Pricing and Simulating Catastrophe Risk Bonds in a Markov-dependent Environment

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Abstract

At present, insurance companies are seeking more adequate liquidity funds to cover the insured property losses related to natural and manmade disasters. Past experience shows that the losses caused by catastrophic events, such as earthquakes, tsunamis, floods, or hurricanes, are extremely high. An alternative method for covering these extreme losses is to transfer part of the risk to the financial markets by issuing catastrophe-linked bonds. In this paper, we propose a contingent claim model for pricing catastrophe risk bonds (CAT bonds). First, using a two-dimensional semi-Markov process, we derive analytical bond pricing formulae in a stochastic interest rate environment with aggregate claims that follow compound forms, where the claim inter-arrival times are dependent on the claim sizes. Furthermore, we obtain explicit CAT bond prices formulae in terms of four different payoff functions. Next, we estimate and calibrate the parameters of the pricing models using catastrophe loss data provided by Property Claim Services from 1985 to 2013. Finally, we use Monte Carlo simulations to analyse the numerical results obtained with the CAT bond pricing formulae.

Keywords: Catastrophe risk bond, Markov-dependent environment, Monte Carlo simulation, Pricing CAT bond.

1. Introduction

Catastrophe risk events such as earthquakes, hurricanes, floods, and manmade disasters occur infrequently, but massive claims over a short period have led to the insolvency of insurance companies. The Insurance Service Office (ISO)'s Property Claim Service (PCS)[1] declared 254 catastrophes that incurred damages of approximately US$112 billion between 1990 and 1996, while the losses due to Hurricane Andrew in 1992 reached US$26 billion[2]. The potentially enormous financial demands on insurance (reinsurance) businesses and the increasing difficulty of covering catastrophic losses by reinsurance mean that it would be useful to introduce a securitization method to protect vulnerable individuals.

[Insert Figure 1 about here]

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[1]ISO's Property Claim Service unit is the internationally recognized authority on insured property losses due to catastrophes in the USA, Puerto Rico, and the US Virgin Islands. It contains information on all the historical catastrophes since 1949, including the states affected, perils, and associated loss estimates. http://www.verisk.com/property-claim-services/.

[2]An illustration of the PCS catastrophe loss data converted to 2014 dollars using the Consumer Prices Index (CPI) in US is given in Figure 1, e.g., the Northridge earthquake (1994) with losses of US$20 billion, 9/11 Terrorist Attacks (2001) with losses of US$25 billion, Hurricane Katrina (2005) with losses of US$50 billion, and Hurricane Sandy with losses of $20 billion. Data from PCS.

CAT risk bonds (or Act-of God bonds) are born for these extreme events and sharing the risk to another level — global financial markets as the only pool of cash large enough to underwrite such losses lies in capital markets and the collection of big investors like pension funds, hedge funds and sovereign wealth funds that normally invest in stocks and bonds. CAT risk bonds are the most popular insurance-linked financial securities and their use has been accelerating in the last decade.

The first experimental transaction was completed in the mid-1990s after Hurricane Andrew and the Northridge earthquake, which incurred insurance losses of US\$15.5 billion and US\$12.5 billion, respectively, by a number of specialized catastrophe-oriented insurance and reinsurance companies in the USA, including AIG, Hannover Re, St Paul Re, and USAA [19]. The CAT bonds market has boomed over the years. The issued capital has increased tenfold within ten years, from less than US\$0.8 billion in 1997 to over US\$8 billion in 2007. The issuers raised more than US\$9 billion of new CAT bonds in 2014[3]. CAT bonds are inherently risky, non-indemnity-based multi-period deals, which pay a regular coupon to investors at end of each period and a final principal payment at the maturity date, if no predetermined catastrophic events occur. A major catastrophe in the secured region before the CAT bond maturity date leads to full or partial loss of the capital.

To bear the catastrophe risk, CAT bonds compensate for a floating London Interbank Offered Rate (LIBOR) coupon plus a premium at a rate between 2% and 20% [12, 19]. We can also refer to a catastrophe as a trigger event, where previous studies categorize five types of triggering variables: indemnity, industry index, modelled loss indices, parametric indices, and hybrid triggers [20, 8]. According to [36, 38], the estimates provided by the PCS are widely accepted as the reference index triggers in financial-market derivatives, including exchange-traded futures and options, CAT bonds, catastrophe swaps, industry loss warranties (ILWs), and other catastrophe-linked instruments. Thus, it is reasonable to use the PCS index losses from the entire property and casualty industry in the USA to estimate the parameters related to aggregate losses for pricing CAT bonds in the present study. We also assume that the CAT loss industry indices are instantaneously measurable and updatable.

Despite the raising popularity, the number of previous studies devoted to CAT bonds pricing is relatively limited. The prediction of catastrophe risks requires an incomplete markets framework to evaluate the CAT bonds price, because the catastrophe risk cannot be replicated by a portfolio of primitive securities, see [22, 11, 10, 50]. In the case of an incomplete market, there is no universal pricing theory that successfully addresses issues such as specification of hedging strategies and price robustness, see [53]. For example, [17, 18] derived an equilibrium pricing model for the uncertain parameters of multi-events risks. Another common technique used in an incomplete market setting is the principle of equivalent utility for obtaining indifferent pricing. [53] calculated the price of a contingent claim under a stochastic interest rate for an exponential utility function. An extension was proposed by [16], who introduced a more complex payment structure. [10] used a time-repeatable representative agent utility. Their approach is based on a model of the term structure of interest rates and a probability structure for catastrophe risks, which assumes that the agent uses a utility function to make choices about consumption streams. They applied their theoretical results to Morgan Stanley, Winterthur, USAA, and Winterthur-style bonds. [54] adopted the [10] framework to price a Greek bond using equilibrium pricing theory with dynamic interest rates. Extensions involving an n financial and m catastrophe risks framework were investigated by [45], which were applied to a structured multi-period coupon earthquakes CAT bond. Recently, [32] considered an over-the-counter insurance contract on catastrophe risk between an insurance company and a hedge-fund. Their approach is also benefited by the utility indifference pricing method, see [53].

Several studies have used stochastic processes to price CAT bonds. Under the assumption of continuous time, one of the approaches is to model the probability of credit default which follows the

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[3]Accessed on 17/12/2015, `http://www.artemis.bm/deal_directory/cat_bonds_ils_issued_outstanding.html`. ARTEMIS is an online website since 1999, Artemis provides news, analysis and data on catastrophe bonds, insurance-linked securities and alternative reinsurance capital.

methodology of pricing credit derivatives in finance. [3] presented a continuous time no-arbitrage price of zero coupon and non-zero coupon CAT bonds that incorporated a compound doubly stochastic Poisson process. The main weakness of this paper is the authors assumed that the arbitrage measure and real world measures coincide. [7] corrected and then applied their results with PCS data to calculate the arbitrage-free price of zero-coupon and coupon CAT bonds. [8] illustrated the value of CAT bonds with loss data provided by PCS when the flow of events was an inhomogeneous Poisson process. These approaches were utilized by [21] for calibrating CAT bonds prices for Mexican earthquakes. [31] obtained a simple closed form CAT bond solution with a LIBOR term structure of interest rate.

The relationship between the loss frequency and loss severity has been extensively discussed in the corresponding literature by implementing a variety of different methodologies. For instance, [35] modelled the loss sizes and intensity dependency using a specific extreme dependence structure with heavy-tailed intensity, and the illustrated examples were from the storm insurance. [51, 52] examined the earthquake data, and they introduced cyclic Poisson and self-exciting models to model the behaviour that smaller earthquakes are more likely to happen after a big earthquake. Also, [23] proposed a self-exciting Hawkes process to model feedback between the loss index and the intensity of the counting process precisely. A more recent paper, [26], proposed two marked point processes: a stochastic hazard rate modulated by a Markov chain and a self-exciting process to inducing clustering in catastrophe events to model the catastrophe options. This approach can capture the fact that the occurrence of one catastrophe event might increase the likelihood of the occurrence of another catastrophe event.

In the present study, we derive CAT bond pricing formulae in a stochastic interest rate environment under an assumption that the occurrence of a localized catastrophe is independent of the global financial market behaviour [10]. We make three main contributions to the area of CAT bond pricing. First, we construct our model in a Markov-dependent environment[4] as an extension of the approach proposed by [38]. For the first time in the CAT bonds area, we model the dependency between the claims sizes and the claim inter-arrival times for the aggregate claims as a semi-Markov process. The main benefit of this extension is the development of a more realistic model, where the both occurrence time before the next claim and the claim severity are partially dependent on the claim intensity, which indicates the seasonality effect of catastrophe events, i.e., a major catastrophe event (e.g. an earthquake) triggers many other catastrophe events (e.g. tsunami, side earthquakes, flood, wildland fire and landslide) in a short period, and some other period of the year tend to have more catastrophes. Second, in order to obtain a more complete example, we structure four different payoff functions (classical zero-coupon and coupon, multi-threshold zero-coupon, and defaultable) and we give analytical formulae for CAT bonds. Third, we apply our theoretical results to construct a CAT bond and then we use PCS data to estimate relevant parameters to obtain analytical solutions, thereby providing clear guidance for practitioners.

The reminder of the paper is organized as follows. Section 2 presents the pricing model of CAT bonds including: assumptions, valuation theory, the interest rate and aggregate claims processes, and payoff functions. Section 3 presents a numerical analysis of the PCS data. In Section 4, we provide a discussion of the results.

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[4]There are many other applications using Markov-dependent structure. [30, 29] provided plenty of applications in queueing theory, insurance mathematics, reliability and maintenance and fluid mechanics. [43, 1, 37] dedicated on modelling and computing Semi-Markov processes in ruin theory. In this paper, we extend on the scope of modelling and pricing catastrophe risk bonds.

2. Modeling CAT Bonds

2.1. Modeling assumptions

In this section, we provide preliminary details of the CAT bond structure, which generalizes and extends the CAT bond pricing approaches in [38]. We price CAT bonds under the following assumptions: (i) an arbitrage-free investment market exists with equivalent martingale measure, (ii) the financial market behaves independently of the occurrence of catastrophes, and (iii) the interest rate changes can be replicated using existing financial instruments.

Let $0 < T < \infty$ be the maturity date of the continuous time trading interval $[0, T]$. The market uncertainty is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$, where $\mathcal{F}_t$ is an increasing family of σ-algebras, which is given by $\mathcal{F}_t = \mathcal{F}_t^{(1)} \times \mathcal{F}_t^{(2)} \subset \mathcal{F}$, for $t \in [0, T]$, where $\mathcal{F}_t^{(1)}$ represents the investment information (e.g. past security prices and interest rates) available to the market at time t and $\mathcal{F}_t^{(2)}$ represents the catastrophic risk information (e.g. insured property losses). The financial risk variables and the catastrophic risk variables can be modelled on $(\Omega^{(1)}, \mathcal{F}^{(1)}, (\mathcal{F}_t^{(1)})_{t \in [0,T]}, \mathbb{P}^{(1)})$ and $(\Omega^{(2)}, \mathcal{F}^{(2)}, (\mathcal{F}_t^{(2)})_{t \in [0,T]}, \mathbb{P}^{(2)})$, respectively. Moreover, define two filtrations $\mathcal{A}^{(1)}$ ($\mathcal{A}_t^{(1)} = \mathcal{F}_t^{(1)} \times \{\emptyset, \Omega_2\}$ for $t \in [0, T]$) and $\mathcal{A}^{(2)}$ ($\mathcal{A}_t^{(2)} = \{\emptyset, \Omega_1\} \times \mathcal{F}_t^{(2)}$ for $t \in [0, T]$). It is proved by Lemma 5.1 of [10] that σ-algebras $\mathcal{A}_t^{(1)}$ and $\mathcal{A}_t^{(2)}$ are independent under the probability measure $\mathbb{P}$. Thus, an $\mathcal{A}_T^{(\kappa)}$ measurable random variable X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ (or an $\mathcal{A}^{(\kappa)}$ adapted stochastic process Y) is said to depend only on the financial risk variables ($\kappa = 1$) or catastrophic risk variables ($\kappa = 2$).

Let us define stochastic processes and random variables with respect to the probability measure $\mathbb{P}$. Denote the CAT bond price process by $\{V_\ell^{(\varrho)}(t) : t \in [0, T]\}$, which is characterized by the aggregate loss process $\{L(t) : t \in [0, T]\}$, and the payoff functions $P_{\mathrm{CAT}}^{(\varrho)}$, where $\ell = 0, 1, 2$ and $\varrho = 1, 2, 3, 4$. For each $t \in [0, T]$, the process $\{N(t) : t \in [0, T]\}$ describes the number of claims that occur until the time t. Define a sequence of dependent pairs $\{(X_k, T_k) : k \in \mathbb{N}^+\}$ be the kth claim occurs during the bond coving period, where X_k and T_k represent the size of the claim and epoch time, respectively. In addition, define the spot interest rate process by $\{r(t) : t \in [0, T]\}$ and $\{W(t) : t \in [0, T]\}$ is a standard Brownian motion.

2.2. Valuation theory

The presence of catastrophic risks that are uncorrelated with the underlying financial risks leads us to consider an incomplete market, and there is no universal theory addresses all aspects of pricing. The benchmark to price uncertain cash flow under an incomplete framework is the representative agent. For valuation purposes, similar to [40], we assume that under the risk-neutral pricing measure $\mathbb{Q}$, the overall economy depends only on financial risk variables. This is a fairly natural approximation because the global economic circumstances in terms of exchange and production are only marginally influenced by localized catastrophes. For more information, see [40, 14, 10, 34, 38]. According to Lemma 5.2 in [10], under the assumption that the aggregate consumption is $\mathcal{A}^{(1)}$ adapted (assumption (ii)), for any random variable X that is $\mathcal{A}_T^{(2)}$ measurable, that

$$E^{\mathbb{Q}}[X] = E^{\mathbb{P}}[X]. \tag{1}$$

Thus, a $\mathcal{A}^{(2)}$ adapted aggregate loss process $\{L(t) : t \in [0, T]\}$ retains its original distributional characteristics after changing from the historical estimated actual probability measure $\mathbb{P}$ to the risk-neutral probability measure $\mathbb{Q}$. And the σ-algebras $\mathcal{A}_T^{(1)}$ and $\mathcal{A}_T^{(2)}$ are independent under the risk-neutral probability measure $\mathbb{Q}$. In an arbitrage-free market (assumption (i)) at any time t, the price of an attainable contingent claim with payoff $\{P(T) : T > t\}$ can be expressed by the fundamental theorem of asset pricing in the following form,

$$V(t) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r(s)ds} P(T) | \mathcal{F}_t), \tag{2}$$

see [13].

2.3. Interest rate process

There are different types of interest rates, such as government and interbank rates. Zero-coupon rates can be either from government rates which are usually deduced by bonds issued by governments or from interbank rates which are exchanged deposits between banks. The interbank rate usually considered as a reference for contracts is the LIBOR (London InterBank Offered Rate) rate, fixed daily in London, and is widely used in the literature. For the purpose of bond prices, all kinds of rates are available. The first stochastic interest rate model was proposed by [39], followed by the pioneering approach of [49] and some other classical models, such as [15, 9, 24, 25, 4].

A typical instantaneous interest rate dynamics proposed by Cox, Ingersol, and Ross (CIR model, [9]) assumed a 'square-root' term in the diffusion coefficient. This model is a benchmark because it provides analytical bonds and bond options pricing. The short-rate dynamics $\{r(t) : t \in [0, T]\}$ under the risk-neutral measure $\mathbb{Q}$ can be expressed as follows,

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), \tag{3}$$

with the condition

$$2k\theta > \sigma^2, \tag{4}$$

where $r(0), k, \theta$, and σ are positive constants. The condition equation (4) guarantees that the process $r(t)$ remains in the positive domain and the origin is inaccessible. Assuming the spot interest rate under the real world measure $\mathbb{P}$ with the form:

$$dr(t) = [k\theta - (k + \lambda_r)r(t)]dt + \sigma\sqrt{r(t)}dW^*(t), \tag{5}$$

where $W^*(t) = W(t) + \int_0^t \frac{\lambda_r \sqrt{r(s)}}{\sigma}ds$ is a Brownian motion under the risk measure $\mathbb{P}$ and λ_r is a constant[5] contributing to the market price of risk. Assuming $\mathbb{Q}$ and $\mathbb{P}$ are equivalent measures, then we compare equation (3) and equation (5), and obtain Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_0^t \frac{\lambda_r^2 r(s)}{\sigma^2}ds + \int_0^t \frac{\lambda_r \sqrt{r(s)}}{\sigma}dW^*(s)\right).$$

The market price of risk process $\lambda_r^*(t)$ is a stochastic process with the functional form

$$\lambda_r^*(t) = \frac{\lambda_r}{\sigma}\sqrt{r(t)}, \qquad t \in [0, T].$$

For detailed information about this transformation, please refer to [38, 42, 46, 34, 44].

According to [6], we can price a pure-discount T-bond at time t by the following equalities:

$$B_{\text{CIR}}(t, T) = A(t, T)e^{-B(t,T)r(t)}, \tag{6a}$$

with

$$A(t, T) = \left[\frac{2he^{(k+\lambda_r+h)(T-t)/2}}{2h + (k + \lambda_r + h)(e^{(T-t)h} - 1)}\right]^{\frac{2k\theta}{\sigma^2}}, \tag{6b}$$

$$B(t, T) = \left[\frac{2(e^{(T-t)h} - 1)}{2h + (k + \lambda_r + h)(e^{(T-t)h} - 1)}\right], \tag{6c}$$

$$h = \sqrt{(k + \lambda_r)^2 + 2\sigma^2}. \tag{6d}$$

---

[5]For the case $\lambda_r = 0$, dynamics equation (3) and equation (5) coincide, where risk neutral world and objective world are identical.

2.4. Aggregate claims process

Now, we describe the CAT bonds payment structure. CAT bonds investors receive premiums if no trigger events occur. In this study, we utilize an insurance industry index trigger to price CAT bonds. This means that investors might lose their capital if the estimated aggregate losses from the whole industry exceed a predetermined level. The aggregate loss process is modelled as a compound distribution process, which is characterized by the frequency (inter-arrival times) and the severity (claim sizes) of catastrophic events, (see [33], [48] and [38]).

2.4.1. Classical Cramer-Lundberg risk model

In the classical actuarial literature, [5] stated that risk models are characterized by the following two stochastic processes: the claim number process, which counts the claims; and the claim amounts process, which determines the losses when a claim occurs. All previous studies of CAT bonds assumed that these two processes are mutually independent. However, because the independence assumption is restrictive in many applications, a more appropriate option, especially for CAT bonds, is to add dependence between the claim sizes and the inter-arrival times in the claims process when modelling the aggregate losses. In this section, for the first time in this area, a CAT bond's aggregate claims process, where the dependency among the characterized processes is described by a semi-Markov risk model. This model was first introduced by [41] and fully developed by [27, 30]. In addition, a special case of this model is introduced where the claim arrival process is a continuous time Markov process with an exponential inter-arrival time.

Define the claim number process $\{N(t) : t \in [0, T]\}$, which follows a Poisson process with parameter $\lambda > 0$, to describe the number of future catastrophes in the insured region. The claim sizes $\{X_k : k \in \mathbb{N}^+\}$, which are independent of the process $\{N(t) : t \in [0, T]\}$, comprise a sequence of positive i.i.d. random variables with a common distribution function $F(x) = \mathbb{P}\{X_k < x\}$, which describes the amount of losses incurred by the kth event. Then, the aggregate loss process $\{L(t) : t \in [0, T]\}$ is modelled by a compound Poisson process, as follows:

$$L(t) = \sum_{k=1}^{N(t)} X_k, \tag{7}$$

with the convention that $L(t) = 0$ when $N(t) = 0$.

2.4.2. A semi-Markov structure

Consider a semi-Markovian dependence structure in continuous time, where the process $\{J_n, n \geq 0\}$ represents the successive type of claims or environment states in the equilibrium measure, which take their values in $J = \{1, ..., m\}$ $(m \in \mathbb{N}^+)$. Define $\{X_n, n \geq 1\}$ as a sequence of successive claim sizes, $X_0 = 0$ a.s. and $X_n > 0, \forall n$, and $\{T_n, n \in \mathbb{N}^+\}$ is the epoch time of the nth claim. Suppose that $0 < T_1 < T_2 < \ldots < T_n < T_{n+1} < \ldots$, $T_0 = U_0 = 0$ a.s., and $U_n = T_n - T_{n-1}$ $(n \in \mathbb{N}^+)$ denotes the sojourn time in state J_{n-1}. Suppose that the trivariate process $\{(J_n, U_n, X_n); n \geq 0\}$ is a semi-Markovian dependence process defined by the following matrix $\mathbf{Q}(= (Q_{ij}), i, j \in J)$,

$$Q_{ij}(t, x) = \mathbb{P}(J_n = j, U_n \leq t, X_n \leq x | (J_k, U_k, X_k), k = 1, 2, ..., n-1, J_{n-1} = i),$$

where the process of successive claims $\{J_n\}$ is an irreducible homogeneous continuous time Markov chain with state space J and transition matrix $\mathbf{P}(= (p_{ij}), i, j \in J)$, where $\lim_{t \to \infty, x \to \infty} Q_{ij}(t, x) = p_{ij}, i, j \in J$. The process changes its state at every claim instance based on the transition matrix $\mathbf{P}$, and an interpretation of this model in terms of CAT bonds is that the arrival time before the next catastrophic event U_{k+1} depends partially on the severity of the previous catastrophic event X_k, for all $k = 0, 1, 2, \ldots$.

Assuming that the random variable J_n, $n \geq 0$ and the two-dimensional random variable (U_n, X_n), $n \geq 1$ are conditionally independent, then

$$
\begin{aligned}
G_{ij}(t, x) &= \mathbb{P}(U_n \leq t, X_n \leq x | J_0, ..., J_{n-1} = i, J_n = j) \\
&= \begin{cases} Q_{ij}(t, x)/p_{ij}, & \text{for } p_{ij} > 0, \\ \mathbb{1}\{t \geq 0\}\mathbb{1}\{x \geq 0\}, & \text{for } p_{ij} = 0, \end{cases}
\end{aligned}
$$

where $\mathbb{1}\{\cdot\}$ represents an indicator function. The random variable J_n, $n \geq 0$ is conditionally dependent on the random variable U_n, $n \geq 1$ and also dependent on the random variable X_n, $n \geq 1$. Denote the distributions of U_n and X_n processes from state i to state j as:

$$
\begin{aligned}
G_{ij}(t, \infty) &= \mathbb{P}(U_n \leq t | J_0, ..., J_{n-1} = i, J_n = j), \\
G_{ij}(\infty, x) &= \mathbb{P}(X_n \leq x | J_0, ..., J_{n-1} = i, J_n = j),
\end{aligned}
$$

for $i, j \in J$, and obtain the following equations by integrating out the condition J_n,

$$
\begin{aligned}
H_i(t, x) &= \mathbb{P}(U_n \leq t, X_n \leq x | J_0, ..., J_{n-1} = i) = \sum_{j=1}^{m} p_{ij} G_{ij}(t, x), \\
H_i(t, \infty) &= \mathbb{P}(U_n \leq t | J_0, ..., J_{n-1} = i), \\
H_i(\infty, x) &= \mathbb{P}(X_n \leq x | J_0, ..., J_{n-1} = i).
\end{aligned}
\tag{8}
$$

Assuming that the sequences $\{U_n, n \geq 1\}$, $\{X_n, n \geq 1\}$ are conditionally independent and given the sequence $\{J_n, n \geq 0\}$, then

$$
G_{ij}(t, x) = G_{ij}(t, \infty) G_{ij}(\infty, x), \qquad \forall t, x \in \mathbb{R}, \forall i, j \in J.
$$

Thus, the semi-Markov kernel $\mathbf{Q}$ can be expressed as

$$
Q_{ij}(t, x) = p_{ij} G_{ij}(t, \infty) G_{ij}(\infty, x), \qquad \forall t, x \in \mathbb{R}, \forall i, j \in J.
$$

Define $Q_{ij}(t, \infty), i, j \in J$ be the kernel of process $\{(J_n, U_n); n \geq 0\}$ and $Q_{ij}(\infty, x), i, j \in J$ be the kernel of process $\{(J_n, X_n); n \geq 0\}$, then

$$
\begin{aligned}
Q_{ij}(t, \infty) &= p_{ij} G_{ij}(t, \infty), & \forall t \in \mathbb{R}, \forall i, j \in J, \\
Q_{ij}(\infty, x) &= p_{ij} G_{ij}(\infty, x), & \forall x \in \mathbb{R}, \forall i, j \in J.
\end{aligned}
$$

In order to calculate the distribution function of the accumulated claims amount, consider the following random walk process, as presented in [30]. Let L_n be the successive total claims amount after the arrival of the nth claim, which is defined as:

$$
L_n = \sum_{k=1}^{n} X_k, \qquad \forall n \geq 1, \forall i, j \in J.
\tag{9}
$$

Then, the joint probability of the process $\{(J_n, T_n, L_n); n \geq 0\}$ is denoted as

$$
\mathbb{P}[J_n = j, T_n \leq t, L_n \leq x | J_0 = i] = Q_{ij}^{*n}(t, x).
$$

This n-fold convolution matrix $\mathbf{Q}^{(n)} (= (Q_{ij}^{(n)}), i, j \in J)$ can be valued recursively by:

$$
\begin{aligned}
Q_{ij}^{*0}(t, x) &= \begin{cases} (1 - G_{ij}(0, \infty))(1 - G_{ij}(\infty, 0)), & \text{if } i = j \\ 0, & \text{elsewhere,} \end{cases} \\
Q_{ij}^{*1}(t, x) &= Q_{ij}(t, x), \qquad \dots \\
Q_{ij}^{*n}(t, x) &= \sum_{l=1}^{m} \int_0^t \int_0^x Q_{lj}^{*(n-1)}(t - t', x - x') dQ_{il}(t', x').
\end{aligned}
$$

Similarly to the processes $\{(J_n, T_n); n \geq 0\}$ and the process $\{(J_n, X_n); n \geq 0\}$ have

$$\mathbb{P}[J_n = j, T_n \leq t | J_0 = i] = Q_{ij}^{*n}(t, \infty) = \sum_{l=1}^{m} \int_0^t Q_{lj}^{*(n-1)}(t - t', \infty) dQ_{il}(t', \infty), \qquad (10)$$

$$\mathbb{P}[J_n = j, L_n \leq x | J_0 = i] = Q_{ij}^{*n}(\infty, x) = \sum_{l=1}^{m} \int_0^x Q_{lj}^{*(n-1)}(\infty, x - x') dQ_{il}(\infty, x').$$

In addition, the transition probabilities are:

$$\mathbb{P}[J_n = j | J_0 = i] = p_{ij}^{*n} = \sum_{l=1}^{m} p_{lj}^{*(n-1)} p_{il},$$

$$\mathbb{P}[L_n \leq x | J_0 = i, J_n = j] = G_{ij}^{*n}(\infty, x) = \begin{cases} \frac{Q_{ij}^{*n}(\infty, x)}{p_{ij}^{*n}}, & \text{for } p_{ij}^{*n} > 0, \\ \mathbb{1}\{x \geq 0\}, & \text{for } p_{ij}^{*n} = 0. \end{cases} \qquad (11)$$

Then, one can obtain the following equation:

$$Q_{ij}^{*n}(t, x) = Q_{ij}^{*n}(t, \infty) G_{ij}^{*n}(\infty, x).$$

Let the counting process $\{N_i(t), t \geq 0\}$ denote the total number of type i claims (claims which fall into state i) that occur in $(0, t]$, for all $i \in J$. Thus, the total number of claims $\{N(t), t \geq 0\}$ that occur in $(0, t]$ is

$$N(t) = \sum_{i=1}^{m} N_i(t),$$

with the convention that $N(0) = 0$, $N_i(0) = 0$. Moreover, define $J_{N(t)}$ as the type of the last claim that occurred before or on t, and thus the aggregate claims process can be expressed as

$$L(t) = L_{N(t)} = \sum_{k=1}^{N(t)} X_k, \qquad (7')$$

which is the same form as the classical aggregate claims process equation (7). Moreover, suppose that the embedded Markov Chain $\{J_n; n \geq 0\}$ is ergodic and that a sequence of unique probabilities $(\Pi_1, ..., \Pi_M)$ exists, which represents the stationary probability distribution, $\Pi_1 + ... + \Pi_M = 1$ and $\Pi_1, ..., \Pi_M \in [0, 1]$. Proposition 2.1 produces the density function of the aggregate loss, which is very useful in the CAT bonds pricing procedure in subsection 2.5.

Proposition 2.1. Let $F_1(t, x)$ denote the probability function that aggregate claims $L(t)$ which are less than or equal to the threshold x, at time t. Then,

$$F_1(t, x) = \sum_{i=1}^{m} \sum_{j=1}^{m} \Pi_i \sum_{n=0}^{\infty} \int_0^t (1 - H_j(t - t', \infty)) d[Q_{ij}^{*n}(t', \infty) G_{ij}^{*n}(\infty, x)].$$

Proof. Starting with the stationary probability for J_0 and states J are equilibrium measure, equation (4.2) in [28] gives

$$F_1(t, x) = \mathbb{P}(\sum_{k=1}^{N(t)} X_k \leq x) = \sum_{i=1}^{m} \sum_{j=1}^{m} \Pi_i \mathbb{P}(\sum_{k=1}^{N(t)} X_k \leq x, J_{N(t)} = j | J_0 = i).$$

Furthermore, according to Chapter 7, equation (3.32) in [30], the following equality holds:

$$\mathbb{P}(\sum_{k=1}^{N(t)} X_k \leq x, J_{N(t)} = j | J_0 = i) = \sum_{n=0}^{\infty} \int_0^t (1 - H_j(t - t', \infty)) dQ_{ij}^{*n}(t', x),$$

and the result follows by simple substitution. $\qquad\qquad \square$

Introduce the SM'/SM model[6] as a particular case of the previous model, and the matrix $\mathbf{G}(t, \infty)(= G_{ij}(t, \infty), i, j \in J)$ is defined as:

$$
G_{ij}(t, \infty) = \begin{cases} 0, & t < 0 \\ 1 - e^{-\lambda_i(t)t}, & t \geq 0. \end{cases}
$$

Thus, the distribution function of the sojourn time depends uniquely on the current state i, which is exponentially distributed with the parameter $\lambda_i(t)$. This parameter is a function depends on time t given in the following equation, and is the same format of equation (38) in [38]:

$$
\lambda_i(t) = a_i + b_i \sin^2(t - c_i) + d_i \exp\{\cos(\frac{2\pi t}{e_i})\}. \tag{12}
$$

Furthermore, assume that the Markov chain jumps to state j at each claim instance with a claim size distribution of $F_j(x) = \mathbb{P}_j(X_k \leq x)$. The idea of having distributions based on states has a practical meaning because a bigger catastrophic event can trigger many other events as side effects. Therefore, the behaviour of the claims (claims sizes and time interval between two claims) depends on the period claims stay[7]. Formally, we have the following assumptions:

$$
G_{ij}(t, \infty) = G_i(t, \infty), G_{ij}(\infty, x) = G_j(\infty, x) = F_j(x), \qquad i, j \in J, t, x > 0.
$$

More precisely, the time before the next claim occurs and the size of the claim depend only on the states current claim stays. While the process $\{J_n, U_n, X_n; n \geq 0\}$ has the following probabilistic structure:

$$
\begin{aligned}
Q_{ij}(t, x) &= \mathbb{P}[J_n = j, U_n \leq t, X_n \leq x | (J_k, U_k, X_k), k = 1, 2, ..., n - 1, J_{n-1} = i] \\
&= \mathbb{P}[J_1 = j, U_1 \leq t, X_1 \leq x | J_0 = i] \\
&= p_{ij} F_j(x)(1 - e^{-\lambda_i(t)t}),
\end{aligned}
$$

$\forall t, x \in \mathbb{R}, \forall i, j \in J$. Thus, J_n, W_n, and X_n are independent of the past given J_{n-1}, and the sequences $\{U_n, n \geq 1\}, \{X_n, n \geq 1\}$ are conditionally independent given the sequence $\{J_n, n \geq 0\}$. Rewrite the equation (10)-(11), and equation (8) as:

$$
\begin{aligned}
Q_{ij}^{*n}(t, \infty) &= (p_{ij}(1 - e^{-\lambda_i(t)t}))^{*n}, \\
G_{ij}^{*n}(\infty, x) &= \frac{(p_{ij} F_j(x))^{*n}}{p_{ij}^{*n}}, \\
H_j(t, \infty) &= \sum_{i=1}^{m} p_{ji}(1 - e^{-\lambda_j(t)t}) = 1 - e^{-\lambda_j(t)t}.
\end{aligned}
$$

Substituting in Proposition 2.1, following corollary can be easily obtained.

Corollary 2.2. At time t, the probability that the total loss amount $L(t)$ is less than or equal to the predefined level D can be computed as:

$$
F_2(t, x) = \sum_{i=1}^{m} \sum_{j=1}^{m} \Pi_i \sum_{n=0}^{\infty} \int_0^t e^{-\lambda_j(t-t')(t-t')} d \left[\frac{(p_{ij}(1 - e^{-\lambda_i(t')t'}))^{*n}(p_{ij} F_j(x))^{*n}}{p_{ij}^{*n}} \right].
$$

---

[6]In queueing theory, SM'/SM means that the distribution of claim arrival time follows non-homogeneous Poisson processes where the parameter depends on the preceding claim in a semi-Markov environment and the parameter of claim amount process depends on the future claim in the same semi-Markov environment, see [30].

[7]For more examples and extensions, the interesting reader can refer to [2].

Remark 1. For $m = 1$, this model is the classical time inhomogeneous Poisson process model with parameter $\lambda(t)$. It is also possible to have the matrix $\mathbf{G}(t, \infty)$ as:

$$G_{ij}(t, \infty) = \begin{cases} 0, & t < 0 \\ 1 - e^{-\lambda_i(t)}t, & t \geq 0, \end{cases}$$

where $\lambda_i(t)$ represents the intensity of the Poisson point process at time t in state i, $i \in J$. Therefore, if we assume that $m = 1$ in this example, the model will reduce to a model that employs the number-of-claims process $\{N(t) : t \in [0, T]\}$ using a nonhomogeneous Poisson process (NHPP) with parameters $\lambda(t) > 0$, as utilized by [38]. One can easily show that the probability of aggregate claims $L(t)$ less than or equal to the threshold x, at time t is equal to:

$$F_0(t, D) = \sum_{n=0}^{\infty} e^{-\lambda(t)t} \frac{(\lambda(t)t)^n}{n!} F^{*n}(x),$$

where $F^{*n}(x) = \mathbb{P}(X_1 + X_2 + \cdots + X_n \leq x)$ denotes the n-fold convolution of F, which is the same as the form given in equation (21) in [38].

2.5. Pricing model for the CAT bonds

In this subsection, we show how to price CAT bonds using the standard tool of a risk-neutral valuation measure with the following payoff functions for T time maturity one-period CAT bonds[8]. Their valuation is a consequence of equation (2).

Defining a hypothetical zero coupon CAT bond at the maturity date, as follows:

$$P_{\text{CAT}}^{(1)} = \begin{cases} Z, & \text{for } L(T) \leq D, \\ \eta Z, & \text{for } L(T) > D, \end{cases} \tag{13}$$

where $L(T)$ is the total insured loss value at the expiry date T, D denotes the threshold value agreed in the bond contract, and η ($\eta \in [0, 1)$) is the fraction of the principle Z, which the bondholders must pay when a trigger event occurs.

The next payoff function with a multi-threshold value is given by the equation

$$P_{\text{CAT}}^{(2)} = \eta_k Z \qquad \forall D_{k-1} < L(T) \leq D_k, \tag{14}$$

where $k = 1, 2, \ldots, h$ with $\eta_1 = 1 > \eta_2 > \cdots > \eta_h \geq 0$ and $D_0 = 0 < D_1 < \cdots < D_h = D$. In general, an investor's rate of return is inversely proportional to the total catastrophe claims.

Another payoff function with a coupon payment at the maturity date, if the trigger has not occurred, is of the form

$$P_{\text{CAT}}^{(3)} = \begin{cases} Z + C, & \text{or } L(T) \leq D, \\ Z, & \text{for } L(T) > D, \end{cases} \tag{15}$$

where $C > 0$ is the coupon payment level.

In order to introduce the final payoff function, consider a CAT bond issuer with the asset value A_{issue} and debt value B_{issue} at the bond maturity time. Define the default risk as the risk when the sponsor is unable to pay their obligations (i.e. the premium of the CAT bond). Thus, a CAT bondholder would not receive the full amount of capital even if the aggregate loss is less than the predetermined level. Let $\{N_{\text{issue}} : N_{\text{issue}} \geq 0\}$ be the number of this issued CAT bond. Furthermore, assume that the issuer's financial situation is independent of the aggregate industry-estimated catastrophic loss

---

[8]We only discuss one-period bonds in this study because multi-period coupon bonds can be treated as a portfolio of zero-coupon bonds with different maturities.

process. If the issuing company obtains sufficient funds to pay the bondholders at the maturity date T, the face value Z will be paid on the condition that a trigger event has not occurred, or a proportion will be paid according to the principle employed. If the issuing company fails to meet its obligation, the bondholders will lose all of their capital. More precisely, the structure of the defaultable payoff function is equal to

$$
P_{\text{CAT}}^{(4)} = \begin{cases} Z, & \text{if } L(T) \leq D \text{ and } A_{\text{issue}} > B_{\text{issue}} + ZN_{\text{issue}} \\ \eta Z, & \text{if } L(T) > D \text{ and } A_{\text{issue}} > B_{\text{issue}} + \eta ZN_{\text{issue}} \\ 0, & \text{otherwise.} \end{cases} \tag{16}
$$

In this case, the payoff of the CAT bond depends not only on the listed catastrophic events, but also on the CAT bond issuing company's financial position. However, at this stage, we are not interested in the performance of the issuing company throughout the trading period.

According to the payoff structures of the CAT bonds equations (13)–(16), the interest rate dynamics equation (3) and the aggregate loss process equation (7), we present the prices of the CAT bonds in Theorem 2.3–2.6. These are the main results of this study.

Zero-coupon CAT bond prices at time t when paying principal Z at the time of maturity T with payoff function equation (13) is shown in Theorem 2.3.

Theorem 2.3. Let $V_\ell^{(1)}(t)$ ($\ell = 0, 1, 2$) be the prices of the T-maturity zero-coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with payoff function $P_{\text{CAT}}^{(1)}$, as defined in equation (13). Then, for a predetermined threshold level D, the value of CAT bond at time t is given by:

$$
V_\ell^{(1)}(t) = B_{\text{CIR}}(t, T)Z(\eta + (1 - \eta)F_\ell(T - t, D)), \qquad \ell = 0, 1, 2,
$$

where $F_\ell(T - t, D)$ represents the accumulated function of the aggregate loss in the alternative models given in Proposition 2.1, Corollary 2.2, and Remark 1, respectively, and the pure discounted bond price $B_{\text{CIR}}(t, T)$ with the CIR interest rate model is given by equation (6).

Proof. [10] suggested that the payoff function is independent of the financial risks variable (interest rate) under the risk-neutral measure $\mathbb{Q}$. Then, according to equation (2), we have

$$
V_\ell^{(1)}(t) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s ds} P_{\text{CAT}}^{(1)}(T)|\mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s ds}|\mathcal{F}_t)\mathbb{E}^{\mathbb{Q}}(P_{\text{CAT}}^{(1)}(T)|\mathcal{F}_t).
$$

Using the result of the zero-coupon bond price with the CIR interest rate model, we have $\mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s ds}) = B_{\text{CIR}}(t, T)$. With equation (1), the above equation can be written as

$$
B_{\text{CIR}}(t, T)\mathbb{E}^{\mathbb{P}}(P_{\text{CAT}}^{(1)}(T)|\mathcal{F}_t).
$$

By simply applying the payoff function equation (13) and rearranging the formula, the CAT bond price can be formulated as

$$
\begin{aligned}
V_\ell^{(1)}(t) &= B_{\text{CIR}}(t, T)\mathbb{E}^{\mathbb{P}}(Z\mathbb{1}\{L(T) \leq D\} + \eta Z\mathbb{1}\{L(T) > D\}|\mathcal{F}_t) \\
&= B_{\text{CIR}}(t, T)(Z\mathbb{P}(L(T) \leq D) + \eta Z\mathbb{P}(L(T) \geq D)) \\
&= B_{\text{CIR}}(t, T)Z(F_\ell(T, D) + \eta(1 - F_\ell(T, D))) \\
&= B_{\text{CIR}}(t, T)Z(\eta + (1 - \eta)F_\ell(T, D)),
\end{aligned}
$$

where $\ell = 0, 1, 2$ and the result follows. $\qquad\square$

Similarly, in the next theorem, we compute the value of the zero-coupon CAT bond at time t when paying principal Z at the time of maturity T, with payoff function equation (14) determining by the amount of the aggregate claims.

Theorem 2.4. Let $V_\ell^{(2)}(t)$ ($\ell = 0, 1, 2$) be the price of the T-maturity zero-coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with the payoff function $P_{\text{CAT}}^{(2)}$, as defined in equation (14). Then, for a predetermined threshold level D, the value of CAT bond at time t is given by:

$$V_\ell^{(2)}(t) = B_{\text{CIR}}(t,T)Z\sum_{k=1}^{h}\eta_k(F_\ell(T-t,D_k) - F_\ell(T-t,D_{k-1})), \qquad \ell = 0, 1, 2,$$

where $F_\ell(T-t,x)$ represents the accumulated function of the aggregate loss in the alternative models given in Proposition 2.1, Corollary 2.2, and Remark 1, respectively, and the pure discounted bond price $B_{\text{CIR}}(t,T)$ with the CIR interest rate model is given by equation (6).

Proof. Similar to the proof in Theorem 2.3, let the payoff function follow equation (14), and can easily obtain that

$$
\begin{aligned}
V_\ell^{(2)}(t) =&B_{\text{CIR}}(t,T)\mathbb{E}^{\mathbb{P}}(\sum_{k=1}^{h}Z\eta_k\mathbb{1}\{D_{k-1} < L(T) \le D_k\}|\mathcal{F}_t)\\
=&B_{\text{CIR}}(t,T)(Z\sum_{k=1}^{h}\eta_k\mathbb{P}(D_{k-1} < L(T) \le D_k))\\
=&B_{\text{CIR}}(t,T)Z\sum_{k=1}^{h}\eta_k(F_\ell(T,D_k) - F_\ell(T,D_{k-1})),
\end{aligned}
$$

where $\ell = 0, 1, 2$ and the result follows. $\qquad\square$

In the next theorem, we show that the value of the coupon CAT bond at time t when paying principal Z and a coupon C at the time to maturity T depends on the payoff function equation (15).

Theorem 2.5. Let $V_\ell^{(3)}(t)$ ($\ell = 0, 1, 2$) be the price of the T-maturity coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with the payoff function $P_{\text{CAT}}^{(3)}$, as defined in equation (15). Then, for a predetermined threshold level D, the value of CAT bond at time t is given by:

$$V_\ell^{(3)}(t) = B_{\text{CIR}}(t,T)(Z + CF_\ell(T-t,D)), \qquad \ell = 0, 1, 2,$$

where $F_\ell(T-t,x)$ represents the accumulated function of the aggregate loss in the alternative models given in Proposition 2.1, Corollary 2.2, and Remark 1, respectively, and the pure discounted bond price $B_{\text{CIR}}(t,T)$ with the CIR interest rate model is given by equations (6).

Proof. Similar to the proof in Theorem 2.3 and if we let the payoff function follow equation (15), we can easily obtain

$$
\begin{aligned}
V_\ell^{(3)}(t) =&B_{\text{CIR}}(t,T)\mathbb{E}^{\mathbb{P}}((Z+C)\mathbb{1}\{L(T) \le D\} + Z\mathbb{1}\{L(T) > D\}|\mathcal{F}_t)\\
=&B_{\text{CIR}}(t,T)((Z+C)\mathbb{P}(L(T) \le D) + Z\mathbb{P}(L(T) \ge D))\\
=&B_{\text{CIR}}(t,T)((Z+C)F_\ell(T,D) + Z(1 - F_\ell(T,D)))
\end{aligned}
$$

where $\ell = 0, 1, 2$ and the result follows. $\qquad\square$

In the next theorem, we show that the price of the zero-coupon CAT bond at time t when paying principal Z at time to maturity T depends on the amount of the aggregate claims, which is also associated with the probability of the issuing company defaulting at time T.

Theorem 2.6. Let $V_\ell^{(4)}(t)$ ($\ell = 0, 1, 2$) be the price of the T-maturity zero-coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with the payoff function $P_{\text{CAT}}^{(4)}$, as defined in equation (16). Then, for a predetermined threshold level D, the value of CAT bond at time t is given by:

$$V_\ell^{(4)}(t) = B_{\text{CIR}}(t, T)Z[\eta + (1 - \eta - \tilde{F}(Z) - \eta\tilde{F}(\eta Z))F_\ell(T - t, D)) + p\tilde{F}(\eta Z)],$$

where $\ell = 0, 1, 2$ and $F_\ell(T - t, D)$ represents the accumulated function of the aggregate loss in the alternative models given in Proposition 2.1, Corollary 2.2, and Remark 1, respectively, and the pure discounted bond price $B_{\text{CIR}}(t, T)$ with the CIR interest rate model is given by equations (6). $\tilde{F}(x)$ denotes the issuing company's default probability at time T and

$$\tilde{F}(x) = \mathbb{P}(\frac{A_{\text{issue}} - B_{\text{issue}}}{N_{\text{issue}}} \le x).$$

Proof. Similar to the proof in Theorem 2.3, we have

$$V_\ell^{(4)}(t) = B_{\text{CIR}}(t, T)\mathbb{E}^{\mathbb{P}}(P_{\text{CAT}}^{(3)}(T)|\mathcal{F}_t).$$

Let the payoff function follow equation (16) and denote $M = \frac{A_{\text{issue}} - B_{\text{issue}}}{N_{\text{issue}}}$. According to the assumption that the default risk and catastrophe risk are independent, i.e. $L(T)$ and M are independent under the measure $\mathbb{P}$, the following equalities hold:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}(P_{\text{CAT}}^{(4)}(T)|\mathcal{F}_t) =& \mathbb{E}^{\mathbb{P}}[Z\mathbb{1}\{L(T) \le D, A_{\text{issue}} > B_{\text{issue}} + ZN_{\text{issue}}\} \\
&+ \eta Z\mathbb{1}\{L(T) > D, A_{\text{issue}} > B_{\text{issue}} + \eta ZN_{\text{issue}} + 0\} \\
=& Z\mathbb{P}(L(T) \le D, M > Z) + \eta Z\mathbb{P}(L(T) > D, M > \eta Z) \\
=& Z\mathbb{P}(L(T) \le D)\mathbb{P}(M > Z) + \eta Z\mathbb{P}(L(T) > D)\mathbb{P}(M > \eta Z),
\end{aligned}
$$

where $\ell = 0, 1, 2$ and the result follows. $\qquad\square$

3. Numerical Application and Analysis

In order to apply pricing formulas to the real world and to obtain the CAT bond prices, we need to compute the exact distribution of the aggregate loss $F_\ell(T, D)$ ($\ell = 0, 1, 2$). However, as in [38], this is extremely difficult to calculate because the closed form solutions of these high-order convolutions are not available. Therefore, we employ Monte Carlo simulations for the analysis and we approximate the CAT bonds prices via numerical computation.

3.1. Data

In actuarial research, an event is referred to as catastrophic if it occurs with a low probability and it causes severe damage. Empirical studies of the catastrophe risks are conducted for the data provided by ISO's PCS unit, which describe insured property losses in the USA caused by catastrophic events over a predetermined threshold that occurred between 1985 and 2013. And then inflation is adjusted for a set of 870 original loss data using the CPI. Figure 2 shows the annual adjusted PCS loss and the total annual number of qualified catastrophes between 1985 and 2013. The 20 most expensive insured CAT losses are listed in Table 1. (An illustration of the individual CAT loss is shown in Figure 1, where the peaks in the figure represent the most costly events.) Thus, we can conclude that the PCS loss data are heavy-tailed, see [38].

[Insert Figure 2 about here]

[Insert Table 1 about here]

13

3.2. Numerical algorithm

In this sub-section, the algorithmic steps for the calculation of the value of the CAT bonds in the previous Section with face value $Z = \text{US\$1}$ at time $t = 0$ are described.

Step 1: Calculate the CAT bond price where the spot interest rate process followed the CIR model, and assuming that the market price of risk λ_r was a constant -0.01. In this experiment, we employ 3-month maturity US monthly Treasury bill data $(1994 - 2013)$[9] to estimate the parameters of the CIR model.

Step 2: In order to analyse the semi-Markov process model[10], assuming that we are working in a two-state $(m = 2)$ environment, i.e., a many claims period (state 1, a stormy season with claim frequency $\lambda_1(t)$, [47]) and a few claims period (a non-stormy season state 2 with claim frequency $\lambda_2(t)$).

Step 3: Fit the distribution of PCS claims losses value by the general extreme value (GEV) distribution and lognormal distribution in two states respectively[11]. PCS claims inter-arrival times distribution parameters are estimated by non-linear least squares for inhomogeneous Poisson processes given in equation (12). And finally calculate the transition probabilities of the claims according to the stormy/non-stormy seasons over the period between 1985 and 2013.

Step 4: Generate 10^5 scenarios in the final step, and obtain the $T \in [0.25, 2.25]$ years maturity zero-coupon CAT bond prices by Monte-Carlo simulations.

3.3. Numerical calculations and discussion

For Step 1, based on the MLE method, we conclude that both the initial short-term interest rate r_0 and the long-term mean interest rate θ were 2.04% annually, the mean-reverting force $k = 0.0984$, and the volatility parameter $\sigma = 4.77\%$.

For Step 2, we arbitrarily define a period as a stormy season (many claims period as State 1) or a non-stormy season (few claims period, as State 2) based on the following conditions, and this is an illustration of one possible method of defining the states in the semi-Markov process. One of the example in nuclear industry given by [2] states that some specific nuclear incidents (or accidents) trend to trigger some other accidents in a short time interval. In this paper, we define a claim occurs within a stormy season if and only if more than two claims hit this area in a month (typically, we use 30 days/month), and an non-stormy season claim if total number of claims within this month is less than or equal to one. By analysing the dates of occurrence for the PCS events data and the frequency table, we can conclude that there were 811 claims occurred in the stormy seasons and 68 claims occurred in the non-stormy season. And the transition probabilities is given in Table 2.

For Step 3, fit the distribution of PCS losses during stormy period (similarly for non-stormy period) by the general extreme value (GEV) distribution with the following parameters: shape parameter $= 0.9273$, location parameter $= 10.2718$, and scale parameter $= 10.6296$, which we compare with the

---

[9]It is not necessary to use a dataset with the same time period as the PCS data because the financial risks and catastrophe risks are independent, [10].

[10]The SM'/SM model is a special case of a general Markov model, so in this part of the application, we considered the SM'/SM model as an example.

[11]The choice of the distribution is very important because it varies the bond price. Readers can refer to [38] for details of the use of MLE to estimate parameters and selecting the best fit model with non-parametric tests. Since the data set used in this paper is the same as used in [38], this study will omit details of the processes used for parameter estimation and the nonparametric tests. As a reminder, GEV distribution with the probability density function (p.d.f.) given by $f(x) = \frac{1}{\sigma_3} \exp\left(-\left(1 + k_3\left(\frac{x-\mu_3}{\sigma_3}\right)\right)^{-\frac{1}{k_3}}\right)\left(1 + k_3\left(\frac{x-\mu_3}{\sigma_3}\right)\right)^{-1-\frac{1}{k_3}}$, when $k \neq 0$; lognormal distribution, with the p.d.f. given by $f(x) = \frac{1}{x\sigma_2\sqrt{2\pi}} \exp\left(-\frac{(lnx-\mu_2)}{2\sigma_2^2}\right)$.

next best fit lognormal distribution with the parameters: $\mu_2 = 2.8586$ and $\sigma_2 = 1.2638$. And the parameters of the model in both states are given in Table 2. Claims intensity follows a non-homogeneous exponential model with parameter given in equation (12), by applying the nonlinear least squares procedure, we conclude that the quantity of loss process could be modelled as an inhomogeneous Poisson process with intensity $\lambda_1(t) = 35.1576 - 0.8740 \sin^2(t - 0.3352) + 1.0338 \exp\{\cos(\frac{2\pi t}{8.2665})\}$ and $\lambda_2(t) = 12.5731 + 1.5258 \sin^2(t - 0.0340) - 0.6736 \exp\{\cos(\frac{2\pi t}{5.3970})\}$. This allows us to model the catastrophic data in changing economic or natural environments. Figure 3 shows a real catastrophe loss trajectory (in black) between 1985 and 2013, and compared with simulated mean of total loss when loss sizes follow a lognormal distribution (in blue) with 95% confidence interval and simulated mean of total loss when loss sizes follow a GEV distribution (in orange) with 95% confidence, in the USA during $2005 - 2013$, for the real data, lognormal distribution simulation and GEV distribution simulation. Since real sample trajectories never fells into 95% confidence interval of lognormal distribution model, we suggest that the GEV distribution fitted better to the long-term real-world aggregate loss process, which coincides with [38] conclusion.

[Insert Figure 3 about here]

[Insert Table 2 about here]

In Step 4, we firstly arbitrarily assume the parameters of payoff function. For the payoff functions equations (13) and (16), we assumed that $\eta = 0.5$ when the aggregate loss $L(T)$ exceeds the threshold level $D \in [434, 5210]$ US\$10 million, i.e., the threshold level in the interval of quarterly to three times the annual average loss. In addition, for the payoff function equation (14), arbitrarily set the following parameters: $h = 3$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\eta_3 = 0.25$, $D_1 = 434$ US\$10 million, $D_2 \in [434, 5210]$ US\$10 million, and $D_3 = \infty$. For a real-world CAT bond, the issuing company might use a multi-threshold payoff structure to reduce the risk of investment and to attract more investors. Furthermore, the probabilities of the issuing company defaulting at time T were $\tilde{F}(Z) = 0.1$ and $\tilde{F}(\eta Z) = 0.05$. Finally, we assume that the coupon payment rate in equation (15) is US\$ 0.1.

Figure 4 illustrates the CAT bond prices for the payoff functions $P_{\text{CAT}}^{(1)}$ with the threshold level D and time to maturity T under the stochastic interest rate assumptions. We show the CAT bond values for the accumulated distributed function of the classical aggregate loss process $F_0(t, D)$ are given in Remark 1, where the loss distribution followed the GEV distribution and the intensity of the claims was a non-homogeneous Poisson process in Figure 4(a). In Figure 4(b), we show the CAT bond prices where the c.d.f. of the aggregate loss process $F_2(t, D)$ follows the SM'/SM model given in Corollary 2.2. With similar settings, Figure 4 and Figures $5 - 7$ illustrate the CAT bonds prices with the GEV distribution for the payoff functions $P_{\text{CAT}}^{(2)}$, $P_{\text{CAT}}^{(3)}$, and $P_{\text{CAT}}^{(4)}$, respectively. Using the payoff function $P_{\text{CAT}}^{(1)}$ as an example, the price differences between the CAT bond prices with the classical and SM'/SM models are shown in Figure 8(a) under the GEV, the NHPP, and stochastic interest rates assumptions. In Figure 8(b), we show how the bond prices are affected by the distribution of the severity of the losses (lognormal and GEV distributions). The differences are particularly evident in the tails (higher threshold level); therefore, a heavy-tailed distribution is a more appropriate choice for modelling catastrophe loss, as demonstrated by [38].

Figures $4 - 7$ show that there are few differences in shape between the different aggregate loss models because we used the same dataset. In general, the CAT bond value rate decreases as the maturity time increases and threshold level decreases. By comparing the different payoff functions, it is clear that CAT bond prices decreased with increasing threshold and when the default risk is added to the payoff function, while the coupon CAT bonds has higher prices compared with the zero-coupon CAT bonds. This indicates that the choice of different payoff functions has a major impact on the CAT bond prices. According to Figure 8(a), the differences in the bond price change significantly, by as much as 3.5% of the face value. And we notice that the prices in our model are slightly higher than those in the model of [38]. This might because our model has more information (longer estimate

period) for both catastrophe risks and financial risks, and this might also because we make the model more realistic by considering the dependency between the claim size and intensity. The values in Figure 4 (V_0^1 and V_2^1) are about 10% higher than the values in Figure 7 (V_0^4 and V_2^4), because CAT risk bonds are less valueable to buy by the investors for bearing extra risks, i.e. default risk. Our results also demonstrate that the choice of the aggregate loss process model affects the bond prices. An illustration of the characteristic CAT bond prices and the value change percentage in terms of [38] model are presented in Table 3. The case using GEV decrease at a faster rate than the log-normal case. This is an interesting result because the mean value of the aggregate loss process GEV distribution was always larger than the mean value of log-normal distribution process, as shown in Figure 3, i.e., there is a higher probability of having larger aggregate loss value if we simulate model by GEV distribution.

[Insert Figure 4 about here]

[Insert Figure 5 about here]

[Insert Figure 6 about here]

[Insert Figure 7 about here]

[Insert Figure 8 about here]

[Insert Table 3 about here]

4. Conclusions

In this study, we develop a contingent claim process to price CAT bonds using models with a risk-free spot interest rate under assumptions of a no-arbitrage market, independently of the financial risks and catastrophe risks, as well as the possibility of replicated interest rate changes with existing financial instruments. Under the risk-neutral pricing measure, bond price formulae is derived for four types of payoff functions (the classic zero coupon, the multi-threshold zero coupon, the coupon, and the defaultable zero coupon payoff functions) when trigger is determined by the aggregate loss process with a semi-Markov-dependent structure. Here the spot interest rate followed CIR model and the inter-arrival time followed an time in-homogeneous Poisson distribution.

The numerical experiments utilized Monte Carlo simulations with data from the PCS loss index in the USA during $1985 - 2013$. The numerical analyses showed that the CAT bond prices decreased as the threshold level decreased, as the time to maturity increased, and with the existence of a default probability. The CAT bond prices increased after the introduction of coupons. Furthermore, we show that the choice of the fitted loss severity distribution has a great impact on the bond prices. The additional dependency between the claim sizes and the claim inter-arrival times is a significant factor when pricing CAT bonds, thereby yielding higher and fairer CAT bond prices.

Although, we propose one way to solve the problem of characterizing the dependency between catastrophe claims, the dependency between the CAT market and the financial market cannot be used within our framework. The problem of the dependency between CAT risks and the financial market risks is very interesting, and it will be addressed in future research.

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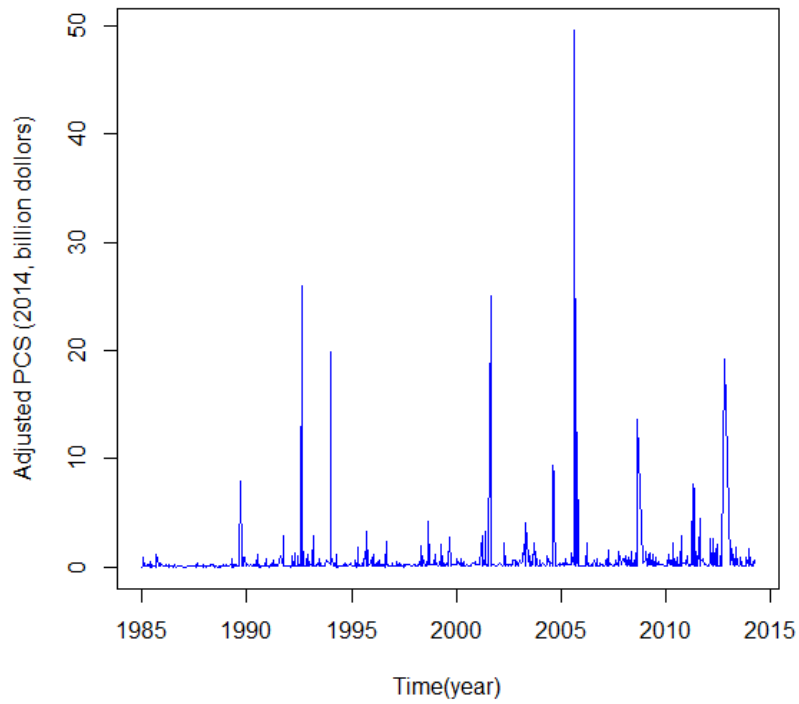
Figures

Figure 1: Graph showing PCS catastrophe loss data for the USA in 1985–2013.
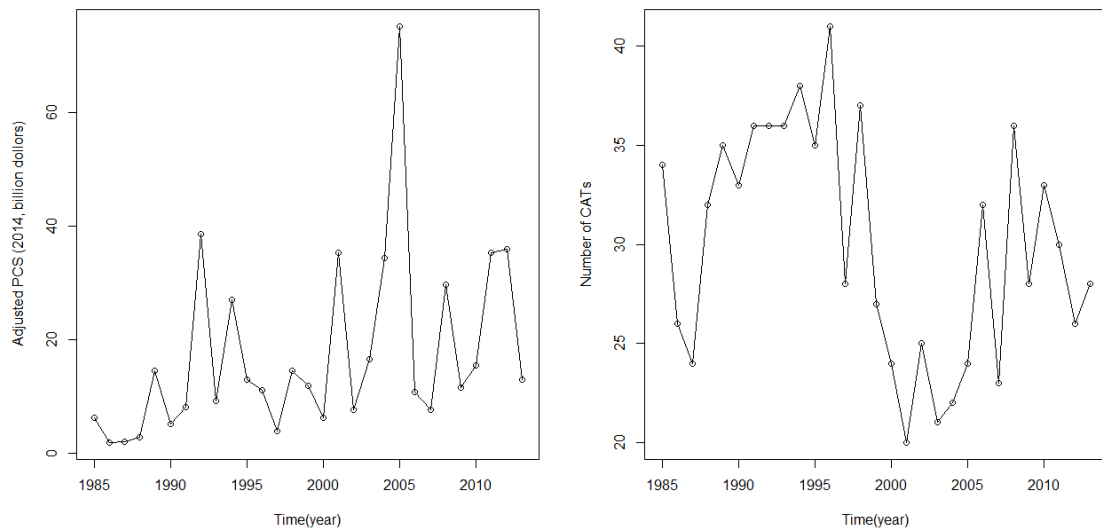

Figure 2: PCS annual catastrophe losses (left) and the number of catastrophes (right) in the USA during 1985 – 2013.
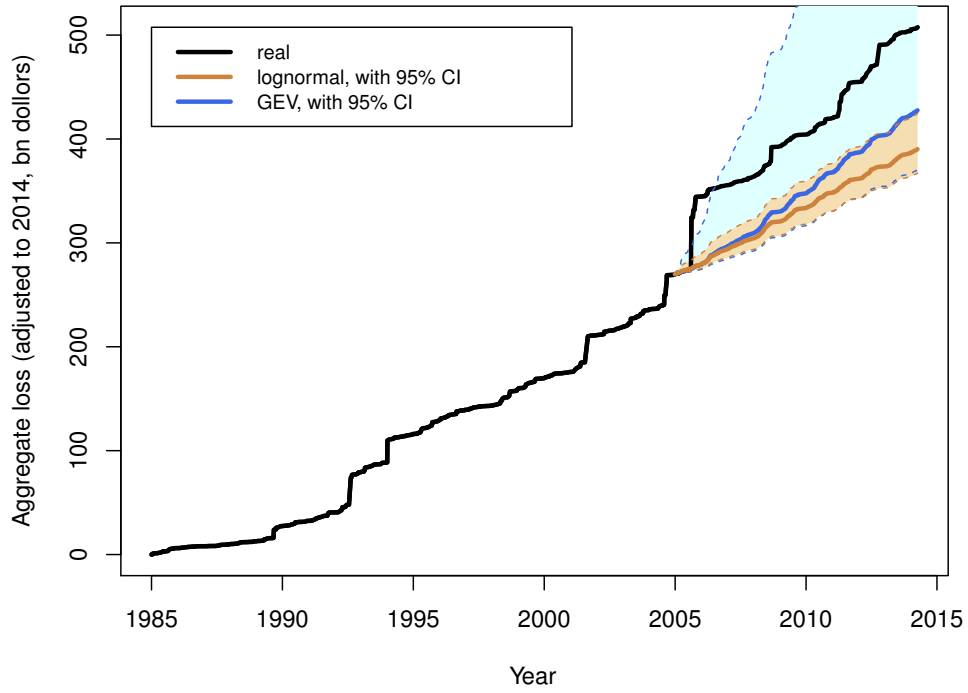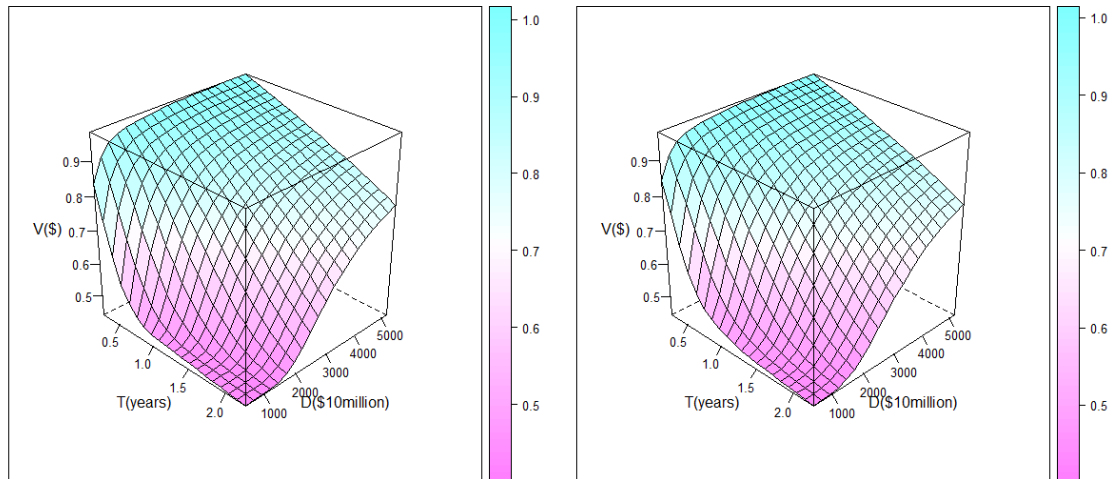
Figure 3: 95% confidence interval for the aggregate loss process in the USA during 1985 – 2013, for the real data, lognormal distribution simulation and GEV distribution simulation.
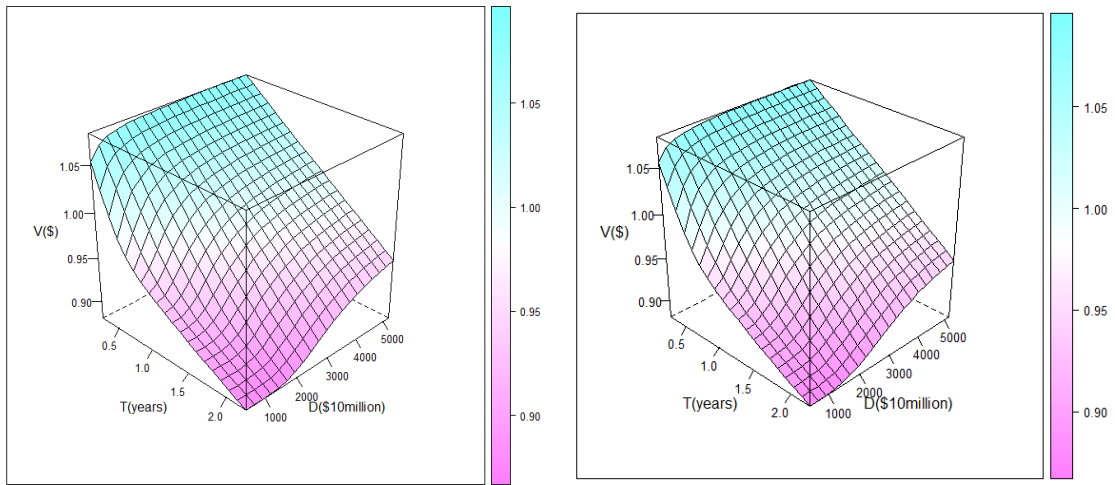

(a) $V_0^{(1)}(t)$ (based on [38] model) with GEV distribution.

(b) $V_2^{(1)}(t)$ (based on our model) with GEV distribution.

Figure 4: CAT bonds prices (z-coordinate axes) for the payoff function $P_{\mathrm{CAT}}^{(1)}$ under the GEV, the NHPP, and stochastic interest rate assumptions. The time to maturity (T) decreases on the left axes and threshold level (D) increases on the right axes.
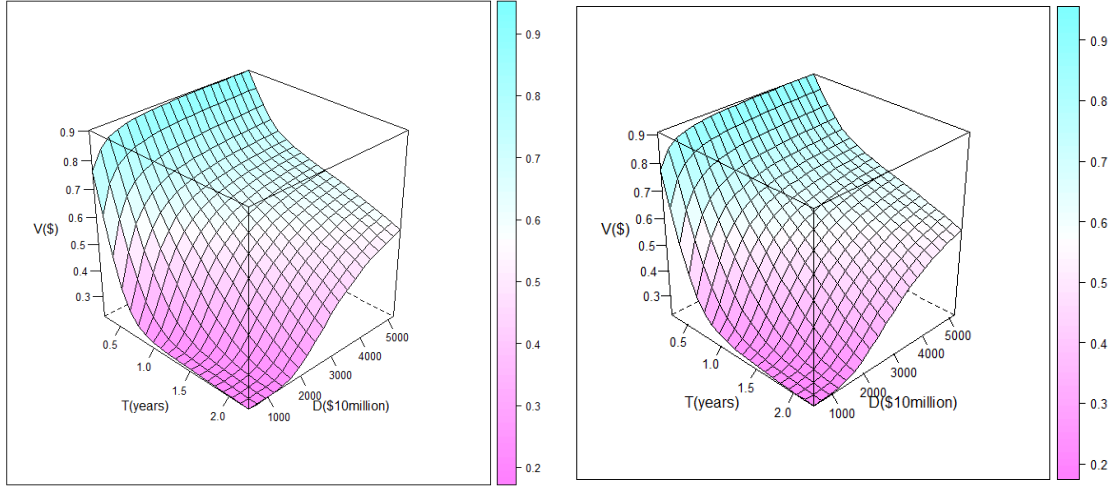
(a) $V_0^{(2)}(t)$ (based on [38] model) with GEV distribution.
(b) $V_2^{(2)}(t)$ (based on our model) with GEV distribution.

Figure 5: CAT bonds prices (z-coordinate axes) for the payoff function $P_{\mathrm{CAT}}^{(2)}$ under the GEV, the NHPP, and stochastic interest rate assumptions. The time to maturity (T) decreases on the left axes and the threshold level (D) increases on the right axes.
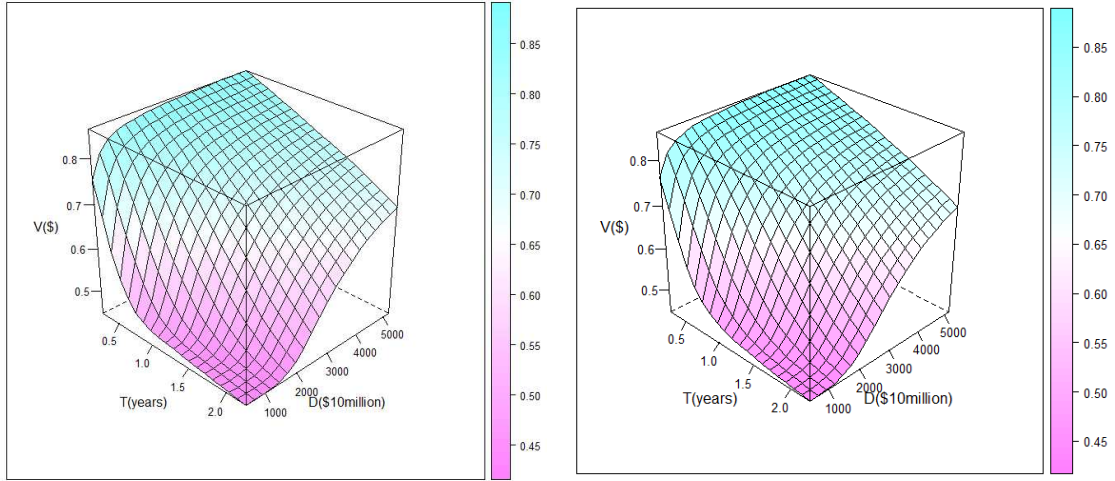

(a) $V_0^{(3)}(t)$ (based on [38] model) with GEV distribution.
(b) $V_2^{(3)}(t)$ (based on our model) with GEV distribution.

Figure 6: CAT bonds prices (z-coordinate axes) for the payoff function $P_{\mathrm{CAT}}^{(3)}$ under the GEV, the NHPP, and stochastic interest rate assumptions. The time to maturity (T) decreases on the left axes and the threshold level (D) increases on the right axes.

(a) $V_0^{(4)}(t)$ (based on [38] model) with GEV distri-
bution.

(b) $V_2^{(4)}(t)$ (based on our model) with GEV distribu-
tion.

Figure 7: CAT bonds prices (z-coordinate axes) for the payoff function $P_{\text{CAT}}^{(4)}$ under the GEV, the NHPP, and stochastic interest rate assumptions. The time to maturity (T) decreases on the left axes and the threshold level (D) increases on the right axes.
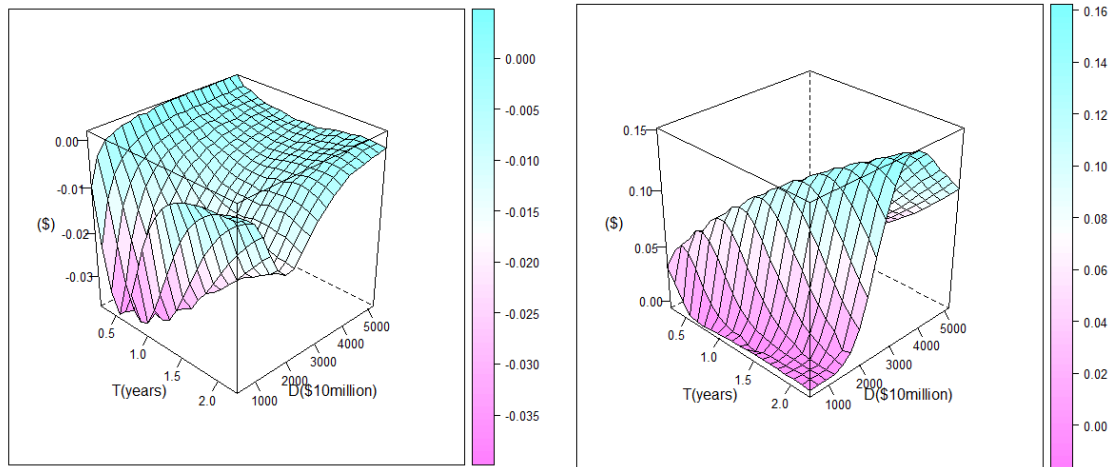

(a) Differences between $V_0^{(1)}$ (based on [38] model) and $V_2^{(1)}$ with GEV.

(b) Differences between the lognormal and GEV dis-
tribution of $V_2^{(1)}$.

Figure 8: Differences (z-coordinate axes) in the CAT bond prices for $P_{\text{CAT}}^{(1)}$ under the GEV (or lognormal), the NHPP, and stochastic interest rate assumptions. The time to maturity (T) decreases on the left axes and the threshold level (D) increases on the right axes.

23

Tables

Table 1: The 20 most costly insured CAT losses in the USA during 1985 – 2013.

Event	Date	PCS loss (US$ billion)	2014 dollars (US$ billion)
Hurricane Katrina	25/08/2005	41.10	49.56
Hurricane Andrew	24/08/1992	15.50	26.02
Terrorist attacks	11/09/2001	18.78	24.97
Northridge Earthquake	17/01/1994	12.50	19.86
Hurricane Sandy	28/10/2012	18.75	19.23
Hurricane Ike	12/09/2008	12.50	13.67
Hurricane Wilma	24/10/2005	10.30	12.42
Hurricane Charley	13/08/2004	7.47	9.32
Hurricane Ivan	15/09/2004	7.11	8.86
Hurricane Hugo	17/09/1989	4.20	7.97
Wind and Thunderstorm Event	22/04/2011	7.30	7.64
Wind and Thunderstorm Event	20/05/2011	6.90	7.22
Hurricane Rita	20/09/2005	5.63	6.79
Hurricane Frances	03/09/2004	4.59	5.73
Hurricane Jeanne	15/09/2004	3.65	4.56
Hurricane Irene	26/08/2011	4.30	4.50
Hurricane Georges	21/09/1998	2.96	4.27
Wind and Thunderstorm Event	02/05/2003	3.21	4.10
Tropical Storm Allison	05/06/2001	2.50	3.32
Hurricane Opal	04/10/1995	2.10	3.25

Table 2: Parameters of the semi-Markov process model.

Parameters		State 1	State 2
GEV distribution	k_3	0.9418	0.7369
	σ_3	10.5250	12.8792
	μ_3	10.0954	11.6703
Lognormal distribution	μ_2	2.8488	2.9875
	σ_2	1.2731	1.1261
Transition probabilities	p_{1j} for $j \in 1, 2$	0.9767	0.0233
	p_{2j} for $j \in 1, 2$	0.3065	0.6935

Table 3: Characteristic CAT bond prices for the payoff function $P_{\text{CAT}}^{(1)}$, and percentage changes of prices in terms of [38] model.

Maturity T	Threshold D	[38] model $V_0^{(1)}$ with GEV	Our model $V_2^{(1)}$ with GEV	lognormal
0.25	434.21	0.8364	0.8462 (1.17%)	0.8783 (5.00%)
0.25	2948.05	0.9729	0.9710 (-0.19%)	0.9830 (1.04%)
0.25	5210.50	0.9788	0.9767 (-0.21%)	0.9827 (0.40%)
0.67	434.21	0.5270	0.5616 (6.56%)	0.5573 (5.73%)
0.67	2948.05	0.9185	0.9225 (0.44%)	0.9590 (4.41%)
0.67	5210.50	0.9415	0.9434 (0.20%)	0.9600 (1.96%)
1.09	434.21	0.4705	0.4831 (2.68%)	0.4817 (2.39%)
1.09	2948.05	0.8502	0.8582 (0.94%)	0.9337 (9.82%)
1.09	5210.50	0.9023	0.9051 (0.31%)	0.9364 (3.79%)
1.51	434.21	0.4583	0.4618 (0.77%)	0.4620 (0.80%)
1.51	2948.05	0.7685	0.7775 (1.17%)	0.8953 (16.49%)
1.51	5210.50	0.8607	0.8630 (0.27%)	0.9142 (6.22%)
2.25	434.21	0.4404	0.4419 (0.34%)	0.4420 (0.37%)
2.25	2948.05	0.6046	0.6143 (1.60%)	0.7458 (23.34%)
2.25	5210.50	0.7751	0.7766 (0.19%)	0.8781 (13.29%)