Optimal taxation and debt with uninsurable risks to human capital accumulation*

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Abstract

We consider an economy where individuals face uninsurable risks to their human capital accumulation, and analyze the optimal level of linear taxes on capital and labor income together with the optimal path of government debt. We show that in the presence of such risks it is beneficial to tax both labor and capital and to issue public debt. We also assess the quantitative importance of these findings, and show that the benefits of government debt and capital taxes both increase with the magnitude of idiosyncratic risks and the degree of relative risk aversion.

Keywords: incomplete markets; Ramsey equilibrium; optimal taxation; optimal public debt.

JEL Classification numbers: D52; D60; D90; E20; E62; H21; O40.

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1 Introduction

Human capital is an important component of wealth both at the individual and aggregate level, and its role has been investigated in various fields in economics. In public finance, Jones, Manuelli and Rossi (1997) show that the zero-capital-tax result of Chamley (1986) and Judd (1985)[1] can be strengthened if human capital accumulation is explicitly taken into account. Specifically, they demonstrate that, in a deterministic economy with human capital accumulation, in the long run not only capital but also labor income taxes should be zero, hence the government must accumulate wealth - that is, public debt be negative - to finance its expenditure.

In this paper we show that the introduction of uninsurable idiosyncratic shocks to the accumulation of human capital drastically changes the result of Jones, Manuelli and Rossi (1997), as it becomes optimal for the government to tax both labor and capital income even in the long run. Thus, our analysis shows how the interaction between market incompleteness and human capital accumulation provides a novel justification for a positive tax rate on capital. The desirability of taxing both capital and labor income then implies a beneficial role of government debt, and so our theory provides also a rationale for the presence of a positive level of government debt, in line with observed evidence.

Our model builds on that of Krebs (2003), who augmented the endogenous growth model of Jones and Manuelli (1990) with uninsurable idiosyncratic risks to human capital accumulation. We assume that there is a unit measure of infinitely-lived individuals, who can invest in three types of assets: government bonds, physical capital, and human capital. The first two assets are risk-free while human capital is risky and there are no insurance markets where this risk can be hedged. As a result, individuals face uninsurable fluctuations in their labor income.

In this environment we study a Ramsey taxation problem, where linear taxes on labor and capital income are chosen so as to maximize a weighted average of individuals' lifetime utility. The model considered turns out to be quite tractable and allows us to derive both analytic and quantitative properties of the solution of the Ramsey problem. It also allows us to maximize the average lifetime utility of individuals, rather than the average of their utility at the steady state (as common in the earlier literature), and thus to take into account also the transition to the steady

[1] Judd and Chamley find that the optimal tax rate on capital is zero in the long run in a deterministic economy with infinitely lived households (see however Werning (2014) for some important qualifications of their findings). The optimality of a zero tax rate on capital in the long run has then been extended by Zhu (1992) to a representative-agent economy with aggregate shocks, and by Karantounias (2013) to economies with more general, recursive preferences. The result by Atkinson and Stiglitz (1976) on uniform commodity taxation theorem provides then a condition under which the optimal tax rate on capital is always zero (not just in the steady state).

state.

Our theoretical findings are summarized as follows. First, taxing labor income is beneficial because it reduces the risk associated with labor income. Indeed, if the government is required to have a balanced budget in every period, the optimal tax rate on labor income is positive and the optimal tax rate on capital income is negative, as long as government purchases are small enough. It is worth noting that at a competitive equilibrium of our model without taxes the ratio of physical-to-human capital is higher than at the first-best allocation (where idiosyncratic shocks are fully hedged), thus there is "over-accumulation" of physical capital.[2] Still, our result shows this does not necessarily mean that capital income should be taxed, on the contrary it should be subsidized under the balanced budget requirement.

Our second result shows the benefits of capital income taxation and government debt. We show that it is beneficial to issue debt whenever the expected rate of return earned by the private sector on its savings, given by a weighted average between the after-tax returns on human and physical capital, exceeds the cost of funds for the government, given by the before-tax return on physical capital. Since human capital is risky while physical capital is safe, this condition is always satisfied when government expenditure is small enough, thus establishing the optimality of positive debt in this case. We also show that at a steady state solution of the Ramsey problem the expected rate of return for the private sector must equal[3] the cost of funds for the government.[4] For this to be possible, the after-tax return on physical capital must be smaller than its before-tax return, and hence the tax rate on capital must be strictly positive in the long run, to pay for the government expenditure and/or for the service of the public debt.

To evaluate the quantitative importance of our findings, we calibrate our model to the U.S. economy. In particular, following Krebs (2003), we set the variance of the shock to human capital so as to match the estimate of the variance of the permanent shock to labor income by Meghir and Pistaferri (2004). We find that at the steady state of the solution of the Ramsey problem, when government expenditure is set at a positive level matching U.S. data, the capital tax rate is sizable (19.64 percent) while the government debt to output ratio is close to 0. The optimal the tax rates turn out to be not very far from our estimate of the current U.S. fiscal policy, while the optimal

[2]See Gottardi, Kajii and Nakajima (2011) for a proof (see also Krebs (2003) for a similar finding). The same over-accumulation result holds in the standard incomplete-market macroeconomic model (see Aiyagari (1994)).

[3]Strictly speaking, when the elasticity of intertemporal substitution is different from one a correction term is also present, capturing the effect of public debt on the saving rate.

[4]It is useful to relate this result to the findings cited above, obtained in a deterministic environment. It is shown that a similar property should hold, but since the return on consumers' savings is not random, this implies that a zero capital tax rate is optimal.

debt level is lower. With the transition path explicitly taken into account, the welfare gains of adopting the optimal policy in our model, with uninsurable shocks to human capital, are relatively modest, and are much smaller, for instance, than the ones obtained in the related deterministic model by Jones, Manuelli, and Rossi (1993). We also find that when instead there is no government expenditure, the optimal level of public debt is quite large, 202 percent of GDP; if in addition the government cannot issue (or purchase) debt, the optimal tax on capital is negative but small (-0.34 percent). These results are in line with the theoretical findings described in the two previous paragraphs. Also, while the benefit of issuing debt proves to be quantitatively quite sizeable, the benefit of taxing labor income relative to capital income so as to reduce human capital risk (our first result) is quantitatively rather small. Finally, we show that the quantitative benefits of debt depend on the size of the risk premium earned by human capital over physical capital, and are then quite sensitive to the agents' degree or risk aversion and the size of the uncertainty.

The idea that uninsurable labor income risks may justify taxing capital income is not new. In the framework of a standard incomplete markets macroeconomic model, where the labor productivity of each individual follows an exogenously specified stochastic process, Aiyagari (1995) also finds that the tax on capital must be positive at the steady state solution of the Ramsey problem.[5] However, in the environment he considered, with no human capital accumulation and endogenous government expenditure, the tax rate on capital must be positive for a steady state to exist. One may then question to what extent the standard incomplete markets model provides a clear support to the view that uninsurable labor income shocks justify capital income taxation.[6] In our model in contrast, with human capital accumulation together with uninsurable income shocks, the existence of a steady state imposes no real restriction on the value of the tax rate on capital and the optimality of a positive tax rate is primarily determined by the comparison of costs and benefits of the tax and debt. A partially different line of argument is pursued by Conesa, Kitao and Krueger (2009) who consider a quantitative overlapping-generations model with uninsurable labor income shocks, allowing also for nonlinear labor income taxes. They find that the optimal capital income tax rate is positive and significant, but the desirability of capital income taxation in their model is primarily due to the effects of the life cycle on agents' behavior and the absence of age-dependent labor income taxes, more than to market incompleteness.

Finally, regarding the desirability of government debt, we should mention a related result obtained by Aiyagari and McGrattan (1998) in a model with uninsurable labor income shocks. However, differently from us they consider an environment where borrowing constraints are binding and

[5] Note that Aiyagari (1995) also examines the problem of maximizing the individuals' lifetime utility.

[6] See also İmrohoroğlu (1998), Domeij and Heathcote (2004) and Açikgöz (2013) for other work along these lines.

derive the optimal policy as solution of the problem of maximizing the agents' steady-state average utility, restricting the labor and capital tax rates to be identical. Because in particular of the latter restriction, the benefits of higher taxes (on labor as well as capital) cannot be separated from those of higher debt.

The rest of the paper is organized as follows. In Section 2 the economy is described, and the benchmark equilibrium without taxes is characterized. Section 3 considers the dynamic Ramsey problem and derives the main theoretical results on the properties of the optimal levels of taxes and government debt. Section 4 describes then the numerical results and discusses the extension of the results to the case where nonlinear taxes are allowed. Section 5 concludes. Most of the proofs are collected in the Online Appendix.

2 The Economy

We consider a competitive economy subject to idiosyncratic shocks. Time is discrete and indexed by $t = 0, 1, 2, \ldots$ The economy consists of consumers, firms producing a homogeneous consumption good using physical capital and human capital as inputs, and the government collecting taxes and issuing debt.

2.1 Consumers

There is a continuum of infinitely lived consumers. In every period each individual is endowed with one unit of raw labor, which he supplies inelastically, and can use his revenue to consume the consumption good and invest in three kinds of assets: a risk-free bond, physical capital and human capital. His level of human capital determines the "efficiency units" of his labor.

Each individual $i \in [0, 1]$ has Epstein-Zin-Weil preferences over random sequences of consumption, which are defined recursively by

$$u_{i,t} = \left\{ (1-\beta)(c_{i,t})^{1-\frac{1}{\psi}} + \beta \left[E_t (u_{i,t+1})^{1-\gamma} \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{\psi}{\psi-1}}, \qquad t = 0, 1, ... \tag{1}$$

where $u_{i,t}$ is the intertemporal utility of individual i evaluated at date t, E_t is the conditional expectation operator at time t, $c_{i,t}$ is his/her consumption in period t, $\beta \in (0, 1)$ is the discount factor, ψ is the elasticity of intertemporal substitution, and γ is the coefficient of relative risk aversion.

Let $b_{i,t-1}$, $k_{i,t-1}$ and $h_{i,t-1}$ denote, respectively, the quantities of risk-free bond, physical capital, and human capital that individual i holds at the end of period $t-1$. To capture the idea that labor income is subject to uninsurable idiosyncratic shocks, we assume that, for each i and t, the human capital of individual i is affected by a random shock, $\theta_{i,t}$ at the beginning of period t. Hence the actual amount of human capital available to individual i is $\theta_{i,t}h_{i,t-1}$.

Each consumer i is initially endowed with a non negative amount $b_{i,-1}$, $k_{i,-1}$ and $h_{i,-1}$ of riskless bond, physical and human capital.[7] Let then $\iota_{k,i,t}$ and $\iota_{h,i,t}$ denote the units of the consumption good invested in, respectively, physical and human capital by individual i in period t. The amount of the two types of capital held at the end of period t, for $t = 0, 1, ...$, is then equal to the amount held at the beginning of the period, less the depreciation, plus the investment:

$$k_{i,t} = \iota_{k,i,t} + (1 - \delta_k)k_{i,t-1} \tag{2}$$

$$h_{i,t} = \iota_{h,i,t} + (1 - \delta_h)\theta_{i,t}h_{i,t-1} \tag{3}$$

where δ_k and δ_h are the depreciation rates of physical and human capital, respectively.

We assume that the variables $\theta_{i,t}$, $i \in [0, 1]$, $t = 0, ...$, are identically and independently distributed across individuals and across periods, with unit mean. We further assume that the law of large number applies, so that the aggregate stock of human capital at the beginning of each period t is not random, that is, the following relation holds with probability one:

$$\int_0^1 \theta_{i,t}h_{i,t-1}\,di = \int_0^1 h_{i,t-1}\,di = H_{t-1}. \tag{4}$$

The idiosyncratic shocks $\theta_{i,t}$ are the only sources of uncertainty. Hence there is no aggregate uncertainty in the economy and the rental rates of the two production factors are deterministic.

Let r_t denote the rental rate of physical capital and w_t the wage rate. Both labor and capital income are subject to linear taxes at the rates $\tau_{h,t}$ and $\tau_{k,t}$ at each date t. In what follows it is convenient to use the notation $\widetilde{r}_t$ and $\widetilde{w}_t$ for the after tax prices, $r_t(1 - \tau_{k,t})$ and $w_t(1 - \tau_{h,t})$.

The flow budget constraint of individual i is given by, for each $t = 0, 1, ...,$,

$$c_{i,t} + k_{i,t} + b_{i,t} + h_{i,t} = R_{k,t}k_{i,t-1} + R_{k,t}b_{i,t-1} + R_{h,t}\theta_{i,t}h_{i,t-1} \tag{5}$$

where $R_{k,t} = 1 - \delta_k + \widetilde{r}_t$, and $R_{h,t} = 1 - \delta_h + \widetilde{w}_t$. Since the (after tax) rate of return on physical capital $R_{k,t}$ is non random, in equilibrium the rate of return on risk-free bonds must be the same.

It is convenient to use $x_{i,t}$ to denote the term on the right hand side of (5), indicating the total wealth of individual i at the beginning of period t after the time t shock $\theta_{i,t}$ has been realized. Individuals may borrow, that is, $b_{i,t}$ can be negative, while the holdings of capital are required to be non-negative: $k_{i,t} \geq 0$ and $h_{i,t} \geq 0$ for all i, t. The amount of borrowing is restricted by the natural debt limit, that prevents consumers from engaging in Ponzi schemes and in this environment

[7] We assume the initial endowment vector $(b_{i,-1},\ k_{i,-1}, h_{i,-1})$ takes only finitely many distinct values across individuals $i \in [0, 1]$.

(where the only source of future income is the revenue from the consumers' accumulated human and physical capital) takes the following form:

$$x_{i,t} \geq 0, \tag{6}$$

for all periods $t = 1, ..$ and all contingencies.

To sum up, given the initial wealth $x_{i,0} > 0$ and a sequence of after tax prices $\{\widetilde{r}_t, \widetilde{w}_t\}_{t=1}^{\infty}$, each individual i maximizes his lifetime utility $u_{i,0}$, defined by (1) subject to the flow budget constraints (5) and the debt limit (6).

The individual's decision depends on the history of idiosyncratic shocks affecting him. Thanks to the specification of the utility function (1) and the assumption that shocks are i.i.d., a tractable characterization of the utility maximizing choices is however possible.

First, it is convenient to rewrite equation (5) as follows:

$$x_{i,t+1} = (1 - \eta_{c,i,t}) \left\{ R_{k,t+1}(1 - \eta_{h,i,t}) + R_{h,t+1}\theta_{i,t+1}\eta_{h,i,t} \right\} x_{i,t} \tag{7}$$

where

$$\eta_{c,i,t} \equiv \frac{c_{i,t}}{x_{i,t}}, \qquad \eta_{h,i,t} \equiv \frac{h_{i,t}}{b_{i,t} + k_{i,t} + h_{i,t}}$$

with initial condition $x_{i,0} > 0$. The individual optimization problem can then be equivalently written as a problem of choosing a sequence of the rate of consumption out of his wealth, $\eta_{c,i,t}$, and of the portfolio composition between human capital and riskless assets (physical capital and risk-free bond), $(\eta_{h,i,t}, 1 - \eta_{h,i,t})$ for every $t = 0, 1, ...$, given $x_{i,0}$.

Define the certainty-equivalent rate of return ρ associated with the after-tax rental rate $\widetilde{r}$ and after-tax wage rate $\widetilde{w}$ as follows:

$$\rho(\widetilde{r}, \widetilde{w}, \eta_h) \equiv \left\{ E\left((1 - \delta_k + \widetilde{r})(1 - \eta_h) + \theta(1 - \delta_h + \widetilde{w})\eta_h\right)^{1-\gamma} \right\}^{\frac{1}{1-\gamma}}. \tag{8}$$

Given the specification of the agents' utility and income, we show in the next lemma that the optimal portfolio at any date, given the prevailing after-tax rates $\widetilde{r}$ and $\widetilde{w}$, is obtained as a solution to the static problem of maximizing $\rho(\widetilde{r}, \widetilde{w}, \eta_h)$ with respect to η_h, independently of all the other choice variables. Given this, it is straightforward to verify that the following simple characterization of the solutions of the individual choice problem obtains:

Lemma 1. *Given $\{\widetilde{r}_t, \widetilde{w}_t\}_{t=1}^{\infty}$ and $x_{i,0}$, for any individual i a utility maximizing sequence of portfolio* *compositions and rates of consumption is characterized by the following rule: for each* $t = 0, 1, ...,$

$$\rho_{t+1} = \max_{\eta'_h \geq 0} \rho(\widetilde{r}_{t+1}, \widetilde{w}_{t+1}, \eta'_h), \tag{9}$$

$$\eta_{h,t} = \arg\max_{\eta'_h \geq 0} \rho(\widetilde{r}_{t+1}, \widetilde{w}_{t+1}, \eta'_h), \tag{10}$$

$$\eta_{c,t} = \left\{ 1 + \sum_{s=0}^{\infty} \prod_{j=0}^{s} \left(\beta^{\psi} \rho_{t+1+j}^{\psi-1} \right) \right\}^{-1}. \tag{11}$$

Moreover, the time t *intertemporal utility level is given by*

$$u_{i,t} = v_t x_{i,t},$$

where $x_{i,t}$ *satisfies (7) and* v_t *satisfies the recursive expression:*

$$v_t^{\psi-1} = (1-\beta)^{\psi} + \beta^{\psi} \rho_{t+1}^{\psi-1} v_{t+1}^{\psi-1}, \tag{12}$$

whose solution is

$$v_t = (1-\beta)^{\frac{\psi}{\psi-1}} \left\{ 1 + \sum_{s=0}^{\infty} \prod_{j=0}^{s} \left(\beta^{\psi} \rho_{t+1+j}^{\psi-1} \right) \right\}^{\frac{1}{\psi-1}}. \tag{13}$$

Since the expressions in (10) and (11) are independent of i, the result implies that all the individuals in the economy choose the same rate of consumption, $\eta_{c,t}$, and the same portfolio, $\eta_{h,t}$, in each period t. The heterogeneity among individuals appears in their level of wealth and hence in their utility level, given by their wealth times a common constant v_t.

Comparing (11) and (13) we see that the utility level per unit of wealth v_t and the consumption share $\eta_{c,t}$ at any date t are related as follows:

$$\eta_{c,t} = (1-\beta)^{\psi} v_t^{1-\psi}. \tag{14}$$

Since ρ is a concave function of η_h, an interior solution of (9) is characterized by the first order conditions:

$$\Phi(\widetilde{r}, \widetilde{w}, \eta_h) \equiv E\Big[\big\{ (1-\delta_k+\widetilde{r})(1-\eta_h) + \theta(1-\delta_h+\widetilde{w})\eta_h \big\}^{-\gamma} \times \big\{ \theta(1-\delta_h+\widetilde{w}) - (1-\delta_k+\widetilde{r}) \big\} \Big] = 0. \tag{15}$$

We shall assume throughout the analysis that the derivatives of the function $\Phi : \mathbb{R}_+^3 \to \mathbb{R}$, defined in equation (15), exhibit the following properties:

Assumption 1. *The derivatives of* $\Phi(.)$*, evaluated at* $\Phi = 0$*, satisfy:*

$$\frac{\partial \Phi}{\partial \widetilde{r}} < 0, \text{ and } \qquad \frac{\partial \Phi}{\partial \widetilde{w}} > 0.$$

The concavity of $\rho(\cdot)$ then implies that $\frac{\partial \Phi}{\partial \eta_h} < 0$. These properties ensure that the consumers' optimal portfolio choice displays the 'normal' comparative statics properties: $\partial \eta_h / \partial \widetilde{r} < 0$ and $\partial \eta_h / \partial \widetilde{w} > 0$. As discussed in the Appendix, these properties are always satisfied when $\gamma \leq 1$. When $\gamma > 1$, they hold under appropriate restrictions on the distribution of θ_i.

2.2 Firms

All firms have identical production technology, described by a Cobb-Douglas production function, so we can proceed as if there is a single representative firm. Letting K_{t-1} and H_{t-1} denote, respectively, the aggregate stock of physical and human capital used by firms at the beginning of period t, the aggregate amount of output produced in period t is:

$$Y_t = F(K_{t-1}, H_{t-1}) = AK_{t-1}^{\alpha} H_{t-1}^{1-\alpha}, \tag{16}$$

where Y_t is the aggregate level of output and A is a constant. The profit maximization condition requires that the marginal productivity of the two factors equal their before-tax rental rates

$$r_t = \frac{\partial F(K_{t-1}, H_{t-1})}{\partial K_{t-1}} \equiv F_{k,t}, \qquad w_t = \frac{\partial F(K_{t-1}, H_{t-1})}{\partial H_{t-1}} \equiv F_{h,t}. \tag{17}$$

2.3 Government

The government purchases an amount of output, G_t, in each period t, financed by collecting taxes and issuing debt. Let B_{t-1} be the government debt outstanding at the beginning of period t. The flow budget constraint of the government at any date t is

$$B_t + F(K_{t-1}, H_{t-1}) - \widetilde{r}_t K_{t-1} - \widetilde{w}_t H_{t-1} = G_t + (1 - \delta_k + \widetilde{r}_t) B_{t-1}, \tag{18}$$

where we used $\widetilde{r}_t$, $\widetilde{w}_t$ and the linear homogeneity of the production function $F(K, H)$ to rewrite the tax revenue $\tau_{k,t} r_t K_{t-1} + \tau_{h,t} w_t H_{t-1}$ and the initial stock of debt B_{-1} is given: $B_{-1} = \int_0^1 b_{i,-1}\, di$.

For a given sequence of the aggregate stocks of physical and human capital and prices $\{K_t, H_t, r_t, w_t\}_{t=0}^{\infty}$, a fiscal policy $\{\widetilde{r}_t, \widetilde{w}_t, B_t\}_{t=0}^{\infty}$ is said to be feasible if the flow budget constraint (18) is satisfied for every $t = 0, 1, \ldots$, and

$$\lim_{t \to \infty} \left[\prod_{j=1}^{t} (1 - \delta_k + \widetilde{r}_j)^{-1} \right] B_t = 0. \tag{19}$$

2.4 Competitive equilibrium

Given a sequence of government purchases, $\{G_t\}_{t=0}^{\infty}$ and a fiscal policy $\{\widetilde{r}_t, \widetilde{w}_t, B_t\}_{t=0}^{\infty}$, a *competitive equilibrium* is defined by a price system $\{r_t, w_t\}_{t=0}^{\infty}$ and a collection of stochastic processes $\{c_{i,t}, x_{i,t},$

$b_{i,t},\, k_{i,t},\, h_{i,t}\}_{t=0}^{\infty}$, $i \in [0,1]$, adapted to the filtration generated by the process of idiosyncratic shocks $\{\theta_{i,t}\}_{t=0}^{\infty}$, such that: (a) for each i, $\{c_{i,t},\, x_{i,t}, b_{i,t},\, k_{i,t},\, h_{i,t}\}_{t=0}^{\infty}$ solves the utility maximization problem of agent i, subject to (5) and (6), given prices, the fiscal policy and the initial endowment $b_{i,-1}$, $k_{i,-1}$, $h_{i,-1}$; (b) firms maximize profits, that is, (17) holds for all $t \geq 0$, with $K_{t-1} = \int_0^1 k_{i,t-1}\, di$ and H_{t-1} satisfying (4); (c) all markets clear:

$$C_t + G_t + K_t + H_t = (1-\delta_k)K_{t-1} + (1-\delta_h)H_{t-1} + F(K_{t-1}, H_{t-1}), \tag{20}$$

where $B_t = \int_0^1 b_{i,t}\, di$, $C_t = \int_0^1 c_{i,t}\, di$; (d) the government policy is feasible, that is (18)-(19) hold.

Let X_t denote the average amount of wealth at the beginning of period t: $X_t \equiv \int_0^1 x_{i,t}\, di$. Recalling that, as shown in Lemma 1, the consumers' optimum is characterized by the sequence $\{\eta_{c,t}, \eta_{h_t}\}_{t=0}^{\infty}$, independent of i, X_t evolves as

$$X_{t+1} = R_{x,t+1}(1-\eta_{c,t})X_t, \qquad t = 0, 1, 2, ... \tag{21}$$

where $R_{x,t+1}$ is the equilibrium average rate of return of individual portfolios: for $t = 0, 1, 2, ...$

$$R_{x,t+1} \equiv (1-\delta_k + \widetilde{r}_{t+1})(1-\eta_{h,t}) + (1-\delta_h + \widetilde{w}_{t+1})\eta_{h,t}. \tag{22}$$

Given X_t, aggregate consumption, physical and human capital are given by

$$C_t = \eta_{c,t} X_t, \tag{23}$$

$$K_t = (1-\eta_{c,t})(1-\eta_{h,t})X_t - B_t, \tag{24}$$

$$H_t = (1-\eta_{c,t})\eta_{h,t} X_t. \tag{25}$$

Hence the aggregate dynamics of a competitive equilibrium is simply determined by the sequence $\{\eta_{c,t}, \eta_{h_t}\}_{t=0}^{\infty}$ and the path of average wealth X_t.

It is also useful to observe that the expected after-tax return on human capital must be greater than that on physical capital for all t:

$$1-\delta_k + \widetilde{r}_{t+1} < 1-\delta_h + \widetilde{w}_{t+1}.$$

This follows from the facts that the investment in human capital is risky and, as shown in Lemma 1, individual consumption $c_{i,t}$ varies positively with $x_{i,t}$, and thus with the idiosyncratic shocks to human capital. Hence the return on human capital is negatively correlated with an agent's marginal rate of substitution.

2.5 Benchmark equilibrium with no taxes

As a benchmark, let us consider the situation where the government does not purchase goods, and does not issue debt nor impose any taxes:

$$G_t = B_t = 0, \quad r_t - \widetilde{r}_t = 0, \quad w_t - \widetilde{w}_t = 0 \qquad \text{for all } t \geq 0, \qquad b_{i,-1} = 0, \quad \text{for all } i \in [0,1]. \tag{26}$$

In this case, the competitive equilibrium has a very simple structure. The aggregate economy is always on a balanced growth path, although each individual's consumption fluctuates stochastically over time.

To see this, notice that, using (26), (17) and the market clearing conditions for physical and human capital, the first order conditions for the consumers' optimal portfolio choice (15) reduce to

$$\Phi\left[F_k(1-\eta_{h,t},\eta_{h,t}),F_h(1-\eta_{h,t},\eta_{h,t}),\eta_{h,t}\right]=0, \tag{27}$$

for every $t = 0, 1, 2, ...$ It is then immediate to verify[8] that there exists a unique $\hat{\eta}_h \in (0, 1)$ satisfying (27), which implies that $\eta_{h,t} = \hat{\eta}_h$ must hold for every t.

Set $\hat{F}_k = F_k(1-\hat{\eta}_h,\hat{\eta}_h)$, $\hat{F}_h = F_h(1-\hat{\eta}_h,\hat{\eta}_h)$, and

$$\hat{\rho}=\rho\left(\hat{F}_k,\hat{F}_h,\hat{\eta}_h\right). \tag{28}$$

The argument above together with Lemma 1 yield the following characterization of the equilibria in this benchmark case:[9]

Proposition 2. *Suppose that Assumption 1 holds. If*

$$\beta^{\psi}\hat{\rho}^{\psi-1}<1,$$

with no government intervention - i.e., under (26) - a unique competitive equilibrium exists, characterized by $\hat{\eta}_h$ *and* $\hat{\eta}_c \equiv 1-\beta^{\psi}\hat{\rho}^{\psi-1}$. *Thus the aggregate variables* C_t, K_t, H_t, *and* X_t *all grow at the same rate*

$$\hat{g}_x=(1-\hat{\eta}_c)\hat{R}_x$$

where

$$\hat{R}_x=(1-\delta_k+\hat{F}_k)(1-\hat{\eta}_h)+(1-\delta_h+\hat{F}_h)\hat{\eta}_h,$$

and

$$\hat{v}\equiv\left[\frac{(1-\beta)^{\psi}}{1-\beta^{\psi}\hat{\rho}^{\psi-1}}\right]^{\frac{1}{\psi-1}}.$$

In what follows we refer to this equilibrium without government purchases or taxes as the *benchmark equilibrium*, and use a hat ($\hat{}$) to denote the value of a variable at this equilibrium.

[8] From (16) we have $F_k(1,0)=F_h(0,1)=0$, $F_k(0,1)=F_h(1,0)=+\infty$, and $\lim_{\eta_h\to 0}F_h(1-\eta_h,\eta_h)\eta_h = \lim_{\eta_h\to 1}F_k(1-\eta_h,\eta_h)(1-\eta_h)=0$. Furthermore, from Assumption 1 and the concavity of $\rho(.)$ with respect to η_h it follows that $\frac{d}{d\eta_h}\Phi\left[F_k(1-\eta_h,\eta_h),F_h(1-\eta_h,\eta_h),\eta_h\right]<0$ whenever $\Phi=0$. The claim then follows.

[9] Krebs (2003) derived analogous properties in a similar environment.

3 Optimal taxation and debt

In this section we study the dynamic Ramsey problem where the optimal tax and debt policy in our environment are determined. As is well known from the work of Chamley (1986) and Judd (1985) and the various other papers which followed, when markets are complete the optimal tax rate on capital income is zero in the steady state. Furthermore, when there is also human capital accumulation, Jones, Manuelli, and Rossi (1997) have shown that both labor and capital income tax should be zero in the long run and hence, with positive government purchases, public debt should be negative. Here we demonstrate that uninsurable risks in human capital accumulation significantly change the nature of optimal taxes. The presence of such risks makes both labor and capital income taxation beneficial, which, in turn, implies that the the optimal amount of government debt is positive as long as government purchases are sufficiently small.

More precisely, the *Ramsey problem* consists in finding the fiscal policy $\{\widetilde{r}_t, \widetilde{w}_t, B_t\}_{t=0}^{\infty}$, satisfying (18) and (19), that maximizes consumers' welfare at the associated competitive equilibrium, as defined in Section 2.4, for a given policy determining the level of government purchases $\{G_t\}_{t=0}^{\infty}$. The resulting equilibrium is then denoted the *Ramsey equilibrium.* As is standard in the literature, we assume that the tax rates, or equivalently the after tax prices, in the initial period are exogenously fixed:

$$\widetilde{r}_0 = \overline{\widetilde{r}}_0, \qquad \text{and} \qquad \widetilde{w}_0 = \overline{\widetilde{w}}_0. \tag{29}$$

We take as social welfare function a weighted average of the lifetime utility of individuals: $\int_0^1 \lambda_i u_{i,0}\, di$, with $\lambda_i \in (0,1)$ for each individual $i \in [0,1]$. By Lemma 1, we have $u_{i,0} = v_0 x_{i,0}$, and therefore

$$\int_0^1 \lambda_i u_{i,0}\, di = v_0 \int_0^1 \lambda_i x_{i,0}\, di = v_0 \Big\{ (1-\delta_k + \overline{\widetilde{r}}_0) \int_0^1 \lambda_i (k_{i,-1} + b_{i,-1})\, di + (1-\delta_h + \overline{\widetilde{w}}_0) \int_0^1 \lambda_i \theta_{i,0} h_{i,-1}\, di \Big\}.$$

Given the initial tax rates (29), the terms in the curly braces are determined independently of the fiscal policy chosen by the government. Thus the government's objective reduces to maximizing the utility coefficient:

$$v_0 = (1-\beta)^{\frac{\psi}{\psi-1}} \left\{ 1 + \sum_{t=0}^{\infty} \prod_{j=0}^{t} \left(\beta^{\psi} \rho_{1+j}^{\psi-1} \right) \right\}^{\frac{1}{\psi-1}} \tag{30}$$

Note that (30) implies that v_0 is strictly increasing in ρ_t for all t, regardless of the value of $\psi > 0$. We should stress that in the Ramsey problem as specified above we are looking for a sequence of tax rates and debt levels which may vary over time and are such to maximize the lifetime utility of agents, not just their steady state utility.

Since the economy considered features accumulation of (physical and human) capital and growth, it is convenient to normalize aggregate variables in terms of the total wealth X_t, for each t

$$k_t \equiv \frac{K_t}{X_t}, \quad h_t \equiv \frac{H_t}{X_t}, \quad b_t \equiv \frac{B_t}{X_t}, \quad g_t \equiv \frac{G_t}{X_{t-1}}.$$

With regard to the public expenditure policy, in what follows we shall assume that it is specified in terms of an exogenous sequence of expenditure levels per unit of total wealth $\{g_t\}_{t=0}^{\infty}$. Given our interest in the optimal taxes and debt in the long run, this specification ensures that the ratio of public expenditure G_t to output Y_t remains the same over time. Since the growth rate of output is endogenously determined, such a property would not be ensured if we adopted a more standard specification where instead $\{G_t\}_{t=0}^{\infty}$ is exogenously given. But we want to emphasize that our main finding on the long run tax rate on capital income does not depend on our assumption that g_t instead of G_t is exogenous. First, it holds without government purchases. Second, as shown in the Appendix, we do obtain a similar result for the case where $\{G_t\}_{t=0}^{\infty}$ is exogenously given.

Once restated in terms of the normalized variables $\{k_t,\ h_t,\ b_t,\ \widetilde{r}_{t+1},\ \widetilde{w}_{t+1},\ \rho_{t+1},\ \eta_{h,t},\ \eta_{c,t},\ R_{x,t+1}\}_{t=0}^{\infty}$, the Ramsey problem can then be formulated as a two-step maximization problem. In the first step, we take as arbitrarily given a sequence $\{\eta_{c,t},\ b_t\}_{t=0}^{\infty}$, and consider the optimal choice of the remaining variables $\{\widetilde{r}_{t+1},\ \widetilde{w}_{t+1},\ \rho_{t+1},\ \eta_{h,t},\ R_{x,t+1},\ k_t,\ h_t\}_{t=0}^{\infty}$. Substituting (24) and (25) into (18), dividing both sides by X_t and using (21), the government budget constraint becomes

$$\begin{aligned} g_{t+1} + (1-\delta_k+\widetilde{r}_{t+1})b_t \qquad\qquad (31)\\ = (1-\eta_{c,t})R_{x,t+1}b_{t+1} + F\left[(1-\eta_{c,t})(1-\eta_{h,t}) - b_t, (1-\eta_{c,t})\eta_{h,t}\right] \\ - \widetilde{r}_{t+1}\left[(1-\eta_{c,t})(1-\eta_{h,t}) - b_t\right] - \widetilde{w}_{t+1}(1-\eta_{c,t})\eta_{h,t} \end{aligned}$$

Conditions (9), (10), (22), and (31) imply that, for each t, the choice of $(\widetilde{r}_{t+1},\ \widetilde{w}_{t+1},\ \eta_{h,t},\ R_{x,t+1})$ only affects ρ_{t+1}, and not ρ_s for any $s \neq t+1$. Therefore, in the first-step problem, v_0 can be maximized by maximizing ρ_{t+1} separately for each t. Note also that among the given sequence $\{\eta_{c,t},\ b_t\}_{t=0}^{\infty}$, ρ_{t+1} is only affected by b_t, b_{t+1}, and $\eta_{c,t}$. Hence the first-step problem reduces to the following static maximization problem:

$$\rho^R(b_t, b_{t+1}, \eta_{c,t}; g_{t+1}) \equiv \max_{\{\widetilde{r}_{t+1}, \widetilde{w}_{t+1}, \eta_{h,t}, R_{x,t+1}\}} \rho(\widetilde{r}_{t+1}, \widetilde{w}_{t+1}, \eta_{h,t}) \qquad (32)$$

subject to (10), (22) and (31).

The second-step problem consists then in the choice of the sequence $\{\eta_{c,t},\ b_t\}_{t=0}^{\infty}$ which maximizes v_0. This problem can be written recursively as:

$$\max_{\{v_{t+1}, b_{t+1}, \eta_{c,t+1}\}_{t=0}^{\infty}} v_0, \qquad (33)$$

subject to (12), (14), and $\rho_{t+1} = \rho^R(b_t, b_{t+1}, \eta_{c,t}; g_{t+1})$.

To derive some insights on the properties of the solution of the dynamic Ramsey problem and hence of the optimal taxes and debt, it is useful to consider first the case where only a subset of policy instruments at a time is available.

3.1 The case of balanced budget: tax labor and subsidize capital

We study first the welfare effects of taxing capital or labor, by examining the case where there are no government purchases nor public debt, thus the budget is balanced at all times:

$$g_t = b_t = 0, \quad \text{for all } t, \tag{34}$$

with $b_{i,-1} = 0$ for all i. Under (34), the Ramsey problem reduces to:

$$\max_{\{\widetilde{r}, \widetilde{w}, \eta_h\}} \rho(\widetilde{r}, \widetilde{w}, \eta_h) \tag{35}$$

subject to (10) and

$$F(1-\eta_h, \eta_h) - \widetilde{r}(1-\eta_h) - \widetilde{w}\eta_h = 0. \tag{36}$$

The solution is time invariant and the economy is always on a balanced growth path, as in the benchmark equilibrium of subsection 2.5. Let us denote the variables at the solution with the superscript o, i.e., v^o, ρ^o, R_x^o, η_c^o, etc.

The balanced budget requirement implies that the tax revenue on one factor equals the subsidy on the other factor. In such a case, taxing labor and subsidizing (physical) capital is welfare improving.

Proposition 3. *Under Assumption 1, at the benchmark equilibrium, social welfare increases if a marginal subsidy on capital (and a corresponding tax on labor) is introduced for all t.*

More specifically, we show in the proof of the proposition that $\left(\frac{\partial \rho}{\partial \widetilde{r}} - \frac{\partial \rho}{\partial \widetilde{w}} \frac{1-\hat{\eta}_h}{\hat{\eta}_h}\right)\Big|_{\widetilde{r}=\hat{F}_k, \widetilde{w}=\hat{F}_h, \eta_h=\hat{\eta}_h} > 0$ and, furthermore, that an increase in $\widetilde{r}$ is equivalent to a decrease in τ_k.[10] As we see from (30), social welfare increases when ρ rises.

Hence, even though taxes are distortionary and there is no need for the government to raise tax revenue, in the incomplete markets environment considered, it is beneficial, for agents' welfare, to tax labor and subsidize physical capital. The intuition for this result is simple. In the benchmark

[10] As shown in the proof of the proposition in the Appendix, the term $(1-\hat{\eta}_h)/\hat{\eta}_h$ equals $-d\widetilde{w}/d\widetilde{r}$, with the relationship between $\widetilde{w}$ and $\widetilde{r}$ implicitly defined by equation (36), evaluated at the benchmark equilibrium with no taxes.

equilibrium (with no taxes) individuals are exposed to the risk in their labor income which they are unable to insure. By taxing risky labor income and using the total revenue of this tax to subsidize the riskless return on capital, that is by setting $\widetilde{r} > \hat{F}_k$ and $\widetilde{w} < \hat{F}_h$, the government can reduce the individual exposure to idiosyncratic risk. Thus, a welfare improvement can be attained by reducing the return of the risky factor and increasing that of the riskless one.[11]

Note that Proposition 3 only characterizes the properties of optimal taxes in a neighborhood of zero. If, in addition, the function $\rho(\widetilde{r}, \widetilde{w}, \eta_h)$ is such that problem (35) has a unique local maximum (that is, the first order approach holds),[12] we can also say that the globally optimal tax rate on physical capital is indeed negative in the environment considered, when government consumption and debt are zero.

The above result tells us that market incompleteness alone does not justify taxing capital income. However, in what follows we show that, once the government is allowed to issue debt, it becomes beneficial to tax capital income (in the long run).

3.2 Desirability of government debt

Next, we turn our attention to the welfare effects of issuing government debt, when markets are incomplete. We show that, starting from a zero level of government debt, increasing the amount of government debt is welfare improving as long as government purchases are small enough.

Consider the allocation obtained as the solution to the Ramsey problem under (34), that is, with no government debt nor expenditure, studied in the previous subsection. We will investigate whether allowing for an arbitrarily small (positive or negative) level of debt at only one date yields a welfare improvement. This amounts to studying the solutions of the Ramsey problem under the following alternative restriction to (34):

$$b_{T+1} = \bar{b}_{T+1}, \qquad b_t = 0, \quad \text{for all } t \neq T+1, \qquad g_t = 0, \quad \text{for all } t, \tag{37}$$

for some given $\bar{b}_{T+1}$, with $b_{i,-1} = 0$ for all i. The next proposition establishes a necessary and sufficient condition under which a positive amount of debt, $\bar{b}_{T+1} > 0$, is welfare improving.

Proposition 4. *Suppose that Assumption 1 hold and consider the Ramsey equilibrium with no public debt nor expenditure, that is, under (34). Then increasing $\bar{b}_{T+1}$ above zero[13] for a given*

[11] A related result has been obtained by Eaton and Rosen (1980) and Barsky, Mankiw and Zeldes (1986). Using a two-period model where the second period labor income is subject to idiosyncratic risk, they consider the effect of a combination of a proportional labor income tax and lump-sum transfers. Such a policy has an insurance effect similar to the one discussed here.

[12] This property is satisfied in all the numerical examples considered in the rest of the paper.

[13] Analogously to the previous Proposition 3, this result characterizes optimal debt in a neighborhood of zero.

period $T+1$ improves the lifetime utility of all individuals if and only if

$$R_x^o > 1 - \delta_k + F_k^o. \tag{38}$$

As argued in Subsection 2.4, since human capital is risky but physical capital is not, by the risk aversion of consumers we have $R_x^o > 1 - \delta_k + \widetilde{r}^o$. Moreover, in Subsection 3.1 we have shown that when $b_t = g_t = 0$ for all t the (locally) optimal tax is negative on capital and positive on labor; if the globally optimal tax on capital have the same - negative - sign, we have $\widetilde{r}^o > F_k^o$. These two properties imply that (38) holds, and so that increasing the amount of government debt in one period is welfare improving.

To gain some intuition for the result in Proposition 4 note that, in the light of (30), whether or not increasing $\bar{b}_{T+1}$ above zero is welfare improving depends on how this change affects the equilibrium values of $\{\rho_{t+1}\}_{t=0}^{\infty}$. As discussed when deriving (32), only ρ_{T+1} and ρ_{T+2} are affected by the change in $\bar{b}_{T+1}$. First note that increasing $\bar{b}_{T+1}$ above zero reduces taxes in period $T+1$ and this increases ρ_{T+1}. The benefit is proportional to the after-tax average rate of return of individual portfolios, R_x^o, at the Ramsey equilibrium with $b_t = g_t = 0$ for all t. This is natural because R_x^o is the average rate that individuals earn using the proceeds from the tax cut in $T+1$. In contrast, the increase in $\bar{b}_{T+1}$ has to be offset by a tax increase in period $T+2$ to redeem the debt. As a result, ρ_{T+2} decreases. The cost is proportional to the (before-tax) rate of return on government debt, $1 - \delta_k + F_k^o$. Whether or not increasing $\bar{b}_{T+1}$ is beneficial depends on the comparison between these two terms and we show in the proof that the benefit of increasing $\bar{b}_{T+1}$ dominates its cost if and only if (38) holds.

We should also point out that the argument of the proof does not use the fact that $g_t = 0$, hence the claim in the proposition extends to the case where $g_t = g > 0$, with the variables R_x^o, F_k^o replaced by the corresponding terms evaluated at the Ramsey equilibrium when $b_t = 0$ and $g_t = g$ for all t (analogously denoted with a superscript g). Naturally, taxes must be appropriately increased so as to satisfy the government budget constraint when $g > 0$ and $b_t = 0$. If the optimal tax on capital becomes positive ($F_k^g > \widetilde{r}^g$), the inequality corresponding to (38) when $g > 0$ holds only if the tax on capital is small relatively to the risk premium earned after-tax by human capital over physical capital.[14]

3.3 Ramsey steady state

Finally, we consider the case where the government can freely choose a fiscal policy $\{\widetilde{r}_{t+1}, \widetilde{w}_{t+1}, B_t\}_{t=0}^{\infty}$ satisfying (18) and (19). We focus in particular on the properties of the Ramsey equilibrium

[14]Note that the expression corresponding to (38) can be rewritten as $F_k^g - \widetilde{r}^g < (\widetilde{w}^g - \delta_h - \widetilde{r}^g + \delta_k)\eta_h^g$.

at a steady state (balanced growth path),[15] and assume so that $g_t = g \geq 0$ for all t.

Proposition 5. *In the steady state of the Ramsey equilibrium the following condition holds:*

$$R_x = (1 - \delta_k + F_k)\left[1 - (1-\psi)\beta^{\psi}\rho^{\psi-2}\rho_{\eta_c}\eta_c\right]^{-1} \tag{39}$$

with $\rho_{\eta_c}^{Rz} \equiv \frac{\partial \rho^R}{\partial \eta_c}$.

In the proof we show, by an analogous argument to the one in the previous subsection, that the beneficial effect on ρ_t and hence also on v_t of an increase in b_t is proportional to R_x, while its cost, due to a reduction of ρ_{t+1}, is proportional to $(1 - \delta_k + F_k)$. There is now an extra term, $(1 - \beta^{\psi}\tilde{\rho}^{\psi-2}(1-\psi)\rho_{\eta_c}\eta_c)^{-1}$, multiplying the latter, which captures the effect on ρ of the change in the saving rate, $1 - \eta_c$. This term arises when b takes a nonzero value.

Condition (39) says that, at a steady state, the (before tax) return on government debt and physical capital, $1 - \delta_k + F_k$, must equal the average rate of return of private consumers' portfolios, R_x, after adjusting for the effect of public debt on the consumers' saving rate. It implies that the steady state tax rate on capital income is strictly positive as long as the effect on the saving rate is not too large.[16]

To see this, consider the case where the elasticity of intertemporal substitution $\psi = 1$, in which case $\eta_{c,t}$ is a constant, equal to $1 - \beta$, so that the effect on the saving rate is zero. Therefore (39) reduces to $R_x = 1 - \delta_k + F_k$. Recalling again the property $R_x > 1 - \delta_k + \tilde{r}$, due to the risky nature of labor income, established in Subsection 2.4, we obtain so $\tilde{r} < F_k$, that is $\tau_k > 0$. By continuity, the same property holds for ψ sufficiently close to unity. These findings are summarized in the following result.

Corollary 6. *When $\psi = 1$, in the steady state of the Ramsey equilibrium we have:*

$$R_x = 1 - \delta_k + F_k \tag{40}$$

and the optimal tax rate on physical capital is positive:

$$\tilde{r} < F_k$$

---

[15] Along a balanced growth path, prices and normalized variables $\{\tilde{r}_t, \tilde{w}_t, b_t, \rho_{t+1}, \eta_{h,t}, \eta_{c,t}, R_{x,t+1}, k_t, h_t\}$ are all constant. Note also that we use the terms "steady state" and "balanced growth path" interchangeably given the fact that a steady state for the normalized variables corresponds to a balanced growth path of the economy. In all the numerical results reported in the next section the solution to the Ramsey problem converges to a steady state.

[16] The steady-state optimal tax rate on capital is indeed strictly positive in all the numerical examples considered in the paper.

3.4 Discussion

The analysis in this section has identified two roles that taxes and government debt can serve in an economy with incomplete markets. The first one is the provision of insurance, that is, it is beneficial to tax a factor that is subject to uninsurable idiosyncratic risks. In our model, labor income is exposed to such risks, which justifies labor income taxes as demonstrated in Proposition 3. This type of benefits exists regardless of whether individual labor productivity is given exogenously or determined endogenously through human capital accumulation.[17]

The second role is the intertemporal allocation of taxes. This corresponds to the benefits of government debt and capital income taxes described in Propositions 4, 5 and Corollary 6. In the presence of uninsurable risks to human capital accumulation, in equilibrium human capital yields a higher expected rate of return than physical capital. Hence without taxes on capital income the rate of return faced by the private sector (R_x) is necessarily greater than the rate of return faced by the government ($1 - \delta_k + F_k$). As long as this is the case, it is cheaper to borrow for the government than for the consumers and increasing government debt is beneficial. The increase in debt requires in turn to increase taxes to pay for servicing the debt. Our findings show that when public debt is at the optimal level the tax rate on capital should be increased until the after tax return on consumers' savings equals the before tax return on public debt (possibly with a correction to take into account the effect of debt on the saving rate). This condition can be interpreted as a kind of "no arbitrage condition" between the government and the private sector.

How robust are this type of benefits? Since the benefits of government debt and capital income taxes we have found arise from the rate of return differential between human and physical capital, they would survive in more general models as long as human capital accumulation is riskier than physical capital. And empirical evidence suggests that human capital does have a higher average rate of return than physical capital (e.g., Card (2001), and Palacios-Huerta, (2003)).

The presence of uninsurable risk clearly plays a key role for these results. If agents were able to trade in complete markets so as to fully hedge all risk, so that human capital would also be, effectively, a safe asset, then we would have $R_k = R_h = R_x$, in which case the condition $R_x = 1-\delta_k+F_k$ implies that $\tau_k = 0$. We should point out however that in our environment market incompleteness alone does not justify capital income taxation, this only happens when the government is allowed to issue debt. As discussed in the Introduction, the reasons for this result are quite different from those of the finding of Aiyagari (1995), since the optimality of capital taxation arises from the comparison of costs and benefits of taxes and debt, rather than from the conditions needed for a steady state to exist. Public debt also plays no role in relaxing the agents' borrowing constraints, since they are

[17] See Gottardi, Kajii, and Nakajima (2014) for the case where individual labor productivity is given exogenously.

never binding in our set-up, hence its optimality does not depend on such role, unlike in Aiyagari and McGrattan (1998).

4 Quantitative analysis

In this section we calibrate our model based on some empirical evidence on the U.S. economy, and examine how market incompleteness affects the structure of optimal taxes and debt.

4.1 Baseline calibration

Suppose that $\theta_{i,t} \in \{1+\overline{\theta}, 1-\overline{\theta}\}$, each occurring with equal probability. The values for the parameters of our model economy, $\{\beta, \psi, \gamma, A, \alpha, \delta_k, \delta_h, g, \tau_k, \tau_h, \overline{\theta}\}$, are set as follows. First, the intertemporal elasticity of substitution ψ is set equal to one and the coefficient of relative risk aversion γ equal to three. Second, the capital share of income α is set to 0.36, and both the depreciation rates of physical and human capital are $\delta_k = \delta_h = 0.06$. Third, the tax rates on capital and labor income are identical, $\tau_k = \tau_h = \tau$. Then the value of the remaining parameters, $\{\beta, A, g, \tau, \overline{\theta}\}$, are determined so that the following features of the U.S. economy are replicated: (i) government purchases are 18 percent of GDP; (ii) government debt is 51 percent of GDP; (iii) the capital-output ratio is 2.7; (iv) the growth rate of GDP is 1.6 percent; (v) the variance of the permanent shock to individual labor earnings is 0.0313. The first four facts are based on Chari, Christiano and Kehoe (1994), and the last one on Meghir and Pistaferri (2004).[18] Here we follow Krebs (2003) in using estimates of permanent shocks to individual labor income to calibrate the variance of the human capital shock, θ. This is based on the fact that i.i.d. human capital shocks imply that labor income follow a logarithmic random walk. The baseline parameter values are summarized in Table 1.

Column (1) of Table 2 reports the fiscal policy and other variables in the balanced-growth equilibrium associated with the baseline parameter values in Table 1. The ratio of government debt to output, the share of government purchases and the growth rate of the economy are pinned down by our calibration assumptions. The capital and labor tax rates are identical by construction and equal to 19.95 percent.[19] We also present the primary surplus of the government and the 'effective'

[18] It is also consistent with the evidence reported by Storesletten, Telmer and Yaron (2004).

[19] Our baseline tax rates (19.95 percent) may be smaller than the values typically used in the literature. For instance, Trostel (1993) uses 40 percent as the benchmark tax rate. The reason for this difference lies in the fact that our model abstracts from transfer payments of the government. As far as the nature of the Ramsey steady state is concerned, this difference is not important, because the Ramsey steady state would not be affected by changing the baseline tax rates, and also because, as discussed in Section 4.4.1, this steady remains unchanged if lump sum transfers independent of the idiosyncratic risks are introduced.

interest rate, defined as the difference between the after-tax interest on government bonds and the growth rate of aggregate output.[20] Finally, we report the human capital premium, defined as the (before-tax) premium in the return on human capital. This variable plays an important role in our set-up to assess the benefits of government debt and capital income taxes and to evaluate the empirical relevance of our model. As reported in the table, this premium in the baseline calibration is 5.28 percent. This is indeed consistent with the evidence given, for instance, by Card (2001) and Palacios-Huerta (2003).

4.2 Results

For the economy described in Table 1 we derive the optimal tax and debt policy, obtained as the solution of the Ramsey problem of maximizing consumers' intertemporal welfare. The tractable nature of the model considered allows us to study also the transitional dynamics of the Ramsey equilibrium, with tax rates which may vary over time. The levels of the fiscal policy and other equilibrium variables at the Ramsey steady state of the economy are reported in the second column of Table 2. In addition, the third and fourth columns report the corresponding values at the Ramsey steady state with no government purchases (that is, when $g = 0$), respectively, when the level of public debt is optimally chosen and when it is constrained to be zero.

Comparing columns (1) and (2), we see that when taxes and public debt are optimally set to maximize consumers' welfare, both the tax rate on labor income and the debt-output ratio are lower than in the baseline policy calibrated on the U.S. economy, while the tax rate on capital income is essentially the same. The data in the last two columns allow then to gain some understanding on the properties of optimal taxes and debt, in light of the results of the previous section.

The case where government purchases are zero is a useful benchmark because, if asset markets were complete, in such case the optimal tax rates (and public debt) would all be zero. Therefore, if we find that non-zero tax rates and debt levels are desirable when $g = 0$, this is entirely due to the distortions caused by market incompleteness. In Proposition 3 we have seen that taxing labor income and subsidizing capital income is beneficial because it reduces the risk faced by individuals. Column (4) in Table 2 illustrates the quantitative significance of this finding: when the government is not allowed to borrow or lend ($b = 0$), the optimal tax rate on labor income is 0.19 percent, and the one on capital income is -0.34 percent. The signs are consistent with the proposition recalled above, but the levels are negligible. On the other hand, column (3) shows that if the government can issue debt the steady-state debt-output ratio is positive, in accord with Proposition 4, and

[20] The government budget constraint (31) implies that, on a balanced growth path, the debt-output ratio is equal to the rate of primary surplus divided by the effective interest rate.

quite large (202.6 percent). In this case the optimal tax rates on labor income and capital income at the steady state are both positive, 4.99 percent and 11.56 percent, respectively, to pay the debt service. Comparing columns (3) and (4), we see that the ability of the government to borrow and lend can significantly affect the structure of optimal taxes. Hence assuming an exogenous amount of government debt, which is often done in the existing literature on optimal taxation with incomplete markets, may provide misleading results.

Finally, comparing columns (2) and (3) in Table 2 we see that the introduction of government purchases and hence the need to pay for them increase the optimal level of both tax rates, to $\tau_k = 19.64$ percent and $\tau_h = 14.88$ percent, and reduce the debt-output ratio to 0.19 percent. Note that, when the level of public debt is optimally chosen, both in columns (2) and (3) the tax rate on capital is positive, as predicted by Corollary 6. To understand the different optimal levels of debt obtained in Columns (2) and (3), we report in Table 3 the value of the variable 'benefits of debt', defined by $R_x^g - (1 - \delta_k + F_k^g)$, evaluated both when $g = 0$ and when g is positive at its baseline level. Proposition 4 and the following discussion show that this variable captures the benefits of issuing debt, and we see that indeed this term is positive when $g = 0$ while it is close to 0 when g equals its baseline value. Furthermore, the significantly lower value of this variable in the second case is primarily explained by the higher level of taxes (in particular on capital) needed to pay for public expenditure: the human capital premium has in fact a similar value in these two cases, as we see in the last row of columns (2) and (4) in Table 2.

How sizable are the benefits associated with the move from the baseline fiscal policy to the Ramsey policy? These benefits can be measured by the rate of permanent increase in consumption of each individual that makes him/her indifferent between the two policies. As can be seen from Lemma 1, this rate is the same for all consumers and is given by the ratio of the values of v_0 under the two policies. Table 4 shows the result. When we only compare the steady states associated with the baseline policy and the Ramsey policy, the welfare gain of adopting the Ramsey policy amounts to an increase of about 8.7 percent in each individual's consumption (in accord with the increase in the growth rate of income reported in Table 2). But this number ignores the cost of transition, where a significant increase in taxes takes place to reduce the debt level. When the transition is taken into account, the gain becomes substantially smaller, 0.85 percent, nevertheless a significant amount.[21] Note that the welfare gains for adopting the Ramsey policy in our model are much smaller than those obtained in a deterministic environment by Jones, Manuelli, and Rossi (1993). This is because, with uninsurable risks to human capital accumulation, the Ramsey steady state is much closer to the baseline U.S. economy.

[21]See the Appendix for a description of the transitional dynamics in the Ramsey equilibrium.

4.3 Sensitivity analysis

In this subsection we examine how the debt-output ratio and the tax rates at the Ramsey steady state vary for different values of the risk aversion coefficient γ, of $\bar{\theta}$, capturing the variance of the idiosyncratic risk, and of the intertemporal elasticity of substitution ψ. For the purpose of normalization, when the values of these parameters are changed we adjust the value of the discount factor β so that the steady-state growth rate under the baseline policy remains equal to 1.6 percent.

Figure 1 plots the results for the changes in risk aversion. We see that the optimal debt-output ratio is very sensitive to the choice of the degree of risk aversion. It is about -100 percent when $\gamma = 1$, and about 70 percent when $\gamma = 5$. Correspondingly the tax rates also vary with risk aversion, the one on capital τ_k much more than that on labor τ_h. To better understand this high sensitivity of the optimal debt level to risk aversion, in Figure 1 we also plotted the primary surplus and the effective interest rate: the decreasing pattern of the latter clearly contributes to amplify the effect on the debt level of the increase in the primary surplus (starting at -4 and going up to 1 percent). More importantly, in Figure 3(a) we plot the values of $R_x^g - (1 - \delta_k + F_k^g)$, the variable benefits of debt, evaluated at the solution of the Ramsey problem subject to the constraint $b = 0$. The pattern of this variable mirrors quite closely that of the optimal level of debt: it varies significantly with γ, increasing when γ rises, and is positive whenever debt is positive. As discussed in Subsection 3.2, the value of this variable is primarily determined by the level of the human capital premium at $b = 0$, $(F_h^g - \delta_h) - (F_k^g - \delta_k)$, together with the optimal level of the tax on capital. For completeness we plot the values of the human capital premium in Figure 3(b), showing that this variable is also increasing in γ and is quite sensitive to the degree of risk aversion. These findings show the importance of the size of the rate of return differential between human and physical capital in determining the desirability of public debt when public expenditure is positive.

Figure 2 then shows that the debt-output ratio varies significantly also with the magnitude of the variance of the idiosyncratic risk. It is negative and large (-200 percent) when there is no idiosyncratic risk ($\mathrm{std}(\theta) = 0$), in accord with the findings mentioned above of the literature on the complete market case. The debt-output ratio gets larger when risk increases, reaching a zero level when $\mathrm{std}(\theta)$ is near its baseline level, 0.1585, and a positive level of about 60 percent when $\mathrm{std}(\theta) = 0.2$. The two tax rates are very close to each other when the idiosyncratic risk is moderate ($\mathrm{std}(\theta) < 0.1$), but when $\mathrm{std}(\theta) > 0.1$ the labor tax rate τ_h proves much less sensitive to changes in the variance of idiosyncratic risk. Most of the increase in the steady state level of debt is then financed with an increase in τ_k. As shown in Figure 3(c) similar properties to the ones found for changes in risk aversion again holds for the pattern of the variable benefits of debt, contributing to explain the high sensitivity of the debt level also in this case.

In contrast, we see in Figure 4 that the value of the intertemporal elasticity of substitution does not affect much the optimal debt-ratio and tax levels. This suggests that the effect of the change in the saving rate, captured by the extra term appearing in (39), is quantitatively small.

4.4 Non Linear tax policies

So far we have restricted our attention to tax policies consisting in proportional taxes on labor and capital income. In this subsection we extend the analysis to allow for other forms of taxation.

4.4.1 Lump sum taxes

We examine first the case where the government can also impose lump sum taxes $\{T_{i,t} : i \in [0,1]\}_{t=0}^{\infty}$, which may depend on the time period t and the identity i of an individual, but not on the realization of idiosyncratic shocks $\{\theta_{i,s}\}_{s=0}^{t}$. A fiscal policy is then given now by $\{\widetilde{r}_t, \widetilde{w}_t, \{T_{i,t} : i \in [0,1]\}, B_t\}_{t=0}^{\infty}$.

Let $\mathrm{TP}_{i,t}$ denote the present-discounted value of the lump sum taxes that individual i has to pay after period t:

$$\mathrm{TP}_{i,t} \equiv \sum_{j=0}^{\infty} \prod_{s=0}^{j} (1 - \delta_k + \widetilde{r}_{t+1+s})^{-1} T_{i,t+1+j},$$

Notice that for this policy to be feasible, it must be

$$R_{k,0}(k_{i,-1} + b_{i,-1} - \mathrm{TP}_{i,-1}) + R_{h,0}\theta_{i,0}h_{i,-1} > 0, \qquad \text{for all } i \in [0,1], \tag{41}$$

to ensure the non negativity of the value of the initial endowment (and hence that the natural debt limit is satisfied).

It is immediate to verify that a competitive equilibrium associated with such policy is also an equilibrium without lump sum taxes where (i) the fiscal policy is given by $\{\widetilde{r}_t, \widetilde{w}_t, \hat{B}_t\}_{t=0}^{\infty}$, with $\hat{B}_t = B_t - \int_0^1 \mathrm{TP}_{i,t}\, di$; and (ii) the consumers' initial endowment of bonds is:

$$\hat{b}_{i,-1} = b_{i,-1} - TP_{i,-1}.$$

Thus the only effect of lump sum taxes is to change the initial distribution of income among consumers. Since the Ramsey steady state does not depend on the initial distribution of income among consumers, the long-run equilibrium is identical with and without lump sum taxes. The only difference is therefore the debt-output ratio, and our previous results on the tax rates at the Ramsey steady state remain valid.

4.4.2 Nonlinear taxes

Next we investigate the effects of allowing for nonlinear taxes on labor income so that, for instance, we may have progressive income taxation.[22] Although a complete analysis of the optimal fiscal policy in this case is beyond the scope of this paper, we can learn something about these effects by considering the simpler case where nonlinear taxes are only allowed in a single period, say $t = 1$. For all the other periods, taxes are restricted to be linear and described as before by $(\widetilde{r}_t, \widetilde{w}_t)$, $t \neq 1$.

Regarding the possible forms of these taxes, we follow Conesa, Kitao and Krueger (2009) in assuming that labor income is taxed according to the following formula:

$$T(\omega_{i,1}) = \tau_{ha}\big[\omega_{i,1} - (\omega_{i,1}^{-\tau_{hb}} + \tau_{hc})^{-\frac{1}{\tau_{hb}}}\big], \tag{42}$$

where $\omega_{i,1} = w_1\theta_{i,1}h_{i,0}$ is the before-tax labor income of individual i in period 1, $T(\omega_{i,1})$ is the labor income tax that he/she must pay, and $(\tau_{ha},\, \tau_{hb},\, \tau_{hc})$ are the parameters describing the tax schedule. Thus a fiscal policy is now described by $\big\{(\tau_{ha},\, \tau_{hb},\, \tau_{hc}),\, \widetilde{r}_1,\, B_1,\, \{\widetilde{r}_t,\, \widetilde{w}_t,\, B_t\}_{t=2}^{\infty}\big\}$.

Suppose the economy is at the Ramsey steady state with linear taxes at the beginning of period 0. The optimal[23] policy for period 1, $\{\tau_{ha}, \tau_{hb}, \tau_{hc}, \widetilde{r}_1, B_1\}$, is computed using a grid search algorithm, while the optimal tax rates after period 2, $\{\widetilde{r}_t, \widetilde{w}_t, B_t\}_{t=2}^{\infty}$, are derived just as before. Let a bar $(\bar{\ })$ over a variable indicate then the value in the Ramsey steady state with linear taxes, and a variable with an asterisk $(*)$ denote its value in the equilibrium where optimal nonlinear labor income taxes are introduced in period $t = 1$.

Consider the case where $g = 0$, the initial wealth $x_{0,i}$ is 1.5 for $i > 0.5$ and 0.5 for $i \leq 0.5$, and the rest of the parameter values are as in Table 1. As shown in column (3) in Table 2, the Ramsey steady state with linear taxes has $\overline{\tau}_k = 0.1156$, $\overline{\tau}_h = 0.0499$, and $\overline{b} = 0.2364$.[24] With nonlinear taxes in period 1, the optimal tax policy at $t = 1$ is instead given by $(\tau^*_{k,1},\, \tau^*_{ha},\, \tau^*_{hb},\, \tau^*_{hc}) = (0.1256, 0.0599, 0.15, 0.7659)$, and $b^*_1 = 0.2355$. With the nonlinear function $T(\omega)$, both the marginal and the average labor income tax rates can vary across individuals, which may enhance risk sharing among them. It turns out, however, that the optimal variation of the marginal and average tax rates across individuals is very small. The lowest and highest values of the marginal tax rate on labor income in period 1, $T'(w_1\theta_{i,1}h_{i,0})$, are 5.65 percent and 5.79 percent, respectively.

[22] Since workers in our set-up are homogenous in terms of the productivity of their investment in human capital, there is no reason to allow for subsidies to human capital accumulation, in addition to taxes on labor income, as in Krueger and Ludwig (2003).

[23] Since non linear taxes may affect the distribution of income among consumers, the specification of the Pareto weights may now matter. In what follows we restrict attention to the case where these weights are identical across individuals: $\lambda_i = 1$ for all i.

[24] Note that $\overline{b}$ is the steady-state value of $b_t = B_t/X_t$ (Table 2 reports the steady-state value of B_{t-1}/Y_t instead).

The average value of the marginal tax rate is about 5.7 percent, that is greater than $\overline{\tau}_h$. Similarly, the lowest and highest value of the average labor income tax rate, $T(w_1\theta_{i,1}h_{i,0})/(w_1\theta_{i,1}h_{i,0})$, are 5.5 percent and 5.68 percent, respectively. Its average across individuals is 5.6 percent. The capital income tax rate, $\tau^*_{k,1}$, is also greater than $\overline{\tau}_k$. The normalized level of debt issued in period 1, b^*_1, is lower than $\overline{b}$, implying that the amount of taxes collected is larger when non-linear taxes are available. We also evaluate the welfare gain of using nonlinear taxes in period 1, measured in date 0 consumption equivalent:[25] the welfare gain is tiny, less than 0.0002 percent of date-0 consumption.

To assess the robustness of the previous finding, we consider various other specifications of the initial wealth distribution $x_{0,i}$, $i \in [0,1]$. But the basic feature remains identical: introducing non-linear taxes results in (i) small variation in both the marginal and average labor income tax rates across individuals; (ii) higher average marginal tax rate on labor income; (iii) higher level of the optimal capital income tax rate in period 1; (iv) small welfare gains. Hence introducing nonlinear taxes does not modify the main qualitative findings of the previous sections, and in particular may even strengthen the benefit of capital income taxation with uninsurable shocks to human capital accumulation.

Our result that allowing for nonlinear labor income taxes increases the optimal capital income tax rate in period 1, $\tau^*_{k,1} > \overline{\tau}_k$, may appear surprising. To understand this, recall that, as reported above, the optimal (average and marginal) tax on labor is higher. The fact that capital income should be taxed more can then be viewed as a way to ensure that the desired ratio between physical and human capital is attained[26], as well as the optimal intertemporal allocation of taxes and hence the optimal debt level.

5 Conclusion

We have studied the Ramsey taxation problem in an incomplete market model with risky human capital accumulation. We have analytically demonstrated the benefits of labor and capital income taxation and of government debt in such an environment. The benefit of labor income taxes emerges for the standard reason: since labor income is subject to uninsurable idiosyncratic risks, taxing it reduces the risks that workers are exposed to. The main contribution of this paper is the finding

[25] It is conventional to measure a gain/loss of a policy reform by a proportional increase/decrease of consumption for all periods. But here, since non-linear taxes are introduced only in one period, we measure its gain by a proportional increase in just one period (period 0).

[26] Note that allowing the government to have more policy instruments does not necessarily make all equilibrium variables closer to the first-best values. In a related context, this point is emphasized in Dávila, Hong, Krusell, and Rios-Rull (2012).

of the benefits of government debt and capital income taxation. They arise because of the rate of return differential between human and physical capital which is, in turn, generated by the difference in risk between them, and implies that it is cheaper to borrow for the government than consumers. Our quantitative results illustrate that the optimal capital-income tax rate in the long run is sizable, while the optimal level of debt is more sensititve to the level of government expenditure and of the risk premium.

In order to keep the model as transparent and tractable as possible, we have made a number of simplifying assumptions. In the environment considered, even though individuals are heterogeneous, they unanimously agree on the fiscal policy that should be implemented. As a result, the optimal policy is determined independently of the wealth distribution and equilibrium aggregate variables are also independent of the distribution. Furthermore, we have considered nonlinear taxes, but only in a limited way, and we have not considered aggregate shocks. Extending the model in these directions is left for our future research.

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Table 1: Baseline parameter values

parameter	value	description
ψ	1	intertemporal elasticity of substitution
γ	3	risk aversion coefficient
A	0.315	coefficient in the production function
α	0.36	share of capital
δ_k	0.06	depreciation rate of physical capital
δ_h	0.06	depreciation rate of human capital
β	0.9511	discount factor
g	0.0256	government purchases as a fraction of total wealth
τ	0.1955	tax rate in the baseline policy ($\tau_{k,t} = \tau_{h,t} = \tau$)
$\overline{\theta}$	0.1585	idiosyncratic shock

Table 2: Steady states

	variables	(1) baseline	(2) Ramsey	(3) $g = 0$	(4) $b = g = 0$
capital tax rate (%)	τ_k	19.95	19.64	11.56	-0.34
labor tax rate (%)	τ_h	19.95	14.88	4.99	0.19
debt-output ratio (%)	$\frac{B_{t-1}}{Y_t}$	51	0.19	202.6	0
primary surplus (%)	$\frac{\tau_k r_t K_{t-1} + \tau_h w_t H_{t-1} - G_t}{Y_t}$	1.55	0.005	7.35	0
effective interest rate (%)	$1 - \delta_k + (1 - \tau_k)F_k - \frac{Y_{t+1}}{Y_t}$	3.03	2.61	3.63	2.69
share of govt purchases (%)	$\frac{G_t}{Y_t}$	18	16.6	0	0
growth rate (%)	$\frac{Y_{t+1}}{Y_t} - 1$	1.6	2.26	3.25	4.81
human capital premium (%)	$(F_h - \delta_h) - (F_k - \delta_k)$	5.28	4.74	2.95	4.86

Table 3: Benefit of government debt (evaluated at $b = 0$)

notation		baseline value of g	$g = 0$
benefit of debt (%)	$R_x^g - (1 - \delta_k + F_k^g)$	0.004	2.75

Table 4: Welfare gain of adopting the Ramsey policy

ignoring transition	considering transition
8.7320	0.8494

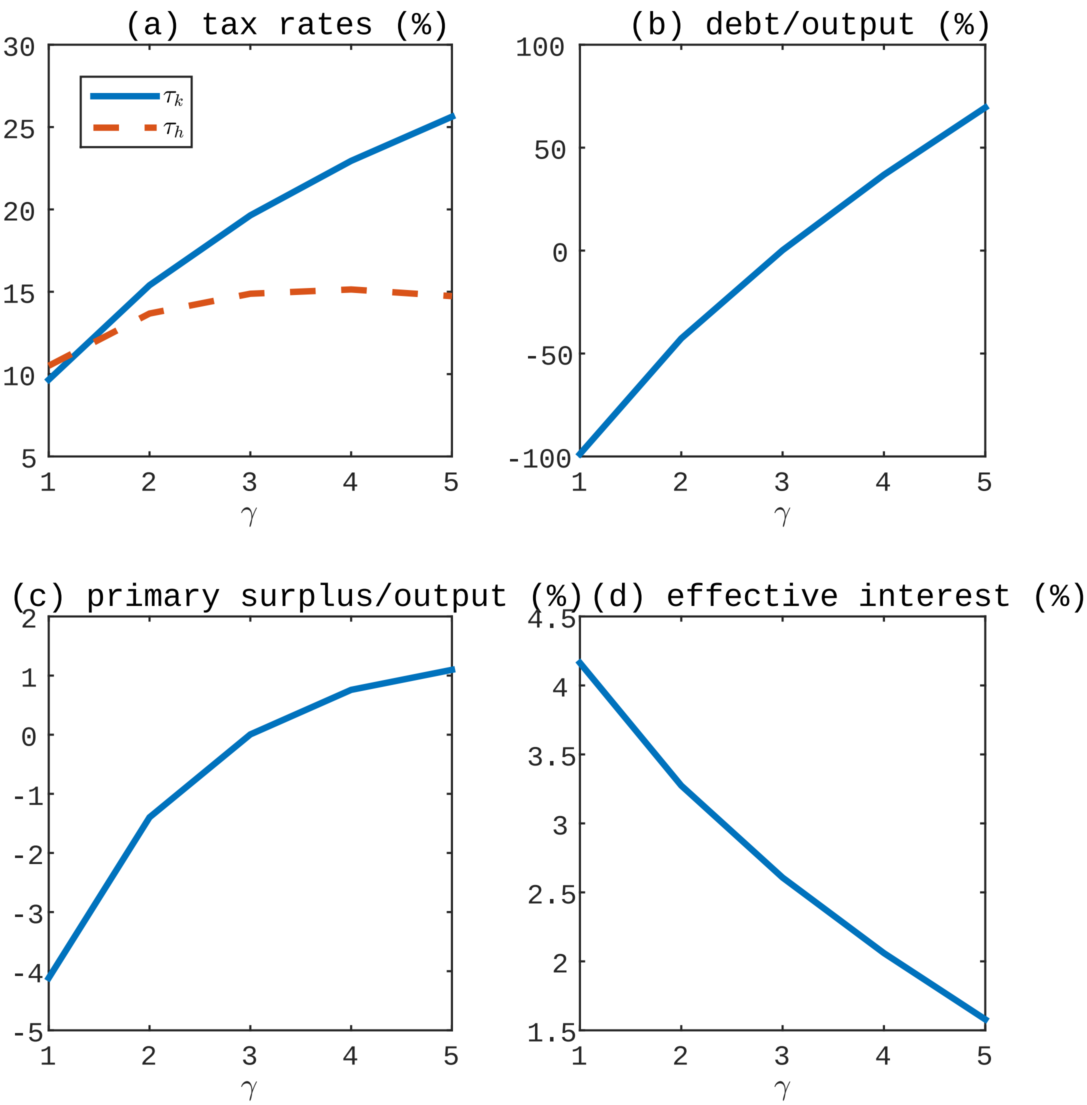

Figure 1: Different values of risk aversion.

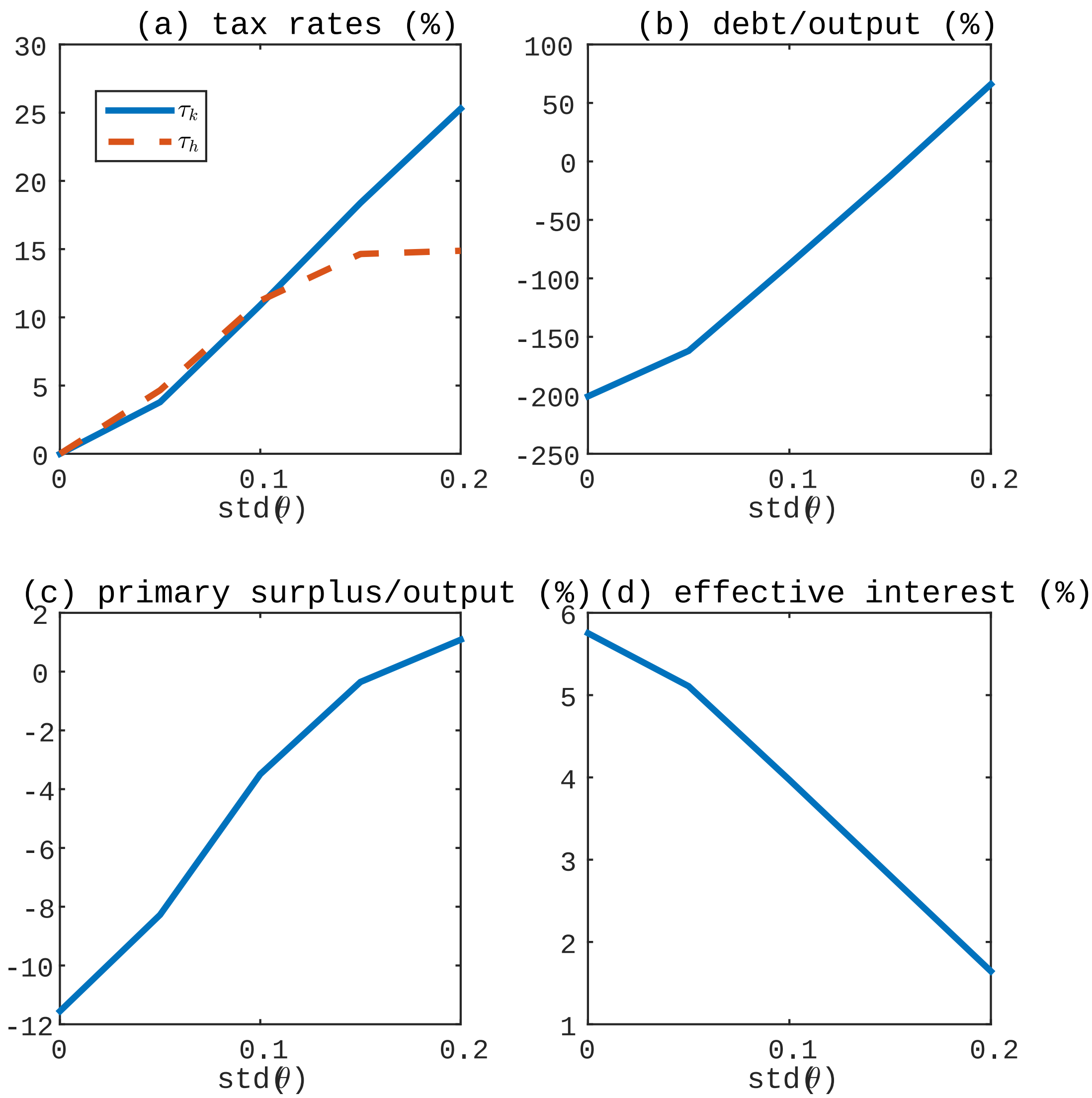

Figure 2: Different values of the idiosyncratic risk.

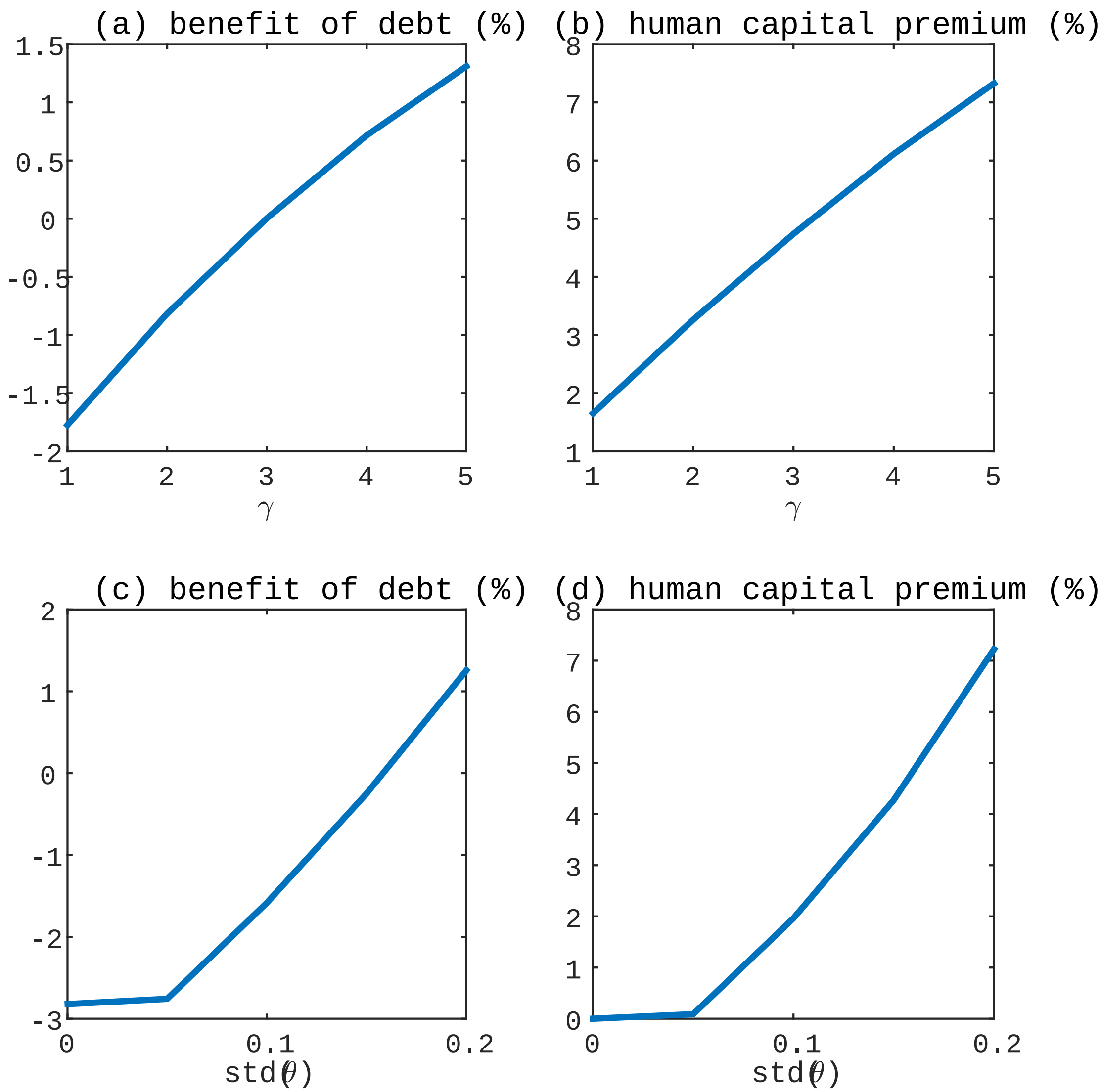

Figure 3: Benefit of government debt and human capital premium evaluated at $b = 0$.

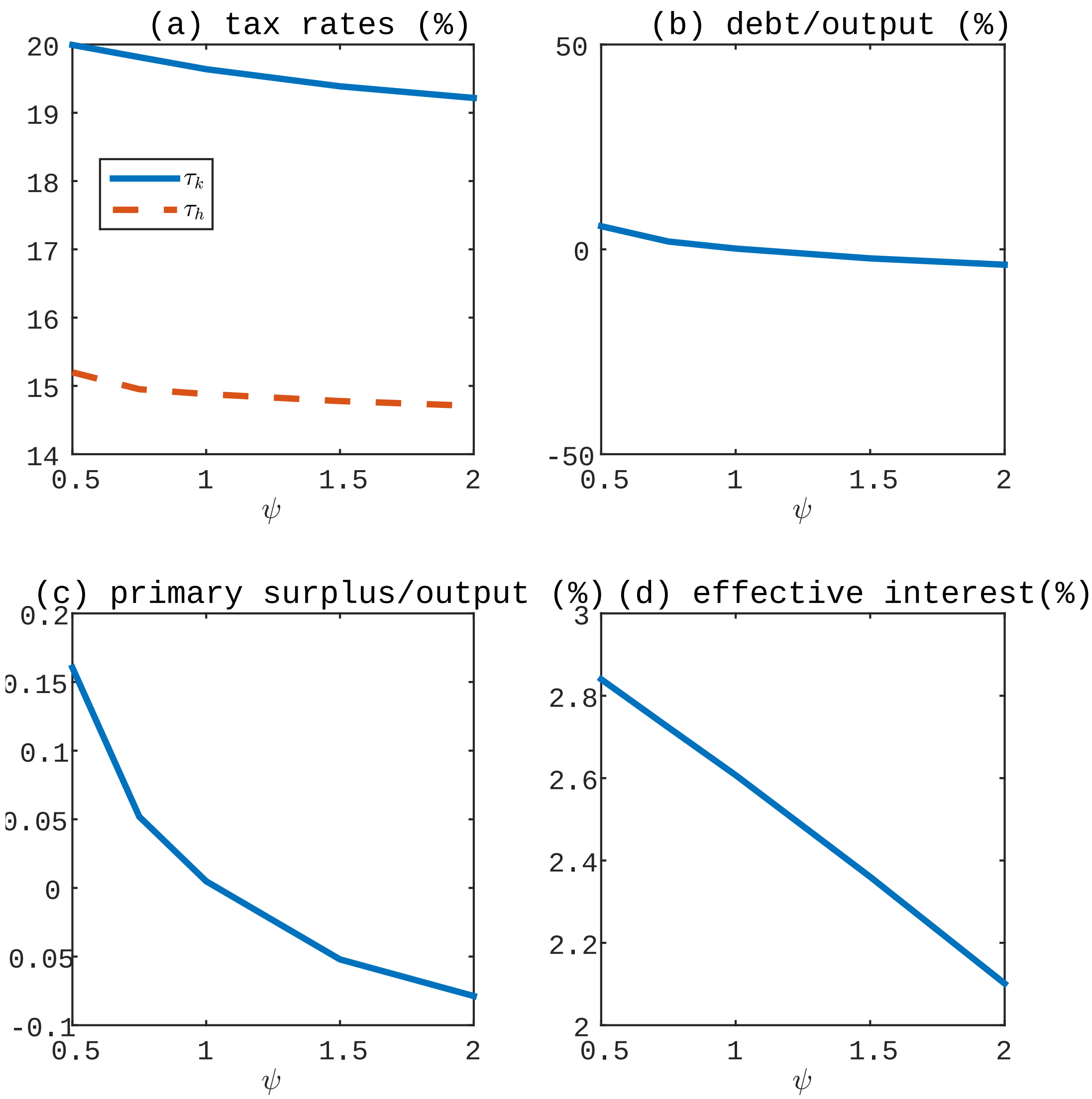

Figure 4: Different values of the intertemporal elasticity of substitution.

Online Appendix

1 Proofs

1.1 Proof of Lemma 1

The proof of this lemma uses an argument similar to Epstein and Zin (1991) and Angeletos (2007). Since the idiosyncratic shocks, $\theta_{i,t}$, are i.i.d. across individuals and across periods, the utility maximization problem of each individual can be expressed as:

$$V_t(x) = \max_{c,\eta_h} \left\{ (1-\beta)c^{1-\frac{1}{\psi}} + \beta \left(E_t[V_{t+1}(x')^{1-\gamma}] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\psi}}}$$

$$\text{s.t.} \qquad x' = (x-c)\left[R_{k,t+1}(1-\eta_h) + R_{h,t+1}\theta'\eta_h\right] \geq 0,$$

$$c \in [0,x], \qquad \eta_h \in [0,1].$$

Here, $V_t(x)$ is the value function for the utility maximization problem of an individual whose total wealth is x at the beginning of period t. We conjecture that there exists a (deterministic) sequence $\{v_t\}_{t=0}^{\infty}$, with $v_t \in \mathbb{R}_+$ for all t, such that

$$V_t(x) = v_t x$$

Using this conjecture and the budget constraint, we obtain

$$\left(E_t[V_{t+1}(x')^{1-\gamma}]\right)^{\frac{1}{1-\gamma}} = v_{t+1}(x-c)\left\{E_t\left[\left(R_{k,t+1}(1-\eta_h) + R_{h,t+1}\theta'\eta_h\right)^{1-\gamma}\right]\right\}^{\frac{1}{1-\gamma}}$$

It follows that in the above maximization problem the individual chooses the portfolio η_h so as to solve the following maximization problem:

$$\eta_h = \arg\max_{\eta_h' \in [0,1]} \left\{E_t\left[\left(R_{k,t+1}(1-\eta_h') + R_{h,t+1}\theta'\eta_h'\right)^{1-\gamma}\right]\right\}^{\frac{1}{1-\gamma}}$$

Let ρ_{t+1} denote the maximized value in this problem. Note that neither η_h nor ρ_{t+1} depends on the initial state x. That is, under the conjectured value function, all individuals would choose the same portfolio and the same certainty-equivalent rate of return.

Given the certainty-equivalent rate of return, ρ_{t+1}, the level of consumption is chosen so as to solve

$$\max_{c\in[0,x]} \left\{(1-\beta)c^{1-\frac{1}{\psi}} + \beta\left[v_{t+1}\rho_{t+1}(x-c)\right]^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}}$$

The first-order condition for this problem is

$$(1-\beta)c^{-\frac{1}{\psi}} = \beta v_{t+1}^{1-\frac{1}{\psi}} \rho_{t+1}^{1-\frac{1}{\psi}} (x-c)^{-\frac{1}{\psi}}$$

which leads to

$$\eta_c = \left\{1 + \left(\frac{\beta}{1-\beta}\right)^{\psi} (v_{t+1}\rho_{t+1})^{\psi-1}\right\}^{-1}$$

where $\eta_c = \frac{c}{x}$.

On the other hand, the Bellman equation implies

$$v_t^{1-\frac{1}{\psi}} = (1-\beta)\eta_c^{1-\frac{1}{\psi}} + \beta \left(v_{t+1}\rho_{t+1}\right)^{1-\frac{1}{\psi}} (1-\eta_c)^{1-\frac{1}{\psi}}$$

This equation and the above first-order condition for c imply that

$$v_t^{\psi-1} = (1-\beta)^{\psi} + \beta^{\psi} v_{t+1}^{\psi-1} \rho_{t+1}^{\psi-1}$$

The bounded solution to this difference equation is

$$v_t = (1-\beta)^{\frac{\psi}{\psi-1}} \left\{1 + \sum_{s=0}^{\infty} \prod_{j=0}^{s} \left(\beta^{\psi} \rho_{t+1+j}^{\psi-1}\right)\right\}^{\frac{1}{\psi-1}}$$

Also, the consumption rate η_c is

$$\eta_{c,t} = (1-\beta)^{\psi} v_t^{1-\psi}$$

It is straightforward to verify that, constructed in this way, $\{V_t(x), \eta_c, \eta_h\}$ indeed characterizes the solution to the utility maximization problem. The rest of the lemma follows immediately.

1.2 Proof of Proposition 3

Totally differentiating constraint (36) of problem (35), we obtain

$$(\tilde{r} - F_k + F_h - \tilde{w})\, d\eta_h - (1-\eta_h)\, d\tilde{r} - \eta_h\, d\tilde{w} = 0.$$

Evaluating this expression at the benchmark equilibrium, where $G_t = B_t = 0$, $\tilde{r}_t = \hat{F}_k$ and $\tilde{w}_t = \hat{F}_h$, for all t, yields

$$(1-\hat{\eta}_h)\, d\tilde{r} + \hat{\eta}_h\, d\tilde{w} = 0.$$

Thus, to satisfy the balanced budget, $\tilde{r}$ and $\tilde{w}$ must satisfy the following relationship around $(\tilde{r}, \tilde{w}) = (\hat{F}_k, \hat{F}_h)$:

$$\frac{d\tilde{w}}{d\tilde{r}} = -\frac{1-\hat{\eta}_h}{\hat{\eta}_h}.$$

Hence the effect of a marginal change in $\tilde{r}$, taking into account the induced change in $\tilde{w}$ via the government budget constraint, is given by $\frac{\partial}{\partial \tilde{r}} - \frac{1-\hat{\eta}_h}{\hat{\eta}_h}\frac{\partial}{\partial \tilde{w}}$ and will be denoted by $\frac{d}{d\tilde{r}}$. Since the lifetime utility is increasing in ρ_t for each t, it suffices to show that $\frac{d\rho}{d\tilde{r}} > 0$.

The envelope theorem implies that $\frac{\partial \rho}{\partial \eta_h} = 0$ at the benchmark equilibrium. It follows that

$$\begin{aligned}\frac{d\rho}{d\tilde{r}} &= \hat{\rho}^{\gamma} E\left[\hat{R}_x(\theta)^{-\gamma}\left\{(1-\hat{\eta}_h) + \theta\hat{\eta}_h\frac{d\tilde{w}}{d\tilde{r}}\right\}\right],\\ &= \hat{\rho}^{\gamma} E\left[\hat{R}_x(\theta)^{-\gamma}(1-\theta)\right](1-\hat{\eta}_h),\end{aligned}$$

where $\hat{R}_x(\theta) \equiv (1-\delta_k+\hat{F}_k)(1-\hat{\eta}_h) + (1-\delta_h+\hat{F}_h)\theta\hat{\eta}_h$. Since $E(\theta) = 1$, we have

$$E\left[\hat{R}_x(\theta)^{-\gamma}(1-\theta)\right] = \mathrm{Cov}(\hat{R}_x(\theta)^{-\gamma}, 1-\theta) > 0,$$

where the inequality follows from the fact that both $\hat{R}_x(\theta)^{-\gamma}$ and $1-\theta$ are decreasing functions of θ. Given that $\hat{\eta}_h < 1$, this proves that $\frac{d\rho}{d\tilde{r}} > 0$.

It remains to show that the after-tax rental rate of capital, $\tilde{r}$, and the tax rate on capital income, τ_k, move in the opposite directions around the benchmark equilibrium. Since $\tau_k = 1 - \frac{\tilde{r}}{F_k}$, we have

$$\frac{d\tau_k}{d\tilde{r}} = \frac{-\hat{F}_k + (-\hat{F}_{kk} + \hat{F}_{kh})\frac{d\eta_h}{d\tilde{r}}}{\hat{F}_k^2}. \tag{43}$$

Differentiating the individual first order conditions (15) yields

$$\left\{\Phi_{\tilde{r}} - \frac{1-\hat{\eta}_h}{\hat{\eta}_h}\Phi_{\tilde{w}}\right\}d\tilde{r} + \Phi_{\eta_h}d\eta_h = 0,$$

so that

$$\frac{d\eta_h}{d\tilde{r}} = \frac{\frac{1-\hat{\eta}_h}{\hat{\eta}_h}\Phi_{\tilde{w}} - \Phi_{\tilde{r}}}{\Phi_{\eta_h}}. \tag{44}$$

Thus we obtain

$$\frac{d\tau_k}{d\tilde{r}} = \frac{1}{\hat{F}_k^2}\frac{-\hat{F}_k\Phi_{\eta_h} + (-\hat{F}_{kk} + \hat{F}_{kh})\left(\frac{1-\hat{\eta}_h}{\hat{\eta}_h}\Phi_{\tilde{w}} - \Phi_{\tilde{r}}\right)}{\Phi_{\eta_h}} < 0,$$

since by Assumption 1 we have $\Phi_{\tilde{w}} > 0$, $\Phi_{\tilde{r}} < 0$, while $\Phi_{\eta_h} < 0$ follows from the strict concavity of $\rho(\tilde{r}, \tilde{w}, \eta_h)$ and $F_{kh} = (1-\alpha)\alpha k^{\alpha-1}h^{-\alpha} > 0$. This completes the proof.

1.3 Proof of Proposition 4

We are interested in the welfare effect of a marginal variation of $\bar{b}_{T+1}$ evaluated at $\bar{b}_{T+1} = 0$, that is the sign of $dv_0/d\bar{b}_{T+1}\big|_{\bar{b}_{T+1}=0}$. Denote the variables solving the Ramsey problem under (37) as $v_t(\bar{b}_{T+1})$, $\rho_t(\bar{b}_{T+1})$, etc.. It is immediate to see that its solution is the same as under (34) for all periods except two,

$$\rho_t(\bar{b}_{T+1}) = \rho^o, \quad \forall t \neq T+1, T+2 \tag{45}$$

Hence from (12) we get $v_t(\bar{b}_{T+1}) = v^o, \quad \forall t \geq T+2$, and $dv_0/dv_T > 0$, so that

$$\left.\frac{dv_0}{db_{T+1}}\right|_{\bar{b}_{T+1}=0} \gtreqless 0 \qquad \Longleftrightarrow \qquad \left.\frac{dv_T}{db_{T+1}}\right|_{\bar{b}_{T+1}=0} \gtreqless 0.$$

We have so[27] $\rho_{T+2}(\bar{b}_{T+1}) = \rho^R(\bar{b}_{T+1}, 0, \eta_{c,T+1}(\bar{b}_{T+1}))$. Recalling again (12), we obtain

$$v_{T+1}(\bar{b}_{T+1}) = \left\{(1-\beta)^\psi + \beta^\psi \rho_{T+2}(\bar{b}_{T+1})^{\psi-1} v_{T+2}(\bar{b}_{T+1})^{\psi-1}\right\}^{\frac{1}{\psi-1}}. \tag{46}$$

Here, note that (45) implies $\partial v_{T+2}/\partial \bar{b}_{T+1} = 0$. In addition, $\partial \rho^R(0,0,\eta_c)/\partial \eta_c = 0$.[28] Differentiating then $v_{T+1}(\bar{b}_{T+1})$ with respect to $\bar{b}_{T+1}$ and evaluating it at $\bar{b}_{T+1} = 0$ yields

$$\left.\frac{dv_{T+1}}{d\bar{b}_{T+1}}\right|_{\bar{b}_{T+1}=0} = \beta^\psi (\rho^{Ro})^{\psi-2} \rho_1^o v^o, \tag{47}$$

where $\rho_1^{Ro} \equiv \partial \rho^R(b, b', \eta_c^o)/\partial b$, evaluated at $b = b' = 0$.[29]

Next, consider the expression analogous to (46) for date T:

$$v_T(\bar{b}_{T+1}) = \left\{(1-\beta)^\psi + \beta^\psi \left(\rho_{T+1}(\bar{b}_{T+1})\right)^{\psi-1} v_{T+1}(\bar{b}_{T+1})^{\psi-1}\right\}^{\frac{1}{\psi-1}}. \tag{48}$$

Its derivative with respect to $\bar{b}_{T+1}$, evaluated at $\bar{b}_{T+1} = 0$, using (47) and again the fact that $\partial \rho^R/\partial \eta_{c,T}\big|_{\bar{b}_{T+1}=b_T=0} = 0$, equals

$$\left.\frac{dv_T}{d\bar{b}_{T+1}}\right|_{\bar{b}_{T+1}=0} = \beta^\psi (\rho^o)^{\psi-2} v^o \left[\rho_2^{Ro} + \beta^\psi (\rho^o)^{\psi-1} \rho_1^{Ro}\right],$$

where $\rho_2^{Ro} \equiv \partial \rho^R(b, b', \eta_c^o)/\partial b'$ evaluated at $b = b' = 0$.

Let us denote then by $\lambda(b, b', \eta_c)$ the Lagrange multiplier on the flow budget constraint for the government in problem (32) and by $\eta_h(b, b', \eta_c)$, $\widetilde{r}(b, b', \eta_c)$, $\widetilde{w}(b, b', \eta_c)$, and $R_x(b, b', \eta_c)$ its solution. Using the envelope property and the fact that b, b' only appear in constraint (31) of the problem,

---

[27] Here and in what follows we omit the dependence of ρ^R on g whenever g_t is constant across periods.

[28] To see this, recall from the definition of $\rho^R(b, b', \eta_c)$ in (32) that η_c affects ρ^R only through the government budget constraint (31). Consider the associated function:

$$\begin{aligned} &f(b, b', \eta_c, \eta_h, \widetilde{r}, \widetilde{w}, R_x) \\ &\qquad \equiv g + (1 - \delta_k + \widetilde{r})b - (1-\eta_c)R_x b' - F\left[(1-\eta_c)(1-\eta_h) - b, (1-\eta_c)\eta_h\right] \\ &\qquad\quad + \widetilde{r}\left[(1-\eta_c)(1-\eta_h) - b\right] + \widetilde{w}(1-\eta_c)\eta_h \end{aligned}$$

We have $\frac{\partial f}{\partial \eta_c}\Big|_{b=b'=0} = 0$ and so, by the envelope theorem we get the claimed property.

[29] The superscript ^{o} indicates, as in the main text, variables evaluated at a solution of the Ramsey problem under the constraint $b_t = g_t = 0$ for all t.

we obtain, when $b_t = g_t = 0$ for all t:[30]

$$\rho_1^{Ro} = -\lambda^o(1 - \delta_k + F_k^o),$$
$$\rho_2^{Ro} = \lambda^o \beta^\psi (\rho^o)^{\psi-1} R_x^o,$$

since

$$\eta_c^o = 1 - \beta^\psi (\rho^o)^{\psi-1}.$$

Therefore,

$$\frac{dv_T}{d\bar{b}_{T+1}} = \xi \left[R_x^o - (1 - \delta_k + F_k^o) \right], \tag{49}$$

where

$$\xi \equiv \beta^{2\psi} (\rho^o)^{2\psi-3} \lambda^o v^o$$

and $\xi > 0$ since $\lambda^o > 0$, as we show next. As argued in Section 3.1, when $b_t = g_t = 0$ for all t, problem (32) reduces to (35).

Let us write the solution to (10) as $\eta_h(\widetilde{r}, \widetilde{w})$. Then the first order conditions for $\widetilde{r}$ and $\widetilde{w}$ in problem (35) are given by

$$0 = \frac{\partial \rho}{\partial \widetilde{r}} - (1 - \eta_h^o)\lambda^o + \left[\frac{\partial \rho}{\partial \eta_h} + \lambda^o(-F_k^o + F_h^o + \widetilde{r}^o - \widetilde{w}^o) \right] \frac{\partial \eta_h}{\partial \widetilde{r}},$$
$$0 = \frac{\partial \rho}{\partial \widetilde{w}} - \eta_h^o \lambda^o + \left[\frac{\partial \rho}{\partial \eta_h} + \lambda^o(-F_k^o + F_h^o + \widetilde{r}^o - \widetilde{w}^o) \right] \frac{\partial \eta_h}{\partial \widetilde{w}}.$$

From the second equation, recalling that under Assumption 1 we have $\frac{\partial \eta_h}{\partial \widetilde{w}} > 0$ and $\frac{\partial \eta_h}{\partial \widetilde{r}} < 0$, we obtain

$$\lambda^o(-F_k^o + F_h^o + \widetilde{r}^o - \widetilde{w}^o) = \frac{-\frac{\partial \rho}{\partial \widetilde{w}} + \eta_h^o \lambda^o}{\frac{\partial \eta_h}{\partial \widetilde{w}}}.$$

Substituting then this equation into the first equation above, and solving for λ^o, we get

$$\lambda^o = \left(1 - \eta_h^o - \frac{\eta_h^o \frac{\partial \eta_h}{\partial \widetilde{r}}}{\frac{\partial \eta_h}{\partial \widetilde{w}}} \right)^{-1} \left(\frac{\partial \rho}{\partial \widetilde{r}} - \frac{\frac{\partial \rho}{\partial \widetilde{w}} \frac{\partial \eta_h}{\partial \widetilde{r}}}{\frac{\partial \eta_h}{\partial \widetilde{w}}} \right) > 0,$$

where the sign of the inequality follows from the fact that $\eta_h^o \in (0,1)$, $\frac{\partial \rho}{\partial \widetilde{r}} > 0$ and $\frac{\partial \rho}{\partial \widetilde{w}} > 0$.

[30] To better understand the form of these expressions, notice that, as we see from (31), a marginal increase of $\bar{b}_{T+1}$ relaxes this constraint at $T+1$ yielding a gain of $\lambda^o \,(1 - \eta_c^o)\, R_x^o$, while tightening this constraint at $T+2$ with a loss of $\lambda^o \beta^\psi (\rho^o)^{\psi-1} (1 - \delta_k + F_k^o)$ (recall that ρ_1^{Ro} is multiplied by $\beta^\psi (\rho^o)^{\psi-1}$ in the expression of dv_T/db_{T+1}). Since $(1 - \eta_c^o) = \beta^\psi (\rho^o)^{\psi-1}$, the comparison of these two reduce to the comparison between R_x^o and $(1 - \delta_k + F_k^o)$.

1.4 Proof of Proposition 5

The Lagrangean for problem (33), using (12) and (14) to substitute for ρ_{t+1} and $\eta_{c,t}$, is

$$v_0 + \sum_{t=0}^{\infty} \lambda_t^v \left\{ (1-\beta)^\psi + \beta^\psi \rho^R (b_t, b_{t+1}, (1-\beta)^\psi v_t^{1-\psi})^{\psi-1} v_{t+1}^{\psi-1} - v_t^{\psi-1} \right\}.$$

The first-order condition with respect to b_{t+1} is then

$$\lambda_t^v \beta^\psi \rho_{t+1}^{\psi-2} \rho_{2,t+1}^R v_{t+1}^{\psi-1} + \lambda_{t+1}^v \beta^\psi \rho_{t+2}^{\psi-2} \rho_{1,t+2}^R v_{t+2}^{\psi-1} = 0, \tag{50}$$

where $\rho_{t+1} \equiv \rho^R(b_t, b_{t+1}, \eta_{c,t})$, $\rho_{2,t+1}^R \equiv \partial \rho^R(b_t, b_{t+1}, \eta_{c,t})/\partial b_{t+1}$, and $\rho_{1,t+2}^R \equiv \partial \rho^R(b_{t+1}, b_{t+2}, \eta_{c,t+1})/\partial b_{t+1}$. The first-order condition for v_{t+1} is

$$\lambda_t^v \beta^\psi \rho_{t+1}^{\psi-1} v_{t+1}^{\psi-2} + \lambda_{t+1}^v \beta^\psi \rho_{t+2}^{\psi-2} \rho_{\eta_c,t+2}^R (1-\beta)^\psi (1-\psi) v_{t+1}^{-\psi} v_{t+2}^{\psi-1} - \lambda_{t+1}^v v_{t+1}^{\psi-2} = 0, \tag{51}$$

where $\rho_{\eta_c,t+2}^R \equiv \partial \rho^R(b_{t+1}, b_{t+2}, \eta_{c,t+1})/\partial \eta_{c,t+1}$.

In a steady-state equilibrium, equation (50) reduces to

$$\rho_2^R + \frac{\lambda_{t+1}^v}{\lambda_t^v} \rho_1^R = 0 \tag{52}$$

and equation (51) to

$$\frac{\lambda_{t+1}^v}{\lambda_t^v} = \beta^\psi \rho^{\psi-1} \left(1 - \beta^\psi \rho^{\psi-1} (1-\beta)^\psi (1-\psi) \frac{\rho_{\eta_c}^R v^{1-\psi}}{\rho} \right)^{-1}, \tag{53}$$

where the term in parenthesis captures the effect on ρ of the change in the savings rate, given by the second term in (51), which only arises (as we saw in foonote 30) when debt is nonzero.

By a similar argument to the one in the proof of Proposition 4 above, at a steady state equilibrium the derivative of ρ^R with respect to b and b' satisfies

$$\begin{aligned} -\frac{\rho_1^R}{\rho_2^R} &= \frac{1-\delta_k+F_k}{(1-\eta_c)R_x} \\ &= \frac{1-\delta_k+F_k}{\beta^\psi \tilde{\rho}^{\psi-1} R_x}, \end{aligned} \tag{54}$$

where, for the second equality, we used again (14), $\eta_c = (1-\beta)^\psi v^{1-\psi}$, and constraint (12), $v^{\psi-1} = (1-\beta)^\psi + \beta^\psi \rho^{\psi-1} v^{\psi-1}$, of problem (33).

Combining (52)-(54) and using again (14), yields the claimed result:

$$R_x = (1-\delta_k+F_k) \left[1 - (1-\psi) \beta^\psi \rho^{\psi-2} \rho_{\eta_c}^R \eta_c \right]^{-1}.$$

2 Sufficient conditions for Assumption 1

Let us rewrite problem (9) more compactly as

$$\max_{\eta_h \geq 0} E\left[u\left(r\left(1-\eta_h\right)+\theta w \eta_h\right)\right],$$

where, with a slight abuse of notation, r denotes $1-\delta_k+\widetilde{r}$, w denotes $1-\delta_h+\widetilde{w}$, and the function $u(.)$ is increasing, concave and with a constant coefficient of relative risk aversion γ. Letting η_h^* be an interior solution of (9), the properties stated in Assumption 1 are equivalent to $\frac{\partial \eta_h^*}{\partial r}<0$ and $\frac{\partial \eta_h^*}{\partial w}>0$, as already noticed in the main text. Setting $R \equiv \theta w-\alpha$, problem (9) may also be written as

$$\max_{\eta_h \geq 0} E\left[u\left(r+R \eta_h\right)\right], \tag{55}$$

when $\alpha=r$. Problem (55) is often referred to as the standard portfolio choice problem. Hereafter, we shall use some results on such problem reported in Gollier (2004).[31]

From Proposition 9 in Gollier (2004) it follows that, when the coefficient of relative risk aversion γ is not larger than one, any first order stochastic improvement in R increases the optimal value of η_h. Since an increase in w induces such an improvement, we conclude that $\frac{\partial \eta_h^*}{\partial w}>0$ if $\gamma \leq 1$.

Note that an increase in r, keeping R (that is, α) constant, constitutes an increase in wealth and so from Proposition 8 in Gollier (2004) it follows that this change induces a decrease in η_h^* if u exhibits decreasing absolute risk aversion. With constant relative risk aversion, u indeed exhibits decreasing absolute risk aversion. There is then a second effect of the increase in r, given by the change in R : an increase in α induces a first order worsening on R and so reduces η_h^* if $\gamma \leq 1$. Hence we conclude that $\frac{\partial \eta_h^*}{\partial r}<0$ if $\gamma \leq 1$.

Having established that the stated properties always hold when $\gamma \leq 1$, we show next that, when $\gamma>1$, they hold for some family of distributions of θ. Assuming that θ is a continuous random variable with density function $g(t)$ differentiable almost everywhere, we shall show below that the stated comparative statics properties hold if both $t \frac{g'(t)}{t}$ and $\frac{g'(t)}{t}$ are non-increasing in t. The condition hold for example when θ is a uniform distribution over some interval, or a Pareto distribution (i.e., the density function is a power function).

To establish the result we build on Proposition 17 in Gollier (2004), stating that, if $u(.)$ is strictly increasing, then any improvement in R in monotone likelihood ratio (MLR) increases the optimal value η_h^* of problem (55). That is, if R and R' are distinct continuous random variables with density f_R and $f_{R'}$ respectively, the optimal value η_h^* under R' is larger than that under R if $f_{R'}(t) / f_R(t)$ is non decreasing in t.

[31] Gollier, C. (2004), "The Economics of Risk and Time," MIT Press.

Since $R = \theta w - \alpha$, $\Pr[R \le z] = \Pr[\theta \le (z+r)/w]$ and so the density function $f(z)$ of R is given by

$$f(z) = \frac{d}{dz}\int_0^{(z+r)/w} g(t)\, dt = \frac{1}{w} g\left(\frac{z+r}{w}\right). \tag{56}$$

So in order to use the above proposition to establish the property $\frac{\partial \eta_h^*}{\partial w} > 0$, it suffices to show that for any $\hat{w} > w$ $\frac{1}{\hat{w}} g\left(\frac{z+r}{\hat{w}}\right) / \frac{1}{w} g\left(\frac{z+r}{w}\right)$ is non decreasing in z. Taking a monotone (logarithmic) transformation and differentiating with respect to z, this condition obtains when

$$\frac{1}{\hat{w}} \frac{g'\left(\frac{z+r}{\hat{w}}\right)}{g\left(\frac{z+r}{\hat{w}}\right)} - \frac{1}{w} \frac{g'\left(\frac{z+r}{w}\right)}{g\left(\frac{z+r}{w}\right)} \ge 0,$$

that is, when

$$\frac{1}{w} \frac{g'\left(\frac{z+r}{w}\right)}{g\left(\frac{z+r}{w}\right)} \text{ is non-decreasing in } w,$$

at any $w > 0$, for given z and r. Since the map $w \mapsto (z+r)/w$ is monotonic and decreasing, setting $t = (r+z)/w$, the condition above can be equivalently stated as

$$t \frac{g'(t)}{g(t)} \text{ is non-increasing in } t.$$

Next, we use the same proposition to derive a condition guaranteeing that $\frac{\partial \eta_h^*}{\partial r} < 0$. Recalling the argument above regarding the effect of increasing r keeping R constant, when $u(.)$ exhibits decreasing absolute risk aversion, it suffices to show that the optimal value of η_h^* decreases as α in $R = w\theta - \alpha$ increases, keeping r fixed. Hence we derive next a condition on $g(t)$ such that a decrease in α induces a MLR improvement: that is, for any $\hat{\alpha} < \alpha$ $\frac{1}{w} g\left(\frac{z+\hat{\alpha}}{w}\right) / \frac{1}{w} g\left(\frac{z+\alpha}{w}\right)$ is non decreasing in z. Arguing analogously as in the previous case, we can show that this property holds if $g'\left(\frac{z+\alpha}{w}\right) / g\left(\frac{z+\alpha}{w}\right)$ is non increasing in α at any $\alpha > 0$, where z and w are fixed. So changing variables we conclude that $\frac{\partial \eta_h^*}{\partial r} < 0$ holds if

$$\frac{g'(t)}{g(t)} \text{ is non-increasing in } t.$$

3 Exogenous government purchases

Here we extend our analysis to the case where the public expenditure policy is specified in terms of an exogenous sequence of absolute levels of expenditure $\{G_t\}_{t=0}^{\infty}$ (rather than per unit of total wealth). We will obtain conditions characterizing the Ramsey steady state which are analogous to those obtained in Proposition 5 and Corollary 6. Hence, also in the case of exogenous G_t, the capital income tax rate must be positive in the long run, as long as the effect on the saving rate is small enough.

When the sequence $\{G_t\}_{t=0}^{\infty}$ is exogenously given, we can no longer use the recursive approach followed in the paper to solve the Ramsey problem in the case where $\{g_t\}_{t=0}^{\infty}$ is exogenously given. We solve instead the problem in a more direct way. Given X_0 and b_0, the Ramsey problem consists in the maximization of v_0 with respect to $\{b_{t+1},\ X_{t+1},\ v_{t+1},\ \widetilde{r}_{t+1},\ \widetilde{w}_{t+1}\}_{t=0}^{\infty}$ subject to

$$
\begin{aligned}
&v_t^{\psi-1} = (1-\beta)^{\psi} + \beta^{\psi}\rho_{t+1}^{\psi-1}v_{t+1}^{\psi-1} \\
&\frac{G_{t+1}}{X_t} + (1-\delta_k+\widetilde{r}_{t+1})b_t = (1-\eta_{c,t})R_{x,t+1}b_{t+1} + F(k_t,h_t) - \widetilde{r}_{t+1}k_t - \widetilde{w}_{t+1}h_t \\
&\frac{X_{t+1}}{X_t} = (1-\eta_{c,t})R_{x,t+1},
\end{aligned}
$$

where $\eta_{h,t}$, $\eta_{c,t}$, ρ_{t+1}, $R_{x,t+1}$, k_t, and h_t are the following functions of $\widetilde{r}_{t+1}$, $\widetilde{w}_{t+1}$, b_t, and v_t:

$$
\begin{aligned}
\eta_{h,t} &= \eta_h(\widetilde{r}_{t+1},\widetilde{w}_{t+1}) \equiv \arg\max_{\eta_h}\ \rho(\widetilde{r}_{t+1},\widetilde{w}_{t+1},\eta_h), \\
\rho_{t+1} &= \rho(\widetilde{r}_{t+1},\widetilde{w}_{t+1}) \equiv \max_{\eta_h}\ \rho(\widetilde{r}_{t+1},\widetilde{w}_{t+1},\eta_h), \\
R_{x,t+1} &= R_x(\widetilde{r}_{t+1},\widetilde{w}_{t+1}) \equiv (1-\delta_k+\widetilde{r}_{t+1})(1-\eta_h(\widetilde{r}_{t+1},\widetilde{w}_{t+1})) + (1-\delta_h+\widetilde{w}_{t+1})\eta_h(\widetilde{r}_{t+1},\widetilde{w}_{t+1}), \\
\eta_{c,t} &= \eta_c(v_t) \equiv (1-\beta)^{\psi}(v_t)^{1-\psi}, \\
k_t &= k(\widetilde{r}_{t+1},\widetilde{w}_{t+1},b_t,v_t) \equiv (1-\eta_c(v_t))(1-\eta_h(\widetilde{r}_{t+1},\widetilde{w}_{t+1})) - b_t, \\
h_t &= h(\widetilde{r}_{t+1},\widetilde{w}_{t+1},v_t) \equiv (1-\eta_c(v_t))\eta_h(\widetilde{r}_{t+1},\widetilde{w}_{t+1}),
\end{aligned}
$$

The Lagrangean for this problem is then:

$$
\begin{aligned}
v_0 + \sum_{t=0}^{\infty}\Bigg[&\lambda_{v,t}\Big\{(1-\beta)^{\psi} + \beta^{\psi}\rho(\widetilde{r}_{t+1},\widetilde{w}_{t+1})^{\psi-1}v_{t+1}^{\psi-1} - v_t^{\psi-1}\Big\} \\
&+ \lambda_{b,t}\Big\{[1-\eta_c(v_t)]R_x(\widetilde{r}_{t+1},\widetilde{w}_{t+1})b_{t+1} + F[k(\widetilde{r}_{t+1},\widetilde{w}_{t+1},b_t,v_t),h(\widetilde{r}_{t+1},\widetilde{w}_{t+1},v_t)] \\
&\qquad - \widetilde{r}_{t+1}k(\widetilde{r}_{t+1},\widetilde{w}_{t+1},b_t,v_t) - \widetilde{w}_{t+1}h(\widetilde{r}_{t+1},\widetilde{w}_{t+1},v_t) - \frac{G_{t+1}}{X_t} - (1-\delta_k+\widetilde{r}_{t+1})b_t\Big\} \\
&+ \lambda_{x,t}\Big\{[1-\eta_c(v_t)]R_x(\widetilde{r}_{t+1},\widetilde{w}_{t+1}) - \frac{X_{t+1}}{X_t}\Big\}\Bigg].
\end{aligned}
$$

The first order conditions for v_t, b_t, and $\widetilde{r}_{t+1}$ are so, respectively,[32]

$$
\begin{aligned}
0 = &-\lambda_{v,t}\frac{v_t^{\psi-2}}{\psi-1} + \lambda_{v,t-1}\frac{\beta^{\psi}}{\psi-1}\rho_{t+1}^{\psi-1}v_{t+1}^{\psi-2} \qquad (57)\\
&+ \lambda_{b,t}\eta_c'(v_t)\Big\{-R_{x,t+1}b_{t+1} - F_{k,t}(1-\eta_{h,t}) - F_{h,t}\eta_{h,t} + \widetilde{r}_{t+1}(1-\eta_{h,t}) + \widetilde{w}_{t+1}\eta_{h,t}\Big\}\\
&- \lambda_{x,t}\eta_c'(v_t)R_{x,t+1},\\
0 = &\ \lambda_{b,t-1}(1-\eta_{c,t-1})R_{x,t} - \lambda_{b,t}(1-\delta_k + F_{k,t}), \qquad (58)\\
0 = &\ (\psi-1)\lambda_{v,t}\beta^{\psi}\rho_{t+1}^{\psi-2}\rho_{r,t+1}v_{t+1}^{\psi-2}\\
&+ \lambda_{b,t}\Big\{(1-\eta_{c,t})R_{x,r,t+1}b_{t+1} + F_{k,t}k_{r,t} + F_{h,t}h_{r,t} - k_t - \widetilde{r}_{t+1}k_{r,t} - \widetilde{w}_{t+1}h_{r,t} - b_t\Big\}\\
&+ \lambda_{x,t}(1-\eta_{c,t})R_{x,r,t+1}, \qquad (59)
\end{aligned}
$$

where $\eta_c'(v_t) \equiv d\eta_c(v_t)/dv_t$, $F_{k,t} \equiv \partial F(k_t,h_t)/\partial k_t$, $F_{h,t} \equiv \partial F(k_t,h_t)/\partial h_t$, $\rho_{r,t+1} \equiv \partial\rho(\widetilde{r}_{t+1},\widetilde{w}_{t+1})/\partial\widetilde{r}_{t+1}$, $R_{x,r,t+1} \equiv \partial R_x(\widetilde{r}_{t+1},\widetilde{w}_{t+1})/\partial\widetilde{r}_{t+1}$, $k_{r,t} \equiv \partial k(\widetilde{r}_{t+1},\widetilde{w}_{t+1},b_t,v_t)/\partial\widetilde{r}_{t+1}$, and $h_{r,t} \equiv \partial h(\widetilde{r}_{t+1},\widetilde{w}_{t+1},v_t)/\partial\widetilde{r}_{t+1}$.

Assuming that G_t grows at an exogenous, constant rate $\gamma_G > 0$, we focus again our attention on a steady state (balanced growth path) where all the variables in equations (57)-(59) remain constant, except for the Lagrange multipliers, $\lambda_{v,t}$, $\lambda_{b,t}$, and $\lambda_{x,t}$ that grow at the same rate:

$$\frac{\lambda_{v,t}}{\lambda_{v,t-1}} = \frac{\lambda_{b,t}}{\lambda_{b,t-1}} = \frac{\lambda_{x,t}}{\lambda_{x,t-1}} \equiv \gamma_\lambda.$$

Since ρ is constant we have $v = (1-\beta)^{\psi}/(1-\beta^{\psi}\rho^{\psi-1})$. Also, $\eta_c = (1-\beta)^{\psi}v^{1-\psi}$, and so

$$\beta^{\psi}\rho^{\psi-1} = 1-\eta_c.$$

It then follows from equation (57) that, along a balanced growth path,

$$\frac{\lambda_{v,t}}{\lambda_{v,t-1}} = (1-\eta_c) + \Lambda\eta_c'(v),$$

where Λ is the term

$$\Lambda \equiv \frac{\psi-1}{v^{\psi-2}}\left[\frac{\lambda_{b,t}}{\lambda_{v,t-1}}\Big\{-R_x b - F_k(1-\eta_h) - F_h\eta_h + \widetilde{r}(1-\eta_h) + \widetilde{w}\eta_h\Big\} - \frac{\lambda_{x,t}}{\lambda_{v,t-1}}R_x\right],$$

a constant given the fact that all Lagrange multipliers grow at the same rate.

We can then use equation (58) to derive the following steady-state condition which is the counterpart of the one in Proposition 5:

$$R_x = (1-\delta_k + F_k)\left[1 + \frac{\Lambda\eta_c'(v)}{1-\eta_c}\right]. \qquad (60)$$

[32] To derive the steady state condition determining the tax rate on capital we do not have to use the first-order conditions with respect to $\widetilde{w}_{t+1}$ or X_{t+1}. But, of course, we would need those conditions to derive all the steady state equilibrium variables.

Just as in the case of a constant, exogenously given level of g, this condition implies that at a Ramsey steady state the average rate of return on consumers' portfolios, R_x, is equal to the before tax return on physical capital (or equivalently the cost of government debt), $1-\delta_k+F_k$, augmented with the effect of public debt on the saving rate, $\Lambda\eta_c'/(1-\eta_c)$. As long as the latter effect is small, we get again $R_x \approx 1-\delta_k+F_k$, which implies that the optimal capital tax rate is positive in the long run: $\tau_k > 0$.

When $\psi = 1$, again the effect on the saving rate valishes, so that condition (60) reduces to

$$R_x = 1-\delta_k+F_k,$$

which is identical to the condition derived in Corollary 6.

4 Algorithm to solve the model numerically

The Ramsey equilibrium for our model can be computed in a straightforward way. The function $\rho^R(b, b', \eta_c)$ is computed as the solution to the maximization problem defined in (32). Then the steady state value of b is obtained by solving equation (39).

The transitional dynamics is computed for the calibrated economy where $\psi = 1$. In this case η_c is constant, so the function above can be written simply as $\rho^R(b, b')$ and (30) simplifies to

$$\ln(v_0) = \sum_{t=0}^{\infty} \beta^{t+1} \ln(\rho_{t+1}).$$

In the dynamic programming formulation, the Ramsey problem (33) can be written as

$$\ln v(b) = \max_{b'} \beta \ln \rho^R(b, b') + \beta \ln v(b').$$

This problem is solved by discretizing the state space and by the value function iteration.

5 Transitional dynamics

The Ramsey equilibrium converges to the steady state only in one period. Figure 1 in this appendix illustrates the transitional dynamics of the Ramsey equilibrium, starting from the "baseline equilibrium" in Table 2 in the main text.

Figure 1: Transitional dynamics of the Ramsey equilibrium starting from the baseline equilibrium.