0.2)	0.0018(0.0001)	0.0286(0.0096)
(1.1,0.4)	-0.0116(0.0195)	0.2162(0.3029)
(0.7,1.4)	0.0005(0.0015)	0.1753(0.7302)
$n = 1000$		
(0.3,0.7)	0.0001(0.0001)	0.0333(0.0394)
(0.7,0.3)	-0.0047(0.0029)	0.0177(0.0251)
(0.9,0.9)	0.0060(0.0021)	0.0109(0.1540)
(0.2,0.2)	-0.0004(7×10^{-5})	0.0064(0.0036)
(1.1,0.4)	-0.0179(0.0108)	0.0462(0.0726)
(0.7,1.4)	0.0001(0.0008)	0.0849(0.1938)

where $\hat{\theta}_i$ is the MLE of θ based on a sample of size n.

Table 2 illustrates *Bias*, and *MSE* values of α and β for the different sample sizes. It can be concluded that as the sample size n increases, the *Bias* and *MSE* decay toward zero.

4. Application

In this section, we analyze two real world data sets to examine the fitness of the proposed model in comparison to other alternative distributions. We use the maximum likelihood method to estimate the parameters of the DWED. The numerical algorithm as explained in the previous section is required to compute MLEs of parameters. The mean and variance of data sets presented in Table 3 which shows the number of European redmites on apple leaves (Gupta and Ong, 2004; Chakraborty and Gupta, 2015; Alamatsaz et al., 2016) are respectively $\bar{x} = 1.146$ and $s^2 = 2.273$. Similarly, the mean and variance of the data sets presented in Table 4 which illustrates the numbers of fires in Greece for the period from 1 July 1998 to 31 August of the same year (Karlis and Xekalaki, 2001; Nekoukhou and Bidram, 2015) are respectively $\bar{x} = 5.398$ and $s^2 = 30.044$. It can be concluded that both of these data sets are overdispersed since the sample variances are greater than the respective sample means. We now compare and discuss fitting the discrete weighted exponential distribution and other distributions to the data sets mentioned above. The alternative distributions considered in this comparison are:

- Exponentiated Geometric (EG) with parameters $\alpha > 0$ and $0 < q < 1$ (Chakraborty and Gupta, 2015);
- Generalized Geometric (GG) with parameters $\alpha > 0$ and $0 < \theta < 1$ (Gomez-Deniz, 2010);
- Kumaraswamy Geometric (KG) with parameters $\alpha > 0$, $\beta > 0$ and $0 < q < 1$ (Akinsete et al., 2014);
- Discrete Generalized Exponential second type (DGE_2) with parameters $\alpha > 0$ and $0 < p < 1$ (Nekoukhou et al., 2013);

- Exponentiated Discrete Weibull (*EDW*) with parameters $\alpha > 0$, $\gamma > 0$ and $0 < p < 1$ (Nekoukhou and Bidram, 2015);
- Negative binomial (*NB*) with parameters $n > 0$, $0 < p < 1$.

The comparison between the discrete weighted exponential distribution and other distributions is performed based on χ^2 goodness-of-fit test, Akaike Information Criterion ($AIC = -2Log(L(\hat{\theta} \mid data)) + 2k$) and Bayesian Information Criterion ($BIC = -2Log(L(\hat{\theta} \mid data)) + kLog(n)$), where k and n are the number of estimated parameters and number of observations, respectively. In the set of competing models, a model is selected as the best fitted model to the data that has the smallest AIC and BIC values.

The MLEs of the parameters of the DWE distribution fitted to the data presented in Table 3 are given by

$$\hat{\alpha} = 0.7356, \quad \hat{\beta} = 0.1554,$$

and the variance-covariance matrix of the MLEs is given by

$$\begin{pmatrix} 0.0374 & 0.0677 \\ 0.0677 & 0.132 \end{pmatrix}.$$

The MLEs of the parameters of the DWE distribution fitted to the data presented in Table 4 are given by

$$\hat{\alpha} = 0.2631, \quad \hat{\beta} = 0.3211,$$

and the variance-covariance matrix of the MLEs is given by

$$\begin{pmatrix} 0.0010 & 0.0066 \\ 0.0066 & 0.0587 \end{pmatrix}.$$

The goodness of fit's statistics including χ^2, AIC and BIC reported in Tables 3 and 4, reveal that the DWE distribution is the best fitted distribution to the data. Since, the computed AIC and BIC of the DWE distribution are the smallest among other alternative distributions.

Table 3: The European redmites on apple leaves dataset and its goodness of fit statistics

Count	Observed	DWE	EG	GG	KG
0	70	68.23	69.87	69.52	69.56
1	38	38.79	37.33	37.55	37.92
2	17	19.85	19.94	20.10	19.62
3	10	10.82	10.65	10.71	10.92
4	9	5.80	5.69	5.69	5.13
5	3	3.18	3.04	3.02	3.34
6	2	1.65	1.62	1.60	1.61
7	1	0.72	0.87	0.85	0.95
≥ 8	0	0.96	0.99	0.96	0.95
Total	150	150	150	150	150
		$\hat{\alpha} = 0.7356$	$\hat{\alpha} = 1.000$	$\hat{\alpha} = 1.027$	$\hat{\alpha} = 0.993$
		$\hat{\beta} = 0.1554$	$\hat{q} = 0.534$	$\hat{\theta} = 0.529$	$\hat{\beta} = 0.550$
					$\hat{q} = 0.3213$
	$\log L$	- 222.382	-222.441	-222.437	-222.440
	χ^2	3.452	3.510	3.545	4.431
	$d.f$	4	4	4	3
	$p - value$	0.485	0.476	0.471	0.218
	AIC	448.765	448.882	448.875	450.881
	BIC	463.186	463.303	463.295	472.512

5. Conclusion

In this paper, we introduce a new discrete distribution for lifetime modelling which originated from a continuous weighted exponential distribution. We present various important distributional and reliability properties and features of this distribution. We illustrate that several discrete distributions, including discrete exponential and geometric distributions can be easily derived from this distribution.We also investigate the efficiency and benefits of using the proposed distribution in practice by applying it to analysis two real world data sets. By comparing the goodness-of-fit measures computed for this distribution with the ones derived for other plausible distributions, we can conclude that the DWE distribution is as good as the EG and GG distributions, but better fitted to the data than the KG distribution. Further investigation is required to explore in which circumstances the proposed model could produce better performance in fitting this model to the corresponding data.

Table 4: The numbers of fires in Greece dataset and its goodness of fit statistics

Count	Observed	DWE	DGE_2	EDW	NB
0	16	14.70	14.54	15.82	14.28
1	13	13.95	15.39	14.44	15.19
2	14	13.40	14.45	13.42	13.49
3	9	12.60	12.88	11.50	12.40
4	11	10.65	11.23	9.48	10.59
5	13	9.25	9.34	8.51	9.84
6	8	8.22	7.85	7.12	8.52
7	4	6.77	7.06	6.65	6.90
8	9	6.67	5.41	5.63	5.85
9	6	4.85	4.23	4.96	4.72
10	3	4.07	3.81	4.46	3.78
11	4	3.43	3.16	3.12	3.20
12	6	2.77	2.47	2.46	2.64
13	0	2.04	2.13	2.14	2.15
14	0	1.67	1.54	2.05	1.85
15	4	1.40	1.25	1.65	1.15
≥ 16	3	6.56	6.26	9.59	6.45
Total	123	123	123	123	123
		$\hat{\alpha} = 0.2631$	$\hat{\alpha} = 1.2547$	$\hat{\alpha} = 1.0806$	$\hat{n} = 1.3360$
		$\hat{\beta} = 0.3211$	$\hat{p} = 0.8224$	$\hat{\gamma} = 1.0929$	$\hat{p} = 0.1983$
				$\hat{p} = 0.8597$	
	$\log L$	- 339.512	-339.852	-339.793	-339.649
	χ^2	19.60	24.44	24.61	23.344
	$d.f$	9	9	8	9
	$p-value$	0.020	0.003	0.001	0.005
	AIC	683.024	683.704	685.585	683.298
	BIC	688.648	689.329	694.022	688.923

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Appendix A. An R code to generate data from a discrete weighted exponential distribution is as follows:

```
rEE<-function(N, alpha , Beta ){
probs = c((alpha/(alpha+Beta)),1 -(alpha/(alpha+Beta)))
dists = runif(N)
data = vector(length=N)
```

```
for(i in 1:N){
  if(dists[i]<probs[1]){
    data[i] = rexp(1,rate=alpha)
   } else {
    data[i] = rgamma(1,shape =2, scale=(1/alpha))
  }
}
return(data)
}
```