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Suboptimal Anisotropic Filtering for Linear Discrete Nonstationary Systems with Uncentered External Disturbance

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Abstract—We consider the robust anisotropic filtering problem for a linear discrete nonstationary system on a finite time interval. We assume that external disturbances acting on the object have anisotropy bounded from above and additionally satisfy two constraints on the moments. Our solution of the filtering problem is based on the boundedness criterion for the anisotropic norm in reverse time and reduces to finding a solution for a convex optimization problem. We illustrate the operation of a suboptimal anisotropic estimator with a numerical example.

Keywords: anisotropic filtering, nonstationary systems, non-centered random disturbances, convex optimization, linear matrix inequalities

1. INTRODUCTION

One fundamental problem in control and signal processing theory is the problem of estimating the system output (in the sense of minimizing a desired quality criterion). Various methods have been applied to solve this problem (for example, Kalman or $\mathcal{H}_2$-filtering and $\mathcal{H}_\infty$-filtering [1–3]), which have used different assumptions about the process model and properties of the disturbing signal. Each of these theories has some flaws, which is why attempts have been made to generalize them, leading, among other things, to the emergence of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$-filtering theory filtering (see, e.g., [4–6]). Solutions of $\mathcal{H}_2$-, $\mathcal{H}_\infty$-, and mixed $\mathcal{H}_2/\mathcal{H}_\infty$-filtering problems are mainly based on solving Riccati equations (see, for example, [7, 8]) and linear matrix inequalities (see, e.g., [9–11]).

For linear discrete *stationary* systems in the absence of prior knowledge about the probability distribution of the external disturbance, a stochastic approach to $\mathcal{H}_\infty$-optimization was proposed in 1994; it took into account possible uncertainty in the disturbance and was later called anisotropic theory [12–17] .

The problem of anisotropic analysis of robust quality for *nonstationary* systems was solved in [18]. A similar result in terms of related Riccati equations in forward and reverse time for the case of an uncentered disturbance has been described in [19]. A solution of the optimal anisotropic filtering problem for linear discrete nonstationary systems on a finite horizon was obtained by

[†] Deceased.

I.G. Vladimirov in 2001 [20]. To find the optimal anisotropic estimator, one has to solve a Riccati difference equation in forward time under a constraint in the form of a special algebraic equation.

Based on the results outlined in [21], the suboptimal anisotropic filtering problem on a finite horizon was solved in [22] in terms of difference linear matrix inequalities in the special case when dimensions of the estimated output and external disturbance are equal. In [23], sufficient conditions for the boundedness of the anisotropic norm in terms of difference linear matrix inequalities in forward time were formulated and proved in the general case. Based on these conditions, the suboptimal anisotropic filtering problem has been posed and solved on a finite time interval in forward time with no restrictions on the first two moments of external disturbance.

In [24], the anisotropic analysis problem and the problem of synthesizing a suboptimal anisotropic control were solved for a nonstationary system with non-centered disturbance, with additional constraints imposed on its first and second moments. A solution to the problem of analysis was obtained in terms of forward time and then used to synthesize the controller. We should immediately note here that it is hard to use the resulting solution (in terms of forward time) to solve the estimator synthesis problem, and therefore one has to obtain a similar solution in reverse time.

This work is organized as follows. Section 2 presents notation and abbreviations and summarizes basic information on anisotropic theory. The criterion for the boundedness of the anisotropic norm for a linear nonstationary system is given in Section 3. The formulation and solution of the anisotropic filtering problem are presented in Section 4. In Section 5, an anisotropic estimator is synthesized using the proposed algorithm in a numerical example.

2. PRELIMINARY INFORMATION

2.1. Abbreviations and Notation

In this work, we use the following abbreviations and notation: $\mathbb{R}^m$—set of vectors of size m with real components; $\mathbb{R}^{n\times m}$—set of matrices of size $n \times m$ with real components; $\mathbb{L}_2^m$—set of square integrable random vectors with values in $\mathbb{R}^m$; $\mathbf{E}[\cdot]$—the expectation operator; $|x|$—Euclidean norm of the vector x; X^{T}—transpose of the matrix or vector X; $0_{n\times m}$ and I_m—zero matrix and unit matrix of sizes $n \times m$ and $m \times m$ respectively (for simplicity 0 and I, if the dimension is clear from context); $A \succ B$ ($A \succeq B$) means that $A - B$ is a positive (nonnegative) definite matrix; tr$A$—trace of a square matrix A; $\det A$ —determinant of a square matrix A; $*$—matrix element symmetric with respect the main diagonal; pdf— probability density function.

2.2. Anisotropic Norm of a Matrix

In this section, we give definitions of anisotropy for a random vector and the anisotropic norm of a matrix, and also introduce mathematical tools for calculating it.

Consider two random vectors $W \in \mathbb{L}_2^m$ and $Z \in \mathbb{L}_2^p$, which are related to each other by means of a matrix $F \in \mathbb{R}^{p\times m}$ as $Z = FW$. The root-mean-square gain of the matrix F is defined as $Q(F,W) = \sqrt{\frac{\mathbf{E}|FW|^2}{\mathbf{E}|W|^2}}$. If there are no additional restrictions on the class of input vectors $W \in \mathbb{L}_2^m$, then the maximum value of $Q(F,W)$ is equal to the $\mathcal{H}_\infty$-norm (that is, the largest singular number) of the matrix F:

$$\sup_{W\in\mathbb{L}_2^m} Q(F,W) = \|F\|_\infty = \sqrt{\max_{i=\overline{1,m}}(\lambda_i(F^{\mathrm{T}}F))}.$$

On the other hand, if W belongs to the class $\mathbb{G}^m(0;\lambda I_m)$ of Gaussian random vectors with zero mean and scalar covariance matrix $\mathrm{cov}(W) = \lambda I_m$ with pdf $p_\lambda(w) = (2\pi\lambda)^{-m/2}\exp\left(-\frac{|w|^2}{2\lambda}\right)$, where

$\lambda > 0$, then the value of $Q(F,W)$ is equal to the scaled $\mathcal{H}_2$-norm (i.e., scaled Frobenius norm) of the matrix F:

$$Q(F,W) = \frac{\|F\|_2}{\sqrt{m}} = \sqrt{\frac{\mathrm{tr}(F^{\mathrm{T}}F)}{m}}, \quad \text{if } W \sim \mathbb{G}^m(0;\lambda I_m) \quad \forall\, \lambda > 0.$$

The first work on anisotropic theory for linear discrete *stationary* systems was the paper [13] (the conference version appeared earlier, in 1994 [12]). It was extended to linear discrete *nonstationary* systems in [18]. Within the framework of the stochastic approach underlying these works, it is assumed that random vectors W have absolutely continuous probability distributions (with pdf f) and finite second moments, and for them anisotropy is defined,

$$\mathbf{A}(W) = \min_{\lambda>0} \mathbf{D}(f\|p_\lambda) = \frac{m}{2}\ln\left(\frac{2\pi e}{m}\mathbf{E}|W|^2\right) - \mathbf{h}(W),$$

and it does not exceed a predefined threshold $a \geqslant 0$, where the minimum of the relative entropy $\mathbf{D}(f\|p_\lambda)$ (or Kullback–Leibler divergence) of the random vector W with pdf f with respect to a Gaussian random vector with pdf p_λ is achieved for $\lambda_* = \mathbf{E}|W|^2/m$. Here $\mathbf{h}(W) = -\mathbf{E}\,[\ln f(x)]$ is the differential entropy of the random vector W.

We additionally require that the following constraints hold for the first and second moments of the random vector W:

$$|\mathbf{E}W| \geqslant \tau, \quad \mathbf{E}(|W - \mathbf{E}W|^2) \leqslant \sigma, \tag{1}$$

where $\tau \geqslant 0$ and $\sigma > 0$ are known numbers. Using the scale invariance property for anisotropy, which asserts that $\mathbf{A}(rW) = \mathbf{A}(W)$ for any non-random number $r \neq 0$, without loss of generality we assume that $\tau^2 + \sigma = 1$. Thus, to define constraints (1) it suffices to know the value of $\tau \in [0;1)$, and the value of the second moment (a random vector scaled in a predefined way) will be bounded by the number $\sigma = 1 - \tau^2$.

The maximum value of the functional $Q(F,W)$ on the set of random vectors with anisotropy $\mathbf{A}(W)$ bounded by $a \geqslant 0$ and constraints (1) on the first and second moments with condition $\tau^2 + \sigma = 1$ is called the (a,τ)-anisotropic norm of matrix F. Below we use the notation

$$|\!|\!|F|\!|\!|_{a,\tau} = \sup_{W \in \mathbb{L}_2^m\colon (1)\,\wedge\,\mathbf{A}(W)\leqslant a} Q(F,W). \tag{2}$$

To work with the (a,τ)-anisotropic norm of a matrix F it is more convenient to use the following formula, equivalent to definition (2):

$$|\!|\!|F|\!|\!|_{a,\tau} = \sup_{\Sigma=\Sigma^T\succ 0}\left\{\sqrt{\mathrm{tr}(\Lambda\Sigma) + \|F\|_\infty^2\tau^2}\,:\ \mathrm{tr}\Sigma = 1-\tau^2,\ -\frac{1}{2}\ln(1-\tau^2) \leqslant -\frac{1}{2}\ln\det(m\Sigma) \leqslant a\right\},$$

where $\Lambda = F^{\mathrm{T}}F$.

The anisotropic norm of a matrix quantitatively reflects its ability to amplify random input vectors, whose distributions and statistical characteristics are known only up to constraints (1) on the first two moments and anisotropy constraint $\mathbf{A}(W) \leqslant a$. The latter constraint is a kind of compromise between two boundary cases: when it is known that the input vector is non-random (i.e., it has a probability distribution completely "concentrated" in the worst possible direction; this is the case of $\mathcal{H}_\infty$-theory) and when it is known that the input vector can with equal probability take values in any direction (i.e., when it has a standard normal distribution up to multiplication by a non-random number; this happens in $\mathcal{H}_2$-theory).

Remark 1. Below, the $(a,0)$-anisotropic norm will be referred to by the notation $|\!|\!|F|\!|\!|_a$, where condition $\tau = 0$ is omitted for brevity.

2.3. Calculation of the Anisotropic Norm of a Nonstationary System Represented in the State Space

Consider an object defined by a linear discrete nonstationary system F on the time interval $k = 0, \dots, N$:

$$F \sim \begin{cases} x_{k+1} = A_k x_k + B_k w_k \\ z_k = C_k x_k + D_k w_k \end{cases} \tag{3}$$

with initial condition $x_0 = 0$. Here $x_k \in \mathbb{R}^n$ is the state vector, $w_k \in \mathbb{R}^{m_w}$ is the vector of external disturbances, $z_k \in \mathbb{R}^{p_z}$ is the output vector. Real matrices A_k, B_k, C_k, D_k, which depend on the discrete time k, have dimensions consistent with the vectors. System (3) at every discrete time step k is defined by the four matrices $F_k = (A_k, B_k, C_k, D_k)$ and associated with the block lower triangular matrix

$$F_{0:N} = \underset{0 \leqslant i,j \leqslant N}{\text{block}} (f_{ij}) \in \mathbb{R}^{p_z(N+1) \times m_w(N+1)}$$

with block elements

$$f_{i,j} = \begin{cases} C_i T_{i,j+1} B_j, & \text{if} \quad i > j \\ D_j, & \text{if} \quad i = j \\ 0, & \text{if} \quad i < j, \end{cases}$$

where $T_{i,j} = A_{i-1} T_{i-1,j}$ and $T_{j,j} = I_n$.

The matrix $F_{0:N}$ defines input-output relations of system F in the form $Z_{0:N} = F_{0:N} W_{0:N}$, where for brevity of notation we have denoted $W_{0:N} = (w_0^{\mathrm{T}}, \dots, w_N^{\mathrm{T}})^{\mathrm{T}}$ and $Z_{0:N} = (z_0^{\mathrm{T}}, \dots, z_N^{\mathrm{T}})^{\mathrm{T}}$. According to the described equivalence in representations, by the norm of system F we mean the corresponding norm of the matrix $F_{0:N}$, in particular, $\|F\|_2 = \|F_{0:N}\|_2$, $|\!|\!|F|\!|\!|_{a,\tau} = |\!|\!|F_{0:N}|\!|\!|_{a,\tau}$, and $\|F\|_\infty = \|F_{0:N}\|_\infty$. To ensure that the strict inequality $\|F\|_2/\sqrt{m} < \|F\|_\infty$ holds for $m = m_w(N+1)$, we assume that inequality $p_z < m_w$ holds.

The following theorem provides formulas for computing the (a,τ)-anisotropic norm $|\!|\!|F|\!|\!|_{a,\tau}$ for a nonstationary system in the state space.

Theorem 1 [19]. *Suppose that the number $a \geqslant -\frac{m}{2}\ln(1-\tau^2)$, where $\tau \in [0;1)$, and system F with representation (3) in the state space are given. Then the (a,τ)-anisotropic norm of system F can be computed as $|\!|\!|F|\!|\!|_{a,\tau} = \mathcal{N}(\mathcal{A}^{-1}(a))$, where*

$$\mathcal{N}(q) = \left(\mathcal{N}_0(q)(1-\tau^2) + \|F\|_\infty^2 \tau^2\right)^{1/2}, \quad \mathcal{N}_0(q) = \frac{\Phi(q) - 1}{q\Phi(q)}, \tag{4}$$

$$\mathcal{A}(q) = \mathcal{A}_0(q) - \frac{m}{2}\ln(1-\tau^2), \quad \mathcal{A}_0(q) = \frac{m}{2}\left(\ln\Phi(q) - \Psi(q)\right), \tag{5}$$

$$\Phi(q) = \frac{1}{m}\sum_{k=0}^{N} \operatorname{tr}(L_k P_k L_k^{\mathrm{T}} + S_k), \quad \Psi(q) = \frac{1}{m}\sum_{k=0}^{N} \ln\det S_k. \tag{6}$$

Families of matrices $\{L_k\}_{k=0}^N$, $\{S_k\}_{k=0}^N$ are related to the solution $\{P_k\}_{k=0}^N$ of the Lyapunov difference equation

$$P_{k+1} = (A_k + B_k L_k) P_k (A_k + B_k L_k)^{\mathrm{T}} + B_k S_k B_k^{\mathrm{T}}, \quad P_0 = 0$$

and with the solution $\{Q_k\}_{k=0}^N$ of the Riccati difference equation

$$Q_k = A_k^{\mathrm{T}} Q_{k+1} A_k + q C_k^{\mathrm{T}} C_k + L_k^{\mathrm{T}} S_k^{-1} L_k, \quad Q_{N+1} = 0$$

by the following expressions:

$$S_k = \left(I_m - B_k^{\mathrm{T}} Q_{k+1} B_k - q D_k^{\mathrm{T}} D_k\right)^{-1},$$

$$L_k = S_k \left(B_k^{\mathrm{T}} Q_{k+1} A_k + q D_k^{\mathrm{T}} C_k\right),$$

where $q = \mathcal{A}^{-1}(a)$.

Remark 2. Since function $\mathcal{N}(q)$ depends on $\|F\|_\infty$, to the expressions given in the theorem it is formally necessary to add the formula for computing $\mathcal{H}_\infty$-norms of system F. And under the assumption that the anisotropic norm $|\!|\!|F|\!|\!|_a$ is bounded, which we will discuss below, we should add the condition that $\|F\|_\infty$ is bounded.

Remark 3. For $\tau = 0$ formulas (4)–(6) define a rule for computing the $(a, 0)$-anisotropic norm of system F, and here $\mathcal{N}(q) = \mathcal{N}_0(q)$, $\mathcal{A}(q) = \mathcal{A}_0(q)$ and $|\!|\!|F|\!|\!|_a = \mathcal{N}_0(\mathcal{A}_0^{-1}(a))$.

3. BOUNDEDNESS CRITERION FOR THE ANISOTROPIC NORM

In this section, we formulate a criterion for the boundedness of the anisotropic norm of a nonstationary system in terms of reverse time. Initially, the boundedness criterion for the anisotropic norm of a nonstationary system was formulated and proved by E.A. Maksimov, and a little later reformulated by Vladimirov in terms of forward time [21]. However, the results obtained by Maksimov were never published due to his untimely death, so we will briefly and without proof (with certain changes) recall them in this section. To begin with, we formulate an auxiliary assertion that follows from Theorem 1 and Remark 3.

Theorem 2 [24]. *Consider a linear discrete nonstationary system F with a model* (3) *in the state space. Suppose that numbers $\tau \in [0; 1)$, $a \geqslant -\frac{m}{2}\ln(1-\tau^2)$, and $\gamma > 0$ are given. The following statements are equivalent:*

- *the (a, τ)-anisotropic norm $|\!|\!|F|\!|\!|_{a,\tau}$ of system F is bounded by γ, i.e., $|\!|\!|F|\!|\!|_{a,\tau} \leqslant \gamma$;*
- *there exist numbers $\gamma_1 > 0$ and $\gamma_2 > 0$ such that $|\!|\!|F|\!|\!|_b \leqslant \gamma_1$, $\|F\|_\infty \leqslant \gamma_2$ and $\gamma_1^2(1-\tau^2) + \gamma_2^2\tau^2 \leqslant \gamma^2$, where $b = a + \frac{m}{2}\ln(1-\tau^2)$.*

It follows from this theorem, in particular, that the boundedness criterion for a nonstationary system with an uncentered disturbance is related to the boundedness criterion for the same system with a centered disturbance (or rather, with a disturbance satisfying condition $\tau = 0$ and an appropriately modified anisotropy constraint) and its extreme case, $\lim\limits_{a\to\infty} |\!|\!|F|\!|\!|_a = \|F\|_\infty$.

The theorem below gives a criterion for the boundedness of the anisotropic norm for a nonstationary system of the form (3) with a centered disturbance in terms of reverse time. Taking into account the notation $F_k = (A_k, B_k, C_k, D_k)$ for the four matrices of system (3), for brevity we introduce the function

$$\Phi(F_k, q, R) = A_k^{\mathrm{T}} R A_k + q C_k^{\mathrm{T}} C_k$$
$$+ \left(A_k^{\mathrm{T}} R B_k + q C_k^{\mathrm{T}} D_k\right)\left(I_m - B_k^{\mathrm{T}} R B_k - q D_k^{\mathrm{T}} D_k\right)^{-1}\left(B_k^{\mathrm{T}} R A_k + q D_k^{\mathrm{T}} C_k\right).$$

We note once again that the foundation for this statement belongs to Maksimov.

Theorem 3. *Consider a linear discrete nonstationary system F with a model* (3) *in the state space. Suppose that numbers $a \geqslant 0$ and $\gamma > 0$ are given. The following statements are equivalent:*

- *the a-anisotropic norm of system F is bounded by the number γ, i.e., $|\!|\!|F|\!|\!|_a \leqslant \gamma$;*

- *there exists a number $q \geqslant 0$ such that the Riccati difference equation $R_k = \Phi(F_k, q, R_{k+1})$, $R_{N+1} = 0$ has a symmetric positive definite solution $R_k = R_k^{\mathrm{T}} \succ 0$ satisfying inequalities*

$$S_k = I_m - B_k^{\mathrm{T}} R_{k+1} B_k - q D_k^{\mathrm{T}} D_k \succ 0,$$

$$\sum_{k=0}^{N} \ln\det S_k \geqslant m \ln(1 - q\gamma^2) + 2a.$$

Proof. The proof of this theorem includes a part of the results from [18, 21], which we do not duplicate here, but rather give the corresponding references. The proof formally consists of four parts. First, the work [21] presents conditions for the boundedness of the anisotropic norm for a nonstationary system in terms of *forward* time, but the course of the proof (up to formula (32)) remains valid in the case considered here. This, among other things, leads to the fact that the direct solution of the problem will be related to the matrix $\Sigma(q) = (I - qF_{0:N}^{\mathrm{T}} F_{0:N})^{-1}$ that characterizes the "worst" possible input disturbance. Second, the matrix $\Sigma(q) = (I - qF_{0:N}^{\mathrm{T}} F_{0:N})^{-1}$ can be factorized as $\Sigma(q) = G_{0:N} G_{0:N}^{\mathrm{T}}$, which leads to the matrix equation $qF_{0:N}^{\mathrm{T}} F_{0:N} + G_{0:N}^{-\mathrm{T}} G_{0:N}^{-1} = I$, which can be represented in a more compact form as

$$\Theta^{\mathrm{T}}\Theta = I, \quad \Theta = \begin{bmatrix} \sqrt{q} F_{0:N} \\ G_{0:N}^{-1} \end{bmatrix}.$$

The matrix Θ can be identified with a system with (A, B, C, D)-representation

$$\Theta \sim \left[\begin{array}{c|c} A_k & B_k \\ \hline \sqrt{q} C_k & \sqrt{q} D_k \\ -S_k^{-1/2} L_k & S_k^{-1/2} \end{array}\right].$$

Third, if we use Lemma 7 from the work [18], then to satisfy condition $\Theta^{\mathrm{T}}\Theta = I$ it suffices to require the existence of a solution of the Riccati equation $R_k = \Phi(F_k, q, R_{k+1})$, $R_{N+1} = 0$, and the auxiliary matrices L_k and S_k introduced above have the form

$$L_k = S_k^{-1}\left(B_k^{\mathrm{T}} R_{k+1} A_k + q D_k^{\mathrm{T}} C_k\right),$$

$$S_k = \left(I_m - B_k^{\mathrm{T}} R_{k+1} B_k - q D_k^{\mathrm{T}} D_k\right).$$

Finally, returning again to [21] (to the final part of the proof), we see that the boundedness condition for the anisotropy of the vector $W_{0:N}$ can be rewritten as

$$\sum_{k=0}^{N} \ln\det S_k \geqslant m \ln(1 - q\gamma^2) + 2a,$$

which completes the proof.

A similar result was given in [24, Theorem 3], but since it was solving the control problem its formulation was given in terms of solving Riccati equations in forward (and not reverse) time.

The following theorem establishes sufficient conditions for the boundedness of the anisotropic norm of a nonstationary system in terms of difference Riccati inequalities in reverse time.

Theorem 4. *Consider a linear discrete nonstationary system F with a model in the state space* (3)*. Suppose also that numbers $\tau \in [0; 1)$, $a \geqslant -\frac{m}{2}\ln(1 - \tau^2)$ and $\gamma > 0$ are given. The following statements are equivalent:*

- *the* (a,τ)*-anisotropic norm* $\|\!|F|\!\|_{a,\tau}$ *of system* F *is bounded by* γ*, i.e.,* $\|\!|F|\!\|_{a,\tau} \leqslant \gamma$*;*
- *there exist numbers* $\gamma_1 > 0$ *and* $\gamma_2 > 0$ *related by condition* $\gamma_1^2(1-\tau^2) + \gamma_2^2\tau^2 \leqslant \gamma^2$*, and a number* $q \geqslant 0$ *such that the Riccati difference inequalities* $P_k \succ \Phi(F_k, q, P_{k+1})$*,* $P_{N+1} = 0$ *and* $\widetilde{P}_k \succ \Phi(F_k, q, \widetilde{P}_{k+1})$*,* $\widetilde{P}_{N+1} = 0$ *have positive definite solutions* $P_k \succ 0$*,* $\widetilde{P}_k \succ 0$*, that satisfy inequalities*

$$Q_k = I_m - B_k^{\mathrm{T}} P_{k+1} B_k - q D_k^{\mathrm{T}} D_k \succ 0,$$
$$\sum_{k=0}^{N} \ln\det Q_k \geqslant m \ln(1-\tau^2) + 2b,$$

where $b = a + \frac{m}{2}\ln(1-\tau^2)$*.*

The proof of the theorem is based on well-known monotonicity properties of the solutions of Riccati difference equations.

The next step, standard within the framework of anisotropic theory, is the transition from exact formulas for calculating the anisotropic norm using the solution of the Riccati equations to its boundedness condition based on Riccati matrix inequalities obtained from the corresponding equations by virtue of their solutions' monotonicity. Schur's lemma is applied to the obtained inequalities, and as a result we get sufficient conditions for the boundedness of the anisotropic norm in terms of linear matrix inequalities. These calculations are rather cumbersome and therefore omitted, but their counterparts related to the solution of the control problem can be traced in [24]. Thus, sufficient conditions for the boundedness of the anisotropic norm of a nonstationary system have the following form.

Theorem 5. *Consider a linear discrete nonstationary system* F *with a model* (3) *in the state space. Suppose also that we are given* $b = a + \frac{m}{2}\ln(1-\tau^2) \geqslant 0$ *and* $\gamma > 0$*. The anisotropic norm of the system satisfies the constraint* $\|\!|F|\!\|_b \leqslant \gamma$ *if there exist a number* $\eta \geqslant 0$ *and such positive definite matrices* M_k *and* Ψ_k *that linear matrix inequalities*

$$\begin{bmatrix} M_k & * & * & * \\ 0 & \eta I_m & * & * \\ M_{k+1}A_k & M_{k+1}B_k & M_{k+1} & * \\ C_k & D_k & 0 & I_p \end{bmatrix} \succ 0 \quad \text{for } k < N, \qquad \begin{bmatrix} M_N & * & * \\ 0 & \eta I_m & * \\ C_N & D_N & I_p \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} \eta I_m - \Psi_k & * & * \\ M_{k+1}B_k & M_{k+1} & * \\ D_k & 0 & I_p \end{bmatrix} \succ 0 \quad \text{for } k < N, \qquad \begin{bmatrix} \eta I_m - \Psi_N & * \\ D_N & I_p \end{bmatrix} \succ 0$$

are feasible, and their solutions additionally satisfy the condition

$$\exp\left(-\frac{2b}{m}\right)(\det \Psi_k)^{1/m_w} \geqslant \eta - \gamma^2.$$

Based on the theorem stated above, we find the coefficients of a suboptimal anisotropic estimator for a nonstationary system.

4. STATEMENT AND SOLUTION OF THE FILTERING PROBLEM

The formulation and solution of the suboptimal anisotropic filtering problem on finite horizon in forward time for the case $\tau = 0$ was presented with proofs, separate consideration of limit cases, and analysis of a numerical example in [23].

Consider a linear discrete nonstationary system F, defined on a finite time interval $k = 0, \dots, N$, of the following form:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k w_k, \\ y_k &= C_{y,k} x_k + D_{y,k} w_k, \\ z_k &= C_{z,k} x_k + D_{z,k} w_k, \end{aligned} \tag{7}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $w_k \in \mathbb{R}^{m_w}$ is the vector of external disturbance, $y_k \in \mathbb{R}^{p_y}$ is the measured output, $z_k \in \mathbb{R}^{p_z}$ is the estimated output. Real matrices A_k, B_k, $C_{y,k}$, $D_{y,k}$, $C_{z,k}$, $D_{z,k}$ have dimensions that are consistent with the vectors. A constraint is imposed on the set of external disturbance signals in the form of the anisotropy level $\mathbf{A}(W) \leqslant a$, and a number $\tau \in (0;1)$ is given.

We will look for the vector estimator system z_k in the form

$$\begin{aligned} \widehat{x}_{k+1} &= (A_k - K_k C_{y,k})\widehat{x}_k + K_k y_k, \\ \widehat{z}_k &= (C_{z_k} - L_k C_{y,k})\widehat{x}_k + L_k y_k, \end{aligned} \tag{8}$$

where $\widehat{x}_k$ is the state of the estimator, and K_k and L_k are matrices to be found.

System $T_{\widetilde{z}w}$, which connects on the time interval $k = 0, \dots, N$ vectors w_k of the external disturbances of system (7) and error vectors $\widetilde{z}_k = z_k - \widehat{z}_k$, has the form

$$\begin{aligned} \widetilde{x}_{k+1} &= (A_k - K_k C_{y,k})\widetilde{x}_k + (B_k - K_k D_{y,k}) w_k, \\ \widetilde{z}_k &= (C_{z,k} - L_k C_{y,k})\widetilde{x}_k + (D_{z,k} - L_k D_{y,k}) w_k, \end{aligned}$$

where $\widetilde{x}_k = x_k - \widehat{x}_k$ is the state error vector.

Problem. For a given system F with model (7) in the state space, number $\tau \in (0;1)$, and anisotropy level $a \geqslant -\frac{m}{2}\ln(1-\tau^2)$, find the parameters of the estimator with model (8) in the state space that minimizes the value of γ, where

$$|\!|\!| T_{\widetilde{z}w} |\!|\!|_{a,\tau} \leqslant \gamma.$$

Let us apply to system $T_{\widetilde{z}w}$ the statement of Theorem 5. Then, in order for the anisotropic norm of the system to be bounded the errors should contain the number $\eta \geqslant 0$ and positive definite matrices M_k and Ψ_k that satisfy linear matrix inequalities

$$\begin{bmatrix} M_k & * & * & * \\ 0 & \eta I_m & * & * \\ M_{k+1}(A_k - K_k C_{y,k}) & M_{k+1}(B_k - K_k D_{y,k}) & M_{k+1} & * \\ C_{z,k} - L_k C_{y,k} & D_{z,k} - L_k D_{y,k} & 0 & I_p \end{bmatrix} \succ 0 \quad \text{for } k < N,$$

$$\begin{bmatrix} M_N & * & * \\ 0 & \eta I_m & * \\ C_{z,N} - L_N C_{y,N} & D_{z,N} - L_N D_{y,N} & I_p \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} M_k & * & * & * \\ 0 & \gamma_2^2 I_m & * & * \\ M_{k+1}(A_k - K_k C_{y,k}) & M_{k+1}(B_k - K_k D_{y,k}) & M_{k+1} & * \\ C_{z,k} - L_k C_{y,k} & D_{z,k} - L_k D_{y,k} & 0 & I_p \end{bmatrix} \succ 0 \quad \text{for } k < N,$$

$$\begin{bmatrix} M_N & * & * \\ 0 & \gamma_2^2 I_m & * \\ C_{z,N} - L_N C_{y,N} & D_{z,N} - L_N D_{y,N} & I_p \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} \eta I_m - \Psi_k & * & * \\ M_{k+1}(B_k - K_k D_{y,k}) & M_{k+1} & * \\ (D_{z,k} - L_k D_{y,k}) & 0 & I_p \end{bmatrix} \succ 0 \quad \text{for } k < N,$$

$$\begin{bmatrix} \eta I_m - \Psi_N & * \\ D_{z,N} - L_N D_{y,N} & I_p \end{bmatrix} \succ 0,$$

and their solutions additionally satisfy the condition

$$\gamma_1^2(1-\tau^2) + \gamma_2^2\tau^2 \leqslant \gamma^2, \quad \exp\left(-\frac{2b}{m}\right)(\det \Psi_k)^{1/m_w} \geqslant \eta - \gamma_1^2.$$

We substitute $M_{k+1}K_k = Y_{k+1}$, and as a result we obtain the following convex optimization problem, according to whose solution estimators K_k and L_k are defined:

$$\gamma^2 \to \min_{M_k, Y_k, \Psi_k, \eta, \gamma_1, \gamma_2}$$

under constraints

$$M_k = M_k^{\mathrm{T}} \succ 0, \quad \Psi_k = \Psi_k^{\mathrm{T}} \succ 0, \quad \eta \geqslant 0,$$

$$\begin{bmatrix} M_k & * & * & * \\ 0 & \xi I_{m_w} & * & * \\ M_{k+1}A_k - Y_{k+1}C_{y,k} & M_{k+1}B_k - Y_{k+1}D_{y,k} & M_{k+1} & * \\ C_{z,k} - L_k C_{y,k} & D_{z,k} - L_k D_{y,k} & 0 & I_{p_z} \end{bmatrix} \succ 0 \quad \text{for } k < N,$$

$$\begin{bmatrix} M_N & * & * \\ 0 & \xi I_{m_w} & * \\ C_{z,N} - L_N C_{y,N} & D_{z,N} - L_N D_{y,N} & I_{p_z} \end{bmatrix} \succ 0,$$

where in the last two families of matrix inequalities, substitutions $\xi = \eta$ and $\xi = \gamma_2^2$ should be made,

$$\begin{bmatrix} \eta I_{m_w} - \Psi_k & * & * \\ M_{k+1}B_k - Y_{k+1}D_{y,k} & M_{k+1} & 0 \\ D_{z,k} - L_k D_{y,k} & 0 & I_{p_z} \end{bmatrix} \succ 0 \quad \text{for } k < N,$$

$$\begin{bmatrix} \eta I_{m_w} - \Psi_N & * \\ D_{z,N} - L_N D_{y,N} & I_{p_z} \end{bmatrix} \succ 0,$$

$$\gamma_1^2(1-\tau^2) + \gamma_2^2\tau^2 \leqslant \gamma^2, \quad \exp\left(-\tfrac{2b}{m}\right)(\det \Psi_k)^{1/m_w} \geqslant \eta - \gamma_1^2.$$

If a solution for this problem has been found, then matrices K_k of the estimator can be found as $K_k = M_{k+1}^{-1} Y_{k+1}$.

5. EXAMPLE

As an example, consider the system

$$x_{1,k+1} = -0.3x_{2,k} + \sin(3k)w_k,$$
$$x_{2,k+1} = 0.2(1+\sin(3k))x_{1,k} - 0.3x_{2,k} - 0.03w_k,$$
$$z_k = 0.5x_{1,k} + 0.5\sin(3k)x_{2,k},$$
$$y_{1,k} = (-2+0.3\sin(5k))x_{1,k} + 0.5x_{2,k} + 0.1\sin(3k)w_k,$$
$$y_{2,k} = x_{2,k} + 0.2w_k$$

with initial state $x_1(0) = 0.26$, $x_2(0) = -0.2$.

We have considered the following data for the calculation: time interval $T = 10$ s, sampling step $\Delta t = 0.1$ s, disturbance anisotropy values for $\tau = 0.03$ are chosen to be equal to $a = 0.05$, $a = 10$,

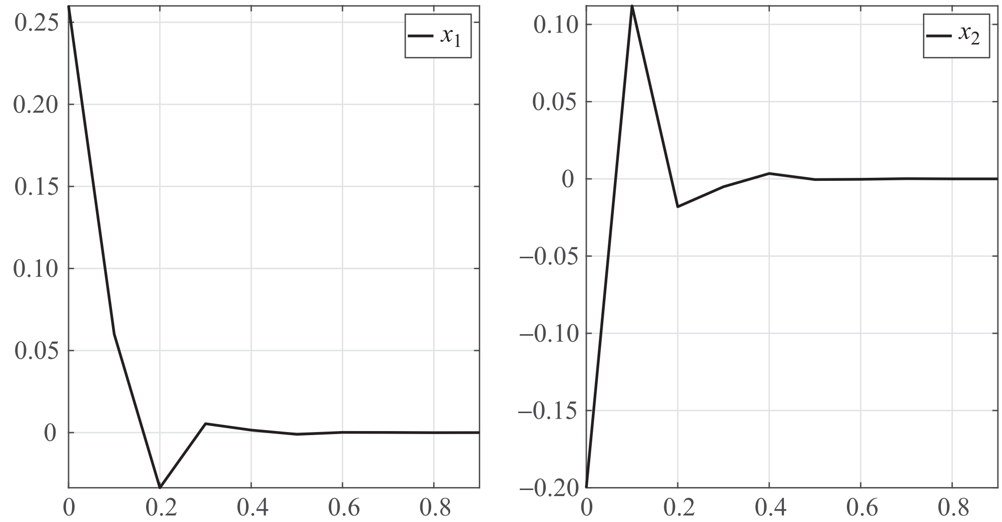

Fig. 1. Dynamics of variables x_1 and x_2 in the absence of external disturbance (only the first second of time is represented).

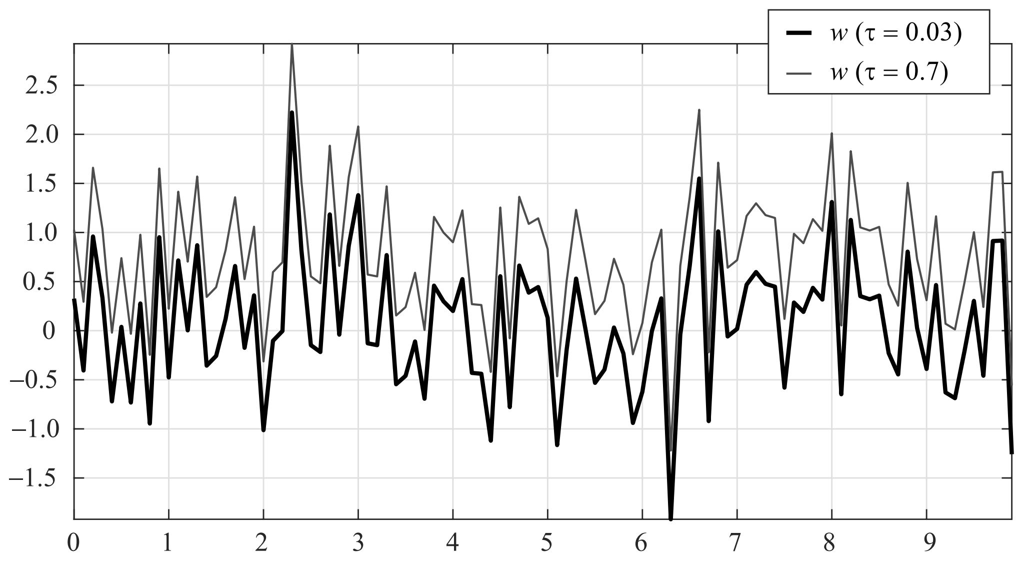

Fig. 2. External disturbance w. Cases $\tau = 0.03$, $a = 10$ and $\tau = 0.7$, $a = 40$.

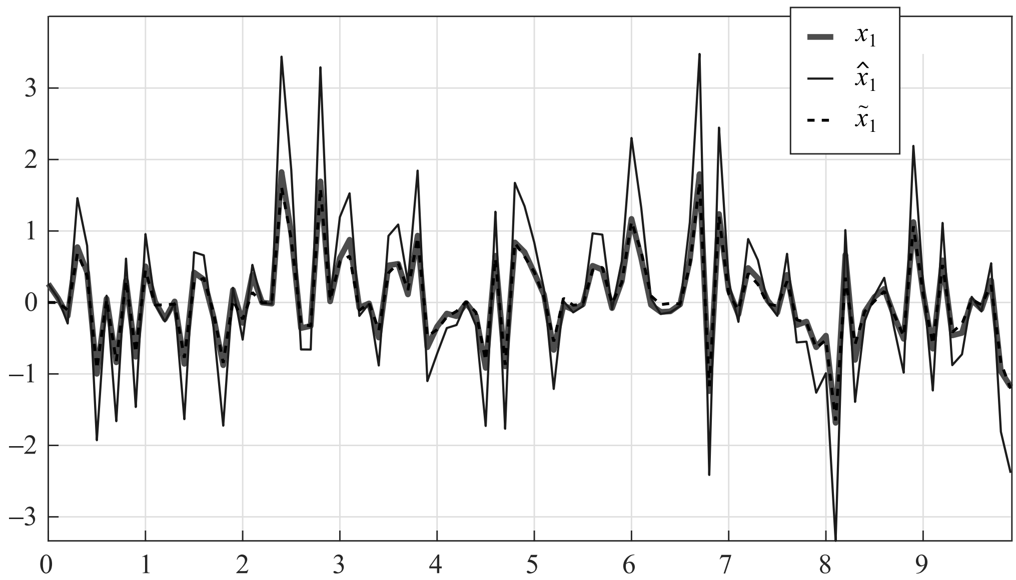

Fig. 3. Dynamics of variable x_1. Cases $\tau = 0.03$, $a = 10$.

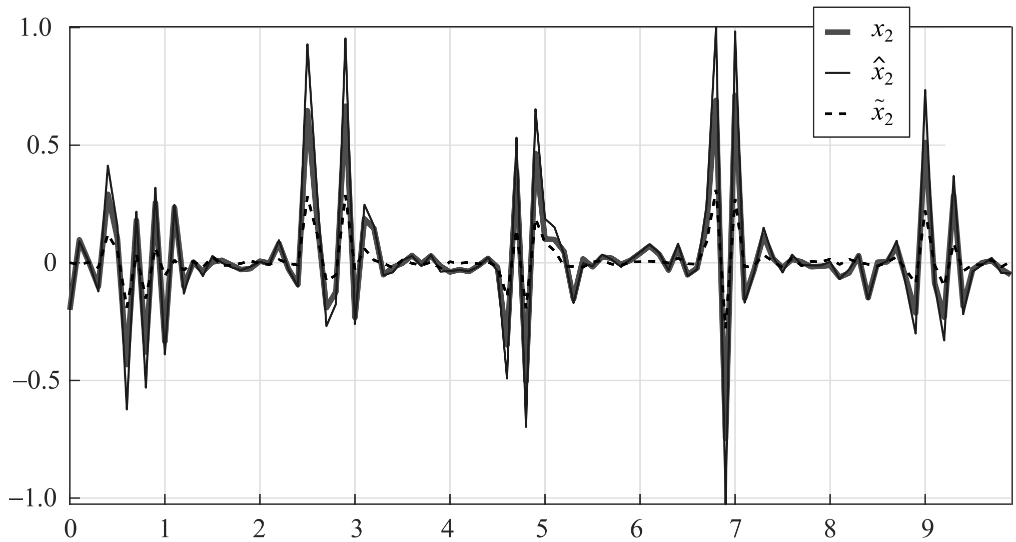

Fig. 4. Dynamics of variable x_2. Cases $\tau = 0.03$, $a = 10$.

and $a = 100$ respectively, disturbance anisotropy values for $\tau = 0.7$ are chosen as $a = 35$, $a = 40$, and $a = 100$ respectively. Computations were made in the MATLAB programming environment using the Yalmip and Sedumi packages [25, 26].

Figure 1 shows system variables x_1 and x_2 (in the first second of time) in the absence of external disturbances. Figure 2 shows the disturbances w used in the simulation for the cases $\tau = 0.03$, $a = 10$ and $\tau = 0.7$, $a = 40$. Figure 3 shows the variable x_1, its estimate and estimation error; Fig. 4, variable x_2, its estimate and estimation error; Fig. 5, output z, as well as its estimate and estimation error. All plots are shown for case $\tau = 0.03$, $a = 10$. Figures 6–8 show similar variables for the case $\tau = 0.7$, $a = 40$. It follows from the figures that the anisotropic estimator adequately estimates the system output both in the presence of (almost) centered disturbance and in the case of a non-centered one.

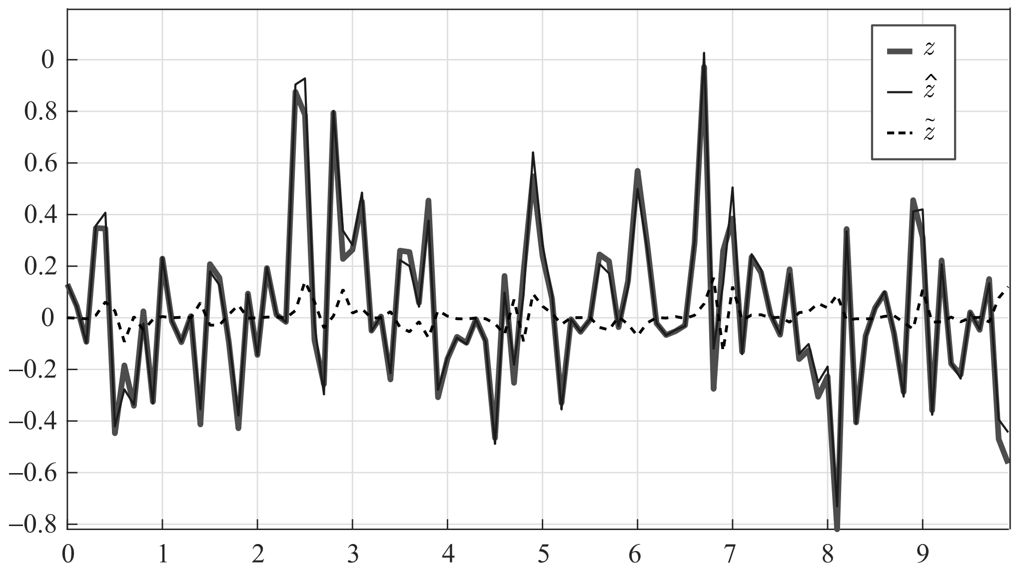

Fig. 5. Dynamics of variable z. Cases $\tau = 0.03$, $a = 10$.

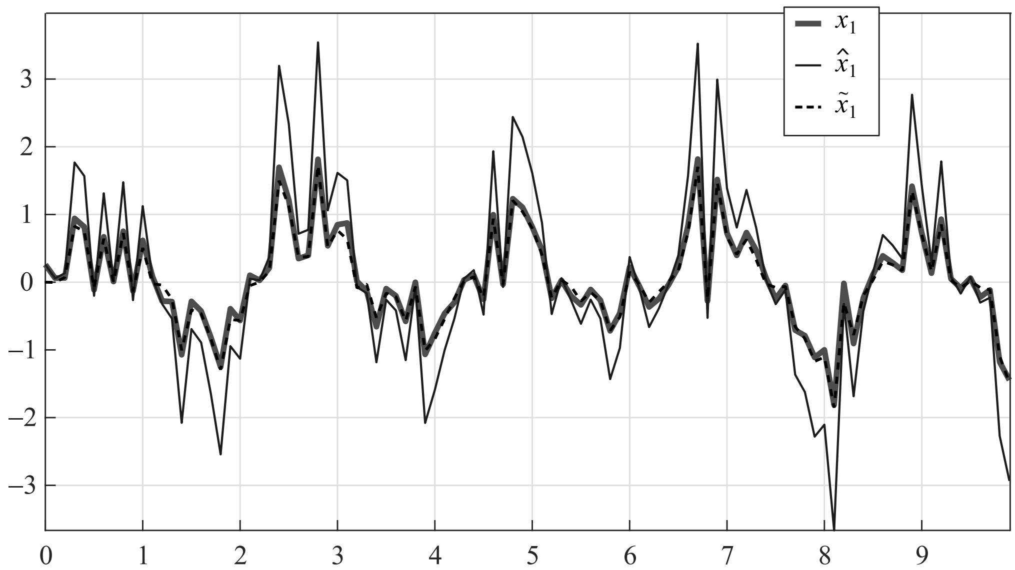

Fig. 6. Dynamics of variable x_1. Cases $\tau = 0.7$, $a = 40$.

To compare the values of the anisotropic norm of the system for different cases with its bounds ($\|T_{\tilde{z}w}\|_2/\sqrt{m} = 0.0511$, $\|T_{\tilde{z}w}\|_\infty = 0.0970$), we present a table that clearly shows that the anisotropic norm has intermediate values between the scaled $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms.

Anisotropic norm values for different pairs (a, τ)

τ	0.03			07		
a	0.05	10	100	35	40	100
$\|T_{\tilde{z}w}\|_{a,\tau}$	0.0513	0.0683	0.0931	0.0791	0.0818	0.0930

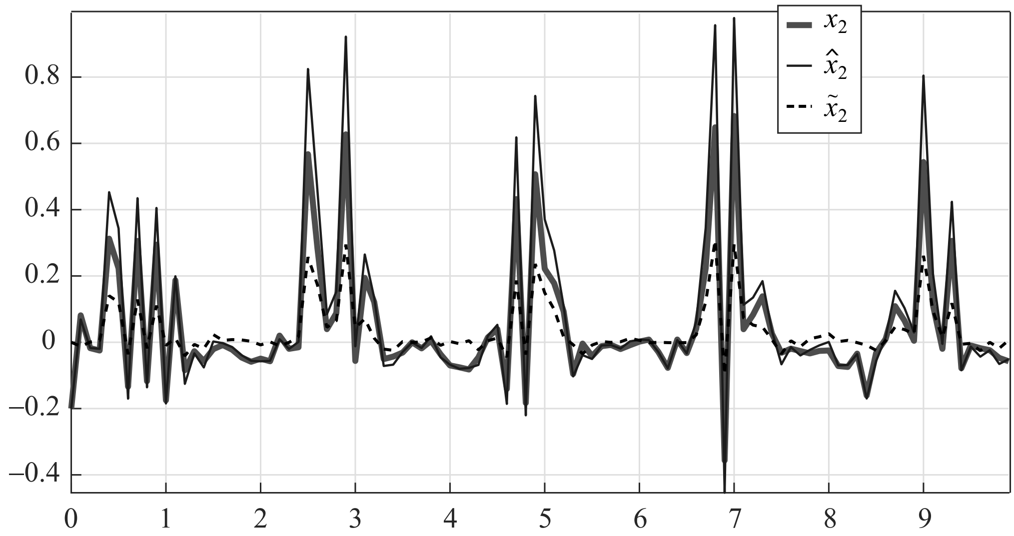

Fig. 7. Dynamics of variable x_2. Cases $\tau = 0.7$, $a = 40$.

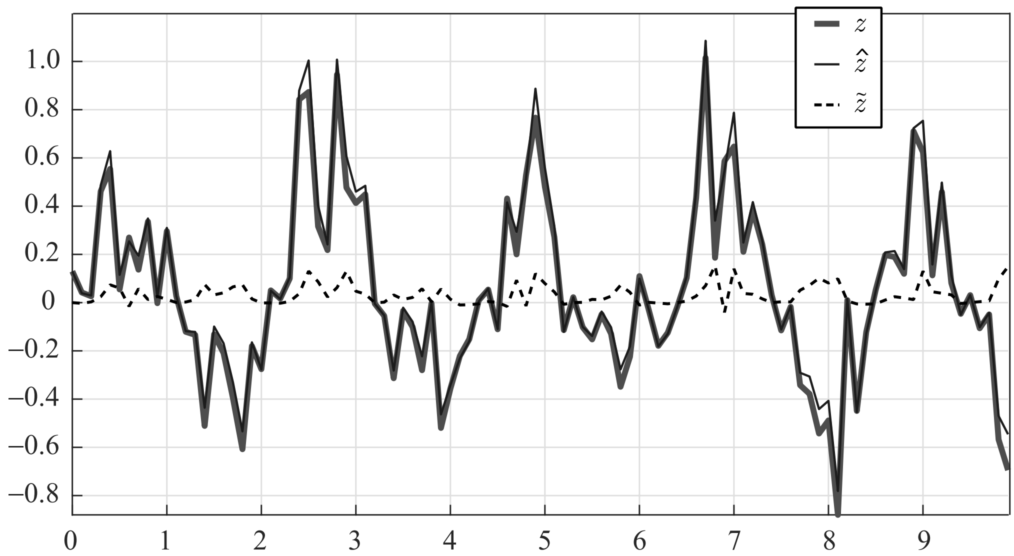

Fig. 8. Dynamics of variable z. Cases $\tau = 0.7$, $a = 40$.

6. CONCLUSION

In this work, we have briefly summarized the results concerning the calculation and boundedness criterion for the anisotropic norm of a nonstationary system. Based on these results, we have presented a solution for the problem of synthesizing a suboptimal anisotropic estimator in reverse time. We have shown that the solution of this problem corresponds to the solution of a certain convex optimization problem. To demonstrate the quality of the estimator for a simple system, we have presented a numerical example.

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