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Nuclear Catastrophe Risk Bonds in a Markov Dependent Environment

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ABSTRACT

The financing of the 2011 Fukushima disaster and the UK Hinkley nuclear power plant investment, respectively by the Japanese, and UK and Chinese governments and the private sector provide a strong motivation for this paper to explore deeper the concept of modeling and pricing Nuclear Catastrophe (N-CAT) risk bonds. Due to the magnitude of the potential liabilities and re-investments needed, the demand to develop a dependable liability coverage product that can be triggered in a case of emergency is required more than ever and it should be considered thoroughly. Thus, in the present paper, under a semi-Markov structure environment to model the relationship between claims severity and intensity, the N-CAT risk bond is further explored under various scenarios supporting further the bond sponsors, allowing them to appreciate more their significance.

Consequently, the new version of the N-CAT risk bond includes several absorbing and transit states to make it more suitable for practitioners. Additionally, this paper employs the two most commonly used interest rate models and considers four types of payoff functions. Finally, two numerical examples illustrate the main findings.

Keyword: Nuclear Power Risk, Catastrophe Risk Bonds, Global Market, Liability, Special Purpose Vehicle, Semi-Markov Environment

INTRODUCTION

Communities often experience different types of natural and human-made disasters such as floods, earthquakes, hurricanes, severe storms, tornadoes, wildfires, heavy snowfalls and human-caused disruptions (including also terrorist attacks and threats) which lead inevitably to numerous governmental declarations and billions of US dollars in losses every year. Furthermore, the welfare impact of such high-levels of disruption does not only depend on the physical characteristics of the event(s) as well as its (their) direct or indirect impacts in terms of lost lives and assets, but also on the aptitude of the economy to absorb, recover, reconstruct and therefore to minimize the aggregate consumption losses in the short and long-run. In practice, losses and recovery costs from those catastrophic (or even cataclysmic) events are typically covered by a combination of utility companies, special insurance schemes and/or governments, for instance a characteristic example is the coverage of the major losses from the 2011 Fukushima disaster primarily by the Japanese government (Conca 2016).

Nuclear power plants are very popular for producing electrical energy in 31 countries (see discussion in (Ayyub et al. 2016)). Among the many developed countries which have nuclear plants, very recently in the UK, Hinkley Point C nuclear power plant got the greenlight from the UK government to be constructed (Farrel and Macalister 2015). It will be the first nuclear power station in a generation which will provide 7% of the country's electricity, and the total investment cost is estimated to be more than US$20 billion. Inevitably, this level of investment brings the risk of nuclear power back into the public eye again. The enquiry which was initially proposed in (Ayyub and Parker 2011) and was then emphasised in (Ayyub et al. 2016): "*how to develop sufficient liability coverage for nuclear power risks?*" . Resources for this purpose are often inadequate and require a cash reserve that could be challenging to maintain. Low penetration rates for insurance leaves it up to individuals, companies and governments to shoulder the financial losses arising from catastrophic events. In emerging markets with non-existent or immature legal regimes, liability can lead to international tensions and potentially wars, particularly in cases of cross-border exposures. Therefore, the potential financial demands on insurance and reinsurance businesses make it appropriate to introduce a mechanism for individuals against nature and man-made disasters.

Catastrophe (CAT) risk bonds (or Act-of God bonds) are securities which are born for these extreme events to share the risk to another level — global financial markets as the only pool of cash large enough to underwrite such losses lies in capital markets and the collection of big investors like pension funds, hedge funds and sovereign wealth funds that normally invest in stocks and bonds. CAT risk bonds show association between the risk deduction for insurers and are an alternative source of capital for insurance companies with large risk transfer needs (Hagendorff et al. 2014). On the other hand, CAT risk bonds' investors enjoy high yield coupon rates and diversification effects on their investment portfolios. Furthermore, the feature of correlation of the traditional stock market allows them to still gain under bad economic conditions and they reduce the barriers to entry and increase the contestability of the reinsurance market (Froot 2001). These lead CAT risk bonds to be the most popular insurance-linked financial securities (ILS) and their use has been accelerating in the last few decades.

Historically, the first experimental transaction was completed in the mid-1990s after Hurricane Andrew and the Northridge earthquake, which incurred insurance losses of US$15.5 billion and US$12.5 billion, respectively, by a number of specialized catastrophe-oriented insurance and reinsurance companies in the USA, including AIG, Hannover Re, St Paul Re, and USAA, (GAO 2002). The CAT risk bond market has boomed over the years. The issued capital has increased tenfold within ten years, from less than US$0.8 billion in 1997 to over US$8 billion in 2007, and the issuers raised more than US$7.8 billion of new CAT risk bonds in 2015 (Artemis 2017a). CAT risk bonds are inherently risky, non-indemnity-based multi-period deals, which pay a regular coupon to investors at the end of each period and a final principal payment at the maturity date, if no predetermined catastrophic events occur. A major catastrophe in the secured region before the CAT risk bonds maturity date leads to full or partial loss of the capital.

The structure of CAT risk bonds, including where the capital flows from one party to another, is presented in Figure 1, see also (Swiss Re Institute 2009). The issuer does not directly issue the CAT risk bond, but uses a Special Purpose Vehicle (SPV) for the transaction. An SPV can be interpreted as a focused insurer whose only purpose is to write one insurance contract. The existence of an SPV, which is equal to a focused one-policy insurer, minimises the frictional cost of capital. Furthermore, sufficient high endowment of the SPV eliminates the counterparty risk. The SPV enters into a reinsurance agreement with a sponsor or counterparty (e.g., an insurer, reinsurer, or government) by issuing CAT risk bonds to investors and receives premiums from the sponsor in exchange for providing a pre-specified coverage. Therefore, sponsors can transfer part of the risks to investors who bear the risk in return for higher expected returns. The SPV collects the capital (principal and premium) and invests the proceeds into a collateral account (trust account, which is typically highly related to short-term securities, e.g., Treasury bonds). The returns generated from collateral accounts are swapped for floating returns based on the London Interbank Offered Rate (LIBOR) in order to immunize the sponsor and the investors from interest rate risk and default risk, (Cummins 2008).

The investors' coupon payments are made up of SPV investment returns, plus the premiums from the sponsor. If no trigger event occurs during the term time of the CAT risk bonds, then the collateral is liquidated at the maturity date of the CAT risk bonds and investors are repaid the principal plus a compensation for bearing the catastrophe risks (solid line in Figure 1). However, if a trigger event occurs before maturity, the SPV will liquidate the collateral required to make the payment and reimburse the counterparty according to the terms of the catastrophe bond transaction, and CAT bond investors will only receive part of the capital (dashed line in Figure 1).

The key parameter of a CAT risk bond transaction is the bond premium. To bear the catastrophe risks, CAT risk bonds carry a 3 to 5 year maturity and compensate for a floating LIBOR coupon plus a premium at a rate between 2% and 20%, see (Cummins 2008; GAO 2002). The main determinants of the CAT risk bond spread/premium is the expected loss, the covered territory, the sponsor, the reinsurance cycle, and the corporate bond spread (Braun 2016). (Galeotti et al. 2013) modelled premiums paid by a sponsor in two parts: the expected value of loss, and a load for risk margin and expenses. They compared the different premium calculation models based on the basis of CAT risk bond contracts issued between April 1999 and March 2009, and recommended the Wang's transformation model (Wang 2004) or the simple linear model to predict CAT risk bond premiums. The key elements of pricing any CAT risk bond are the loss exceedance curve and the triggers. Only when a pre-specified condition is met (e.g., a predetermined events occurs and the loss exceeds a predetermined level), investors begin to lose their investment, and those conditions are triggers. Triggers can be structured in many ways from a sliding scale of actual losses experienced by the issuer (indemnity) to a trigger which is activated when industry wide losses from an event hit a certain point (industry index trigger) to an index of weather or disaster conditions, which means actual catastrophe conditions above a certain severity will trigger a loss (parametric index trigger) etc., see (Swiss Re Institute 2009; Hagedorn et al. 2009; Burnecki et al. 2011; Johnson 2013) among others. A few CAT risk bonds use the indemnity trigger type because it is subject to the highest degree of moral hazard, due to the fact that the loss is controlled by the sponsor (Hagendorff et al. 2014). (Swiss Re Institute 2009) illustrated the relationship between transparency and basis risk for various types of CAT risk bond triggers, also in Figure 2, and investors prefer to buy the bonds with better transparency while sponsors want to minimise the basis risks.

CAT risk bonds can be structured to provide per-occurrence cover, so exposure to a single major loss event (currently US$ 12,932.41 million which accounts for 55.6%) or to provide annual aggregate cover, exposure to multiple event triggers over each annual risk-period (Woo 2004; Artemis 2017b). Some CAT risk bonds transactions work on a multiple loss approach and so are only triggered (or portions of the deals are) by second and subsequent events. This means that sponsors can issue a deal that will only be triggered by a second landfalling hurricane to hit a certain geographical location, for example.

Despite the rising popularity, the number of previous studies devoted to CAT risk bond modeling and pricing is relatively limited. Some notable models have been based on: quasi Monte Carlo (Vaugirard 2003; Albrecher et al. 2004) and indifference pricing techniques (Young 2004), entropy based models (Ling and Jun 2009), a simple robust model (Jarrow 2010), a representative agent pricing approach (Cox and Pedersen 2000; Shao et al. 2015), premium calculation models (Galeotti et al. 2013), a mixed approximation method (Ma and Ma 2013), a Bayesian pricing model (Ahrens et al. 2014), a cluster analysis approach (Constantin et al. 2014), a multifactor pricing model (Gomez and Carcamo 2014), modeling using multifractal processes (Hainaut and Boucher 2014), fuzzy based approaches (Nowak and Romaniuk 2013b; Nowak and Romaniuk 2017), and with Cox-Ingersoll-Ross interest rate models (Nowak and Romaniuk 2016).

Some notable applications have included: modeling of tropical cyclones (Daneshvaran and Morden 2004), systemic risks in agriculture for the case of Georgia cotton (Vedenov et al. 2006), transportation assets and feasibility analysis for bridges (Sircar et al. 2009), calibration using Chinese earthquake loss data (Wu and Zhou 2010), models for earthquakes(Penalva Zuast 2002; Zimbidis et al. 2007; Tao et al. 2009; Härdle and Cabrera 2010; Ahrens et al. 2014; Shao et al. 2015), modeling of tornado occurrence in the USA (Hainaut and Boucher 2014), exposure to currency exchange risk (Lai et al. 2014), seismic risk management of insurance portfolio (Goda 2015), hedging of flood losses (Tetu et al. 2015), and temperature-based agricultural applications (Karagiannis et al. 2016) among others.

Recently, Shao et al. (Shao et al. 2017) modeled the dependence of the claim inter-arrival time on the claim size for the aggregate claims as a semi-Markov process. As it has been discussed in (Shao et al. 2017), there are quite a few applications where the Markov-dependent structure has been applied. For instance, (Janssen and Manca 2007; Janssen and Limnios 1999) provided plenty of applications in queueing theory, insurance mathematics, reliability and maintenance and fluid mechanics. (Reinhard 1984; Asmussen and Rolski 1992; Lu and Li 2005) focus on modeling and computing Semi-Markov processes in ruin theory. Moreover, (Ayyub et al. 2016) proposed nuclear catastrophe risk bonds (also known as N-CAT) for the very first time, addressing the nuclear liability conventions and the current liability limitations, for more details see (Ayyub and Parker 2011). This N-CAT risk bond utilised the indemnity trigger with lowest basis risks to the sponsor, however, it has lowest transparency for investors. In order to prevent the intent of manipulating the N-CAT risk bonds prices by deliberately triggering a nuclear catastrophe, the N-CAT risk bonds writer should specify in the contract that man-made accidents directly caused by the reimbursement beneficiaries (normally government) are excluded. Although very unlikely, this extra term in the N-CAT risk bonds contract provides safeguard against such behaviour.

In the present paper, a complete analysis of N-CAT risk bonds is presented by implementing three main extensions compared with the previous papers (Ayyub et al. 2016; Shao et al. 2017). First, the authors embed a flexible interest rate model framework. Thus, a sensitivity analysis based on the classical Vasicek and Cox-Ingersoll-Ross models is provided (Nowak and Romaniuk 2013a). Then, the authors construct model in a Markov-dependent environment (Shao et al. 2017) and the authors generalise the transition matrix with w transit states and r absorbing states (Ayyub et al. 2016). Finally, by employing four payoff functions including an issuer default model, two illustrative numerical examples are provided.

The contents of this paper are organized as follows. The Modelling N-CAT Risk Bond section presents the pricing model of CAT risk bonds including: assumptions, probability structure, valuation method, interest rate processes, aggregate claims processes, and the payoff functions. Explicit closed form solutions are shown in Theorems 2.1 to 2.4. The section Numerical Examples: Analysis and Discussion illustrates the numerical examples of the N-CAT risk bonds pricing formulae and compares the effect size for varying interest rates, time to maturity and threshold levels, accordingly.

MODELLING N-CAT RISK BOND

Following closely (Cox and Pedersen 2000; Shao et al. 2015; Ayyub et al. 2016; Shao et al.

2017), in the present paper, the N-CAT risk bonds is priced under the following assumptions: (i) an arbitrage-free investment market exists with an equivalent martingale measure, (ii) the financial market behaves independently of the occurrence of catastrophes, and (iii) the interest rate changes can be replicated using existing financial instruments.

Probabilistic structure and valuation theory

Let $0 < T < \infty$ be the maturity date of the continuous time trading interval $[0, T]$. The market uncertainty is defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, where $\mathcal{F}_t$ is an increasing family of σ-algebras, which is given by $\mathcal{F}_t = \mathcal{F}_t^{(1)} \times \mathcal{F}_t^{(2)} \subset \mathcal{F}$, for $t \in [0, T]$, where $\mathcal{F}_t^{(1)}$ represents the investment information (e.g., past security prices and interest rates) available to the market at time t and $\mathcal{F}_t^{(2)}$ represents the catastrophic risk information (e.g., insured property losses). The financial risk variables and the catastrophic risk variables can be modelled on $\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \left(\mathcal{F}_t^{(1)}\right)_{t\in[0,T]}, \mathbb{P}^{(1)}\right)$ and $\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \left(\mathcal{F}_t^{(2)}\right)_{t\in[0,T]}, \mathbb{P}^{(2)}\right)$, respectively. Moreover, define two filtrations $\mathcal{A}^{(1)}$ ($\mathcal{A}_t^{(1)} = \mathcal{F}_t^{(1)} \times \left\{\emptyset, \Omega^{(2)}\right\}$ for $t \in [0, T]$) and $\mathcal{A}^{(2)}$ ($\mathcal{A}_t^{(2)} = \left\{\emptyset, \Omega^{(1)}\right\} \times \mathcal{F}_t^{(2)}$ for $t \in [0, T]$). It is proved by Lemma 5.1 (Cox and Pedersen 2000) that the σ-algebras $\mathcal{A}_t^{(1)}$ and $\mathcal{A}_t^{(2)}$ are independent under the probability measure $\mathbb{P}$. Thus, an $\mathcal{A}_T^{(\kappa)}$ measurable random variable X on $\left(\Omega = \Omega^{(1)} \times \Omega^{(2)}, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ (or an $\mathcal{A}^{(\kappa)}$ adapted stochastic process Y) is said to depend only on the financial risk variables ($\kappa = 1$) or catastrophic risk variables ($\kappa = 2$).

The presence of catastrophic risks that are uncorrelated with the underlying financial risks leads us to consider an incomplete market, and there is no universal theory addressing all aspects of pricing (Young 2004). The benchmark to price uncertain cash flow under an incomplete framework is the representative agent. For valuation purposes, similar to (Merton 1976), the authors assume that under the risk-neutral pricing measure $\mathbb{Q}$, the overall economy depends only on financial risk variables. This is a fairly natural approximation because the global economic conditions and other securities traded on capital markets are only marginally influenced by localized catastrophes, for more information and justification see (Cox and Pedersen 2000; Merton 1976; Doherty 1997; Lee and Yu 2002; Ma and Ma 2013; Gürtler et al. 2016). According to Lemma 5.2 (Cox and Pedersen 2000), under an assumption that the aggregate consumption is $\mathcal{A}^{(1)}$ adapted (assumption (ii)), for any random variable X that is $\mathcal{A}_T^{(2)}$ measurable,

$$E^{\mathbb{Q}}[X] = E^{\mathbb{P}}[X]. \tag{1}$$

Thus, a $\mathcal{A}^{(2)}$-adapted aggregate loss process $\{L(t) : t \in [0, T]\}$ retains its original distributional characteristics after changing from the historical estimated actual probability measure $\mathbb{P}$ to the risk-neutral probability measure $\mathbb{Q}$. The σ-algebras $\mathcal{A}_T^{(1)}$ and $\mathcal{A}_T^{(2)}$ are independent under the risk-neutral probability measure $\mathbb{Q}$. In an arbitrage-free market (assumption (i)) at any time t, the price of an attainable contingent claim with payoff $\{P(T) : T > t\}$ can be expressed by the fundamental theorem of asset pricing in the following form:

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r(s)ds} P(T)|\mathcal{F}_t\right), \tag{2}$$

see (Delbaen and Schachermayer 1994). Similar to (Shao et al. 2017), the authors assume particular types of payoff functions. Thus, the authors denote the CAT risk bonds price process by $\left\{V^{(\varrho)}(t) : t \in [0, T]\right\}$, which is characterized by the aggregate loss process $\{L(t) : t \in [0, T]\}$, and the payoff functions $P_{CAT}^{(\varrho)}$, where $\varrho = 1, 2, 3, 4$. For each $t \in [0, T]$, the process $\{N(t) : t \in [0, T]\}$ describes the number of claims that occur until time t. In addition, define the spot interest rate process by $\{r(t) : t \in [0, T]\}$ and let $\{W(t) : t \in [0, T]\}$ be a standard Brownian motion.

Interest rate process

There are different types of interest rates, such as government and interbank rates. Zero-coupon rates can be either from government rates which are usually deduced by bonds issued by governments or from interbank rates which are exchanged deposits between banks. The most important interbank rate usually considered as a reference for contracts is the LIBOR rate, fixed daily in London. For the purpose of bond prices, all kinds of interest rate models are feasible. The first stochastic interest rate model was proposed by (Merton 1973), followed by the pioneering approach of (Vasicek 1977) and some other classical models, such as (Dothan 1978; Cox et al. 1985; Ho and Lee 1986; Hull and White 1990; Black et al. 1990). This section provides analysis for two spot interest rate dynamics: Vasicek and Cox-Ingersoll-Ross (CIR) models, with the explicit solution for the value of zero-coupon bonds, which are widely used in the financial literature. Other forms of spot interest rates can also be used in the N-CAT risk bond model, but require computational calculations for the zero-coupon bond value.

Vasicek model

The instantaneous short rate has the following stochastic process under the risk-neutral measure $\mathbb{Q}$:

$$dr(t) = a(b - r(t))dt + \sigma dW(t), \tag{3}$$

where $\{W(t) : t \in [0, T]\}$ is a standard Wiener process under $\mathbb{Q}$. The terms a and b are, respectively, mean reversion speed and mean reversion level of the short rate. The price of a zero-coupon bond at time t with maturity time T is:

$$B_V(t, T) = A(t, T)e^{-B(t,T)r(t)}, \tag{4}$$

where

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}, \tag{5}$$

$$A(t, T) = \exp\left(\frac{(B(t, T) - T + t)\left(a^2 b - \sigma^2/2\right)}{a^2} - \frac{\sigma^2 B(t, T)^2}{4a}\right). \tag{6}$$

Cox–Ingersoll–Ross (CIR) model

The short-rate dynamics $\{r(t) : t \in [0, T]\}$ under the risk-neutral measure $\mathbb{Q}$ can be expressed as follows:

$$dr(t) = k\left[\theta - r(t)\right]dt + \sigma\sqrt{r(t)}dW(t), \tag{7}$$

with the condition

$$2k\theta > \sigma^2, \tag{8}$$

where $\{W(t) : t \in [0, T]\}$ is a standard Brownian motion, and $r(0)$, k, θ and σ are positive constants. CIR model is an extension of Vasicek model where the standard deviation factor changes over time. It also fixes the Vasicek model shortcoming on theoretically possibility of a negative interest rate. (Nowak and Romaniuk 2013a) compared the CAT bond prices under the assumption of the spot interest rate described by the Vasicek, Hull-White, and CIR models. Readers can refer to (Brigo and Mercurio 2007) for more information on interest rate dynamics. Assume a constant $\lambda_r(t)$ which represents the market price of risk, and price a pure-discount T-bond at time t by the following equalities:

$$B_{CIR}(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{9}$$

where

$$A(t,T) = \left[\frac{2\gamma e^{(k+\lambda_r+\gamma)(T-t)/2}}{2\gamma + (k+\lambda_r+\gamma)\left(e^{(T-t)h} - 1\right)}\right]^{\frac{2k\theta}{\sigma^2}}, \tag{10}$$

$$B(t,T) = \left[\frac{2\left(e^{(T-t)\gamma} - 1\right)}{2\gamma + (k+\lambda_r+\gamma)\left(e^{(T-t)\gamma} - 1\right)}\right], \tag{11}$$

$$\gamma = \sqrt{(k+\lambda_r)^2 + 2\sigma^2}. \tag{12}$$

Aggregate claims process

In the classical actuarial literature, (Bowers Jr. et al. 1986) stated that risk models are characterised by the following two stochastic processes: the claim number process (or frequency), which counts the claims; the claim amounts process or severity, which determines the size of losses when a claim occurs. Previous literature on CAT risk bonds assumed that these two processes are mutually independent. However, because the independence assumption is restrictive in many applications, the relationship between the claim sizes and the inter-arrival times between the events process is considered when modeling the aggregate losses of CAT risk bonds, and the first experimental analysis was conducted by (Shao et al. 2017; Ayyub et al. 2016). This paper completes those models, and introduces additional flexibility for practitioners to implement in a real world CAT risk bond deal.

The aggregate loss process $\{L(t) : t \in [0, T]\}$ is defined as a function of two independent variables, claim number process $\{N(t) : t \in [0, T]\}$ and claim sizes $\{X_n : n \in \mathbb{N}^+\}$:

$$L(t) = \sum_{n=1}^{N(t)} X_n, \tag{13}$$

with the convention that $L(t) = 0$ when $N(t) = 0$, and $X_0 = N(0) = 0$ almost surely (a.s.). The value of the total loss process $L(t)$ is typically calculated by the bond issuer to determine whether or not it met the predetermined level of the trigger event specified in the bond contract.

Similar to (Ayyub et al. 2016; Shao et al. 2017), the authors also consider a semi-Markovian dependence structure in continuous time, where the process $\{J_n, n \geq 0\}$ represents the successive type of claims or environment states taking their values in $J = \{1, \ldots, w, w+1, \ldots, w+r\}$. However, this is an extension of (Ayyub et al. 2016) to a more general case with $(w + r)$ states. For notational convenience, denote $W = \{1, 2, \ldots, w\}$, and $O = \{w+1, w+2, \ldots, w+r\}$, therefore, $J = W + O$. Here states W are called the work of the system, referring to the incident and accident risks events; and states O (absorbing states) are defined as the failure of the system, where the N-CAT risk bonds system terminates when a major accident risk event occurs, leading the bonds to exercise immediately. the authors call states O absorbing states because once the system reaches those states, the system is unable to escape and will stay there forever. Bond issuers can structure multiple absorbing states in their contract to establish a CAT risk bonds which will exercise immediately in different predetermined situations. The transition matrix $\mathbf{P} = \left(p_{ij}, i, j \in J\right)$ can be written as

$$\mathbf{P} = \begin{pmatrix} \mathbf{W} & \mathbf{R} \\ \mathbf{0} & \mathbf{I}_r \end{pmatrix} = \begin{pmatrix} p_{11} & \cdots & p_{1w} & p_{1(w+1)} & \cdots & \cdots & p_{1(w+r)} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ p_{w1} & \cdots & p_{ww} & p_{w(w+1)} & \cdots & \cdots & p_{w(w+r)} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \vdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{14}$$

where $\sum_{j=1}^{w+r} p_{ij} = 1, i \in J$, and $\mathbf{I}_r$ is an r-by-r identity matrix. To interpret this claim based N-CAT risk bond structure more precisely: the bond should not be exercised before expiration if and only if all events occurred stay in the incident or accident level (the event state i stays in the period of work of the system W), and the probability having the next event in state j ($j \in J$) is p_{ij}. If a major accident occurs (a state O event), the N-CAT risk bond contract will terminate immediately, i.e. the system will stay in the state O. Also, $J_{N(T)}$ is the state where the last claim stays at the exercise date.

Define $\{T_n, n \in \mathbb{N}^+\}$ to be the epoch time of the nth claim. Suppose that $0 < T_1 < T_2 < \ldots < T_n < T_{n+1} < \ldots, T_0 = U_0 = 0$ a.s., and let $U_n = T_n - T_{n-1}$ ($n \in \mathbb{N}^+$) denote the sojourn time in state J_{n-1}. Assume that the trivariate process $\{(J_n, U_n, X_n)\,; n \geq 0\}$ is a semi-Markovian dependency process defined by the matrix $\mathbf{Q} = \left(Q_{ij}, i, j \in J\right)$:

$$Q_{ij}(t, x) = \mathbb{P}\left(J_n = j, U_n \leq t, X_n \leq x \middle| \left(J_k, U_k, X_k\right), k = 1, 2, \ldots, n-1, J_{n-1} = i\right). \tag{15}$$

Assuming that the random variable $J_n, n \geq 0$ and the two-dimensional random variable $(U_n, X_n), n \geq 1$ are conditionally independent, then

$$G_{ij}(t,x) = \mathbb{P}(U_n \le t, X_n \le x | J_0, \ldots, J_{n-1} = i, J_n = j)$$
$$= \begin{cases} Q_{ij}(t,x)/p_{ij}, \text{ for } p_{ij} > 0, \\ \mathbb{1}\{t \ge 0\}\mathbb{1}\{x \ge 0\}, \text{ for } p_{ij} = 0, \end{cases} \tag{16}$$

where $\mathbb{1}\{\cdot\}$ denotes an indicator function. Denote now

$$G_{ij}(t,\infty) = \mathbb{P}(U_n \le t | J_0, \ldots, J_{n-1} = i, J_n = j), \tag{17}$$
$$G_{ij}(\infty,x) = \mathbb{P}(X_n \le x | J_0, \ldots, J_{n-1} = i, J_n = j), \tag{18}$$
$$H_i(t,x) = \mathbb{P}(U_n \le t, X_n \le x | J_0, \ldots, J_{n-1} = i) = \sum_{j=0}^{m} p_{ij} G_{ij}(t,x), \tag{19}$$
$$H_i(t,\infty) = \mathbb{P}(U_n \le t | J_0, \ldots, J_{n-1} = i), \tag{20}$$
$$H_i(\infty,x) = \mathbb{P}(X_n \le x | J_0, \ldots, J_{n-1} = i). \tag{21}$$

Assuming that the sequences $\{U_n, n \ge 1\}$, $\{X_n, n \ge 1\}$ are conditionally independent and given the sequence $\{J_n, n \ge 0\}$, then

$$G_{ij}(t,x) = G_{ij}(t,\infty)G_{ij}(\infty,x), \forall t,x \in \mathbb{R}, \forall i,j \in J. \tag{22}$$

Thus, the semi-Markov kernel $\mathbf{Q}$ can be expressed as the following product

$$Q_{ij}(t,x) = p_{ij}G_{ij}(t,\infty)G_{ij}(\infty,x), \forall t,x \in \mathbb{R}, \forall i,j \in J. \tag{23}$$

Let L_n be the successive total claims amount after the arrival of the nth claim. Then, the joint probability of the process $\{(J_n, T_n, L_n); n \ge 0\}$ can be denoted as

$$\mathbb{P}[J_n = j, T_n \le t, L_n \le x | J_0 = i] = Q_{ij}^{*n}(t,x), \tag{24}$$
$$\mathbb{P}[J_n = j, T_n \le t, L_{n-1} \le x | J_0 = i] = \tilde{Q}_{ij}^{*n}(t,x), \tag{25}$$

where $i, j \in J$. It is crucial to introduce the process $\widetilde{Q}^{*n}_{ij}(t, x)$ because when a major accident occurs (a state O event), the N-CAT risk bonds need to be exercised immediately regardless of the size of this particular event. These two n-fold convolution matrices $\mathbf{Q}^{*n} = \left(Q^{*n}_{ij}, i, j \in J\right)$ and $\widetilde{\mathbf{Q}}^{*n} = \left(\widetilde{Q}^{*n}_{ij}, i \in J, j \in 0\right)$ can be valued recursively by the following two parts:

$$Q^{*0}_{ij}(t, x) = \begin{cases} \left[1 - G_{ij}(0, \infty)\right] \left[1 - G_{ij}(\infty, 0)\right], \text{ if } i = j, \\ 0, \text{ elsewhere,} \end{cases} \tag{26}$$

$$Q^{*1}_{ij}(t, x) = Q_{ij}(t, x), \quad \dots \tag{27}$$

$$Q^{*n}_{ij}(t, x) = \mathbb{P}\left[J_n = j, \dots, J_1 = W, L_n \le x, T_n \le t | J_0 = i\right] = \sum_{k=1}^{w} \int_0^t \int_0^x Q^{*(n-1)}_{kj}\left(t - t', x - x'\right) dQ_{ik}\left(t', x'\right). \tag{28}$$

and

$$\widetilde{Q}^{*0}_{ij}(t, x) = 0, \tag{29}$$

$$\widetilde{Q}^{*1}_{ij}(t, x) = Q_{ij}(t, \infty), \quad \dots \tag{30}$$

$$\widetilde{Q}^{*n}_{ij}(t, x) = \mathbb{P}\left[J_n = 0, J_{n-1} = W, \dots, J_1 = W, L_{n-1} \le x, T_n \le t | J_0 = i\right] = \sum_{k=1}^{w} \int_0^t Q^{*(n-1)}_{ik}\left(t - t', x\right) d\left(Q_{kj}\left(t', \infty\right)\right). \tag{31}$$

Moreover, suppose that a sequence of probabilities $(\Pi_1, \dots, \Pi_{w+r})$ exists (assume that $\Pi_{w+1} = \cdots = \Pi_{w+r} = 0$, a.s.), representing the starting probability distribution for the embedded Markov Chain $\{J_n; n \ge 0\}$, $\Pi_1 + \Pi_2 + \cdots + \Pi_w = 1$ and $\Pi_1, \Pi_2, \cdots, \Pi_w \in [0, 1]$.

The following probabilities are essential for pricing N-CAT risk bonds. At time t, for the predetermined threshold level D ($D \geq 0$),

$$
\begin{aligned}
F_1(t,D) &= \mathbb{P}\left(L(t) \leq D, J_{N(T)} = W\right) = \sum_{i=1}^{w}\sum_{j=1}^{w}\Pi_i \sum_{n=0}^{\infty}\int_0^t \left(1 - H_j\left(t-t',\infty\right)\right) dQ_{ij}^{*n}\left(t',D\right), && (32)\\
F_2(t,D) &= \mathbb{P}\left(L(t) \leq D, J_{N(T)} = O\right) = \sum_{i=w+1}^{w+r}\sum_{i=1}^{w}\Pi_i \sum_{n=0}^{\infty} Q_{ij}^{*n}(t,D), && (33)\\
F_3(t,D) &= \mathbb{P}\left(J_{N(T)} = O\right) = \sum_{i=w+1}^{w+r}\sum_{i=1}^{w}\Pi_i \sum_{n=0}^{\infty} \widetilde{Q}_{ij}^{*n}(t,D), && (34)\\
F_4(t,D) &= \mathbb{P}\left(L(t) > D, J_{N(T)} = W\right) = 1 - F_1(t,D) - F_3(t,D), && (35)\\
F_5(t,D) &= \mathbb{P}\left(L(t) \leq D\right) = F_1(t,D) + F_2(t,D), && (36)
\end{aligned}
$$

which are the probability of a total loss less than the threshold level and that the last event is not a major accident, the probability of a total loss less than the threshold level and that the last event is a major accident, the probability of a major accident occurring, the probability of a total loss greater than the threshold level and that the last event is not a major accident, and the probability of a total loss less than the threshold level, respectively. In the N-CAT risk bonds pricing model, the process changes its state at every claim instance based on the transition matrix $\mathbf{P}$, with the claim size distribution dependent on the future state. While, the arrival time before the next catastrophic claim U_n depends on the severity of the current event X_n, for all $n = 0, 1, 2, \ldots$.

Payoff functions

This section illustrates the most common payoff functions for CAT risk bonds, (Shao et al. 2017) and two-trigger type payoff structure for T time maturity one-period CAT risk bonds. This paper only discusses one-period bonds in this paper because multi-period coupon bonds can be treated as a portfolio of zero-coupon bonds with different maturities. Define a hypothetical zero coupon N-CAT risk bonds with face value Z at the maturity date, as follows:

$$
P_{CAT}^{(1)} = \begin{cases} Z, \text{ for } L(T) \leq D, \\ \eta Z, \text{ for } L(T) > D, \end{cases} \tag{37}
$$

where $L(T)$ is the total insured loss value at the expiry date T, D denotes the threshold value agreed in the bond contract, and η ($\in [0, 1)$) is the fraction of the principle Z, which the bondholders must pay when a trigger event occurs.

The next payoff function with a multi-threshold value is given by

$$P_{CAT}^{(2)} = \sum_{k=1}^{l} \eta_k Z \quad \forall D_{k-1} < L(T) \leq D_k, \tag{38}$$

where $\eta_1 = 1 > \eta_2 > \cdots > \eta_l \geq 0$ and $D_0 = 0 < D_1 < \cdots < D_l = D$. In general, an investor's rate of return is inversely proportional to the total catastrophe claims.

Another payoff function with a coupon payment at the maturity date, if the trigger has not occurred, is of the form

$$P_{CAT}^{(3)} = \begin{cases} Z + C, \text{ for } L(T) \leq D, \\ Z, \text{ for } L(T) > D, \end{cases} \tag{39}$$

where $C > 0$ is the coupon payment level.

The two-trigger type payoff function is defined by the following structure:

1. If at expiry time T, $L(T) \geq D$ ($D \geq 0$) and $J_{N(T)} = W$, that is, the total loss is greater than a predefined level and no major accident occurred prior to T, the bond holder will lose part of the capital and receive $\eta_2 Z (\eta_2 > 0)$;
2. If a major accident (state O event) ($J_k \in \{w + 1, w + 2, \ldots, w + r\}$) occurs before the expiry date T, the N-CAT risk bonds expires immediately and the bond holder will receive a partial amount of their principle $\eta_3 Z$ (normally $0 < \eta_3 < \eta_2$);
3. Otherwise the bond holder will receive the face value Z.

Formally, the payoff function described above is given mathematically by

$$P_{CAT}^{(4)} = \begin{cases} Z, \text{ for } L(T) \leq D \text{ and } J_{N(T)} = W, \\ \eta_2 Z, \text{ for } L(T) > D \text{ and } J_{N(T)} = W, \\ \eta_3 Z, J_{N(T)} = O. \end{cases} \tag{40}$$

According to (Shao et al. 2017), the bondholders' payoffs are also determined by the bond issuers' leverage ratio which is the indicator of the financial risk. In this paper, assume that F_{De} is the probability of a certain financial institute defaulting in a given period, while bondholders receive 0 if their bond seller is unable to repay their obligation, which is the worst case scenario.

Pricing N-CAT risk bonds

This section derives the price of N-CAT risk bonds using the standard tool of a risk-neutral valuation measure with the payoff functions mentioned above. N-CAT risk bond prices at time t paying principal Z at time to maturity T are given in the following Theorems 2.1 to 2.4, see also (Shao et al. 2017; Ayyub et al. 2016; Cox and Pedersen 2000).

Theorem 2.1. *Let $V^{(1)}(t)$ be the prices of the T-maturity zero-coupon N-CAT risk bond with face value Z under the risk-neutral measure $\mathbb{Q}$ at time t with payoff function $P_{CAT}^{(1)}$, as defined in Eq.* (37). *Then,*

$$V^{(1)}(t) = B(t,T)Z\left(\eta + (1-\eta)\left(F_1(T-t,D) + F_2(T-t,D)\right)\right)\left(1 - F_{De}\right), \tag{41}$$

where $F_1(T-t,D)$ and $F_2(T-t,D)$ represent the probabilities given in Eqs. (32) *and* (33)*, respectively, pure discounted bond price $B(t,T)$ is the zero-coupon bond value, and F_{De} is the probability of a bond issuer defaulting.*

Proof. (Cox and Pedersen 2000) stated that the payoff function is independent of the financial risks variable (interest rate) under the risk-neutral measure $\mathbb{Q}$. Then, according to Eq. (2),

$$V^{(1)}(t) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds} P_{CAT}^{(1)}(T)|\mathcal{F}_t\right) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}|\mathcal{F}_t\right)\mathbb{E}^{\mathbb{Q}}\left(P_{CAT}^{(1)}(T)|\mathcal{F}_t\right). \tag{42}$$

Using the closed form solution of the zero-coupon bond price, $\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\right) = B(t,T)$ as discussed in Interest Rate Process section, where $B(t,T) = B_V(t,T)$ or $B_{CIR}(t,T)$ in this paper, and this can be easily substituted depending on the choice of the interest rate model. Together with Eq. (1), the above equation can be rewritten as

$$B(t,T)\mathbb{E}^{\mathbb{P}}\left(P_{CAT}^{(1)}(T)|\mathcal{F}_t\right). \tag{43}$$

By simply applying the payoff function Eq. (37) and rearranging the formula, the N-CAT risk bond price can be formulated as

$$\begin{aligned} V^{(1)}(t) =& B(t,T)\mathbb{E}^{\mathbb{P}}\left[(Z\mathbb{1}\{L(T) \le D\} + \eta Z\mathbb{1}\{L(T) > D\})(1 - F_{De})|\mathcal{F}_t\right] \\ =& B(t,T)\left(Z\mathbb{P}\left(L(T) \le D\right) + \eta Z\mathbb{P}\left(L(T) \ge D\right)\right)(1 - F_{De}) \\ =& B(t,T)Z\left(F_5(T,D) + \eta\left(1 - F_5(T,D)\right)\right)(1 - F_{De}), \end{aligned} \tag{44}$$

The result follows by some rearrangement. □

The proofs of Theorems 2.2 and 2.3 follow the same procedure of Theorem 2.1.

Theorem 2.2. *Let $V^{(2)}(t)$ be the prices of the T-maturity zero-coupon N-CAT risk bond with face value Z under the risk-neutral measure $\mathbb{Q}$ at time t with payoff function $P_{CAT}^{(2)}$, as defined in Eq.* (38). *Then,*

$$V^{(2)}(t) = B(t,T)Z\sum_{k=1}^{l}\eta_k\left(F_5\left(T-t, D_k\right) - F_5\left(T-t, D_{k-1}\right)\right)(1 - F_{De}), \tag{45}$$

where $F_5(T-t, D)$ represents the probabilities given in Eq. (36)*, pure discounted bond price $B(t,T)$ is the zero-coupon bond value, and F_{De} is the probability of a bond issuer defaulting.*

Theorem 2.3. *Let $V^{(3)}(t)$ be the prices of the T-maturity coupon N-CAT risk bond with face value Z under the risk-neutral measure $\mathbb{Q}$ at time t with payoff function $P^{(3)}_{CAT}$, as defined in Eq.* (39)*. Then,*

$$V^{(3)}(t) = B(t,T)\left(Z + CF_5(T-t,D)\right)(1-F_{De}), \tag{46}$$

where $F_5(T-t,D)$ represents the probabilities given in Eq. (36)*, pure discounted bond price $B(t,T)$ is the zero-coupon bond value, and F_{De} is the probability of a bond issuer defaulting.*

Theorem 2.4. *Let $V^{(4)}(t)$ be the value of the T-maturity zero-coupon N-CAT risk bond with face value Z under the risk-neutral measure $\mathbb{Q}$ at time t with payoff function $P^{(4)}_{CAT}$, as defined in Eq.* (40)*. Then,*

$$V^{(4)}(t) = B(t,T)Z\left(\eta_2 + (1-\eta_2)F_1(T-t,D) + (\eta_3-\eta_2)F_3(T-t,D)\right)(1-F_{De}), \tag{47}$$

where $F_1(T-t,D)$ and $F_3(T-t,D)$ represent the probabilities given in Eqs. (32) *and* (34)*, respectively, pure discounted bond price $B(t,T)$ is the zero-coupon bond value, and F_{De} is the probability of a bond issuer defaulting.*

Proof. Similar to the proof of Theorem 2.1, we have:

$$\begin{aligned}
V^{(4)}(t) =& B(t,T)\mathbb{E}^{\mathbb{P}}\left(P^{(4)}_{CAT}(T)|\mathcal{F}_t\right)\\
=& B(t,T)\mathbb{E}^{\mathbb{P}}[(Z\mathbb{1}\left\{L(T)\le D, J_{N(T)}=W\right\} + \eta_2 Z\mathbb{1}\left\{L(T)>D, J_{N(T)}=W\right\}\\
&+\eta_3 Z\mathbb{1}\left\{J_{N(T)}=O\right\})(1-F_{De})|\mathcal{F}_t]\\
=& B(t,T)Z(\mathbb{P}\left(L(T)\le D, J_{N(T)}=W\right) + \eta_2\mathbb{P}\left(L(T)>D, J_{N(T)}=W\right)\\
&+\eta_3\mathbb{P}\left(J_{N(T)}=O\right))(1-F_{De})\\
=& B(t,T)Z\left(F_1(T,D)+\eta_2F_4(T,D)+\eta_3F_3(T,D)\right)(1-F_{De}).
\end{aligned} \tag{48}$$

The result follows by some rearrangement. □

NUMERICAL EXAMPLES: ANALYSIS AND DISCUSSION

Due to limitations in obtaining real data for the determination and calibration of some of the many parameters involved in the pricing process of the N-CAT risk bond, thus for illustration purposes of the theoretical findings, the following two numerical examples are discussed. In the insurance industry, bond issuers can judge their situation and choose a suitable model (more accurately: number of states and the payoff structure) applicable for them. It is important to note that the choice of the model and distribution are crucial in N-CAT risk bond pricing because they affect the bond price significantly. However, the method of selecting a better model and the numerical algorithm are beyond the scope of this paper. Interested readers can refer to (Shao et al. 2017) for more details.

A generalized example

This section considers a general example of N-CAT risk bonds, applying the pricing formula in the previous section as an illustration.

As in (Ayyub et al. 2016), in this particular example, the authors adopt the same number of states and distributions. Thus, there are 4 states in the period of work of the system ($w = 4$) and 1 absorbing state ($r = 1$). The inter-arrival time distribution $G_{ij}(t, \infty)$ is defined to be a Poisson process with parameter λ_i, determined by the state where the system starts. Here, arbitrarily choose $\lambda_i = 10, 30, 5, 20$ for $i = 1, 2, 3, 4$, respectively. The claim size distribution $G_{ij}(\infty, x)$ is assumed to follow a lognormal distribution with mean μ_j and variance σ_j, determined by the state where the system ends. Similarly, assume that $\mu_j = 2, 1, 2.5, 3, 1.5$ and $\sigma_j = 1, 0.8, 1.5, 1.2, 1.5$ for $j = 1, 2, 3, 4, 5$, respectively. Moreover the transition matrix $\mathbf{P} = (p_{ij})$ is given by

$$\mathbf{P} = \begin{pmatrix} 0.397 & 0.3 & 0.2 & 0.1 & 0.003 \\ 0.4 & 0.096 & 0.3 & 0.2 & 0.004 \\ 0.4 & 0.4 & 0.199 & 0.1 & 0.002 \\ 0.2 & 0.2 & 0.5 & 0.098 & 0.001 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{49}$$

and the stationary distribution $(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = (0.348, 0.261, 0.264, 0.127)$. The parameters of the interest rate model are calibrated from the same data set as in (Shao et al. 2017) (i.e., 3-month maturity US monthly treasury bill data for the period of 1994–2013). Assume that in both models, the initial short-term interest rate r_0 is 0.3% and the market price of risk λ_r is a constant -0.01. The parameters of the Vasicek model are $\{a, b, \sigma\} = \{0.0790, 3.48\%, 1.28\%\}$, and the parameters of the CIR model are $\{k, \theta, \sigma\} = \{0.0388, 3.78\%, 5.32\%\}$.

To analyze the N-CAT risk bond price sensitivity in terms of the maturity and threshold level, this paper calculates the bond values with the face value of US$1,000 for $T = [0.5, 2]$ years to maturity and threshold level $D = [100, 1600]$ in millions of US$. The payoff function's parameters are $\eta = 0.5$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\eta_3 = 0.25$, $C = 0.1$ and $F_{De} = 0.1\%$. For the case when $P^{(2)}_{CAT}$, $D_1 = 100$ US$ in millions, D_2 varies within the range of $[100, 1600]$, and $D_3 = \infty$.

A benefit of the CIR model over the Vasicek model to analyze the spot interest rate is that CIR prevents the interest rate falling below zero, which applies to the majority of the real world interest rate situations. However, according to Figures 3 and 4, the actual differences in the zero-coupon bond prices, implementing the formula in the Interest Rate Process section, between those two models are relatively small. N-CAT risk bond issuers can employ other interest rate models in their contract, but the change is highly likely to be non-significant compared to the bond value proposed in this paper, see also (Nowak and Romaniuk 2013a). Therefore, from this point onwards, only results based on the CIR interest rate model will be shown to save space.

Figure 5 illustrates the value of the N-CAT risk bonds with face value US$1000 for the payoff functions $P^{(1)}_{CAT}$, $P^{(2)}_{CAT}$, $P^{(3)}_{CAT}$ and $P^{(4)}_{CAT}$ with threshold level D and time to maturity T under the stochastic interest rate assumptions. Comparing across the sub-figures in Figure 5, the bond values depend heavily on the choice of the payoff function. The value of a zero-coupon bond is normally less than its face value, as is indicated in Figures 5a, 5b, 5d; while for a coupon bond, the bond value is greater than the face value, see Figure 5c. For all payoff functions, the prices of N-CAT risk bonds ($V(t)$) decrease with the increase of the time to maturity (T); while the bond prices ($V(t)$) increase with the increase of the threshold level (D).

Another example featuring the International Nuclear and Radiological Event Scale (INES)

The International Atomic Energy Agency (IAEA) introduced a worldwide tool – the INES Scale – for communicating the safety significance or damage severity of nuclear and radiological events, see (International Atomic Energy Agency (IAEA) 2013) and Figure 6 for more details. The pyramid on the left-hand side of Figure 6 classifies nuclear-related events on the scale of level 0 to level 7, and the severity of an event falling within one level is about ten times greater than in the previous level. While the right-hand side of Figure 6 generally describes the events in terms of a range of impacts, including people and the environment, radiological barriers and control and defence-in-depth. NCAT risk bond issuers are encouraged to use the INES Scale as a general guidance in bond contract design. In this example, assume the number of states corresponds to the INES Scale level (5 states in the period of work of the system $w = 5$ for risk levels 1 to 5, and 2 absorbing states $r = 2$ for risk levels 6 and 7).

The inter-arrival time distribution $G_{ij}(t, \infty)$ is defined to be a Poisson process with parameter λ_i, determined by the state where the N-CAT risks bonds system starts. Here, arbitrarily choose $\lambda_i = 5, 20, 10, 30, 40$ for $i = 1, 2, 3, 4, 5$, respectively. Again, the authors omitted the analysis of the effect on CAT risk bonds between different $G_{ij}(t, \infty)$ distributions. The claim size distribution $G_{ij}(\infty, x)$ is assumed to follow a lognormal distribution with mean μ_j and variance σ_j, determined by the state where the system ends. Similarly, assume that $\mu_j = 0, 1, 2, 3, 4, 5, 6$ and $\sigma_j = 0.25, 0.5, 1, 1.5, 2, 10, 20$ for $j = 1, 2, 3, 4, 5, 6, 7$, respectively. In general, claims with a higher risk level (or which are more severe) tend to receive more losses, and have more chance to experience an extreme event. Moreover the transition matrix $\mathbf{P} = (p_{ij})$ is given by

$$\mathbf{P} = \begin{pmatrix} 0.4989 & 0.25 & 0.15 & 0.06 & 0.04 & 1\times10^{-3} & 1\times10^{-4} \\ 0.25 & 0.3978 & 0.2 & 0.1 & 0.05 & 2\times10^{-3} & 2\times10^{-4} \\ 0.3 & 0.2 & 0.2967 & 0.1 & 0.1 & 3\times10^{-3} & 3\times10^{-4} \\ 0.35 & 0.25 & 0.15 & 0.1956 & 0.05 & 4\times10^{-3} & 4\times10^{-4} \\ 0.35 & 0.3 & 0.15 & 0.1 & 0.0945 & 5\times10^{-3} & 5\times10^{-4} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{50}$$

and the stationary distribution $(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5) = (0.3, 0.25, 0.2, 0.15, 0.1)$. Therefore, in this example, there is only a small chance to have a level 6 or level 7 severity accident compared to the chances of having a level 1 to 5 severity event. Additionally, the parameters of the interest rate model and the payoff functions are the same as in the previous example.

The value of the N-CAT risk bonds with face value US$1000 for the payoff functions $P_{CAT}^{(1)}$, $P_{CAT}^{(2)}$, $P_{CAT}^{(3)}$ and $P_{CAT}^{(4)}$ with threshold level $D = 1000$ and time to maturity $T = 1$ under the stochastic interest rate assumptions are 961.51, 780.77, 1088.10 and 953.05, respectively. Comparing with the previous example (980.51, 636.87, 1091.90 and 969.76, respectively), the CAT risk bond prices in the new example are lower than in the previous example as it is more risky (includes more risks with higher possible losses). Similarly, with additional coupon payment, the bond price $V^{(3)}$ is more valuable than the face value; the simple zero-coupon price $V^{(1)}$ depreciates in value with extra layers of discount on face value ($P_{CAT}^{(2)}$) and additional default risks ($P_{CAT}^{(4)}$) in the payoff functions, as a result of $V^{(2)}$ and $V^{(4)}$, respectively.

CONCLUSIONS

This paper set out to explore the concept of modeling N-CAT risk bonds under various scenarios, and to help bond sponsors to set a fair price in their contract. The motivation behind this work was to protect those liability limited regions against the huge economic losses caused by the nuclear power plant faults. Moreover, there is increasing attention in this area because of the 2011 Fukushima disaster and the UK Hinkley nuclear power plant. In our approach a complete N-CAT risk bond model is proposed as an easily applicable solution for practitioners, filling the gap between the theoretical study and the real world application.

The aggregate claims process is one of the most popular indicators as the CAT risk bond trigger. This paper employs a semi-Markov structure to model the dependence of the claim intensity on the severity. In addition, this is the very first paper which includes absorbing states in the Markov process and presents a generalised model with w transit states to indicate the work of the system and r absorbing states to indicate the stop of the system in the CAT risk bonds literature. In any real world application, bond issuers can use any interest rate model they prefer to obtain a pure zero-coupon bond value. However, this might require numerical approximation, because there is not always a closed form solution for a given interest rate model. This paper employed the two most commonly used interest rate models as illustrations. Moreover, four types of payoff structure are proposed in this paper. It is proved that given the same time to maturity and threshold level, different payoff structures can suggest significantly different prices. Additionally, the driving factors of the N-CAT risk bond value are the length of the CAT risk bond contract and the level of the trigger threshold value, i.e., the longer the time to maturity and the smaller the threshold level, the lower the value of the bond. This work is also applicable to other catastrophe risks events.

Although, in the present paper, a model is proposed with the flexibility of different interest rates, aggregate claims, payoff structures and the underlying distributions, the relationship between the nuclear power risks and the financial market risks is not considered in our framework. In the literature (Gürtler et al. 2016; Ragin and Halek 2016) examined the impact of natural catastrophes and financial crises on the CAT risk bond premiums. It would be interesting though to consider the case of terrorism as a future extension of the current model (Allison 2005; Kunreuther et al. 2005). This is still a very challenging area to address in future research.

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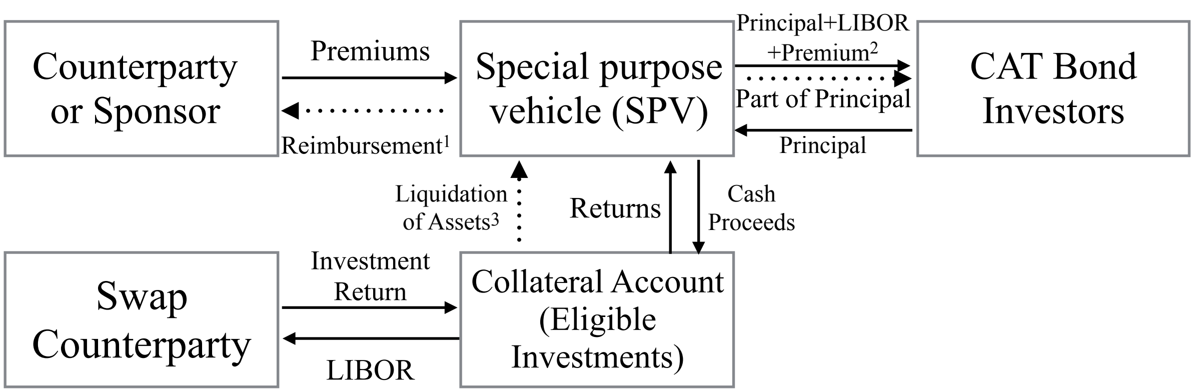


[1] Event Contingent
[2] At maturity
[3] Event contingent or at maturity

Fig. 1. The process for CAT risk bonds is described.

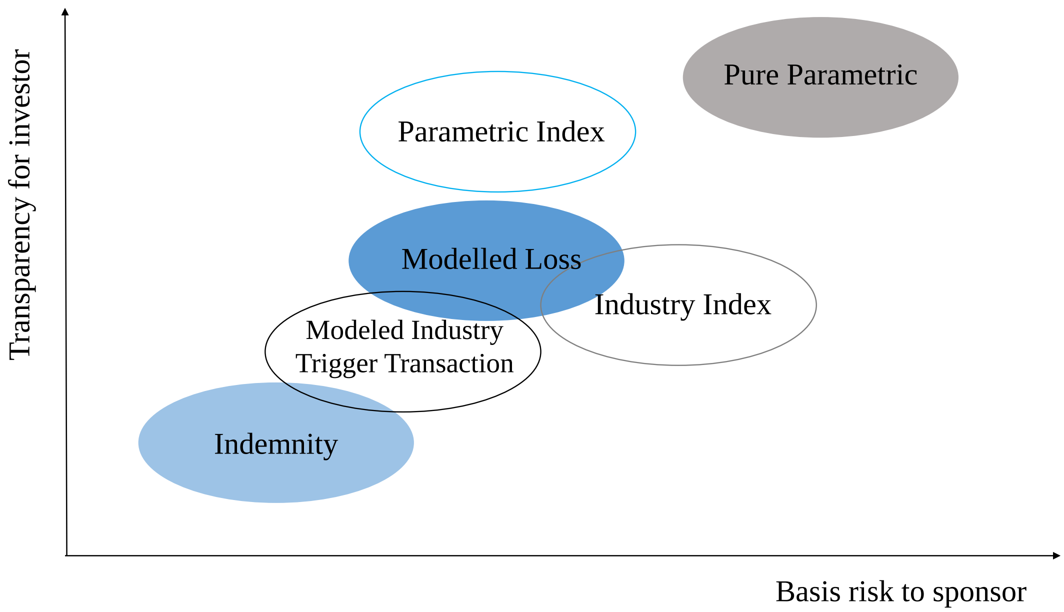


Fig. 2. The transparency and basis risk for various types of triggers.

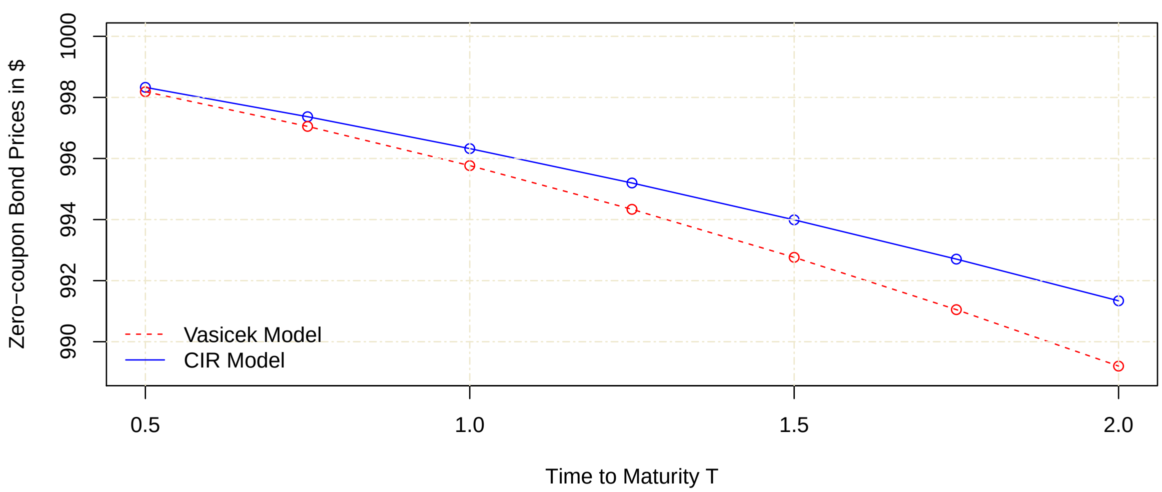


Fig. 3. Zero-coupon bond prices with the face value US$1000, time to maturity between 0.5 years to 2 years, when interest rate follows a Vasicek or CIR model.

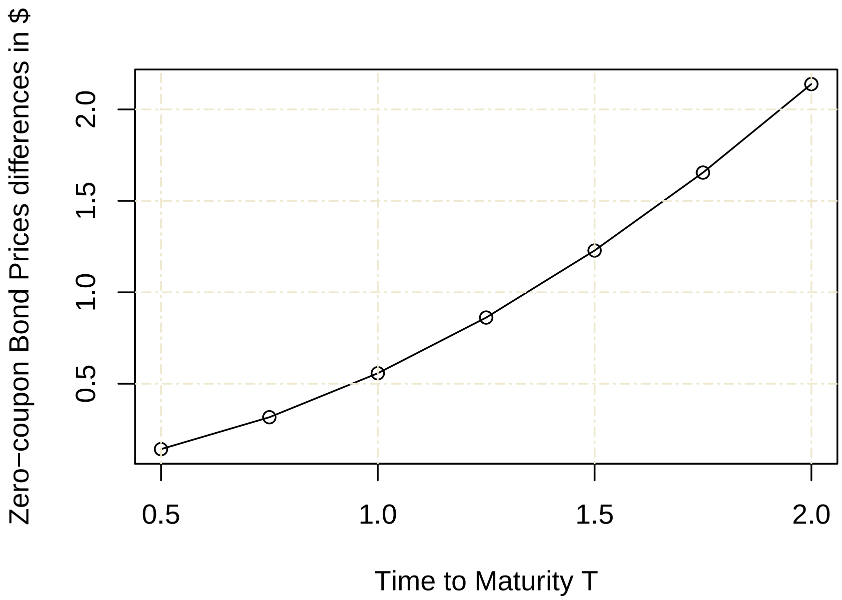
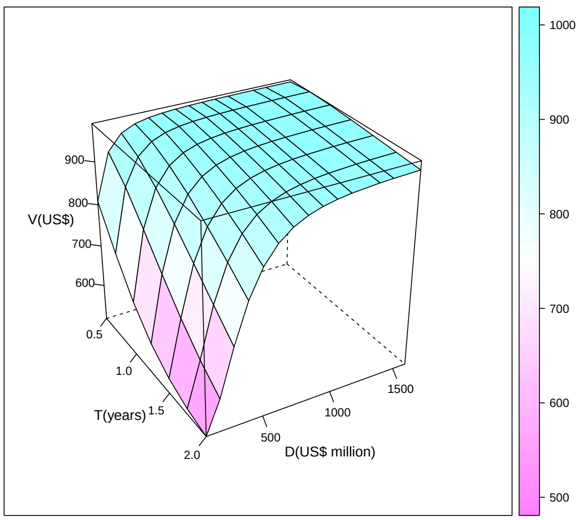
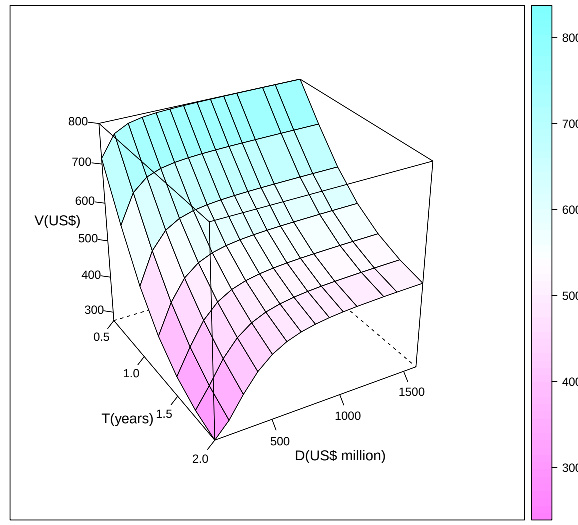


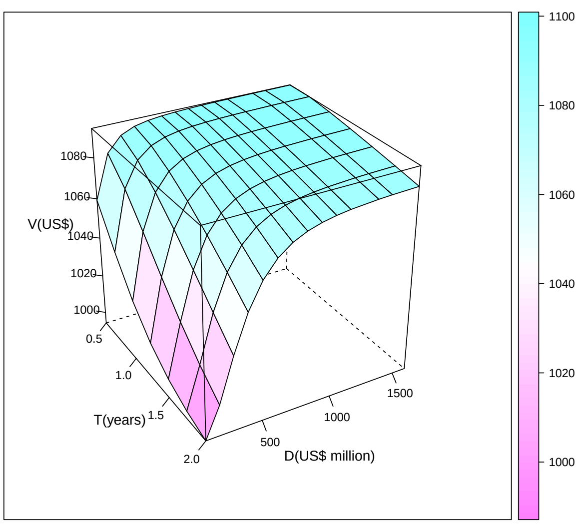
Fig. 4. The relative change between the two prices presented in Fig.3.



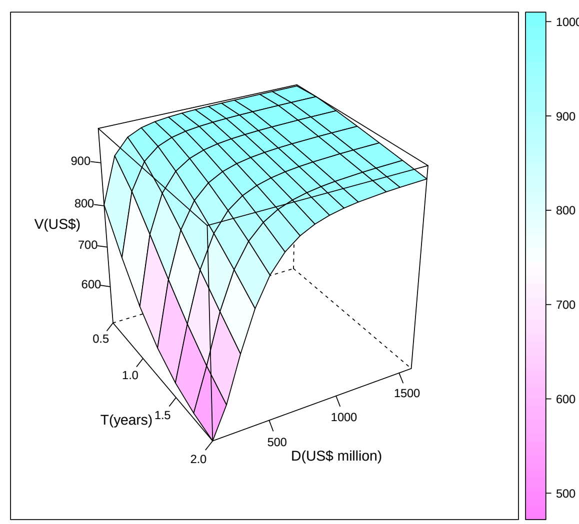
(**a**) The price of N-CAT risk bonds $V^{(1)}(t)$ featuring the payoff function $P^{(1)}_{CAT}$.



(**b**) The price of N-CAT risk bonds $V^{(2)}(t)$ featuring the payoff function $P^{(2)}_{CAT}$.



(**c**) The price of N-CAT risk bonds $V^{(3)}(t)$ featuring the payoff function $P^{(3)}_{CAT}$.



(**d**) The price of N-CAT risk bonds $V^{(4)}(t)$ featuring the payoff function $P^{(4)}_{CAT}$.

Fig. 5. Value of N-CAT risk bonds (z-coordinate axes) with face value US$1000 under the lognormal, the non-homogenous Poisson process and CIR interest rate assumptions. Here, time to the maturity (T) decreases by the left axes and threshold level (D) increases by the right axes.

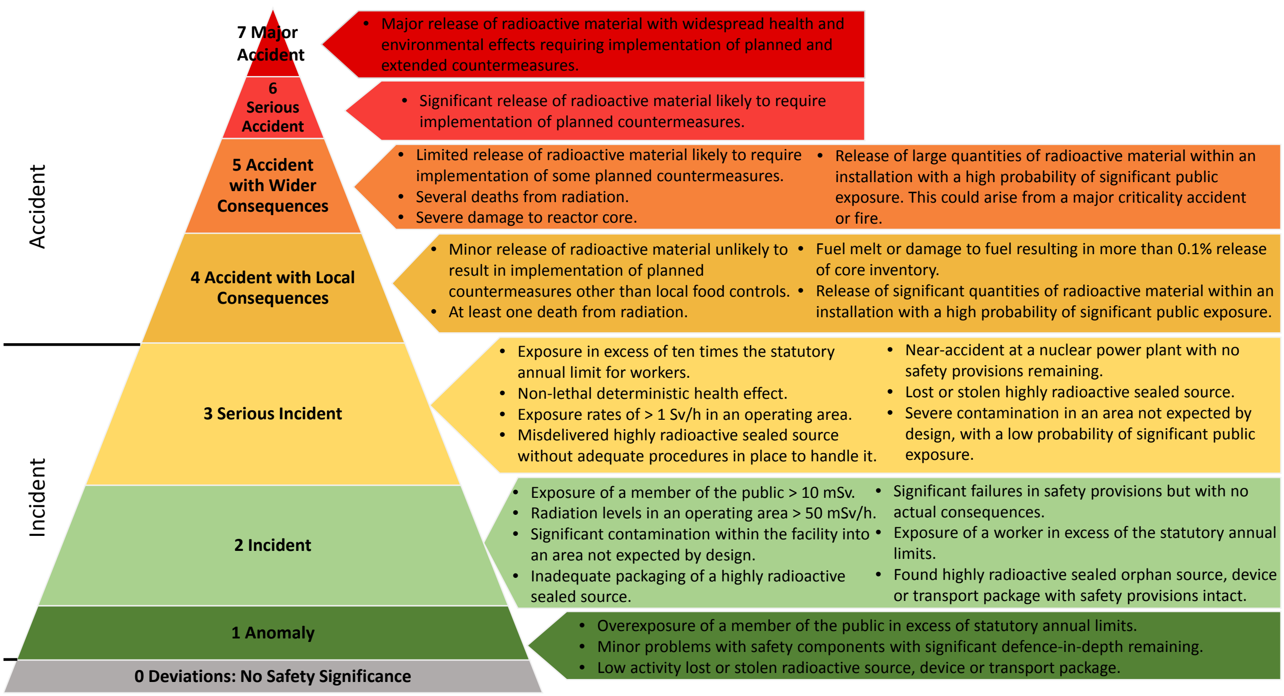


Fig. 6. The international nuclear and radiological event scale (INES) and general description.