Homopolar oscillating-disc dynamo driven by parametric resonance

Priede, J. , Avalos-Zuñiga, R. and Plunian, F.

Author pre-print (submitted version) deposited in CURVE May 2012

Original citation & hyperlink:

Priede, J. , Avalos-Zuñiga, R. and Plunian, F. (2010) Homopolar oscillating-disc dynamo driven by parametric resonance. Physics Letters A, volume 374 (4): 584–587.

http://dx.doi.org/10.1016/j.physleta.2009.11.022

Homopolar oscillating-disc dynamo driven by parametric resonance

Jānis Priede
Applied Mathematics Research Centre, Coventry University, Coventry, CV1 5FB, United Kingdom

Raúl Avalos-Zúñiga
Universidad Autónoma Metropolitana, D.F., México

Franck Plunian
Université Joseph Fourier, CNRS, LGIT, Grenoble, France

We use a simple model of Bullard-type disc dynamo, in which the disc rotation rate is subject to harmonic oscillations, to analyze the generation of magnetic field by the parametric resonance mechanism. The problem is governed by a damped Mathieu equation. The Floquet exponents, which define the magnetic field growth rates, are calculated depending on the amplitude and frequency of the oscillations. Firstly, we show that the dynamo can be excited at significantly subcritical disc rotation rates when the latter is subject to harmonic oscillations with a certain frequency. Secondly, at supercritical mean rotation rates, the dynamo can also be suppressed but only in narrow frequency bands and at sufficiently large oscillation amplitudes.

In dynamo experiments, a high driving power is necessary to achieve the self-excitation of the magnetic field. The ensuing liquid metal flow is usually strongly turbulent. In both Riga and Karlsruhe dynamo experiments [1, 2], the turbulent fluctuations were partly inhibited by the internal walls, whereas in the Cadarache experiment [3], the absence of such walls resulted in large-scale flow fluctuations [4]. The effect of flow fluctuations on the dynamo threshold has been addressed in several recent studies [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Solving the kinematic dynamo problem for a given non-stationary flow usually governed by the Navier-Stokes equations shows that turbulence generally has an adverse effect on the dynamo excitation, unless the fluctuations are strong enough to drive the dynamo by themselves without any mean flow. In the latter case, we speak of a fluctuation dynamo [15, 16], whose experimental implementation seems hardly feasible because of the high excitation threshold. However the possibility that fluctuations excite the magnetic field by the parametric resonance mechanism [17] can not be excluded. Parametric resonance has been proposed in the somewhat different context of spiral galaxies as promoter of bisymmetric magnetic field structure [18, 19, 20, 21].

In this paper, we use a simple model of the Bullard-type disc dynamo [22] to show that the magnetic field can indeed be excited by the parametric resonance mechanism, even when relatively small harmonic oscillations are added to significantly subcritical disc rotation rates.

Consider a Bullard-type disc dynamo [22] which consists of a solid conducting disc rotating with a generally time-dependent angular velocity $z(t)$ about its axis, and a wire twisted around the axle and connected by sliding contacts to the rim of the disc and the axle as shown in Fig. 1. The disc is assumed to be segmented so that azimuthal current can flow only at its rim. This corresponds to the modification of the Bullard disc dynamo suggested by Moffatt in order to eliminate exponential

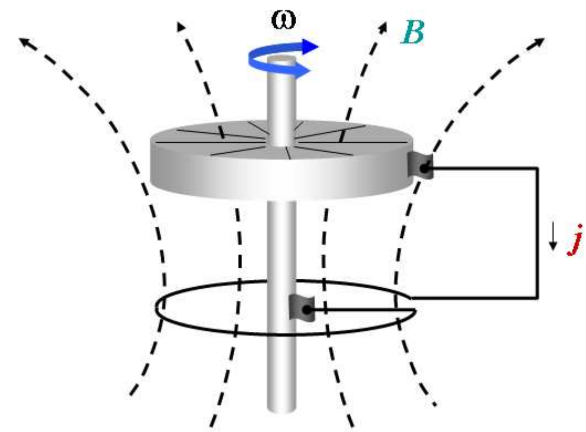

Figure 1: Sketch of a homopolar disc dynamo.

growth of the magnetic field in the limit of a perfectly conducting disc [23]. The system is described by the following set of dimensionless equations (for details see Ref. [23])

$$\begin{aligned} \dot{x} &= r(y-x), \\ \dot{y} &= xz + mx - (m+1)y, \\ \dot{z} &= g\,[1 + x(mx-(m+1)y)] - kz, \end{aligned} \tag{1}$$

where x and y are magnetic fluxes through the loop made by the wire and the rim of disc, respectively; z is the dimensionless angular velocity of the disc; r accounts for the resistance of the disc relative to that of the loop, and m characterizes the relative mutual inductance of the disc and the loop; the dot stands for the time-derivative d/dt. The disc is driven by a generally time-dependent torque g, and braked by a viscous-type friction characterized by the coefficient k which is necessary for the structural stability of the system [24]. Henceforth, we assume the friction to be strong with respect to the inertia of the

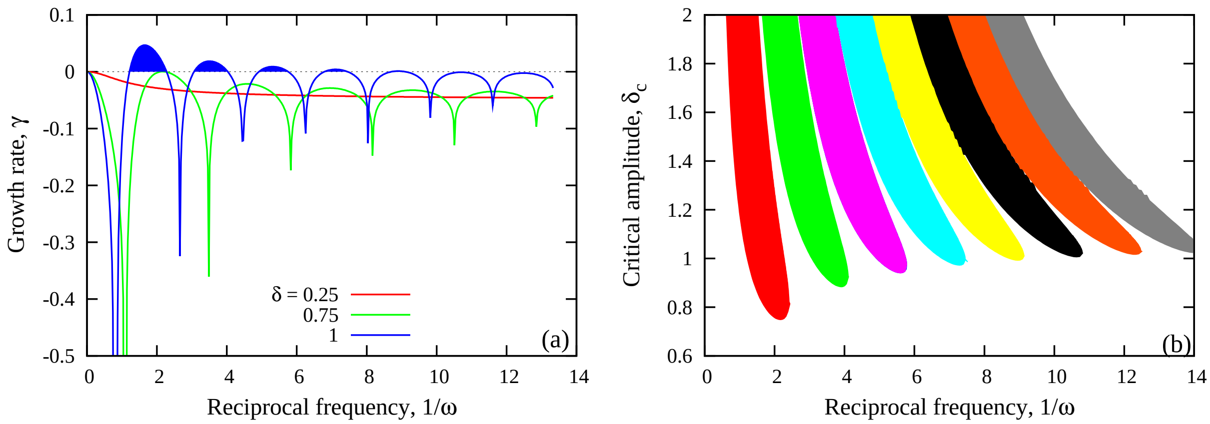

Figure 2: Growth rate γ versus the reciprocal frequency (a) and the critical amplitude δ_c versus the frequency (b) for the marginal mean rotation rate at $\alpha_0 = 0$.

disc accounted for by $\dot{z}$ in Eq. (1), which, thus, results in $z = z_0 \left[1 + x(mx - (m+1)y)\right]$, where $z_0 = g/k$. The remaining two 1st-order ODEs in (1) can be combined into a single 2nd-order Duffing-type equation [7] with a non-linear friction

$$\ddot{x} + (1 + \beta x^2)\dot{x} - \alpha x + \lambda x^3 = 0, \qquad (2)$$

where x and t are rescaled by $(m+1+r)$ and $(m+1+r)^{-1}$, respectively, and $\alpha = r(z_0-1)/(m+1+r)^2$, $\beta = z_0(m+1)(m+1+r)$, and $\lambda = rz_0$. Further, we focus on the evolution of small initial perturbations of the magnetic field characterized by $x \ll 1$, for which Eq. (2) can linearized by setting $\beta = \lambda = 0$. Then the only remaining parameter α depends directly on the deviation of the disc rotation rate from its critical value $\alpha = 0$. For $\alpha > 0$, a small initial magnetic field starts to grow exponentially provided that the disc rotates steadily [22]. In this study, we are interested in how the generation of the magnetic field is affected by the unsteadiness of the disc rotation

$$\alpha = \alpha_0 + \delta\cos(\omega t), \qquad (3)$$

which besides the mean part α_0 contains also an oscillatory component with the amplitude δ and the circular frequency ω. Then the linearized Eq. (2) reduces to a damped Mathieu equation

$$\ddot{x} + \dot{x} - (\alpha_0 + \delta\cos(\omega t))x = 0.$$

Using the substitution $x(t) = \exp(-t/2)\chi(\omega t/2)$, the equation above can be transformed into the canonical Mathieu equation

$$\ddot{\chi} + \left[a - 2q\cos(2\tau)\right]\chi = 0, \qquad (4)$$

where $a = -(1+4\alpha_0)/\omega^2$, $q = 2\delta/\omega^2$ and $\tau = \omega t/2$. According to Floquet theory, a particular solution to Eq. (4) can be written as $\chi(\tau) = \exp(i\nu\tau)f(\tau)$, where $f(\tau)$ is a π-periodic function and ν is the Floquet exponent—both dependent on the parameters a and q. According to this solution, the amplitude of the magnetic field $x(t)$ evolves exponentially in time with the maximum growth rate $\gamma = \left(|\Im\left[\omega\nu\right]| - 1\right)/2$, where the modulus accounts for the time-reflection symmetry of Eq. (4) [26]. Thus, the amplitude of the magnetic field grows exponentially when $\gamma > 0$, whilst the marginal state is defined by $\gamma = 0$. We use the Maple computer algebra software to calculate Floquet exponent which defines the growth rate γ depending on the amplitude δ and the frequency ω. Next, we find the critical oscillation amplitude δ_c as the function of frequency ω for fixed values of the mean rotation α_0 by solving numerically equation $|\Im\left[\omega\nu\right]| = 1$ corresponding to $\gamma = 0$. Eventually, we determine the minimal oscillation amplitude $\delta_{\min}$ and the corresponding frequency at which an exponentially growing magnetic field first appears. The corresponding numerical results are presented and discussed below.

We start with a marginal mean disc rotation rate $\alpha_0 = 0$, which corresponds to the dynamo excitation threshold when the disc rotates steadily, i.e., $\gamma = 0$ when $\delta = 0$. As seen in Fig. 2(a), the disc oscillations about the critical rotation rate with a sufficiently small amplitude ($\delta = 0.25$) brings the growth rate to a constant level below zero as the frequency is reduced (reciprocal frequency increased). As the oscillation amplitude increases, first, the growth rate splits into separate frequency bands whose width decreases as $\sim 1/\omega$ for $\omega \to 0$. In order to show this increasingly fine-scale structure of the growth rate as $\omega \to 0$, we use the reciprocal frequency which is proportional to the period of disc oscillations. Secondly, the growth starts to increase with the oscillation amplitude and approaches zero again at $\delta \approx 0.75$ for a certain critical frequency. Further increase in the oscillation amplitude to $\delta = 1$ results in the appearance of several frequency bands with positive growth rates ($\gamma > 0$) which are shown as filled regions in Fig. 2(a). The critical amplitude δ_c at which the growth rate turns zero, is shown for $\alpha_0 = 0$ in 2(b) against the oscillation frequency. Marginal state with $\gamma = 0$ corresponds to the boundaries

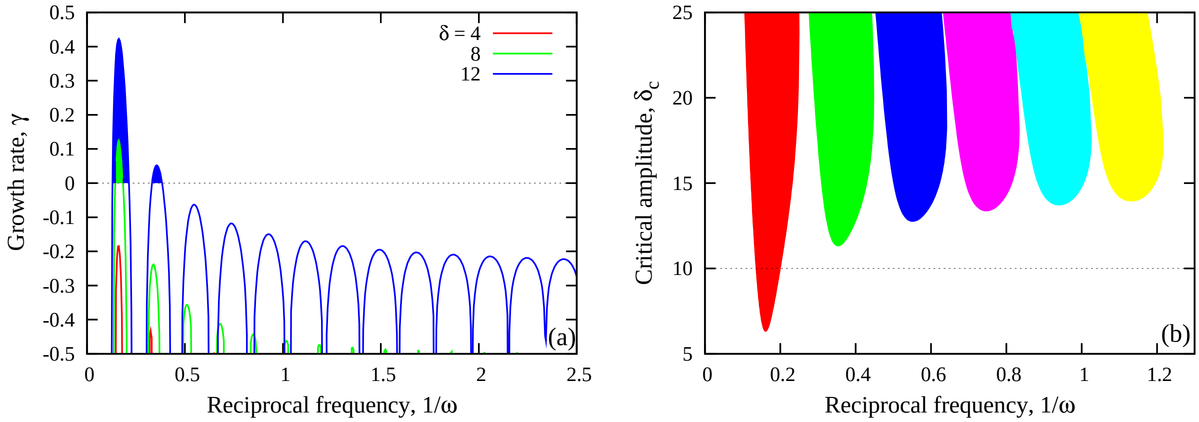

Figure 3: Growth rate γ versus the reciprocal frequency (a) and the critical amplitude δ_c versus the frequency for a strongly subcritical mean rotation rate at $\alpha_0 = -10$.

of the separate frequency bands shown by different colors. Growth rate is positive corresponding to the dynamo action inside the filled frequency bands which approach each other closely as the oscillation amplitude increases. Note that only a certain number of first instability bands are shown in Fig. 2(b). There is an infinite sequence of similar instability bands of decreasing width as $\omega \to 0$. Thus, the range of unstable frequencies, which is intervened by infinitely many, much narrower stability bands, extends down to $\omega = 0$, whereas it is bounded from above by the first instability band. The critical oscillation amplitude, which is the lowest for the first instability band, rises to an asymptotic value depending on α as $\omega \to 0$. The minimal value of the critical oscillation amplitude and the corresponding frequency at which it occurs are plotted in Fig. 4 versus subcritical (negative) values of α.

The reduction of disc rotation rate to a moderately subcritical value of $\alpha_0 = -1$, results in the increase of the minimal oscillation amplitude necessary for the dynamo action ($\gamma > 0$) up to $\delta_c \approx 2$. It means that the maximum disc rotation rate (3) during the oscillation cycle temporally exceeds the critical value $\alpha = 0$ for a steady rotation. This, however, changes when the disc rotation rate is reduced further down to $\alpha_0 = -10$, for which the growth rates and the corresponding generation bands are shown in Fig. 3. In this case, a positive growth rate band is seen in Fig. 3(a) already at $\delta_c = 8$. This implies that $\alpha < 0$ during the whole cycle of disc oscillation. Thus, the dynamo appears at a maximum disc rotation rate below its critical value for a steady rotation. There is a range of frequencies seen Fig. 3(b), at which a subcritical growth of the magnetic field is possible with the the oscillation amplitude $\delta_c < -\alpha_0 = 10$. As seen in Fig. 4, such a subcritical excitation of magnetic field appears already at $\alpha_0 \lesssim -4$, where the minimal required oscillation amplitude $\delta_{\min}$ becomes smaller than $-\alpha_c$, which is shown by the dashed line. Note that for a strongly subcritical α_0 and steadily rotating disc, an initial perturbation of the magnetic field oscillates with the period $O(1/\sqrt{-\alpha_0})$ which is much shorter than the characteristic damping time $O(1)$. Thus, for such α_0 the damping of magnetic field oscillations becomes relatively slow that enables parametric excitation of the magnetic field by a relatively weak modulation of the disc rotation rate.

For $\alpha_0 < -0.25$, the bands of unstable frequencies are associated with the subharmonics of Mathieu equation defined by $a^{1/2} = 1, 2, 3, \ldots$ in Eq. (4) [26]. Thus, the most unstable band corresponds to the first subharmonic with $\omega_1 = \sqrt{-4\alpha_0 - 1}$ which is seen in Fig. 4 to approximate ω_c well for $\omega \gtrsim 1$. In addition, it is interesting to note that the critical frequency, ω_c, at which $\delta_{\min}$ occurs, is numerically very close to $\delta_{\min}$ itself in the whole range of $\alpha_0 < 0$.

For a supercritical disc rotation rate with $\alpha_0 = 1$, the growth rate γ, which is seen in Fig. 5(a) to be positive for a steadily rotating disc ($\delta = 0$), first reduces as the oscillation amplitude is increased to $\delta = 2$. Moreover, γ

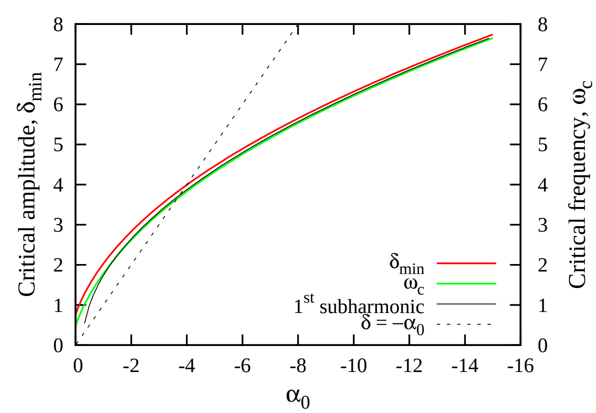

Figure 4: Minimal oscillation amplitude and critical frequency versus subcritical rotation rate parameter ($\alpha < 0$).

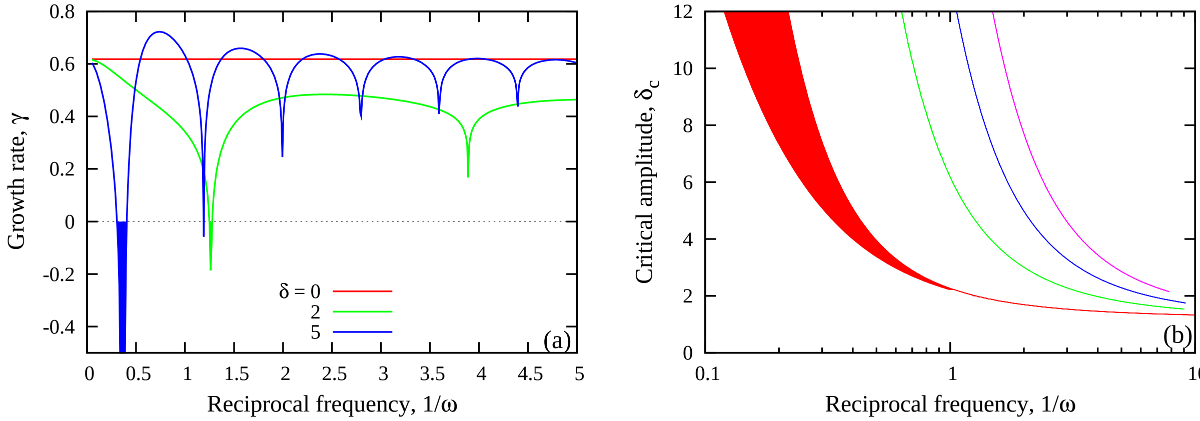

Figure 5: Growth rate γ versus the reciprocal frequency (a) and the critical modulation amplitude δ_c versus the frequency for a slightly supercritical mean rotation rate at $\alpha_0 = 1$. Filled curves correspond to suppressed dynamo.

is seen to become negative within certain, relatively narrow frequency bands. These negative growth rate bands ($\gamma < 0$), where the dynamo is suppressed by the disc oscillations, are shown in Fig. 5(a) by filled curves. As seen in Fig. 5(b), the width of the first suppressed frequency band noticeably increases while the whole band shifts towards the high-frequency range, i.e., low reciprocal frequencies, as the oscillation amplitude increases. At the same time, the width of the subsequent frequency bands, where the dynamo is suppressed, remains very small, while the spacing between the adjacent suppression bands reduces in terms of the reciprocal frequency.

In conclusion, we have shown in this study that dynamo can be excited by the parametric resonance at considerably low velocities of the conducting medium when the latter is subject to harmonic oscillations. Even though this was demonstrated for a simple model Bullard-type disc dynamo, we expect that similar mechanism may also be relevant for more realistic dynamos.

[1] A. Gailitis et al., Phys. Rev. Lett. **84**, 4365 (2000); Phys. Rev. Lett. **86**, 3024 (2001).
[2] R. Stieglitz and U. Müller, Phys. Fluids **13**, 561 (2001).
[3] R. Monchaux et al., Phys. Rev. Lett. **98**, 044502 (2007).
[4] A. de la Torre and J. Burguete, Phys. Rev. Lett. **99,** 054101 (2007)
[5] S. Fauve and F. Pétrélis, in *Peyresq lectures on nonlinear phenomena*, ed. J. Sepulchre, (World Scientific, Singapore, 2003), pp. 1-64.
[6] C. Normand, Phys. Fluids **15**, 1606 (2003).
[7] N. Leprovost, B. Dubrulle, F. Plunian, *Magnethohydrodynamics* **40**, 1 (2004).
[8] N. Leprovost and B. Dubrulle, Eur. Phys. J. B **44**, 395 (2005).
[9] J-P. Laval, P. Blaineau, N. Leprovost, B. Dubrulle and F. Daviaud Phys. Rev. Lett. **96**, 204503 (2006).
[10] F. Petrelis and S. Fauve, Europhys. Lett. **76**, 602-608 (2006).
[11] R. Volk, Ph. Odier and J.-F. Pinton, Phys. Fluids **18**, 085105 (2006).
[12] M. Peyrot, F. Plunian, and C. Normand, Phys. Fluids **19**, 054109 (2007).
[13] M. Peyrot, A. Gilbert, and F. Plunian, Phys. Plasmas **15**, 122104 (2008).
[14] U. Müller and R. Stieglitz, Phys. Fluids **21**, 034108 (2009).
[15] A. A. Schekochihin et al., New J. Phys. **9**, 300 (2007).
[16] R. Stepanov and F. Plunian, Astrophys. J **680**, 809 (2008).
[17] L. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, Oxford, 1969).
[18] M. Chiba and M. Tosa, Mon. Not. R. Astr. Soc. **244**, 714 (1990).
[19] D. Schmitt and G. Rüdiger, Astron. Astrophys. **264**, 319 (1992).
[20] K.M. Kuzanyan and D.D. Sokoloff, Astrophys. Space Sc. **208**, 245 (1993).
[21] D. Moss, Astron. Astrophys. **308**, 381 (1996).
[22] E. C. Bullard, Proc. Camb., Phil., Soc. **51**, 744 (1955).
[23] H. K. Moffatt, Geophys. Astrophys. Fluid Dynam. **14**, 147 (1979).
[24] R. Hide, Geophys. Res. Let. **22**, 1057 (1995).
[25] F. Plunian, Ph. Marty, and A. Alemany, Proc. R. Soc. Lond. A **454**, 1835 (1997).
[26] C. M. Bender, and S. A. Orsag, *Advanced mathematical methods for scientists and engineers* (McGraw-Hill, New York, 1978), Sec. 11.4.