

# Spaces of paths and the path topology

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## Spaces of paths and the path topology

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## Spaces of paths and the path topology

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The natural topology on the space of causal paths of a space-time depends on the topology chosen on the space-time itself. Here we consider the effect of using the path topology on space-time instead of the manifold topology, and its consequences for how properties of space-time are reflected in the structure of the space of causal paths. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4963144>]

### I. INTRODUCTION AND CONTEXT

What is the topology of space-time?

This question has several answers, each of which has its attractions.

The first, and most obvious answer is that since space-time is a Lorentz manifold, the topology is the manifold topology. This answer enables us to make use of the apparatus of differential topology and (pseudo-)Riemannian geometry and has provided enormous benefits in the attempt to understand gravitation and the structure of the universe.

But if we take the attitude that a space-time is not a differentiable manifold endowed with additional geometric structure, but that the geometric structure is as fundamental as the set of points, then there are some attractive alternatives, which use the topology to encode the geometry in a more fundamental way.

The first of these was Zeeman's construction<sup>1</sup> of the fine topology on Minkowski space (the finest topology which induces the standard topology on each timelike straight line and spacelike hypersurface) which encodes the information of the causal and linear structure of Minkowski space. This was generalized to curved space-time by Göbel<sup>2</sup> by taking the finest topology with respect to which all timelike geodesics and spacelike hypersurfaces inherit the standard topology. If we regard the (pseudo-)metric structure of the space-time as fundamental, this topology has some compelling properties: in particular, it determines the metric up to an overall constant. On the other hand, it does not admit a countable neighbourhood basis, which makes it very difficult to work with in particular cases.

We could reasonably also consider an alternative approach, in which we do not require timelike geodesics and spacelike hypersurfaces to inherit the standard topology, but rather require that all timelike paths do. (Note that throughout, we will use the terminological convention that a curve is a function from some interval to space-time, and a path is the image of a curve.) This is the approach taken by Hawking, King, and McCarthy.<sup>3</sup> This topology is rather more tractable than the Zeeman or Göbel topologies as it admits a countable neighbourhood basis. For each  $x \in M$ , and each open neighbourhood  $U$  of  $x$ , we denote by  $I(p, U)$  the set of points connected to  $p$  by a timelike path lying in  $U$ , and by  $K(p, U)$  the set  $I(p, U) \cup \{x\}$ . Choose some arbitrary Riemannian metric  $h$  on  $M$ , and denote by  $B_\epsilon(x)$  the open ball of all points within distance  $\epsilon$  of  $x$  with respect to  $h$ . Finally, for an open convex normal neighbourhood,  $U$ , of  $x$ , let  $L_U(x, \epsilon) = K(p, U) \cap B_\epsilon(x)$ . The sets of the form  $L_U(\epsilon, x)$  are then a basis for the path topology. Indeed, by choosing  $\epsilon = 1/n$  for  $n \in \mathbb{N}$ , we obtain a countable neighbourhood basis, so that the path topology is first countable.

The path topology on  $M$  is of particular interest if one's primary concern is the causal, rather than the metric structure of  $M$ . The continuous curves are precisely the Feynman paths, and the

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path topology induces the discrete topology on null and spacelike sets. Furthermore, the path topology determines both the differentiable and conformal structure of  $\mathcal{M}$ .

There has been recent interest in these alternative topologies, in particular showing some surprising consequences for the connectedness of space-time;<sup>4-6</sup> more recently it has been argued that the use of limit curve theorems in the study of singularities must take proper account of the topology of space-time.<sup>7</sup>

In this note we will consider the consequences of using the path topology on space-time to induce a topology on a space of causal paths in a strongly causal space-time. We will see that the situation is rather different for timelike paths and null geodesics (as one might expect, since the path topology induces the discrete topology on a null geodesic), and that these differences present some difficulties, but also some subtleties.

## II. TOPOLOGY ON PATH SPACES

So let  $\mathcal{M}$  denote a space-time, i.e., a time-orientable Lorentz manifold, and let  $\mathcal{P}$  be a set of paths (i.e., the set of images of a set of continuous curves) in  $\mathcal{M}$ . We will, of course, have some particular sets of paths, such as timelike or causal in mind, but it will be convenient to set up the general situation first and then specialize to the particular cases of interest. Note that as we are considering the path, rather than the curve, the notion of future or past pointing does not apply directly to the path: however, when we consider timelike or causal paths we will assume throughout that a curve whose image is a given path has a future-directed parameterization.

So let  $\gamma \in \mathcal{P}$ , let  $x \in \gamma$ , and let  $U$  be an open neighbourhood of  $x$  (in whichever topology on  $\mathcal{M}$  is under consideration). Then the set  $\{\gamma' \in \mathcal{P} | \gamma' \cap U \neq \emptyset\}$  is a neighbourhood of  $\gamma$ . We take the topology on  $\mathcal{P}$  obtained by using these neighbourhoods as a sub-basis, and recall that a neighbourhood basis is given by the set of all finite intersections of these sets.<sup>8</sup>

As usual,<sup>9</sup> we can say that  $\gamma_n$  converges to  $\gamma$  if for any neighbourhood of  $\gamma$ , all but finitely many  $\gamma_n$  lie in this neighbourhood, and that  $\gamma$  is a limit curve of  $\gamma_n$  if  $\gamma_n$  has a subsequence that converges to  $\gamma$ .

We then denote by  $\mathcal{P}_M$  the space  $\mathcal{P}$  equipped with the topology obtained by using the usual manifold topology on  $\mathcal{M}$ , and by  $\mathcal{P}_P$  the space  $\mathcal{P}$  equipped with that obtained by using the path topology on  $\mathcal{M}$ .

Furthermore, we denote by  $\mathcal{T}$  the space of endless timelike paths in  $\mathcal{M}$  and by  $\mathcal{C}$  the space of endless causal paths in  $\mathcal{M}$ . Note that as we consider continuous paths, without imposing differentiability, there need not be a tangent vector to a path  $\gamma$  in  $\mathcal{T}$ . But even if the path is differentiable, the tangent need not be timelike: any portion of such a path can be null, as long as it is not a null geodesic. We only require that all points on  $\gamma$  be timelike related to one another. We now wish to consider the relationships between  $\mathcal{T}_P$ ,  $\mathcal{T}_M$ ,  $\mathcal{C}_P$ , and  $\mathcal{C}_M$ .

## III. THE SPACE OF TIMELIKE PATHS

First, we establish the equivalence of the topologies induced on  $\mathcal{T}$  by the path and manifold topologies for  $\mathcal{M}$ .

Let  $\gamma \in \mathcal{T}$  and let  $\mathcal{U}$  be a neighbourhood of  $\gamma$  in  $\mathcal{T}_M$ . Then there exist sub-basis elements  $\mathcal{U}_1 \dots \mathcal{U}_n$  arising from points  $x_1, \dots, x_n \in \gamma$  and  $M$ -open neighbourhoods  $U_i$  of  $x_i$  as described above. Now, denote by  $\mathcal{V}_i$  the set of all elements of  $\mathcal{T}$  which pass through  $K(x_i, U_i)$ , and by  $\mathcal{V}$  the intersection of the  $\mathcal{V}_i$ . Then  $\mathcal{V}$  is a  $P$ -neighbourhood of  $\gamma$  which is a subset of  $\mathcal{U}$ .

Conversely, let  $\mathcal{V}$  be a neighbourhood of  $\gamma$  in  $\mathcal{T}_P$ . Then there exist sub-basis elements  $\mathcal{V}_1 \dots \mathcal{V}_n$  arising from points  $x_1, \dots, x_n \in \gamma$  and  $P$ -open neighbourhoods  $V_i$  of  $x_i$  as described above. Now, each  $V_i$  is of the form  $K(x_i, W_i)$  for some  $M$ -open neighbourhood  $W_i$  of  $x_i$ . So let  $y_i \in \gamma \cap W_i$ , and let  $U_i$  be an open neighbourhood of  $y_i$  inside  $K(x_i, W_i)$ . Finally, denote by  $\mathcal{W}_i$  the set of all  $\gamma \in \mathcal{T}$  which pass through  $U_i$ , and by  $\mathcal{W}$  the intersection of the  $\mathcal{W}_i$ . This gives an  $M$ -neighbourhood of  $\gamma$  which is a subset of  $\mathcal{V}$ .

Then each  $P$ -neighbourhood of  $\gamma$  lies inside an  $M$ -neighbourhood, and vice versa, giving us the result.

*Proposition 1.*  $\mathcal{T}_P$  is homeomorphic to  $\mathcal{T}_M$ .

We see then that if  $\gamma_n \in \mathcal{T}$ , convergence in the  $P$ -topology is equivalent to convergence in the  $M$ -topology;  $\gamma_n \rightarrow \gamma$  in  $\mathcal{T}_M$  if and only if  $\gamma_n \rightarrow \gamma$  in  $\mathcal{T}_P$ . In addition, it is clear from the definition of neighbourhoods that if  $\gamma_n \rightarrow \gamma$ , then for each  $x \in \gamma$  there is a sequence  $x_n \in \gamma_n$  such that  $x_n \rightarrow x$ , in both topologies. This has the surprising consequence.

*Proposition 2.* If  $\gamma_n \rightarrow \gamma$  in  $\mathcal{T}_M$ , then for each  $x \in \gamma$ , there exist  $x_n \in \gamma_n$  such that  $x_n \rightarrow x$  in the path topology on  $M$ .

This is certainly encouraging: it tells us that if we consider timelike paths, the notion of convergence is unaffected by the choice of whether we use the manifold or the path topology on  $M$ . This might at first sight seem plausible given that the path topology was chosen so to induce the standard topology on timelike paths, but we should recall that this choice of topology can have extreme consequences for the relationships between these timelike paths, as we see from the fact that there are no non-trivial homotopies of timelike paths in  $M$  equipped with the path topology.<sup>4-6</sup>

However, we must now extend consideration to the more problematic case of causal paths. Since the path topology induces the discrete topology on null geodesic segments, we might expect their inclusion to complicate the situation, and indeed it does.

#### IV. THE SPACE OF CAUSAL PATHS

So, we now consider  $C$ , and the relationship between  $C_P$  and  $C_M$ .

By the same argument as before, if  $\gamma_n$  converges to  $\gamma$  in  $C_P$ , it must also converge in  $C_M$ .

Furthermore, if  $\gamma_n$  in  $C$  converges to  $\gamma$  in  $C_M$  and  $\gamma \in \mathcal{T}$ , then again, we must also have  $\gamma_n \rightarrow \gamma$  in  $C_P$ . The difficulty arises when  $\gamma$  contains a null geodesic segment.

Let  $x \in \gamma$ , and let some neighbourhood of  $x$  in  $\gamma$  (in the manifold topology) be a null geodesic segment. Then there exists  $\gamma_n$ , a sequence of timelike paths such that  $\gamma_n \rightarrow \gamma$  in  $C_M$ , but not in  $C_P$ .

We can exhibit such a sequence as follows. Let  $U$  be an open convex normal  $M$ -neighbourhood of  $x$ , and let  $p$  and  $q$  be points in  $\gamma \cap U$ , and on the future and past light cones of  $x$ , respectively. Further, let  $p_n$  be a sequence of points in  $U$  on the future null cone of  $x$ , but not on  $\gamma$ , such that  $p_n \rightarrow p$ ; similarly let  $q_n$  lie in  $U$  on the past null cone, but not on  $\gamma$ , and such that  $q_n \rightarrow q$ .

Then since  $p_n$  and  $q_n$  are connected by a piecewise null geodesic which is not a null geodesic,  $p_n \in I^+(q_n)$  for all  $n$ . Define  $\gamma$  to be the causal path consisting of the null geodesic segment from  $q$  to  $p$  and otherwise an arbitrary timelike path.

Now take some convex normal neighbourhood,  $V$  of  $p$ . For  $p_n$  sufficiently close to  $p$ ,  $\gamma$  will enter  $I^+(p_n, V)$ , so we can connect  $p_n$  to  $p'_n \in \gamma$  in such a way that also  $p'_n \rightarrow p$  (again, in the  $M$ -topology). We can carry out the same procedure for  $q_n$  and  $q$ . Patching these segments together, we take  $\gamma_n$  to coincide with  $\gamma$  after  $p'_n$  and before  $q'_n$ , to connect  $p'_n$  and  $q'_n$  to  $p_n$  and  $q_n$  as described above, respectively, and to be the timelike geodesic segment between  $q_n$  and  $p_n$ .

Then clearly each  $\gamma_n$  is timelike,  $\gamma_n \rightarrow \gamma$  in  $C_M$ , but no  $\gamma_n$  intersects  $K(x, U)$ , and so  $\gamma_n$  does not converge to  $\gamma$  in  $C_P$ .

We therefore see the following proposition.

*Proposition 3.* The path topology on  $M$  induces a strictly finer topology on  $C$  than the manifold topology does.

Now it is clear that  $\mathcal{T}_M$  is dense in  $C_M$ ; but we now know that convergence of a sequence in  $C_M$  does not imply its convergence in  $C_P$ . Nevertheless, we can see that if  $\gamma \in C$ , there exists a sequence  $\gamma_n$  in  $\mathcal{T}$  such that  $\gamma_n \rightarrow \gamma$  in  $C_P$ .

To see this, choose a Riemannian metric  $h$  on  $M$  and denote by  $B_r(x)$  the  $h$ -ball of radius  $r$  centred on  $x$ , by abuse of notation let  $\gamma$  be the image of  $\gamma : (-\infty, \infty) \rightarrow M$ , and let  $K_n$  be the image  $\gamma$  restricted to the interval  $[-n, n]$ . Finally, define

$$U_n = \bigcup_{x \in K_n} B_{1/n}(x).$$

Then by pushing each point of  $K_n$  a sufficiently small distance in a timelike direction we can obtain<sup>10</sup> a path which is everywhere timelike and intersects each  $B_{1/n}(x)$ ; let  $\gamma_n$  be this path extended arbitrarily outside  $U_n$ .

Then if  $x$  is any point on  $\gamma$ , and  $U$  is any  $M$ -open neighbourhood of  $x$ , for sufficiently large  $n$  the  $B_{1/n}$  ball lies inside  $U$ , and so for sufficiently large  $n$ , all  $\gamma_n$  intersect  $K(x, U)$ . Thus  $\gamma_n$  is a sequence of curves in  $\mathcal{T}$  which converges to  $\gamma$  in  $C_P$ .

This establishes the following proposition.

*Proposition 4.  $\mathcal{T}_P$  is dense in  $C_P$ .*

From this we now see that if  $\gamma \in \mathcal{T}$ , the open neighbourhoods of both topologies agree; but if  $\gamma \in C \setminus \mathcal{T}$ , there are more  $P$ -neighbourhoods than  $M$ -neighbourhoods. In passing we can also note that since any causal path can be approximated arbitrarily closely in the  $M$ -topology by a piecewise null geodesic path,  $C \setminus \mathcal{T}$  is also dense in  $C$ , in both topologies.

## V. CONSEQUENCES

So, the principal difference between  $C_P$  and  $C_M$  is the neighbourhoods of curves with null geodesic segments; but of course, null geodesic segments are highly important in causality, since they locally determine the causal structure. Also, since piecewise null geodesic paths are dense in  $C$ , as noted above, this may lead to significant difference between the options when we try to understand how causal properties of space-time are reflected in the topology of  $C$ .

We begin by considering the characterization of global hyperbolicity of  $\mathcal{M}$ . It is well known that  $\mathcal{M}$  is globally hyperbolic iff for any  $x, y \in \mathcal{M}$ , the space of causal paths connecting  $x$  and  $y$  is compact,<sup>11</sup> and has also been shown that this is equivalent to these spaces of causal paths being Hausdorff, and to the space of all causal paths, or, equivalently, the space of all timelike paths in  $\mathcal{M}$  being Hausdorff.<sup>12</sup> By a slight abuse of notation, we will use  $\mathcal{T}$  and  $C$  to denote the spaces of timelike and spacelike paths from  $x$  to  $y$  in  $\mathcal{M}$ .

We should note that the definition of the topology used in Ref. 11 was introduced by Geroch in Ref. 13 and is slightly different from the definition of the manifold topology used here and in Ref. 12. Call this the Geroch topology. So let  $x, y \in \mathcal{M}$ , let  $A, B$  be open sets such that  $x \in A$  and  $y \in B$ , and let  $\gamma$  be a causal path connecting  $x$  to  $y$ . Finally, let  $R$  be an ( $M$ -)open set in  $\mathcal{M}$  which contains  $\gamma$ . Then the set of causal paths from  $x$  to  $y$  which intersect  $R$  clearly contains the set of causal paths from  $x$  to  $y$  which lie inside  $R$ , and so an  $M$ -neighbourhood of  $\gamma$  contains a Geroch-neighbourhood. Conversely, we can cover  $\gamma$  by finitely many sufficiently small causally convex open sets, where sufficiently small means that each set lies inside  $R$  and that no causal path can leave one of the sets and re-enter it. Then the intersection of all  $M$ -neighbourhoods of  $\gamma$  corresponding to these sets gives an  $M$ -neighbourhood of  $\gamma$  lying inside the Geroch-neighbourhood. Thus the two descriptions impose the same topology on this space of causal paths.

Now, since  $C_P$  is finer than  $C_M$ , it follows that if  $C_M$  is Hausdorff, then  $C_P$  is also Hausdorff. On the other hand, if  $C_M$  is not Hausdorff then it follows from the arguments of Ref. 12 that  $\mathcal{T}_M$  is not Hausdorff; but  $\mathcal{T}_M = \mathcal{T}_P$ , so that  $\mathcal{T}_P$  is not Hausdorff, and so  $C_P$  is also not Hausdorff. We thus have the following proposition.

*Proposition 5.  $C_M$  is Hausdorff if and only if  $C_P$  is Hausdorff; consequently,  $\mathcal{M}$  is globally hyperbolic if and only if  $C_P$  is Hausdorff.*

Thus the equivalence of global hyperbolicity to the space of causal paths being Hausdorff is maintained when we shift from the manifold topology on  $\mathcal{M}$  to the path topology. But this does not work for the compactness criterion.

Let  $C$  now be the space of all causal paths from  $x$  to  $y$  in  $\mathcal{M}$ , where  $y \in I^+(x)$ . Recall that for any path  $\gamma$  in  $C$  with a null geodesic segment we can find  $\gamma_n \in \mathcal{T}$  such that  $\gamma_n \rightarrow \gamma$  in  $C_M$ , but not in  $C_P$ . Since  $C_P$  is finer than  $C_M$ , if  $\gamma_n \rightarrow \gamma' \neq \gamma$  in  $C_P$ , then we must also have  $\gamma_n \rightarrow \gamma'$  in  $C_M$ . Clearly this holds for any subsequence of  $\gamma_n$ . Thus if  $C_M$  is Hausdorff, and hence compact,  $C_P$  contains a

sequence with no convergent subsequence, and therefore<sup>8</sup> cannot be compact. If  $C_M$  is not compact, then clearly  $C_P$  is not compact. We thus have the following proposition.

*Proposition 6.* If  $x, y \in M$  with  $y \in I^+(x)$ , and  $C$  is the space of causal paths from  $x$  to  $y$ , then  $C_M$  is compact if and only if  $C_M$  is Hausdorff if and only if  $C_P$  is Hausdorff; however,  $C_P$  is never compact.

Finally, we consider the limit curve theorem. There are numerous versions of this,<sup>14</sup> but roughly speaking all say that if  $\gamma_n$  is a sequence of causal paths and  $x_n \in \gamma_n$ , with  $x$  a limit point of  $\{x_n\}$ , then there is an endless causal path  $\gamma$  through  $x$  which is a limit curve of  $\{\gamma_n\}$ , all in the manifold topology.

Let  $p_n$  be a sequence of null vectors in three dimensional Minkowski space, such that  $p_n \rightarrow p$  in the usual topology, and let  $\gamma_n$  be the null geodesic through the origin with tangent  $p_n$ , and  $\gamma$  to be the null geodesic through the origin with tangent  $p$ . Since the origin lies on each  $\gamma_n$ , we can take  $x_n$  to be the origin for all  $n$ , so we have a sequence  $x_n \in \gamma_n$ , and the origin is a limit point of this sequence. Clearly,  $\gamma$  is the unique limit curve of  $\{\gamma_n\}$  in  $C_M$ .

But if  $U$  is any open set (not containing the origin), and  $p \in \gamma \cap U$ , then for all  $n$ ,  $\gamma_n \cap K_p(U) = \emptyset$ , so that  $\gamma$  is not a limit curve of  $\{\gamma_n\}$  in  $C_P$ . But then  $\{\gamma_n\}$  has no limit curve in  $C_P$ , and in particular no limit curve containing the origin.

Hence the limit curve theorem fails to hold if we replace the manifold topology by the path topology throughout its statement.

## VI. CONCLUSION

This suggests that although the path topology is of great interest from the point of view of encapsulating the differentiable and causal structure of space-time, it is nevertheless inappropriate for at least some important aspects of the study of causal structure, where the manifold topology remains both technically easier to work with and fruitful.

## ACKNOWLEDGMENTS

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