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A robust IDA-PBC approach for handling uncertainties in underactuated mechanical systems

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***Abstract*—Interconnection and damping assignment passivity based control (IDA-PBC) is a method that has been developed to (asymptotically) stabilize nonlinear systems formulated in port-controlled Hamiltonian (PCH) structure. This method has gained increasing popularity and has been successfully applied to a wide range of dynamical systems. However, little is known about the robustness of this method in response to the effects of uncertainty which could result from disturbances, noises, and modeling errors. This paper explores the possibility of extending some energy shaping methods, taking into account the robustness aspects, with the aim of maintaining (asymptotic) stability of the system in the presence of perturbations which inevitably exist in any realistic applications. We propose constructive results on robust IDA-PBC controllers for underactuated mechanical systems that are quite commonly found in practice and have the most challenging control problems within this context. The proposed results extend some existing methods and provide a new framework that allows the implementation of integral and input-to-state stability controllers to underactuated mechanical systems. The results are validated on two physical systems: an inertia wheel pendulum and a rotary inverted pendulum that represent separable and nonseparable PCH systems, respectively.**

***Index Terms*—Hamiltonian systems, nonlinear systems, passivity-based control, robust control, input-to-state stability, underactuated systems.**

I. INTRODUCTION

Control design methods for systems described by port-controlled Hamiltonian (PCH) model have been developed in several works (see [1] and references therein). Adopting the PCH structure that geometrically describes a large class of nonlinear models gives a number of advantages such as the obvious relation between the dynamics and the energy of the system, the energy conservative property that makes the model marginally stable to start with, and the coupling between the non-damping and the damping elements. However, this modeling approach results in exclusion of important ingredients of the system's dynamics such as the frictions. Hence, relying only on the pure PCH model, often results in a controller that works very well in simulation, but needs further adjustment in implementation [1], [2].

Besides the issue of modeling, complexity of systems, nonlinearities, and demand for control accuracy have made control design problems more challenging [3]. System's perturbations such as measurement noise, disturbances and model uncertainties are common problems that affect the performance of the control systems in real applications. This motivates the establishment of the robust control paradigm, with the adaptive and integral control among the main approaches. Broadly speaking, the integral action control is the most popular approach to deal with such effects, that so far has kept the dominance of PID controller in practice.

The interconnection and damping assignment passivity-based control (IDA-PBC) [4] is a physically inspired control design method that invokes the principles of *energy shaping* and *dissipation*, formulated for systems described by PCH models. The main objective of this method is to stabilize the dynamical system by rendering its closed-loop *passive* (by shaping its energy) with a desired storage function (which is a proper Lyapunov function) [5]. Furthermore, the system can be asymptotically stabilized if it can be rendered strictly (output) passive by means of damping injection [6]. While IDA-PBC controller is theoretically proven to asymptotically stabilize classes of PCH systems; in real applications, the effect of disturbances, uncertainties or reference signal may deteriorate the performance of the control system [7], and the closed-loop system is more likely to suffer from steady-state errors or even instability. Apparently, when it comes to parametric uncertainties, the real-time implementation of control system requires a real-time and reasonably accurate estimate of these uncertainties. Thus, the main objective of this paper is to investigate the robust stabilization of *perturbed* PCH systems to encounter the effects of system's uncertainties.

A solution to deal with robustness issue of PCH systems has been recently reported in [7], while the problem of robustification of IDA-PBC for *fully-actuated* mechanical systems has been recently addressed in [8]. However, it is well recognized that *underactuated* mechanical systems represent the most challenging class of PCH systems. This is due to restrictions on control authority on all degrees of freedom, and restrictions on the extensions on the PCH structure. This implies that any extensions in the system coordinates, such as adding integral action, must preserve the PCH structure matrix as well as, preserve the passivity and (asymptotic) stability of the closed-loop system. In our earlier work [9], we proposed a novel framework to incorporate integral control for underactuated mechanical system. The design, though dedicated for separable class of mechanical systems where the mass matrix is constant, was the first toward solving the problem of robustification of IDA-PBC for this class of systems.

This paper provides some extensions to results presented in [9] and proposed some novel results in robust IDA-PBC that extend [10]. First, we propose an integral control for non-separable underactuated systems, which are the most complicated class due to non-constant mass matrix, while actually represent the largest set of underactuated systems [11]. Then, we address the problem of matched and unmatched disturbances for both separable and nonseparable systems.

The rest of the paper is organized as follows. In Section II, a brief review on IDA-PBC design is presented. Section III introduces the problem under consideration. The first main result, namely the integral control of underactuated mechanical systems is discussed in Section IV. The robustness of underactuated mechanical systems under the presence of matched and unmatched time-varying disturbances is discussed in Section V Finally, the results are validated using two interesting application examples; an inertia wheel pendulum which is a separable underactuated mechanical systems in Section VI, and the rotary inverted pendulum, which is a nonseparable underactuated mechanical systems in Section VII. The paper is then concluded in Section VIII where final comments and directions for future research are provided.

II. PRELIMINARIES

The set of real and natural numbers (including 0) are denoted respectively by $\mathbb{R}$ and $\mathbb{N}$. Given an arbitrary matrix G, we denote the transpose of G by $G^\top$. $G^\perp$ denotes the full rank left annihilator

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of G, i.e. $G^{\perp}G = 0$. We denote an $n \times n$ identity matrix with I_n. For a vector $x \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, we denote the Euclidean norm as $|x|$ and $|A|$, respectively, where $|x|^2 = x^\top x$. Furthermore, the weighted norm is denoted as $\|x\|_A := x^\top A x$. A vector e_i is a unit vector with unity i^{th} element, or a basis vector for the Euclidean space. For any continuous function $H(i,j)$, the gradient is $\nabla_i H(i,j) := \partial H(i,j)/\partial i$. We use Young's inequality $\zeta\eta|y||z| \leq \frac{\zeta^2}{2}|y|^2 + \frac{\eta^2}{2}|z|^2$ with positive constants ζ and η. We use a standard stability and passivity definitions for nonlinear systems [3]. Due to space limit, the arguments of functions are often dropped whenever they are clear from the context.

A. Port-Controlled Hamiltonian Systems

Consider a standard mechanical system whose dynamics are represented in a Port-Controlled Hamiltonian (PCH) form:

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & I_n\\ -I_n & 0\end{bmatrix}\begin{bmatrix}\nabla_q H\\ \nabla_p H\end{bmatrix} + \begin{bmatrix}0\\ G(q)\end{bmatrix}u, \qquad y = G^\top(q)\nabla_p H \tag{1}$$

where $q \in \mathbb{R}^n,\ p \in \mathbb{R}^n$ are the states, u and $y \in \mathbb{R}^m$, $m \leq n$, are the input and output variables, respectively. If $m = n$ the system is called *fully-actuated*, whereas if $m < n$ it is called *underactuated*. The Hamiltonian function, which is the total energy of the system, is defined as the sum of the kinetic energy and the potential energy

$$H(q,p) = K(q,p) + V(q) = \frac{1}{2}p^\top M^{-1}(q)p + V(q), \tag{2}$$

where $M(q) > 0$ is the symmetric inertia matrix and $V(q)$ is the potential energy function. The PCH system is called *separable* if M is constant, or otherwise it is called *non-separable*.

B. Review on IDA-PBC Design

We briefly review the general procedure of the IDA-PBC design as has been proposed for instance in [1], [4], [12]. Given a PCH system (1), IDA-PBC design procedure consists of two parts, which correspond to its design steps; the energy shaping and the damping injection

Energy shaping

The main objective of IDA-PBC is to stabilize the PCH system by state-feedback controller. This is achieved by replacing the interconnection matrix and the energy function (Hamiltonian) of the system with a *desired* ones while preserving the PCH form of the total system in closed-loop. This can be mathematically expressed as

$$\begin{bmatrix}0 & I_n\\ -I_n & 0\end{bmatrix}\begin{bmatrix}\nabla_q H\\ \nabla_p H\end{bmatrix} + \begin{bmatrix}0\\ G(q)\end{bmatrix}u_{es} = \begin{bmatrix}0 & M^{-1}M_d\\ -M_dM^{-1} & J_2(q,p)\end{bmatrix}\begin{bmatrix}\nabla_q H_d\\ \nabla_p H_d\end{bmatrix}. \tag{3}$$

The desired total energy in closed-loop is assigned to be

$$H_d(q,p) = K_d(q,p) + V_d(q) = \frac{1}{2}p^\top M_d^{-1}(q)p + V_d(q), \tag{4}$$

with $M_d = M_d^\top > 0$ the desired inertia matrix and $V_d(q)$ the desired potential energy, such that H_d has an isolated minimum at the desired equilibrium point q_e, i.e.

$$q_e = \arg\min H_d(q) = \arg\min V_d(q). \tag{5}$$

The following conditions are required so that (5) holds:

Condition 2.1: Necessary extremum assignment: $\nabla_q V_d(q_e) = 0$.

Condition 2.2: Sufficient minimum assignment: $\nabla_q^2 V_d(q_e) > 0$, i.e. the Hessian of the function at the equilibrium point is positive.

Equation (3) constitutes the matching conditions of the IDA-PBC method [4], which is a set of PDEs in the form of

$$G^{\perp}\{\nabla_q H - M_dM^{-1}\nabla_q H_d + J_2M_d^{-1}p\} = 0, \tag{6}$$

with $J_2 = -J_2^\top$ a free parameter. PDEs (6) can be separated into two elements; kinetic energy PDEs (dependent on p) and potential energy PDEs (independent of p), respectively:

$$G^{\perp}\{\nabla_q(p^\top M^{-1}p) - M_dM^{-1}\nabla_q(p^\top M_d^{-1}p) + 2J_2M_d^{-1}p\} = 0 \qquad G^{\perp}\{\nabla_q V - M_dM^{-1}\nabla_q V_d\} = 0. \tag{7}$$

If these sets of PDEs (7) are solved, or in other words M_d, V_d and J_2 are obtained, then u_{es} is given by

$$u_{es} = G^{\ddagger}\left(\nabla_q H - M_dM^{-1}\nabla_q H_d + J_2M_d^{-1}p\right), \tag{8}$$

with

$$G^{\ddagger} = (G^\top G)^{-1}G^\top. \tag{9}$$

Note that this $G^{\ddagger}$ is not a pseudo-inverse of G.

Damping injection

The next task after finding u_{es} is to find the damping injection (dissipation) controller, which is

$$u_{di} = -K_vG^\top\nabla_p H_d, \tag{10}$$

with $K_v = K_v^\top > 0$ is the damping gain, to add the damping to the closed-loop system that ensures asymptotic stabilization to the desired equilibrium. u_{di} is applied via a negative feedback of the passive output to achieve asymptotic stability, provided that the system is *zero-state detectable*. The system (1) is called *zero-state observable* if $u(t) = y(t) = 0,\ \forall\, t \geq 0 \implies (q(t),p(t)) = (q_e, 0)$. It is *zero-state detectable* if $u(t) = y(t) = 0,\quad \forall t \geq 0 \implies \lim_{t\to\infty}(q(t),p(t)) = (q_e,0)$. Thus, the total IDA-PBC controller is

$$u_{ida} = u_{es} + u_{di}. \tag{11}$$

Given a PCH system (1), by applying the controller (11) we obtain the following preserved PCH dynamics

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & M^{-1}M_d\\ -M_dM^{-1} & J_2 - R_d\end{bmatrix}\begin{bmatrix}\nabla_q H_d\\ \nabla_p H_d\end{bmatrix}, \qquad y_d = G^\top(q)\nabla_p H_d, \tag{12}$$

where $R_d = GK_vG^\top > 0$ is the damping matrix.

III. Problem Formulation

While the controller (11) guarantees (asymptotic) stability of the equilibrium $(q_e, 0)$, it might not satisfy some performance criteria. This work addresses the problem of finding a dynamic controller v, that enters the system at the same port as the controller (11), to eliminate steady-state errors and/or reject disturbances. Hence, the total closed-loop control law is $u = u_{ida} + v$. For the PCH formulation, the inclusion of the integral control (IC) depends on how the inputs act on the states [7]. We call the states that receive direct action from the input as *passive* outputs, or *non-passive* otherwise.

The idea of applying IC on the *passive* outputs of PCH systems has been proposed in [1]. The application of IC for fully-actuated systems has been shown in [8]. In [9], the authors were the first to propose a framework toward introducing the IC for underactuated mechanical systems, which is the most challenging class of systems. A novel method that allows the implementation of integral control to (separable) underactuated mechanical systems within PCH framework using IDA-PBC has been presented in that earlier work. In order to develop the result, the following assumption is used.

Assumption 3.1: We assume a stabilizing IDA-PBC controller (11) has been obtained for the underactuated PCH system (1), i.e. the

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system (1) is (*asymptotically*) stable with the state feedback controller (11), and the desired (closed-loop) energy function is given by (4). The asymptotic stability proof is established by calculating the time derivative of (4) along trajectories of (12), which satisfies

$$\dot{H}_d \leq -\lambda_{min}\{K_v\}|G^\top M_d^{-1}x_p|^2 \leq 0. \tag{13}$$

Thus, asymptotic stability is concluded using the arguments used in the proof of [13, Proposition 1] and [14, Proposition 1] by applying the detectability condition and invoking Barbashin-Krasovskii's theorem [6]. This also implies that the partial differential equations (PDEs), also called the matching equations, have already been solved and we are interested in incorporating the dynamic controller v as an extra term to the IDA-PBC controller, to eliminate steady-state errors and/or reject external disturbances. Therefore, extending the system (12), we introduce the dynamics

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & M^{-1}M_d\\ -M_dM^{-1} & J_2 - R_d\end{bmatrix}\begin{bmatrix}\nabla_q H_d\\ \nabla_p H_d\end{bmatrix} + \begin{bmatrix}d_1\\ d_2\end{bmatrix} + \begin{bmatrix}0\\ G\end{bmatrix}v. \tag{14}$$

where $d_1 \in \mathbb{R}^n$, $d_2 \in \mathbb{R}^n$, are the unmatched and matched time-varying bounded disturbance, respectively. First, we present the design of an integral control law v without disturbances i.e. $d_1 = d_2 = 0$, then we extend the results to deal with disturbances adopting the ISS formalism.

In this paper we present our results for general underactuated mechanical systems, extending the results in [9] which dealt with the separable mechanical systems. As we explained in Subsection II-A, for non-separable systems both the original inertia matrix M and the desired inertia matrix M_d are non-constant, but functions of q instead. Thus, their derivatives are non zero and need to be taken into account in the construction of the control law. As expected, more complicated control laws are obtained as a consequence. In the next sections, we first introduce the integral control for underactuated mechanical systems, and then we consider the effect of disturbances on both separable and non-separable systems.

IV. Integral Control for underactuated Mechanical Systems using IDA-PBC

This section presents the construction of a stabilizing *integral* (dynamic) control law for both separable and non-separable underactuated mechanical systems.

A. Integral control

Proposition 4.1: Consider the PCH dynamics (14), with $d_1 = d_2 = 0$ and G constant, satisfying Assumption 3.1, in closed-loop with the integral control

$$\begin{aligned}
v &= G^\ddagger\Big(\frac{M_dM^{-1}}{2}\sum_{i=1}^{n} e_i p^\top \nabla_{q_i} M_d^{-1} p + (J_2 - R_d)M_d^{-1}\mathcal{K}x_v\\
&\quad - \mathcal{K}\dot{x}_v - \frac{M_dM^{-1}}{2}\sum_{i=1}^{n} e_i (p+\mathcal{K}x_v)^\top \nabla_{x_{q_i}} M_d^{-1}(p+\mathcal{K}x_v)\Big)\\
\dot{x}_v &= (M^{-1}\mathcal{K})^\top \nabla_{x_q}\tilde{H} \qquad (15)\\
\nabla_{x_q}\tilde{H} &= \nabla_{x_q}\tilde{V} + \frac{1}{2}\sum_{i=1}^{n} e_i x_p^\top \nabla_{x_{q_i}} M_d^{-1}x_p,
\end{aligned}$$

with the desired Hamiltonian function

$$\tilde{H} = \frac{1}{2}x_p^\top M_d^{-1}(x_q)x_p + \frac{1}{2}x_v^\top x_v + \tilde{V}(x_q), \tag{16}$$

such that the total control input takes the form

$$u = u_{ida} + v, \tag{17}$$

with u_{ida} given in (11) and

$$\mathcal{K} = GK_iG^\top, \tag{18}$$

with the integral gain matrix $K_i = K_i^\top > 0$. Then $(q,p,x_v) = (q_e, 0, 0)$ is an asymptotically stable equilibrium of the closed-loop system (14)-(17). Introducing the state transformation

$$x_q = q; \qquad x_p = p + \mathcal{K}x_v, \tag{19}$$

with $x_q \in \mathbb{R}^n$, $x_p \in \mathbb{R}^n$, and $x_v \in \mathbb{R}^n$, we can realize the augmented closed-loop system preserving the PCH form:

$$\begin{bmatrix}\dot{x}_q\\ \dot{x}_p\\ \dot{x}_v\end{bmatrix} = \begin{bmatrix}0 & M^{-1}M_d & -M^{-1}\mathcal{K}\\ -M_dM^{-1} & J_2 - R_d & 0\\ (M^{-1}\mathcal{K})^\top & 0 & 0\end{bmatrix}\begin{bmatrix}\nabla_{x_q}\tilde{H}\\ \nabla_{x_p}\tilde{H}\\ \nabla_{x_v}\tilde{H}\end{bmatrix}. \tag{20}$$

■

Remark 4.1: As already mentioned in Section III, the energy shaping procedure has been already done through the design of a stabilizing controller using IDA-PBC. Thus, the Hamiltonian function (16) is exactly the same as (4) but rewritten in the new coordinates and the term $\frac{1}{2}x_v^\top x_v$ is added to accommodate the additional coordinate x_v. i.e no energy shaping for the kinetic and potential functions at this stage but is assumed being shaped in advance through the IDA-PBC procedures.

Remark 4.2: It is well-known that the control law u_{ida} is equivalent to the proportional and derivative (PD) controller. Thus, the total control input (15) consists of the integral control v and u_{ida} is truly a nonlinear PID controller. We first introduced this concept in [9].

Proof of Proposition 4.1: The time derivative of the Hamiltonian function (16) along the trajectories of (20) is

$$\begin{aligned}
\dot{\tilde{H}} &= \nabla_{x_q}\tilde{H}^\top \dot{x}_q + \nabla_{x_p}\tilde{H}^\top \dot{x}_p + \nabla_{x_v}\tilde{H}^\top \dot{x}_v\\
&= \nabla_{x_q}\tilde{H}^\top M^{-1}M_d\nabla_{x_p}\tilde{H} - \nabla_{x_q}\tilde{H}^\top M^{-1}\mathcal{K}\nabla_{x_v}\tilde{H}\\
&\quad - \nabla_{x_p}\tilde{H}^\top M_dM^{-1}\nabla_{x_q}\tilde{H} + \nabla_{x_p}\tilde{H}^\top J_2\nabla_{x_p}\tilde{H}\\
&\quad - \nabla_{x_p}\tilde{H}^\top R_d\nabla_{x_p}\tilde{H} + \nabla_{x_v}\tilde{H}^\top (M^{-1}\mathcal{K})^\top \nabla_{x_q}\tilde{H}
\end{aligned}$$

Rearranging and rewriting some terms with a transpose yields:

$$\begin{aligned}
\dot{\tilde{H}} &= -\nabla_{x_p}\tilde{H}^\top M_dM^{-1}\nabla_{x_q}\tilde{H} + \Big(\nabla_{x_p}\tilde{H}^\top M_dM^{-1}\nabla_{x_q}\tilde{H}\Big)^\top\\
&\quad + \nabla_{x_p}\tilde{H}^\top J_2\nabla_{x_p}\tilde{H} - \nabla_{x_p}\tilde{H}^\top R_d\nabla_{x_p}\tilde{H}\\
&\quad + \nabla_{x_v}\tilde{H}^\top \mathcal{K}M^{-1}\nabla_{x_q}\tilde{H} - \Big(\nabla_{x_v}\tilde{H}^\top \mathcal{K}M^{-1}\nabla_{x_q}\tilde{H}\Big)^\top \qquad (21)\\
&= \nabla_{x_p}\tilde{H}^\top J_2\nabla_{x_p}\tilde{H} - \nabla_{x_p}\tilde{H}^\top R_d\nabla_{x_p}\tilde{H}\\
&= -x_p^\top M_d^{-1}R_dM_d^{-1}x_p \leq -\lambda_{min}\{K_v\}|G^\top M_d^{-1}x_p|^2 \leq 0.
\end{aligned}$$

Note that because $J_2 = -J_2^\top$, the term $\nabla_{x_p}\tilde{H}^\top J_2\nabla_{x_p}\tilde{H}$ is equal to zero. It follows that the system (20) has a stable equilibrium at $(q_e, 0, 0)$. Furthermore, asymptotic stability is concluded using the arguments used in the Remark (3.1). Thus, given $M(x_q) > 0$ and $M_d(x_q) > 0$, $\nabla_{x_q}\tilde{H} \equiv 0$ is only true if the system converges to its equilibrium point q_e. Notice that from (19), $x_q = q \implies \nabla_{x_q}\tilde{H} = \nabla_q H_d$. Thus,

$$\nabla_q H_d \equiv 0 \implies x_q = q_e \text{ and } x_v = 0,$$

as H_d is the energy function of the pre-assumed *asymptotically* stable dynamics (see Remark 4.1). This proves that the equilibrium $(q_e, 0, 0)$ of the augmented system is *asymptotically* stable. Next, we verify the coincidence of the position and momenta states of system (14) with their corresponding states in (20).

(i) For the position states q, we have

$$\begin{aligned}
\dot{q} &\equiv \dot{x}_q\\
M^{-1}p &\equiv M^{-1}M_d\nabla_{x_p}\tilde{H} - M^{-1}\mathcal{K}\nabla_{x_v}\tilde{H}\\
&= M^{-1}M_dM_d^{-1}x_p - M^{-1}\mathcal{K}x_v\\
&= M^{-1}(p + \mathcal{K}x_v) - M^{-1}\mathcal{K}x_v.
\end{aligned}$$

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(ii) For the momenta states p,

$$\begin{aligned}\dot{p} &= -M_d M^{-1}\nabla_q H_d + (J_2 - R_d)\nabla_p H_d + Gv \\ &\equiv \dot{x}_p - \mathcal{K}\dot{x}_v \\ &= -M_d M^{-1}\nabla_{x_q}\tilde{H} + (J_2 - R_d)\nabla_{x_p}\tilde{H} - \mathcal{K}\dot{x}_v.\end{aligned}$$

Using the change of coordinates (19) and rearranging:

$$\begin{aligned}&- M_d M^{-1}\nabla_q V_d - \frac{1}{2}M_d M^{-1}\sum_{i=1}^{n} e_i p^\top \nabla_{q_i} M_d^{-1} p \\ &+ (J_2 - R_d)M_d^{-1}p + Gv = -M_d M^{-1}\nabla_{x_q}\tilde{V} \\ &- \frac{1}{2}M_d M^{-1}\sum_{i=1}^{n} e_i (p + \mathcal{K}x_v)^\top \nabla_{x_{q_i}} M_d^{-1}(p + \mathcal{K}x_v) \\ &+ (J_2 - R_d)M_d^{-1}p + (J_2 - R_d)M_d^{-1}\mathcal{K}x_v - \mathcal{K}\dot{x}_v\end{aligned}$$

Rearranging further, we obtain:

$$\begin{aligned}Gv = &\frac{M_d M^{-1}}{2}\sum_{i=1}^{n} e_i p^\top \nabla_{q_i} M_d^{-1} p + (J_2 - R_d)M_d^{-1}\mathcal{K}x_v \\ &- \frac{M_d M^{-1}}{2}\sum_{i=1}^{n} e_i (p + \mathcal{K}x_v)^\top \nabla_{x_{q_i}} M_d^{-1}(p + \mathcal{K}x_v) \\ &- \mathcal{K}\dot{x}_v.\end{aligned} \quad (22)$$

Notice that $M_d M^{-1}\nabla_q V_d = M_d M^{-1}\nabla_{x_q}\tilde{V}$ as $q = x_q$ from (19) and $V_d(q) = \tilde{V}(x_q)$ from Remark 4.1. Finally, the integral control v in (15) is obtained by pre-multiplying (22) with the term $G^\ddagger$ given in (9), which completes the proof. ■

V. ISS FOR UNDERACTUATED MECHANICAL SYSTEMS USING IDA-PBC

In this section we present our results on input-to-state stability (ISS) stabilization of underactuated mechanical systems with time-varying disturbances employing IDA-PBC method to obtain the stabilizing controller. The theory of input-to-state stability (ISS) introduced in [15] is an extension of the Lyapunov stability theory to deal with systems with inputs. ISS combines the Lyapunov stability notion and the *bounded-input-bounded-output (BIBO)* stability notion [16], and it is a central tool in nonlinear systems analysis that studies the influence of inputs and disturbances on a system, and the robustness of the system with respect to such influences. Consider a PCH system (14), the objective is to provide a control design method to stabilize the system, in a certain sense, subject to matched, $d_2 \in \mathbb{R}^n$, and unmatched, $d_1 \in \mathbb{R}^n$, time-varying bounded disturbances. The terms matched and unmatched refer to whether the disturbances enter the system through the same channel/ports as the control or not. We first discuss the case of matched disturbances and prove the integral input-to-state stability (iISS) variant of the stability property, and then we provide a more general result on ISS to deal with both types of disturbances. We use the definition of ISS and its variants as stated in [16, Section 3.3], [17, Remark 2.4] and [18].

A. iISS for time-varying matched disturbance

Interestingly, the system (14) subjects to a matched disturbance d_2 ($d_1 = 0$) is *naturally* iISS using the integral control (17) proposed in Proposition 4.1. Rewriting the PCH form (20) to include the disturbance as:

$$\begin{bmatrix}\dot{x}_q \\ \dot{x}_p \\ \dot{x}_v\end{bmatrix} = \begin{bmatrix}0 & M^{-1}M_d & -M^{-1}\mathcal{K} \\ -M_d M^{-1} & J_2 - R_d & 0 \\ (M^{-1}\mathcal{K})^\top & 0 & 0\end{bmatrix}\begin{bmatrix}\nabla_{x_q}\tilde{H} \\ \nabla_{x_p}\tilde{H} \\ \nabla_{x_v}\tilde{H}\end{bmatrix} + \begin{bmatrix}0 \\ d_2 \\ 0\end{bmatrix}. \quad (23)$$

The iISS can be proven by taking

$$\tilde{H} = \frac{1}{2}x_p^\top M_d^{-1} x_p + \frac{1}{2}x_v^\top x_v + \tilde{V}(x_q) \quad (24)$$

as a candidate iISS-Lyapunov function. The Lyapunov derivative along the trajectories of (23) is computed as

$$\begin{aligned}\dot{\tilde{H}} &= \nabla_{x_p}\tilde{H}^\top \dot{x}_p + \nabla_{x_v}\tilde{H}^\top \dot{x}_v + \nabla_{x_q}\tilde{H}(x_q)^\top \dot{x}_q \\ &= -(M_d^{-1}x_p)^\top R_d M_d^{-1}x_p + (M_d^{-1}x_p)^\top d_2 \\ &\leq -\|(M_d^{-1}x_p)^\top G\|_{K_v}^2 + (M_d^{-1}x_p)^\top d_2.\end{aligned} \quad (25)$$

Using the Young's inequality as defined in Section II, rewritten as $-\zeta|y|^2 + \eta|y||z| \leq -\frac{\zeta}{2}|y|^2 + \frac{\eta^2}{2\zeta}|z|^2$, it yields

$$\begin{aligned}\dot{\tilde{H}} &\leq -\frac{\lambda_{min}\{K_v\}}{2}|G^\top M_d^{-1}x_p|^2 + \frac{1}{2\lambda_{min}\{K_v\}}|d_2|^2 \\ &\leq -\alpha(|x_p|) + \sigma(|d_2|),\end{aligned} \quad (26)$$

with $\alpha,\ \sigma \in \mathcal{K}_\infty$. Without lose of generality, assuming K_v a diagonal matrix, then $\lambda_{min}\{K_v\}$ is the smallest diagonal elements (individual gains) in K_v. The inequality (26) proves that the system is smoothly dissipative. Furthermore, the system is zero-state detectable from the output $G^\top M_d^{-1}x_p$. This can be shown as follows, from (23), (25) and

$$d_2 \equiv 0, \implies \dot{\tilde{H}} \leq -\frac{\lambda_{min}\{K_v\}}{2}|G^\top M_d^{-1}x_p|^2,$$

which implies the detectability condition invoking the arguments in Assumption 3.1. Thus, all conditions of the iISS property [18] are satisfied, which proves that the closed-loop PCH system is iISS with respect to the matched disturbances.

B. ISS for time-varying matched and unmatched disturbances

In this subsection, we show the more general case when both matched and unmatched disturbances are present. The following proposition provides a constructive ISS control design method, requiring a change of coordinates on both the positions and momenta states to establish the ISS property.

Proposition 5.1: Consider the system (14) with time-varying bounded disturbances $d_1(t)$ and $d_2(t)$, in closed-loop with the dynamic controller

$$\begin{aligned}v = G^\ddagger\Big(&M_d(q)M^{-1}(q)\nabla_q H_d - M^{-1}(q)p - 2M^{-1}(q)\mathcal{K}\nabla_{x_q}\tilde{H} \\ &- 2M_d(x_q)M^{-1}(q)\nabla_{x_q}\tilde{H} - (J_2 - R_d)M_d^{-1}(q)p \\ &+ (J_2 - R_d)M_d^{-1}(x_q)p + 2(J_2 - R_d)M_d^{-1}(x_q)\mathcal{K}x_v \\ &- 2M_d(x_q)M^{-1}(q)x_v\Big) \\ \dot{x}_v = M&^{-1}(q)\mathcal{K}\nabla_{x_q}\tilde{H} + \frac{1}{2}M^{-1}(q)p,\end{aligned} \quad (27)$$

with the desired Hamiltonian function

$$\tilde{H} = \frac{1}{2}x_p^\top M_d^{-1}(x_q)x_p + \frac{1}{2}x_v^\top x_v + \tilde{V}(x_q), \quad (28)$$

such that the total control input takes the form

$$u = u_{ida} + v, \quad (29)$$

with u_{ida} given in (11). Introducing the state transformation

$$x_q = q - x_v; \qquad x_p = \frac{1}{2}p + \mathcal{K}x_v \quad (30)$$

where $x_q \in \mathbb{R}^n,\ x_p \in \mathbb{R}^n,\ x_v \in \mathbb{R}^n$, and $\mathcal{K}$ as defined in (18), the closed-loop dynamics in new variables $x := [x_q\ x_p\ x_v]$ can be written as

$$\begin{bmatrix}\dot{x}_q \\ \dot{x}_p \\ \dot{x}_v\end{bmatrix} = \begin{bmatrix}-M^{-1}(q)\mathcal{K} & M^{-1}(q)M_d(x_q) & -M^{-1}(q)\mathcal{K} \\ -M_d(x_q)M^{-1}(q) & J_2 - R_d & -M_d(x_q)M^{-1}(q) \\ (M^{-1}(q)\mathcal{K})^\top & M^{-1}(q)M_d(x_q) & -M^{-1}(q)\mathcal{K}\end{bmatrix} \times \begin{bmatrix}\nabla_{x_q}\tilde{H} \\ \nabla_{x_p}\tilde{H} \\ \nabla_{x_v}\tilde{H}\end{bmatrix} + \begin{bmatrix}d_1 \\ \frac{d_2}{2} \\ 0\end{bmatrix}. \quad (31)$$

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Then the closed-loop system (14), (27)-(28) is ISS with respect to the disturbances d_1 and d_2 and the function (28) is an ISS-Lyapunov function for the system. ■

Remark 5.1: The assumption stated in Remark 4.1 also applies to the Hamiltonian function (28).

Proof of Proposition 5.1: First, we show the coincidence of the position and momenta states of system (14) with their corresponding states in (31).

For the position states q, from (30) we have

$$\begin{aligned} \dot{q} &\equiv \dot{x}_q + \dot{x}_v \\ M^{-1}p + d_1 &\equiv -M^{-1}\mathcal{K}\nabla_{x_q}\tilde{H} + M^{-1}x_p - M^{-1}\mathcal{K}x_v + d_1 \\ &\quad + M^{-1}\mathcal{K}\nabla_{x_q}\tilde{H} + M^{-1}x_p - M^{-1}\mathcal{K}x_v \\ &= 2M^{-1}x_p - 2M^{-1}\mathcal{K}x_v + d_1 \\ &= 2M^{-1}\left(\frac{1}{2}p + \mathcal{K}x_v\right) - 2M^{-1}\mathcal{K}x_v + d_1 \\ &= M^{-1}p + d_1. \end{aligned} \tag{32}$$

Thus, the left-hand side and right-hand side of (32) are equal. For the momenta p,

$$\begin{aligned} \dot{p} &= -M_d M^{-1}\nabla_q H_d + (J_2 - R_d)\nabla_p H_d + Gv + d_2 \\ &\equiv 2\dot{x}_p - 2\dot{x}_v \\ &\equiv -2M_d M^{-1}\nabla_{x_q}\tilde{H} + 2(J_2 - R_d)\nabla_{x_p}\tilde{H} \\ &\quad - 2M_d M^{-1}\nabla_{x_v}\tilde{H} - 2M^{-1}\mathcal{K}\nabla_{x_q}\tilde{H} \\ &\quad - 2M^{-1}M_d\nabla_{x_p}\tilde{H} + 2M^{-1}\mathcal{K}\nabla_{x_v}\tilde{H} + d_2. \end{aligned}$$

Substituting (30) and rearranging:

$$\begin{aligned} &- M_d M^{-1}\nabla_q H_d + (J_2 - R_d)M_d^{-1}(q)p + Gv = \\ &- 2(M^{-1}\mathcal{K} + M_d M^{-1})\nabla_{x_q}\tilde{H} + 2(J_2 - R_d)M_d^{-1}\mathcal{K}x_v \\ &- 2M_d M^{-1}x_v - M^{-1}p + (J_2 - R_d)M_d^{-1}(x_q)p \\ Gv &= M_d M^{-1}\nabla_q H_d - 2(M^{-1}\mathcal{K} + M_d M^{-1})\nabla_{x_q}\tilde{H} \\ &- (J_2 - R_d)M_d^{-1}(q)p + (J_2 - R_d)M_d^{-1}(x_q)p \\ &- M^{-1}p + 2(J_2 - R_d)M_d^{-1}\mathcal{K}x_v - 2M_d M^{-1}x_v \end{aligned} \tag{33}$$

Finally, pre-multiplying both sides of (33) with $G^{\ddagger}$ we obtain the control law (27). Moreover, consider (28) as a candidate ISS-Lyapunov function, its time-derivative along the trajectories of (31) along with (30) is given by

$$\begin{aligned} \dot{\tilde{H}} &= \nabla_{x_q}\tilde{H}(x_q)^\top \dot{x}_q + \nabla_{x_p}\tilde{H}^\top \dot{x}_p + \nabla_{x_v}\tilde{H}^\top \dot{x}_v \\ &= \nabla_{x_q}\tilde{H}^\top\Big(- M^{-1}\mathcal{K}\nabla_{x_q}\tilde{H} + M^{-1}M_d\nabla_{x_p}\tilde{H} - M^{-1}\mathcal{K}\nabla_{x_v}\tilde{H} \\ &\quad + d_1\Big) + \nabla_{x_p}\tilde{H}^\top\Big(- M_d M^{-1}\nabla_{x_q}\tilde{H} + (J_2 - R_d)\nabla_{x_p}\tilde{H} \\ &\quad - M_d M^{-1}\nabla_{x_v}\tilde{H} + \frac{d_2}{2}\Big) \\ &\quad + \nabla_{x_v}\tilde{H}^\top(M^{-1}\mathcal{K}\nabla_{x_q}\tilde{H} + M^{-1}M_d\nabla_{x_p}\tilde{H} - M^{-1}\mathcal{K}\nabla_{x_v}\tilde{H}). \end{aligned}$$

Following the same procedures as in the proof of Proposition 4.1 by rewriting some terms with a transpose and canceling equal terms with opposite signs and rearranging we obtain:

$$\begin{aligned} \dot{\tilde{H}} &= -\nabla_{x_q}\tilde{H}^\top M^{-1}\mathcal{K}\nabla_{x_q}\tilde{H} + \nabla_{x_q}\tilde{H}^\top d_1 - \nabla_{x_p}\tilde{H}^\top R_d\nabla_{x_p}\tilde{H} \\ &\quad + \frac{1}{2}\nabla_{x_p}\tilde{H}^\top d_2 - \nabla_{x_v}\tilde{H}^\top M^{-1}\mathcal{K}\nabla_{x_v}\tilde{H} \\ &= -\nabla_{x_q}\tilde{H}^\top M^{-1}\mathcal{K}\nabla_{x_q}\tilde{H} + \nabla_{x_q}\tilde{H}^\top d_1 - (M_d^{-1}x_p)^\top R_d M_d^{-1}x_p \\ &\quad - \nabla_{x_v}\tilde{H}^\top M^{-1}\mathcal{K}\nabla_{x_v}\tilde{H} + \frac{1}{2}(M_d^{-1}x_p)^\top d_2 \\ &\le -\|\nabla_{x_q}\tilde{H}\mathcal{K}\|^2_{M^{-1}} - \|(M_d^{-1}x_p)^\top G\|^2_{K_v} - \|\nabla_{x_v}\tilde{H}\mathcal{K}\|^2_{M^{-1}} \\ &\quad + \nabla_{x_q}\tilde{H}^\top d_1 + \frac{1}{2}(M_d^{-1}x_p)^\top d_2. \end{aligned}$$

Applying the Young's inequality, rewritten as $-\zeta|y|^2 + \eta|y||z| \le -\frac{\zeta}{2}|y|^2 + \frac{\eta^2}{2\zeta}|z|^2$ and using the constant ρ such that $\rho I_n \le M^{-1}(q)$, gives

$$\begin{aligned} \dot{\tilde{H}} &\le -\frac{\rho}{2}|\nabla_{x_q}\tilde{H}\mathcal{K}|^2 + \frac{1}{2\rho}|d_1|^2 - \frac{\lambda_{min}\{K_v\}}{2}|G^\top M_d^{-1}x_p|^2 \\ &\quad + \frac{1}{8\lambda_{min}\{K_v\}}|d_2|^2 - \rho|\nabla_{x_v}\tilde{H}\mathcal{K}|^2 \\ &\le -\alpha(|x_q, x_p, x_v|) + \sigma(|d|), \end{aligned} \tag{34}$$

with $d = [d_1 \ d_2]^\top$. Now, from (34) and the fact that $\tilde{H}$ is positive definite, proper and has an isolated minimum (5) due to Remark 5.1, all conditions of the ISS property from [16, Section 3.3] and [17, Remark 2.4] are satisfied, which completes the proof. ■

VI. Example for separable PCH systems: the inertia wheel pendulum

The Quanser inertia wheel pendulum (IWP) module [19] is used to illustrate our proposed results. The module consists of an unactuated planar inverted pendulum with an actuated symmetric disk/wheel attached to its end, which is free to rotate about an axis parallel to the axis of rotation of the pendulum. The system has two degrees-of-freedom; the angular position of the pendulum q_1 and the angular position of the wheel q_2. Only the wheel is actuated by a motor, hence the system is underactuated. The dynamic equations of the IWP system can be written in a PCH form (1) with $n = 2$, $m = 1$ and

$$M = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_2 \end{bmatrix} \ G = e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ V(q_1) = k_3\left(1 + \cos(q_1)\right). \tag{35}$$

with the motor torque as the control input u, $k_1 = m_p l_{c_1}^2 + m_w l^2 + I_p + I_w$, $k_2 = I_w$ and $k_3 = g(m_p l_{c_1} + m_w l)$. The value of the model parameters are: $m_p = 0.2164$, $m_w = 0.085$, $l = 0.2346$, $l_{c_1} = 0.1173$, $I_p = 2.233 \times 10^{-4}$, $I_w = 2.495 \times 10^{-5}$ and $g = 9.81$.

A. IDA-PBC Stabilizing Controller

To start with, an asymptotically stabilizing controller is constructed using IDA-PBC design procedures proposed in [2]. The main objective is to first obtain a continuous control law to swing up the pendulum by spinning the wheel and to stabilize it at its upward position $q = (0, q_2)$ for any $q_2 \in [0,\ 2\pi]$. By fixing M_d to be a constant matrix of the form

$$M_d = \Delta\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = \Delta\begin{bmatrix} m_1 & \left(\frac{k_2}{k_1}\right)m_1 + \varepsilon \\ \left(\frac{k_2}{k_1}\right)m_1 + \varepsilon & m_3 \end{bmatrix}, \tag{36}$$

where $\varepsilon > 0$, $\Delta = k_1 k_2 - k_2^2$ and having $G^\perp = [1 \ 0]$, the desired Hamiltonian (4) is obtained as

$$\begin{aligned} H_d &= \frac{1}{2}p^\top M_d^{-1}p + V_d(q) \\ V_d(q) &= -k_3\gamma_1\cos(q_1) + \frac{1}{2}K_p(\varepsilon k_1\gamma_1 q_1 + q_2)^2, \end{aligned} \tag{37}$$

with $\gamma_1 = \frac{1}{k_2(m_2 - m_1)}$ and $K_p > 0$ the gain of the energy shaping controller

$$u_{es} = \gamma_2\sin(q_1) + k_p\gamma_3(\varepsilon k_1\gamma_1 q_1 + q_2), \tag{38}$$

with $\gamma_2 = -k_3\gamma_1(m_2k_2 - m_3k_2)$; $\gamma_3 = -\varepsilon k_1\gamma_1(m_2k_2 - m_3k_2) - (-m_2k_2 + m_3k_1)$. The damping injection controller is

$$u_{di} = -k_v\frac{\Delta}{\Delta_d}(-m_2p_1 + m_1p_2), \tag{39}$$

with $\Delta_d = \det(M_d) = \Delta^2(m_1m_3 - m_2^2)$ and $K_v > 0$ the damping injection controller gain.

B. ISS Controller

Following the ISS controller design presented in Proposition 5.1, the control input is obtained as

$$
\begin{aligned}
u = u_{ida} &+ c_1(\sin(q_1) - 2\sin(q_1 + x_{v_1})) + \frac{1}{\Delta}(k_2 p_1 - k_1 p_2) \\
&- (c_2 + c_3)(\varepsilon k_1 \gamma_1 (q_1 + 2x_{v_1}) + q_2) - \frac{2k_v k_i m_1 \Delta}{\Delta_d} x_{v_2} \\
&- \frac{2k_p k_i k_1}{\Delta}(\varepsilon k_1 \gamma_1 q_1 + q_2) - 2k_2(m_2 - m_3)x_{v_1} \\
&- 2(k_1 m_3 - k_2 m_2)x_{v_2}
\end{aligned} \tag{40}
$$

with $c_1 = \gamma_1 k_2 k_3 (m_2 - m_3), c_2 = \varepsilon \gamma_1 k_1 k_2 k_p (m_2 - m_3), c_3 = k_p(k_1 m_3 - k_2 m_2)$.

C. Simulations

The ISS IDA-PBC controllers designed for the IWP system are implemented in MATLAB and Simulink environment to evaluate the performance of the control system. In all simulations, the initial condition $[q_0,\ p_0] = [\pi, 0, 0, 0]$ for the system is used.

1) ISS simulations: The ISS control law (40) has been implemented on the IWP system for *unmatched* disturbances case (the most complicated case) with the design parameters $m_1 = 0.4$, $m_3 = 5$, $\epsilon = 1$. The disturbance vector is selected as $d = \lambda \tanh(\dot{p})$. Here, we have selected two different sets of controller parameters ($k_p = 1.1,\ k_v = 5.6 \times 10^{-5},\ k_{i1} = 1.5,\ \rho_1 = 0.09 \times 10^{-12}$) and ($k_p = 0.4,\ k_v = 5.6 \times 10^{-5},\ k_{i2} = 1.5,\ \rho_2 = 0.09 \times 10^{-11}$), in response to two different disturbance gains ($\lambda = 60$) and ($\lambda = 90$), respectively. The simulation results are shown in Figures 1 and 2. We can see the convergence of all states to their desired values with reasonable transients. These figures also show that for relatively high disturbances ($\lambda = 90$), we have to select a large value of ρ to enlarge the domain of attraction and thus the system is ultimately bounded. This follows the proof of Proposition 5.1. Notice that we have decreased the proportional gain k_p to make sure that the maximum torque does not exceed the actuator limit.

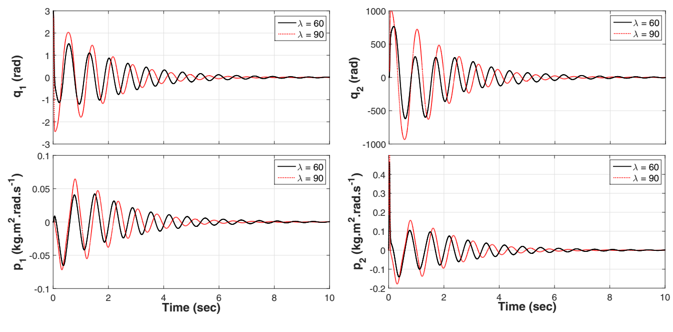

Fig. 1. Response with *unmatched* disturbance.

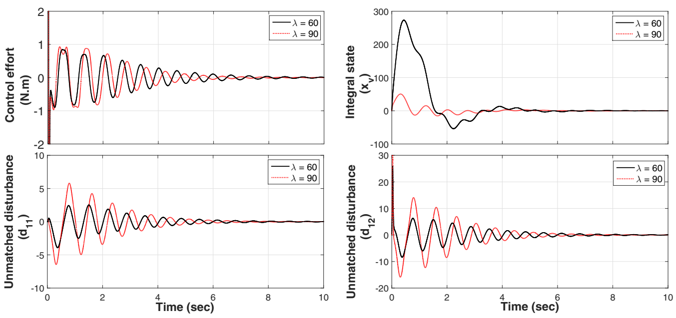

Fig. 2. Control input, update law and disturbance input with *unmatched* disturbance.

VII. Example for non-separable PCH systems: the rotary inverted pendulum

In this example we test the control design to a rotary pendulum system [20], whose dynamics can be represented as a non-separable PCH system with $n = 2$, $m = 1$ and

$$
M(q) = \begin{bmatrix} \gamma & -\sigma\cos(q_1) \\ -\sigma\cos(q_1) & \rho + \gamma\sin^2(q_1) \end{bmatrix}, \quad \text{and} \tag{41}
$$

$$
M^{-1}(q) = \frac{1}{\Delta}\begin{bmatrix} \rho + \gamma\sin^2(q_1) & \sigma\cos(q_1) \\ \sigma\cos(q_1) & \gamma \end{bmatrix}, \tag{42}
$$

where $\Delta = \det(M) = \gamma\rho + \gamma^2\sin^2(q_1) - \sigma^2\cos^2(q_1)$, with the positive constants $\gamma = J_p + \frac{1}{4}m_p L_p^2$, $\sigma = \frac{1}{2}m_p L_p L_r$, $\rho = J_r + m_p L_r^2 + \frac{1}{4}m_r L_r^2$, and $\kappa = \frac{1}{2}m_p g L_p$. The potential energy of the system is

$$
V(q_1) = \kappa\left(1 + \cos(q_1)\right). \tag{43}
$$

A. IDA-PBC Stabilizing Controller

The control objective is to asymptotically stabilize the rotary inverted pendulum at its unstable equilibrium point $q_e = (0, q_2)$ for any $q_2 \in [0, 2\pi]$. Following the control design procedure in [6], the desired inertia matrix is assigned as

$$
M_d(q) = \Delta \begin{bmatrix} (\cos(q_1) + \epsilon) & \frac{-\sigma\cos(q_1)(\cos(q_1)+\epsilon)}{\gamma} \\ \frac{-\sigma\cos(q_1)(\cos(q_1)+\epsilon)}{\gamma} & m_3 \end{bmatrix} \tag{44}
$$

and the closed-loop potential function $V_x = V_d$ is computed as

$$
\begin{aligned}
V_d(q) = \lambda_1 &\Big(-\epsilon\lambda_2 \tanh^{-1}\left(\lambda_2\cos(q_1)\right) + \ln\left(\cos(q_1) + \epsilon\right)\Big) \\
&+ \frac{k_p}{2} q_2^2,
\end{aligned} \tag{45}
$$

where $\lambda_i,\ i = 1, 2$ are constants. Therefore, the gradient of the desired potential energy function V_x is computed as

$$
\begin{bmatrix} \nabla_{q_1} V_d \\ \nabla_{q_2} V_d \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1 \sin(q_1)\epsilon(\sigma^2+\gamma^2)}{\gamma(\rho+\gamma)-(\sigma^2+\gamma^2)\cos^2(q_1)} - \frac{\lambda_1\sin(q_1)}{(\epsilon+\cos(q_1))} \\ \lambda_1 \sin(q_1) K_p q_2 \end{bmatrix}. \tag{46}
$$

Thus, the IDA-PBC controller that asymptotically stabilizes the pendulum at its upright equilibrium $q_e = (0, 0)$ is obtained as

$$
\begin{aligned}
u_{ida} = &-\frac{\sigma\cos(q_1)}{\gamma}\Big(\gamma m_3 - \left(\rho + \gamma\sin^2(q_1)\right)\left(\epsilon + \cos(q_1)\right)\Big) \\
&\times \Bigg(\frac{\epsilon\lambda_1\lambda_2^2\sin(q_1)}{1 - \lambda_2^2\cos^2(q_1)} - \frac{\lambda_1\sin(q_1)}{\epsilon + \cos(q_1)} + \frac{\mathcal{B}_1}{2}p_1^2 + \mathcal{B}_2 p_1 p_2 \\
&+ \frac{\mathcal{B}_3}{2}p_2^2 \Bigg) - \Bigg(\gamma m_3 - \frac{\sigma^2\cos^2(q_1)\left(\epsilon + \cos(q_1)\right)}{\gamma}\Bigg) k_p q_2 \\
&- j_2 \frac{\Delta}{\gamma\Delta_d}\Big(\gamma m_3 p_1 + \sigma\cos(q_1)\left(\epsilon + \cos(q_1)\right)p_2\Big) \\
&- \frac{k_v\Delta\left(\cos(q_1) + \epsilon\right)}{\gamma\Delta_d}\left(\sigma\cos(q_1)p_1 + \gamma p_2\right),
\end{aligned} \tag{47}
$$

where

$$
\begin{aligned}
\Delta_d &= \det(M_d) \\
&= \frac{\Delta^2}{\gamma^2}\left(\cos(q_1) + \epsilon\right)\Big(m_3\gamma^2 - \sigma^2\cos^2(q_1)\left(\cos(q_1) + \epsilon\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_1 = \frac{m_3\sin(q_1)}{(\gamma\Delta_d)^2}\Big(&-2\gamma^2\Delta_d\cos(q_1)(\gamma^2 + \sigma^2) \\
&+ \Delta^3\left(m_3\gamma^2 - \epsilon\sigma^2\cos^2(q_1) - \sigma^2\cos^3(q_1)\right) \\
&- \sigma^2\Delta^3\cos(q_1)\left(\epsilon + \cos(q_1)\right)\left(2\epsilon + 3\cos(q_1)\right)\Big),
\end{aligned}
$$

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$$\mathcal{B}_2 = -\frac{\sigma \sin(q_1)\big(\epsilon + \cos(q_1)\big)}{\gamma \Delta_d \big(m_3\gamma^2 - \epsilon\sigma^2\cos^2(q_1) - \sigma^2\cos^3(q_1)\big)} \times\Big(\sigma^2\Delta\cos^2(q_1)\big(2\epsilon{+}3\cos(q_1)\big){+}\big(2\cos^2(q_1)(\gamma^2{+}\sigma^2) + \Delta\big)\big(m_3\gamma^2 - \epsilon\sigma^2\cos^2(q_1) - \sigma^2\cos^3(q_1)\big)\Big),$$

$$\mathcal{B}_3 = -\frac{\sin(q_1)\cos(q_1)\big(\epsilon + \cos(q_1)\big)}{\Delta_d\big(m_3\gamma^2 - \epsilon\sigma^2\cos^2(q_1) - \sigma^2\cos^3(q_1)\big)} \times\Big(2(\gamma^2+\sigma^2)\big(m_3\gamma^2 - \epsilon\sigma^2\cos^2(q_1) - \sigma^2\cos^3(q_1)\big) + \sigma^2\Delta\big(2\epsilon + 3\cos(q_1)\big)\Big).$$

B. Integral Controller

Now, we proceed to compute $\frac{1}{2}\sum_{i=1}^{n} e_i x_p^\top \nabla_{q_i} M_d^{-1} x_p$ of the integral control term. We have $M_d(q_1) = M_d(x_{q_1})$ as $q = x_q$ from (19), thus, from (44), we have

$$M_d^{-1}(q_1) = \begin{bmatrix} \frac{\Delta m_3}{\Delta_d} & \frac{\Delta\sigma\cos(q_1)(\cos(q_1)+\epsilon)}{\Delta_d\gamma} \\ \frac{\Delta\sigma\cos(q_1)(\cos(q_1)+\epsilon)}{\Delta_d\gamma} & \frac{\Delta(\epsilon+\cos(q_1))}{\Delta_d} \end{bmatrix}.$$

Hence, $\nabla_{q_i} M_d^{-1}$ is computed as

$$\nabla_{q_i} M_d^{-1} = \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_2 & \mathcal{B}_3 \end{bmatrix}. \tag{48}$$

As M_d depends only on q_1 and independent from q_2, we obtain

$$\frac{1}{2}\sum_{i=1}^{n} e_i x_p^\top \nabla_{q_i} M_d^{-1} x_p = \begin{bmatrix} \frac{1}{2}\mathcal{B}_1 x_{p_1}^2 + \mathcal{B}_2 x_{p_1} x_{p_2} + \frac{1}{2}\mathcal{B}_3 x_{p_2}^2 \\ 0 \end{bmatrix},$$

and $\nabla_{x_q}\tilde{H}$ in (15) is computed as

$$\begin{aligned} \nabla_{x_q}\tilde{H} &= \nabla_{x_q} V_x + \frac{1}{2}\sum_{i=1}^{n} e_i x_p^\top \nabla_{q_i} M_d^{-1} x_p \\ \begin{bmatrix} \nabla_{x_{q_1}}\tilde{H} \\ \nabla_{x_{q_2}}\tilde{H} \end{bmatrix} &= \begin{bmatrix} \nabla_{x_{q_1}}\tilde{V} + \frac{1}{2}\mathcal{B}_1 x_{p_1}^2 + \mathcal{B}_2 x_{p_1} x_{p_2} + \frac{1}{2}\mathcal{B}_3 x_{p_2}^2 \\ \nabla_{x_{q_2}}\tilde{V} \end{bmatrix}, \end{aligned} \tag{49}$$

where $\nabla_{x_{q_1}}\tilde{V}$ and $\nabla_{x_{q_2}}\tilde{V}$ are from (46) but rewritten in x coordinates. Thus, $\dot{x}_v$ in (15) is computed as

$$\begin{aligned} \dot{x}_v &= (M^{-1}\mathcal{K})^\top \nabla_{x_q}\tilde{H} \\ &= \begin{bmatrix} k_i & 0 \\ 0 & k_i \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\rho+\gamma\sin^2(q_1)}{\Delta} & \frac{\sigma\cos(q_1)}{\Delta} \\ \frac{\sigma\cos(q_1)}{\Delta} & \frac{\gamma}{\Delta} \end{bmatrix} \begin{bmatrix} \nabla_{x_{q_1}}\tilde{H} \\ \nabla_{x_{q_2}}\tilde{H} \end{bmatrix} \\ \begin{bmatrix} \dot{x}_{v_1} \\ \dot{x}_{v_2} \end{bmatrix} &= \begin{bmatrix} 0 \\ \frac{k_i\sigma\cos(q_1)}{\Delta}\nabla_{x_{q_1}}H_x + \frac{k_i\gamma}{\Delta}\nabla_{x_{q_2}}H_x \end{bmatrix}, \end{aligned} \tag{50}$$

Substituting (50) into (15), we obtain the integral controller as

$$\begin{aligned} v &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & j_2 \\ -j_2 & -k_v \end{bmatrix} \\ &\times \begin{bmatrix} \frac{\Delta m_3}{\Delta_d} & \frac{\Delta\sigma\cos(q_1)(\cos(q_1)+\epsilon)}{\gamma\Delta_d} \\ \frac{\Delta\sigma\cos(q_1)(\cos(q_1)+\epsilon)}{\gamma\Delta_d} & \frac{\Delta(\epsilon+\cos(q_1))}{\Delta_d} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{v_1} \\ x_{v_2} \end{bmatrix} \\ &- \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_{v_1} \\ \dot{x}_{v_2} \end{bmatrix} \\ &= -\left(\frac{j_2\Delta\sigma\cos(q_1)}{\gamma\Delta_d} + \frac{k_v\Delta}{\Delta_d}\right)(\epsilon + \cos(q_1))x_{v_2} - \dot{x}_{v_2}, \end{aligned} \tag{51}$$

where from (49) and (50)

$$\begin{aligned} \dot{x}_{v_2} &= \frac{k_i\sigma\cos(q_1)}{\Delta}\left(\frac{1}{2}\mathcal{B}_1 x_{p_1}^2 + \mathcal{B}_2 x_{p_1} x_{p_2} + \frac{1}{2}\mathcal{B}_3 x_{p_2}^2\right) \\ &+ \frac{k_i\sigma\cos(q_1)}{\Delta}\left(\frac{\lambda_1\sin(q_1)\epsilon(\sigma^2+\gamma^2)}{\gamma(\rho+\gamma) - (\sigma^2+\gamma^2)\cos^2(q_1)}\right) \\ &- \frac{\lambda_1 k_i\sigma\cos(q_1)\sin(q_1)}{\Delta(\epsilon+\cos(q_1))} + \frac{k_i\gamma}{\Delta}(\lambda_1\sin(q_1)k_p q_2). \end{aligned} \tag{52}$$

Remark 7.1: Notice that in (52), $\dot{x}_{v_2}$ contains ***both*** terms $\nabla_{x_{q_1}}H_x$ and $\nabla_{x_{q_2}}H_x$ which are the non-passive components of $n = 2$ coordinates q_1 and q_2. Thus, the produced ***single*** controller v $(m = 1)$ contains the integrals of both terms, namely, $\nabla_{x_{q_1}}H_x$ and $\nabla_{x_{q_2}}H_x$. This is an advantage of this method which could provide integral action on both, (n), coordinates using a single, (m), controller though this is an underactuated system.

C. Simulations

The values of the parameters from the Quanser QUBE-servo rotary inverted pendulum [20] are used, i.e. $m_p = 0.024$, $L_p = 0.129$, $J_p = 3.33 \times 10^{-5}$, $m_r = 0.095$, $L_r = 0.085$ and $J_r = 5.72 \times 10^{-5}$. The effect of uncertainties is evaluated by adding a constant *unmatched* disturbance $d = 2$ Nm to the *unactuated* coordinate (the pendulum). First, the simulations are carried out without including the dynamic integral control action, i.e. using u_{ida} only, where its gains were selected as $k_p = 0.005$, $k_v = 2 \times 10^{-5}$, $m_3 = 65$, $\epsilon = 1.1$ with the initial conditions $[q_0, p_0] = [\frac{\pi}{2}, 1.2, -0.2 \times 10^{-3}, -0.5 \times 10^{-3}]$. Figure 3 shows the time history of the position states q_1 and q_2. As expected, the performance of the controlled system is highly deteriorated by the action of big disturbances.

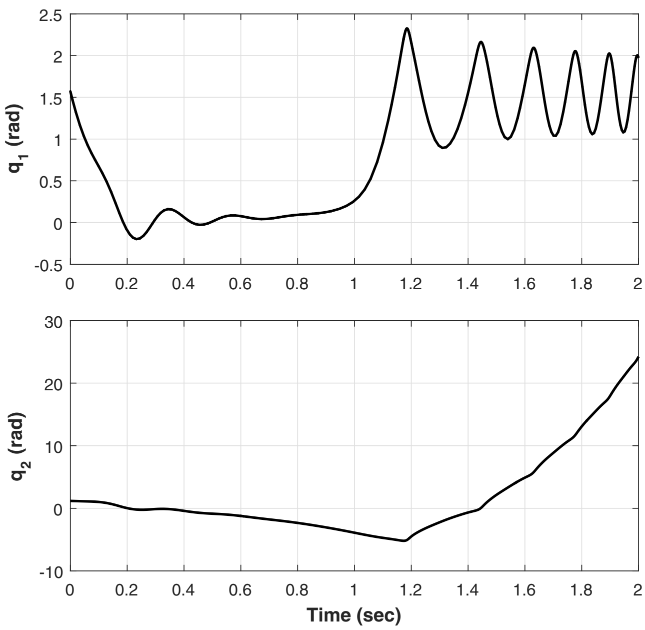

Fig. 3. Top graph: Time history of the position state q_1. Bottom graph: Time history of the position state q_2. (With u_{ida} only).

The rotary inverted pendulum is then simulated, under the same conditions and disturbances, by implementing the integral controller (51), (52) along with the IDA-PBC controller (47). The value of the integral gain is selected as $k_i = 1 \times 10^{-6}$. Figure 4 depicts the time histories of the pendulum and arm angular positions. It is clear that the states converge to their desired positions. Figure 5 (top) shows the time histories of the dynamic extension state x_v and Figure 5 (middle) and (bottom) show the integral control and total control, respectively. It is evident that this controller reject the disturbances achieving a good performance with acceptable effort.

VIII. Conclusion

In this paper, we have proposed a set of control design methods, which are the various extension of the IDA-PBC method, to deal

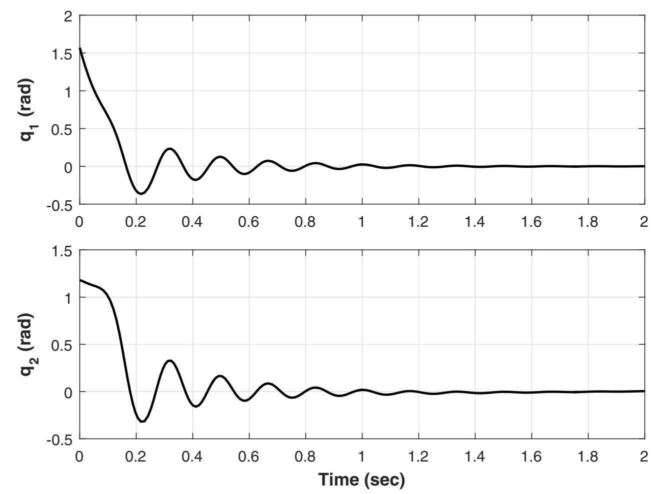

Fig. 4. Top graph: Time history of the position state q_1. Bottom graph: Time history of the position state q_2. (With u_{ida} and integral controller).

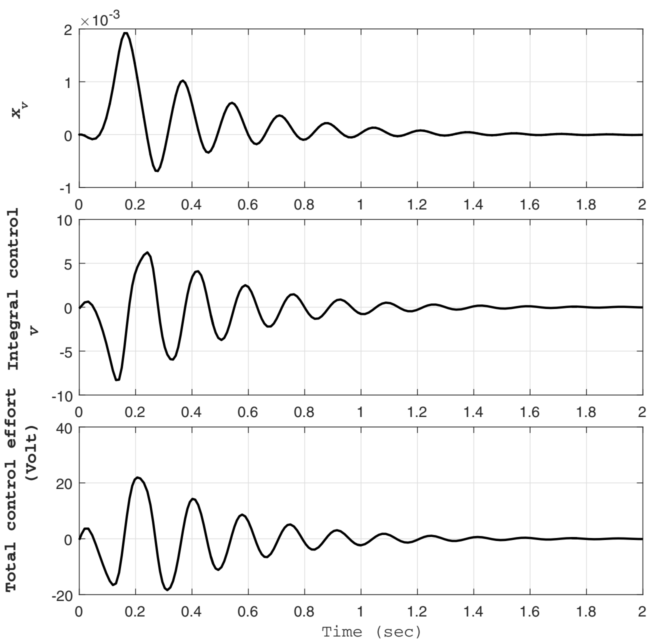

Fig. 5. Top graph: Time history of the dynamic extension x_v. Middle graph: Time history of the dynamic controller v. Bottom graph: Time history of the total control $u_{ida} + v$.

with several robustness-related issues for underactuated mechanical systems within PCH framework. In particular, IDA-PBC method along with a dynamic state-feedback controller that involves integral controller is used to improve the robustness of the closed-loop system. We have successfully extended our results in [9] to deal with the non-separable PCH systems, which are more realistic class of systems in practice. Although as expected this has resulted in more complex control law as the derivative of the inertia matrices M and M_d are needed to be taken into account, our proposed method keeps the procedure clean and systematic. The matched and unmatched disturbance rejection problems are solved using the IC controller with a particular change of coordinates that involves adding some damping terms. These results ensure that the ISS property is satisfied.

The approaches have been validated using two interesting illustrative examples, an inertia wheel pendulum and rotary inverted pendulum which are separable and nonseparable underactuated systems. The effectiveness of the proposed controllers has been shown through numerical simulations. The simulation results demonstrate that the system is robust with respect to different perturbations, preserving the PCH structure, retaining the (asymptotic) stability with high performance. While only two example are presented as illustration, other systems belong to PCH class of systems (see for instance [11]) are possible to apply our results to. Future research direction include a general adaptive control scheme that considers uncertainties in the potential and kinetic energy function.

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