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Classification of Solutions to the Plane Extremal Distance Problem for Bodies With Smooth Boundaries

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ABSTRACT

The determination of the contact points between two bodies with analytically described boundaries can be viewed as the limiting case of the extremal point problem, where the distance between the bodies is vanishing. The advantage of this approach is that the solutions can be computed efficiently along with the generalized state during time integration of a multibody system by augmenting the equations of motion with the corresponding extremal point conditions. Unfortunately these solutions can degenerate when one boundary is concave or both boundaries are non-convex. We present a novel method to derive degeneracy and non-degeneracy conditions that enable the determination of the type and codimension of all degenerate solutions that can occur in plane contact problems involving two bodies with smooth boundaries. It is shown that only divergence bifurcations are relevant and thus we can simplify the analysis of the degeneracy by restricting the system to its one dimensional center-manifold. The resulting expressions are then decomposed by applying the Multinomial Theorem resulting in a computationally efficient method to compute explicit expressions for the Lyapunov coefficients and transversality conditions. Furthermore, a procedure to analyze the bifurcation behavior qualitatively at such solution points based on the Tschirnhaus transformation is given and demonstrated by examples. The application of these results enables in principle the continuation of all solutions simultaneously beyond the degeneracy as long as their number is finite.

1 Introduction

The proper treatment of contact problems involving complex shaped bodies in computational dynamics still poses a challenge in the scientific community. In this domain, formulations and algorithms are required that allow for a precise determination of the contact forces while being computationally very efficient at the same time. This distinguishes the application in multibody simulations from contact problems in other fields, like in virtual reality where the precision requirements

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are less strict (an overview is found e.g. in Lin and Gottschalk [1] or Kockara et al. [2]) or in finite-element analysis where the computation time is less critical, but the spatial distribution of the contact stresses must be determined (see e.g. Wriggers [3]). The main sub-problems in contact simulation are collision detection, contact kinematics and dynamics, i.e., the determination of the contact forces from the kinematic quantities. This article addresses the first two problems by investigating the augmentation of the equations of motions with the analytic extremal point conditions which neither relies on tessellation nor on global optimization or recursive interval subdivision methods as used by Snyder et al. [4], and thus enables in principle the precise and efficient computation of potential contact points. Unfortunately, when concave boundaries of the contacting bodies are involved the problem in general is not uniquely solvable any more and a changing number of solutions might occur. We derive here a method to compute explicit expressions that enable the determination of the solution behavior when a non-hyperbolic solution is encountered during the time-integration. We state these for important cases and demonstrate their application by giving five examples related to commonly encountered bifurcation types. For the sake of simplicity we restrict our analysis to bodies with smooth boundaries and the plane case, but in principle the corresponding expressions for spatial contact problems can be derived analogously.

2 Plane Contact

In the case of regular parametric boundaries $\mathbf{c}_1(s_1)$ and $\mathbf{c}_2(s_2)$, the conditions for extremal points of the plane contact problem[1] are basically derived by differentiating the squared distance from $\mathbf{c}_1(s_1)$ to $\mathbf{c}_2(s_2)$ with respect to the curve parameters s_1 and s_2. In the separation the representation Eqs. (1) stated by Pfeiffer and Glocker [5] is better suited for dynamic simulation,

$$\begin{bmatrix} \left({}_g\mathbf{c}_2(s_2,\mathbf{r}_2,\varphi_2) - {}_g\mathbf{c}_1(s_1,\mathbf{r}_1,\varphi_1)\right)\cdot {}_g\mathbf{t}_1(s_1,\varphi_1) \\ {}_g\mathbf{n}_1(s_1,\varphi_1)\cdot {}_g\mathbf{t}_2(s_2,\varphi_2) \end{bmatrix} = \mathbf{0} \tag{1}$$

where ${}_g\mathbf{n}_i$ are the outwards pointing normals and ${}_g\mathbf{t}_i$ the tangent vectors of body i. The configuration is given by the time-dependent orientations φ_i and translation vectors $\mathbf{r}_i$ of the bodies' coordinate systems with respect to a common reference frame g. The solution of the set of non-linear Eqs. (1) for the curve parameters s_1 and s_2 yields the contact points and the extremal distance between the bodies can be calculated. The transformation of the boundaries ${}_b\mathbf{c}$ represented in a local frame b attached to the body to the coordinate system g is given by

$${}_g\mathbf{c}_i = \mathbf{A}_i \cdot {}_b\mathbf{c}_i + \mathbf{r}_i\ , \tag{2}$$

where

[1] Actually there is an additional condition to ensure that the outsides/insides of the boundary are always matched, but this condition is not required for tracking the solutions if the initial conditions comply with it.

$$\mathbf{A}_i = \begin{bmatrix} \cos(\varphi_i) & -\sin(\varphi_i) \\ \sin(\varphi_i) & \cos(\varphi_i) \end{bmatrix}$$

is a plane rotation matrix. Without loss of generality we choose the common frame g to be the body-fixed coordinate system attached to the first body in which the boundary of this body is described. Then Eqs. (1) simplify to

$$\mathbf{f}(\mathbf{s},\mathbf{r},\varphi) = \begin{bmatrix} (\mathbf{c}_2(s_2,\mathbf{r},\varphi) - \mathbf{c}_1(s_1)) \cdot \mathbf{t}_1(s_1) \\ \mathbf{n}_1(s_1) \cdot \mathbf{t}_2(s_2,\varphi) \end{bmatrix} = \mathbf{0} \tag{3}$$

where $\mathbf{r} = [x,y]^\mathsf{T}$ and φ are used to express body 2 in body 1's coordinate system. The indexes indicating the reference coordinate system have been omitted for the sake of a compact notation. From Eq. (3) it is obvious that there are only three independent control parameters, namely x, y and φ, covering any relative location and orientation of the two bodies.

The advantage of Eqs. (1) and Eqs. (3) is that they can be used to augment the equations of motion of a multibody system. The solution can then be integrated along with these given valid initial conditions [6]. It has been shown by Hartmann [7] that this method is very efficient in terms of computation time.

3 Augmented Equations of Motions

One possible approach to obtain the equations of motions of a multibody system augmented with the extremal point tracking equations is shown in Shabana [6], where the parameters of the boundaries $\mathbf{s} = \begin{bmatrix} s_1 & s_2 \end{bmatrix}^\mathsf{T}$ are added to the vector of generalized coordinates $\mathbf{q}$ to obtain the new coordinates $\bar{\mathbf{q}} = \begin{bmatrix} \mathbf{q}^\mathsf{T} & \mathbf{s}^\mathsf{T} \end{bmatrix}^\mathsf{T}$. Equations (3) are appended to the set of original constraint equations $\tilde{\mathbf{C}}$ of the multibody system which with the substitutions $\mathbf{r} = \mathbf{r}(\mathbf{q})$ and $\varphi = \varphi(\mathbf{q})$ yield

$$\mathbf{C}(\mathbf{q},\mathbf{s},t) = \begin{bmatrix} \tilde{\mathbf{C}}(\mathbf{q},t) \\ \mathbf{f}(\mathbf{q},\mathbf{s}) \end{bmatrix} = \mathbf{0} \tag{4}$$

The augmented equations of motion now take the form

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{C_q}^\mathsf{T} \\ \mathbf{0} & \mathbf{0} & \mathbf{C_s}^\mathsf{T} \\ \mathbf{C_q} & \mathbf{C_s} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{s}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{0} \\ \mathbf{Q}_d \end{bmatrix} \tag{5}$$

The bold subscipts denote the Jacobians of the constraint vector $\mathbf{C}$ with respect to the indicated vectors, $\mathbf{M}$ is the generalized mass matrix, $\boldsymbol{\lambda}$ are Lagrange multipliers and $\mathbf{Q}$ is the vector of generalized forces. The vector of the generalized constraint forces $\mathbf{Q}_d$ is obtained by differentiating $\mathbf{C}(\mathbf{q}(t),\mathbf{s}(t),t)$ twice with respect to time and separating the terms of the left side in the third equation of Eqs. (5):

$$\mathbf{Q}_d = -\mathbf{C}_{tt} - \left((\mathbf{C_q\dot{q}})_\mathbf{q} + (\mathbf{C_s\dot{s}})_\mathbf{q}\right)\mathbf{\dot{q}} - 2\mathbf{C}_{t\mathbf{q}}\mathbf{\dot{q}} - \left((\mathbf{C_q\dot{q}})_\mathbf{s} + (\mathbf{C_s\dot{s}})_\mathbf{s}\right)\mathbf{\dot{s}} - 2\mathbf{C}_{t\mathbf{s}}\mathbf{\dot{s}} \tag{6}$$

Given valid initial conditions, the integration of Eqs. (5) enable the tracking of the curve parameters associated with an extremal point over time. But when $\mathbf{C_s}$ becomes degenerate, the integration of the equations of motion usually cannot continue and we discuss next how the solution behaves in such a case.

4 Solution Behaviour

The classification of solutions is of particular interest. The focus is on degenerate solutions as they are the cause for structural changes of solutions. The transitions that are associated with a change in the number of solutions are the problem to be addressed, because they prevent the continous tracking of extremal points with standard ODE-solvers.

One way to approach the classification problem is to apply the methods from Dynamical System Theory (see e.g. [8] or [9]). The extremal distance points are fixed points or singularities of the differential equations $\mathbf{\dot{s}} = \mathbf{f}(\mathbf{s})$ with $\mathbf{f}$ defined by Eqs. (3). If a fixed point has no eigenvalues with zero real part, the point is said to be hyperbolic or non-degenerate. In this case, the classification can be conducted based on the linear term of the Taylor-expansion of Eqs. (3), as the flow of the system they define is topologically conjugate to the flow of the corresponding linear system according to the Theorem of Hartmann-Grobmann [8].

A pre-requisite for a structural change is that the solution to Eqs. (3) becomes non-hyperbolic and then it cannot be classified by inspection of the linear term of its Taylor-expansion [8]. The goal is to find a classification and the conditions under which the different events occur, as well as the bifurcation types.

For different degeneracy conditions the system will be restricted to its center manifold [10]. The bifurcation types can be determined by comparison with their known normal forms. In the following we assume that $\mathbf{f}$ is in the form of Eqs. (3).

4.1 Degeneracy Conditions

The existence of a unique solution $\mathbf{s}$ to Eqs. (3) is guaranteed if and only if its Jacobian with respect to $\mathbf{s}$

$$\mathbf{J}_{\mathbf{f},\mathbf{s}} = \begin{bmatrix} -\frac{\partial \mathbf{c}_1}{\partial s_1}\cdot\mathbf{t}_1 + (\mathbf{c}_2 - \mathbf{c}_1)\cdot\frac{\partial \mathbf{t}_1}{\partial s_1} & \frac{\partial \mathbf{c}_2}{\partial s_2}\cdot\mathbf{t}_1 \\ \frac{\partial \mathbf{n}_1}{\partial s_1}\cdot\mathbf{t}_2 & \mathbf{n}_1\cdot\frac{\partial \mathbf{t}_2}{\partial s_2} \end{bmatrix} \tag{7}$$

has full rank, as stated by the inverse function theorem. Without loss of generality the closed boundary curves $\mathbf{c}$ are parametrized by unit arc-length, i.e. the normal and tangent vectors have unit length, in counter-clockwise order. Some simple transformations and the application of the Frenet formulas [11] to Eq. (7) yield

$$\mathbf{J_{f,s}} = \begin{bmatrix} -1-\kappa_1\,(\mathbf{c}_2-\mathbf{c}_1)\cdot\mathbf{n}_1 & \mathbf{t}_1\cdot\mathbf{t}_2 \\ \kappa_1\,\mathbf{t}_1\cdot\mathbf{t}_2 & -\kappa_2\,\mathbf{n}_1\cdot\mathbf{n}_2 \end{bmatrix}, \tag{8}$$

where κ_1 and κ_2 are the signed curvatures of the curves $\mathbf{c}_1$ and $\mathbf{c}_2$ respectively. Note that the signs of the Frenet formulas [11] have been modified to match the different definition of the normal direction.

In the case the $\mathbf{c}_i$ are locally extremal points of the squared distance function, the tangent vectors of the two distinct bodies are parallel and so are the normals. Moreover for physically meaningful solutions the scalar product of the unit tangents as well as the unit normals is minus unity. Then Eq. (8) becomes

$$\mathbf{J_{f,s}} = \begin{bmatrix} -1-\kappa_1\,d & -1 \\ -\kappa_1 & \kappa_2 \end{bmatrix}, \tag{9}$$

where d is the signed scalar distance from point $\mathbf{c}_1$ to $\mathbf{c}_2$.

The Jacobian in Eq. (9) is rank-deficient if and only if its determinant vanishes as already stated by Pfeiffer and Glocker [5]:

$$\det(\mathbf{J_{f,s}}) = -\kappa_1 - \kappa_2 - \kappa_1\kappa_2\,d = 0 \tag{10}$$

For the contact of a convex with a non-concave curve, there always exists a unique solution as long as the bodies are not penetrating, i.e. $d \geq 0$. But in general, solutions can degenerate. This case occurs when the curvatures at the extremal points κ_1 and κ_2 and the distance d between them are such that the solution is located on the surface shown in Fig. 1.

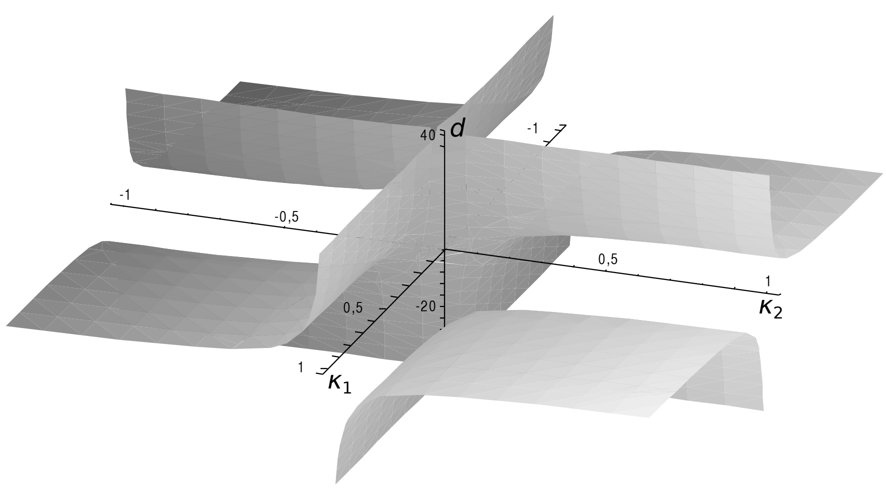

Fig. 1. Visualization of the degeneracy condition

For the further discussion the eigenvalues of the Jacobian $\mathbf{J_{f,s}}$ will be required which are

$$\lambda_{1,2} = \frac{1}{2}\,\mathrm{tr}\left(\mathbf{J_{f,s}}\right) \pm \frac{1}{2}\sqrt{\mathrm{tr}\left(\mathbf{J_{f,s}}\right)^2 - 4\,\det\left(\mathbf{J_{f,s}}\right)} \tag{11}$$

where $\mathrm{tr}\left(\mathbf{J_{f,s}}\right) = -1 - \kappa_1 d + \kappa_2$ is the trace of $\mathbf{J_{f,s}}$ in Eq. (9).

4.2 Restriction to the Center Manifold

In the case $\det\left(\mathbf{J_{f,s}}\right) = 0$ and $\mathrm{tr}\left(\mathbf{J_{f,s}}\right) \neq 0$ it follows from Eq. (11) that the non-zero eigenvalue of $\mathbf{J_{f,s}}$ is $\lambda_1 = \mathrm{tr}\left(\mathbf{J_{f,s}}\right)$. Eq. (10) is satisfied when either of the following applies:

1. $\kappa_1 = \kappa_2 = 0$
2. $d = -\frac{\kappa_1+\kappa_2}{\kappa_1\cdot\kappa_2}$ and $\kappa_1 \neq 0$, $\kappa_2 \neq 0$

Following the procedure in Kuznetsov [10] to obtain the parameter-dependent system restricted to its center manifold we will first extend the dynamical system artificially to account for the control parameters x, y and φ resulting in

$$\dot{\mathbf{p}} = \mathbf{g}\left(\mathbf{p}\right) \tag{12}$$

where

$$\begin{aligned}\mathbf{p} &= \begin{bmatrix} s_1 \; s_2 \; x \; y \; \varphi \end{bmatrix}^\mathsf{T} \\ \mathbf{g} &= \begin{bmatrix} \mathbf{f}^\mathsf{T} \; 0 \; 0 \; 0 \end{bmatrix}^\mathsf{T} .\end{aligned}$$

Next, Eqs. (12) are expanded into a Taylor-series at the origin[2] $\mathbf{p} = \mathbf{0}$ and a linear transformation is applied such that the linear terms take Jordan canonical form. After substitution of $\mathbf{f}$ by Eq. (3) the Jacobian of Eqs. (12) in case 1 is given by

$${}_1\mathbf{J_g} = \begin{bmatrix} -1 & -1 & t_1^x & t_1^y & \frac{\partial \mathbf{c}_2}{\partial \varphi}\cdot \mathbf{t}_1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{13}$$

[2] without loss of generality it is assumed that the fixed point $\mathbf{p}_0$ is at the origin. A simple change of coordinates $\mathbf{p} = \tilde{\mathbf{p}} + \mathbf{p}_0$ is applied if this is not the case.

having the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 0$ with algebraic multiplicity four and geometric multiplicity three. For case 2 we get

$$
{}_2\mathbf{J_g} = \begin{bmatrix} \frac{\kappa_1}{\kappa_2} & -1 & t_1^x & t_1^y & \frac{\partial \mathbf{c}_2}{\partial \varphi} \cdot \mathbf{t}_1 \\ -\kappa_1 & \kappa_2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{14}
$$

with the eigenvalues $\lambda_1 = \left(\kappa_1 + {\kappa_2}^2\right)/\kappa_2$ and also $\lambda_2 = 0$ with algebraic multiplicity four and geometric multiplicity three. Thus the real Jordan form of Eq. (13) and Eq. (14) is given by

$$
\hat{\mathbf{J}}_\mathbf{g} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}
$$

Hence, after applying the transformation $\mathbf{p} = \mathbf{P}\hat{\mathbf{p}}$ the dynamical system of Eqs. (12) takes the form

$$
\dot{\hat{\mathbf{p}}} = \hat{\mathbf{J}}_\mathbf{g}\hat{\mathbf{p}} + \mathbf{P}^{-1}\left[\sum_{i=2}^{\infty} \mathbf{g_i}\left(\mathbf{P}\hat{\mathbf{p}}\right)\right] = \hat{\mathbf{J}}_\mathbf{g}\hat{\mathbf{p}} + \sum_{i=2}^{\infty} \hat{\mathbf{g}}_\mathbf{i}\hat{\mathbf{p}}\,, \tag{16}
$$

where $\hat{\mathbf{p}}$ are the new coordinates, the $\mathbf{g_i}$ collect the terms of i-th order of $\mathbf{g}$ and $\mathbf{P}$ is the transformation matrix to Jordan normal form that differs for case 1 and 2.

A significant simplification of (16) can be achieved by (recursively) applying the following relations to the Taylor-coefficients $\mathbf{g}$ and $\mathbf{g_i}$:

$$
\begin{aligned}
&\frac{\partial \mathbf{c}_1(s_1)}{\partial s_1} &&= \mathbf{t}_1(s_1) && \frac{\partial \mathbf{c}_2(s_2,\mathbf{r},\varphi)}{\partial s_2} &&= \mathbf{t}_2(s_2,\varphi) \\
&\frac{\partial \mathbf{t}_1(s_1)}{\partial s_1} &&= -\kappa_1(s_1)\,\mathbf{n}_1(s_1) && \frac{\partial \mathbf{n}_1(s_1)}{\partial s_1} &&= \kappa_1(s_1)\,\mathbf{t}_1(s_1) \\
&\frac{\partial \mathbf{t}_2(s_2,\varphi)}{\partial s_2} &&= -\kappa_2(s_2)\,\mathbf{n}_2(s_2,\varphi) && \frac{\partial \mathbf{n}_2(s_2,\varphi)}{\partial s_2} &&= \kappa_2(s_2)\,\mathbf{t}_2(s_2,\varphi) \\
&\frac{\partial \mathbf{c}_2(s_2,\mathbf{r},\varphi)}{\partial x} &&= \begin{bmatrix}1 & 0\end{bmatrix}^{\mathsf{T}} && \frac{\partial \mathbf{c}_2(s_2,\mathbf{r},\varphi)}{\partial y} &&= \begin{bmatrix}0 & 1\end{bmatrix}^{\mathsf{T}} \\
&\frac{\partial \mathbf{t}_2(s_2,\varphi)}{\partial \varphi} &&= -\mathbf{n}_2(s_2,\varphi) && \frac{\partial \mathbf{n}_2(s_2,\varphi)}{\partial \varphi} &&= \mathbf{t}_2(s_2,\varphi) \\
&\frac{\partial^3 \mathbf{c}_2}{\partial \varphi^3}(s_2,\varphi) &&= -\frac{\partial \mathbf{c}_2}{\partial \varphi}(s_2,\varphi) && \frac{\partial^4 \mathbf{c}_2}{\partial \varphi^4}(s_2,\varphi) &&= (\mathbf{c}_2-\mathbf{r})(s_2,\varphi)
\end{aligned}
\tag{17}
$$

The first row are the definitions of the tangent vectors, the next two rows are the Frenet formulas [11] modified to match our definition of the normal vector, the fourth row is explained by Eq. (2), and the last two rows are derived from Eq. (2) considering the fact that the k-th derivative of a 2d-rotation matrix with respect to its parameter φ describes a rotation by $\varphi + k\cdot\pi/2$. As clearly seen from Fig. 2, additionally the following holds at the extremal point $\mathbf{p} = \mathbf{0}$ with $i, j = 1,2$ and $i \neq j$:

$$
\begin{aligned}
&\mathbf{t}_i\cdot\mathbf{n}_j = 0 \quad && \mathbf{t}_j\cdot\mathbf{t}_j = 1 \quad && \mathbf{t}_i\cdot\mathbf{t}_j = -1 \quad && \mathbf{n}_j\cdot\mathbf{n}_j = 1 \quad && \mathbf{n}_i\cdot\mathbf{n}_j = -1 \\
&\mathbf{c}_2-\mathbf{c}_1 = \mathbf{d} && \mathbf{t}_i\cdot\mathbf{d} = 0 && \mathbf{n}_1\cdot\mathbf{d} = d && \mathbf{n}_2\cdot\mathbf{d} = -d &&
\end{aligned}
\tag{18}
$$

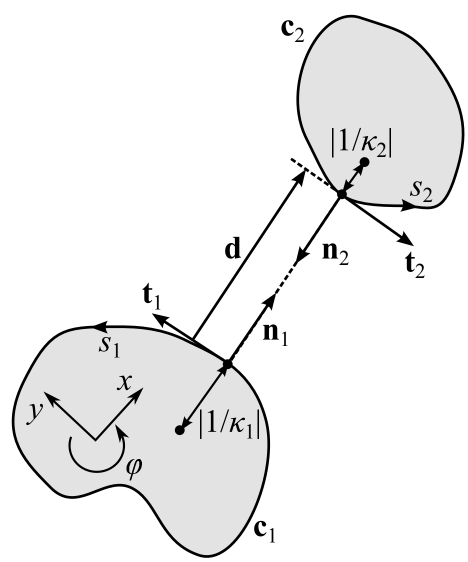

Fig. 2. Boundary curves and quantities at extremal points.

According to Kuznetsov [10] the system defined by Eqs. (12) exhibits the same bifurcation behaviour on its invariant center manifold[3], but the problem of studying the dynamics is reduced to analyzing a one-dimensional system, as the Jacobian of

[3]the Reduction Principle states that the original system is locally topologically equivalent to a system consisting of the restriction to the center manifold and only the linear terms of the remaining equations, which are decoupled.

the original system $\mathbf{f}$, without accounting for the parameter dependency, has a single critical eigenvalue[4]. Unfortunately, the computation of an exact center manifold equation $\hat{p}_1 = h(\hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5)$ is not possible in general. But if we choose a polynomial ansatz

$$\hat{p}_1 = h(\hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5) \approx \sum_{\iota_2,\iota_3,\iota_4,\iota_5=0} \left(k_{\iota_2\iota_3\iota_4\iota_5} \prod_{i=2}^{5} \hat{p}_i{}^{\iota_i} \right) , \quad 2 \leq \sum_{i=2}^{5} \iota_i \leq N \tag{19}$$

where $\hat{p}_i$ is the i-th component of vector $\hat{\mathbf{p}}$, it can be approximated up to any order N. According to [9] h must satisfy the quasilinear partial differential equation

$$\frac{\partial h}{\partial \hat{p}_2} \cdot \dot{\hat{p}}_2 (h, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5) - \dot{\hat{p}}_1 (h, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5) = 0 \tag{20}$$

for its graph to be an invariant center manifold, where $\dot{\hat{p}}_1$ and $\dot{\hat{p}}_2$ are the first respectively second equation of Eqs. (16). The coefficients k in Eqs. (19) are found by equating the coefficient of each monomial in Eq. (20) to zero which results in a nonlinear system of equations. For higher-order approximations this is a tedious procedure. But fortunately, for the dynamical system restricted to the center manifold, which is the second row of Eqs. (16) with the substitution Eq. (19), we have to compute only the Lyapunov values c_n of the monomials $\hat{p}_2{}^n$ to determine the codimension of the bifurcation, and hence its type, and the coefficients c_{nm} of the monomials $\hat{p}_m \hat{p}_2{}^n$, $m = 3 \ldots 5$ to investigate the unfolding of the bifurcation if the family given by Eqs. (3) is in general position with respect to the bifurcation surface, as shown in Shilnikov [12]. Hence we write the dynamical system restricted to the center manifold in the form of Eq. (21), where ψ_h collects all the remaining terms.

$$\dot{\hat{p}}_2 = \psi_2 (\hat{p}_2) + \sum_{m=3}^{5} \hat{p}_m \psi_m (\hat{p}_2) + \psi_h (\hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5) \tag{21}$$

with

$$\begin{aligned} \psi_2 (\hat{p}_2) &= \sum_{n=2}^{\infty} c_n \cdot \hat{p}_2{}^n \\ \psi_m (\hat{p}_2) &= \sum_{n=0}^{\infty} c_{nm} \cdot \hat{p}_2{}^n, \qquad c_{03} = 1,\ c_{04} = c_{05} = 0 \end{aligned}$$

In the following we derive a much more efficient method for the computation of the required coefficients. Only considering the pure $\hat{p}_2{}^n$ terms, i.e. terms not containing control parameters as factors in Eq. (16) and Eq. (19), we get

[4] critical eigenvalue refers to eigenvalues λ with $\mathrm{Re}(\lambda) = 0$

$$
\begin{aligned}
\psi_2 &= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\left(b_{j_1 j_2}\hat{p}_2{}^{j_2}\left(\sum_{\eta=2}^{\infty}k_\eta \hat{p}_2{}^{\eta}\right)^{j_1}\right)\\
&= \sum_{j_1,j_2=0}^{\infty}\left[b_{j_1 j_2}\,\hat{p}_2{}^{j_2}\left(\sum_{l_1+l_2+\ldots=j_1}\left[\binom{j_1}{l_1,l_2,\ldots}\prod_{\eta=2}^{\infty}\left(k_\eta\,\hat{p}_2{}^{\eta}\right)^{l_\eta}\right]\right)\right]
\end{aligned}
\tag{22}
$$

where $b_{j_1 j_2}$ are the Taylor-coefficients of the terms $\hat{p}_1{}^{j_1}\hat{p}_2{}^{j_2}$ in $\dot{\hat{p}}_2$ and k_η are the constants $k_{\eta 000}$ from Eq. (19). The last expression was derived by applying the Multinomial Theorem and from Eq. (19) we recognize that $k_1 = 0$ and thus $l_1 = 0$[5]. Analogously we can compute the expression

$$
\begin{aligned}
\psi_m =& \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\left(b_{j_1 j_2 m}\hat{p}_2{}^{j_2}\left(\sum_{\eta=2}^{\infty}k_\eta \hat{p}_2{}^{\eta}\right)^{j_1}\right)+\\
&\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\left(j_1 b_{j_1 j_2}\hat{p}_2{}^{j_2}\left(\sum_{\nu=1}^{\infty}k_{\nu m}\hat{p}_2{}^{\nu}\right)\left(\sum_{\eta=2}^{\infty}k_\eta \hat{p}_2{}^{\eta}\right)^{j_1-1}\right)\\
=& \sum_{j_1,j_2=0}^{\infty}\hat{p}_2{}^{j_2}\left[b_{j_1 j_2 m}\left(\sum_{l_1+l_2+\ldots=j_1}\left[\binom{j_1}{l_1,l_2,\ldots}\prod_{\eta=2}^{\infty}\left(k_\eta\,\hat{p}_2{}^{\eta}\right)^{l_\eta}\right]\right)+\right.\\
&\left. j_1 b_{j_1 j_2}\left(\sum_{\nu=1}^{\infty}k_{\nu m}\hat{p}_2{}^{\nu}\right)\left(\sum_{l_1+l_2+\ldots=j_1-1}\left[\binom{j_1-1}{l_1,l_2,\ldots}\prod_{\eta=2}^{\infty}\left(k_\eta\,\hat{p}_2{}^{\eta}\right)^{l_\eta}\right]\right)\right]
\end{aligned}
\tag{23}
$$

for the remaining relevant terms in Eq. (21), where $b_{j_1 j_2 m}$ are the Taylor-coefficients of the terms $\hat{p}_1{}^{j_1}\hat{p}_2{}^{j_2}\hat{p}_m$ in $\dot{\hat{p}}_2$ and $k_{\nu m}$ are the constants $k_{\nu\iota_3\iota_4\iota_5}$ from Eq. (19) with ι_m set to one and the ι_k, $k = [3,4,5]\setminus m$ set to zero. The first term in Eq. (23) originates from the terms that are already linear in the control parameter $\hat{p}_m$ in Eq. (16) and do not contain powers of the other control parameters. In the substituted center manifold approximation given by Eq. (19), only the terms which do not contain control parameters as factors are accounted for, as all others would destroy the linearity in the control parameters when multiplied with $\hat{p}_2{}^{j_2}\hat{p}_m$ and hence must be considered in ψ_h. The second term in Eq. (23) originates from the terms which do not contain any control parameter $\hat{p}_m, m = 3\ldots5$ in Eq. (16). In this case the linear dependency on one of the control parameters is established through the substitution given by Eq. (19). The two factor form substituted for $\hat{p}_1$ is the portion of the center manifold approximation that contains only the terms linear[6] in $\hat{p}_m$, all other terms are considered either in ψ_2 if not dependent on any control parameter or in ψ_h if a nonlinear dependency on the control parameters exists.

The first equation in Eq. (16) is stated in an analog form after the center manifold approximation has been substituted:

$$
\dot{\hat{p}}_1 = \zeta_2\left(\hat{p}_2\right)+\sum_{m=3}^{5}\hat{p}_m\,\zeta_m\left(\hat{p}_2\right)+\zeta_h\left(\hat{p}_2,\hat{p}_3,\hat{p}_4,\hat{p}_5\right)
\tag{24}
$$

with

[5] as the product in the last expression of Eq. (22) must not disappear for $k_1 = 0$, $l_1 = 0$ follows.
[6] note that $\hat{p}_m$ has been factored out and does not appear in Eq. (23) but can be seen in Eq. (21)

$$\zeta_2(\hat{p}_2) = \sum_{n=2}^{\infty} d_n \cdot \hat{p}_2{}^n$$
$$\zeta_m(\hat{p}_2) = \sum_{n=1}^{\infty} d_{nm} \cdot \hat{p}_2{}^n$$

In Eq. (24) the functions ζ_2 and ζ_m have the same form as in Eq. (22) and Eq. (23) respectively. Only the coefficients b are replaced by the coefficients a indicating that they denote now the corresponding Taylor-coefficients in $\dot{\hat{p}}_1$, i.e. the first row of Eqs. (16).

For the coefficients $a_{j_1 j_2}$, $b_{j_1 j_2}$, $a_{j_1 j_2 m}$ and $b_{j_1 j_2 m}$ the following holds which is easily recognized from Eq. (15):

$$\begin{aligned} a_{00} = a_{01} = b_{00} = b_{10} = b_{01} &= 0 \\ a_{003} = a_{004} = a_{005} = b_{004} = b_{005} &= 0 \\ b_{003} &= 1 \end{aligned}$$

In the following we present a procedure to compute the coefficients c_n and c_{nm} in Eq. (21).

4.2.1 Coefficients of the $\hat{p}_2{}^n$ monomials

The vector field defined by Eqs. (16) restricted to the center manifold takes the form of Eq. (21). First we determine the coefficient c_N for the order N term from Eq. (22).

1. find all R combinations of integers $j_2, l_\eta \geq 0$ with $\eta = 2 \ldots N-1$ that satisfy the condition for the N-th order term[7]:

$$j_{2,r} + \sum_{\eta=2}^{N-1} (\eta \cdot l_{\eta,r}) = N\,, \qquad r = 1 \ldots R$$

   This is a linear Diophantine equation that can be solved by brute-force[8] or more efficiently by e.g. the algorithm presented in [13].

2. determine j_1 for each combination:

$$j_{1,r} = \sum_{\eta=2}^{N-1} l_{\eta,r}$$

3. the coefficient $c_{N,r}$ for combination r is:

[7] the combinations with $l_1 \neq 0$ need not be considered, as $k_1 = 0$ and thus the corresponding c_n vanishes.
[8] i.e. by trying all possible values of $j_2 \leq N$ and $l_i \leq N/i$.

$$c_{N,r} = b_{j_{1,r} j_{2,r}} \binom{j_{1,r}}{l_{2,r}, l_{3,r} \dots} \prod_{\eta=2}^{N-1} k_\eta{}^{l_{\eta,r}}$$

4. summation of $c_{N,r}$ yields c_N:

$$c_N = \sum_{r=1}^{R} c_{N,r}$$

The above equation still contains the unknowns k_η, $\eta = 2 \dots N-1$ and we will show later how to compute these.

Next, we determine the coefficient c_{NM} of the term linear in $\hat{p}_M$, where $3 \leq M \leq 5$, and of N-th order in $\hat{p}_2$.

4.2.2 Coefficients of the $\hat{p}_m \hat{p}_2{}^n$ monomials

1. find all R_1, R_2 combinations of integers $j_2, \nu, l_\eta \geq 0$ that satisfy the conditions for the $N+1$-st order term:

$$\begin{aligned} j_{2,r_1} + \sum_{\eta=2}^{N} (\eta \cdot l_{\eta,r_1}) &= N\,, \qquad r_1 = 1 \dots R_1 \\ j_{2,r_2} + \nu_{r_2} + \sum_{\eta=1}^{N} (\eta \cdot l_{\eta,r_2}) &= N\,, \qquad r_2 = 1 \dots R_2 \end{aligned}$$

These are again linear Diophantine equations.

2. determine j_1 for each combination:

$$\begin{aligned} j_{1,r_1} &= \sum_{\eta=2}^{N} l_{\eta,r_1} \\ j_{1,r_2} &= \sum_{\eta=2}^{N} l_{\eta,r_2} + 1 \end{aligned}$$

3. the coefficient $c_{NM,r}$ for combination r is:

$$\begin{aligned} c_{1,NM,r_1} &= b_{j_{1,r_1} j_{2,r_1} M} \binom{j_{1,r_1}}{l_{2,r_1}, l_{3,r_1} \dots} \prod_{\eta=2}^{N} k_\eta{}^{l_{\eta,r_1}} \\ c_{2,NM,r_2} &= b_{j_{1,r_2} j_{2,r_2}} k_{\nu_{r_2} M} \binom{j_{1,r_2} - 1}{l_{2,r_2}, l_{3,r_2} \dots} \prod_{\eta=2}^{N} k_\eta{}^{l_{\eta,r_2}} \end{aligned}$$

4. summation of $c_{NM,r}$ yields c_{NM}:

$$c_{NM} = \sum_{r_1=1}^{R_1} c_{1,NM,r_1} + \sum_{r_2=1}^{R_2} c_{2,NM,r_2}$$

The above equation still contains the unknowns k_η, $\eta = 2\dots N$ and $k_{\nu_{r_2}M}$ and we will show next how to compute these.

4.2.3 Constants k_i

The coefficients k_n of the monomials $\hat{s}_2{}^n$ appearing in the center manifold approximation given by Eq. (19) that are required to compute c_N are found by inspection of Eq. (20) with the substitutions stated in Eq. (21) and Eq. (24). In general, the coefficients k_2 through k_N are required for the determination of a $N+1$-order term's coefficient. We follow a recursive procedure computing k_i starting with $i = 2$, where each iteration follows an analog procedure as above.

1. for the first term of Eq. (20) find all R_1 combinations of integers $\iota_2 \geq 2,\ j_2, l_\eta \geq 0$ where ι_2 is the corresponding exponent from Eq (19) that satisfies the condition for the N-th order term:

$$\iota_{2,r_1} - 1 + j_{2,r_1} + \sum_{\eta=2}^{N-2} (\eta \cdot l_{\eta,r_1}) = N\,, \qquad r_1 = 1\dots R_1$$

   for the second term of (20) we find all R_2 combinations of integers $j_2, l_\eta \geq 0$ that satisfy the condition for its N-th order term:

$$j_{2,r_2} + \sum_{\eta=2}^{N} (\eta \cdot l_{\eta,r_2}) = N\,, \qquad r_2 = 1\dots R_2$$

   These are again linear Diophantine equations that can be solved as previously described.

2. determine j_1 for each combination separately for the first and second term:

$$\begin{aligned} j_{1,r_1} &= \sum_{\eta=2}^{N-2} l_{\eta,r_1} \\ j_{1,r_2} &= \sum_{\eta=2}^{N} l_{\eta,r_2} \end{aligned}$$

3. the coefficients d_{1,N,r_1} for combination r_1 in the first term of Eq. (20) is:

$$d_{1,N,r_1} = k_{\iota_{2,r_1}} \iota_{2,r_1} b_{j_{1,r_1} j_{2,r_1}} \binom{j_{1,r_1}}{l_{2,r_1}, l_{3,r_1}, \dots} \prod_{\eta=2}^{n-1} k_\eta{}^{l_{\eta,r_1}}$$

   and the coefficients d_{2,N,r_2} for combination r_2 in the second term of Eq. (20) is:

$$d_{2,N,r_2} = a_{j_{1,r_2} j_{2,r_2}} \binom{j_{1,r_2}}{l_{2,r_2}, l_{3,r_2}, \dots} \prod_{\eta=2}^{n} k_\eta{}^{l_{\eta,r_2}}$$

4. summation of d_{1,N,r_1} and d_{2,N,r_2} yields the overall coefficients for the N-th order term and the condition to solve for the

unknown k_N, using the results $k_n,\ n = 1 \dots N-1$ from the previous iterations:

$$\sum_{r_1=1}^{R_1} d_{1,N,r_1} - \sum_{r_2=1}^{R_2} d_{2,N,r_2} = 0$$

4.2.4 Constants k_{im}

A procedure analogous to the above results in the following expressions.

1.

$$\begin{aligned}
j_{2,r_1} + \sum_{\eta=2}^{N} (\eta \cdot l_{\eta,r_1}) = N\,, \qquad & r_1 = 1 \dots R_1 \\
j_{2,r_2} + \nu_{r_2} + \sum_{\eta=2}^{N} (\eta \cdot l_{\eta,r_2}) = N\,, \qquad & r_2 = 1 \dots R_2 \\
\iota_{2,r_3} - 1 + j_{2,r_3} + \sum_{\eta=2}^{N} (\eta \cdot l_{\eta,r_3}) = N\,, \qquad & r_3 = 1 \dots R_3 \\
\iota_{2,r_4} - 1 + j_{2,r_4} + \nu_{r_4} + \sum_{\eta=2}^{N} (\eta \cdot l_{\eta,r_4}) = N\,, \qquad & r_4 = 1 \dots R_4
\end{aligned}$$

2.

$$\begin{aligned}
j_{1,(r_1,r_3)} &= \sum_{\eta=2}^{N} l_{\eta,(r_1,r_3)} \\
j_{1,(r_2,r_4)} &= \sum_{\eta=2}^{N} l_{\eta,(r_2,r_4)} + 1
\end{aligned}$$

3.

$$\begin{aligned}
d_{1,NM,r_1} &= a_{j_{1,r_1} j_{2,r_1} M} \binom{j_{1,r_1}}{l_{2,r_1}, l_{3,r_1} \dots} \prod_{\eta=2}^{N} k_\eta{}^{l_{\eta,r_1}} \\
d_{2,NM,r_2} &= j_{1,r_2} a_{j_{1,r_2} j_{2,r_2}} k_{\nu_{r_2} M} \binom{j_{1,r_2} - 1}{l_{2,r_2}, l_{3,r_2} \dots} \prod_{\eta=2}^{N} k_\eta{}^{l_{\eta,r_2}} \\
d_{3,NM,r_3} &= \iota_{2,r_3} k_{\iota_{2,r_3} M} b_{j_{1,r_3} j_{2,r_3}} \binom{j_{1,r_3}}{l_{2,r_3}, l_{3,r_3} \dots} \prod_{\eta=2}^{N} k_\eta{}^{l_{\eta,r_3}} + \\
&\qquad \iota_{2,r_3} k_{\iota_{2,r_3}} b_{j_{1,r_3} j_{2,r_3} M} \binom{j_{1,r_3}}{l_{2,r_3}, l_{3,r_3} \dots} \prod_{\eta=2}^{N} k_\eta{}^{l_{\eta,r_3}} \\
d_{4,NM,r_4} &= \iota_{2,r_4} k_{\iota_{2,r_4}} j_{1,r_4} b_{j_{1,r_4} j_{2,r_4}} k_{\nu_{r_4} M} \binom{j_{1,r_4} - 1}{l_{2,r_4}, l_{3,r_4} \dots} \prod_{\eta=2}^{N} k_\eta{}^{l_{\eta,r_4}}
\end{aligned}$$

4. solve the following equation for k_{NM} using the previous results for $k_{nM},\ n = 1 \dots N-1$ and $k_n,\ n = 1 \dots N$:

$$-\sum_{r_1=1}^{R_1} d_{1,NM,r_1} - \sum_{r_2=1}^{R_2} d_{2,NM,r_1} + \sum_{r_3=1}^{R_3} d_{3,NM,r_1} + \sum_{r_4=1}^{R_4} d_{4,NM,r_1} = 0$$

The procedure described above allows the symbolic computation of the coefficients c_N and c_{NM} utilizing a Computer Algebra System on a standard PC up to some significant order N.

4.3 Bifurcation Types

By knowing the coefficients c_n of the terms without control parameters and c_{nm} of terms in which the control parameters appear only linearly, we can investigate the bifurcation behavior of the solutions to Eqs. (3) when they become degenerate. The degree of degeneracy depends on the number of degeneracy conditions satisfied. If Eq. (10) holds it is not sufficient to examine the linear terms of the system's Taylor expansion at the solution in order to determine its stability. This is one degeneracy condition. Further conditions of degeneracy are a vanishing trace of the system's Jacobian given by Eq. (9) and zero Lyapunov values up to an order $N \geq 2$ [12].

The basic type of the bifurcation is determined by the degeneracy conditions satisfied, but how the bifurcation is unfolded depends on the control parameters of the system defined by Eqs. (3). A sufficient condition for a versal unfolding to exist is that the family is in general position with respect to the bifurcation surface, i.e. the matrix

$$\left.\left[\begin{array}{ccc} \frac{\partial \hat{p}_2}{\partial \hat{p}_3} & \cdots & \frac{\partial \hat{p}_2}{\partial \hat{p}_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial^z \hat{p}_2}{\partial \hat{p}_3 \partial \hat{p}_2^{z-1}} & \cdots & \frac{\partial^z \hat{p}_2}{\partial \hat{p}_M \partial \hat{p}_2^{z-1}} \end{array}\right]\right|_{\hat{p}_2 \cdots \hat{p}_M = 0} = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & \ldots \\ c_{13} & c_{14} & c_{15} & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \\ c_{(z-1)3} & c_{(z-1)4} & c_{(z-1)5} & 0 & \ldots \end{array}\right] \tag{25}$$

must have maximal rank in the case of one zero eigenvalue of the Jacobian given by Eq. (9) according to [12]. The degree of degeneracy[9] is denoted by z and the number of control parameters is $M-2$. Since there are only three control parameters in our problem, bifurcations at solutions with $z > 3$ are not versally unfolded in general.

The expressions for c_{nm} in Eq. (21) simplify considerably if the y-axis of body 1's coordinate system is parallel to the common normal at the extremal point and the origin of body 2's coordinate system is located on the boundary of body 2 at the extremal point. Then terms with factors $\partial c_2/\partial \varphi$ and many of the terms containing components of **t** and **n** vanish. This is always permitted since the solution behavior must not depend on the choice of the bodies' coordinate systems. However, the analytic expression for c_n and c_{nm} are still too lengthy to be stated here and thus only special cases and examples are shown in the following.

4.3.1 Bifurcations at Points of Vanishing Curvature

The coefficients of the monomials $\hat{p}_2{}^N$ and $\hat{p}_M \hat{p}_2{}^N$ have been computed for the case $\kappa_1 = \kappa_2 = 0$ by applying the procedure described above up to sixth order terms.

It is found that the rank of the upper-left 3×3 submatrix of Eq. (25) is at most 2. After applying above mentioned coordinate transformations it can be shown that $c_{24} = c_{25}$ and $c_{34} = c_{35}$ which proves this fact. Hence bifurcations at

[9] or codimension

solutions with $z > 2$ generally are not versally unfolded by the given control parameters.

Compact expressions[10] for the Lyapunov values and the coefficients c_{nm} can be derived under the condition $\frac{\partial^i \kappa_j}{\partial {s_j}^i} = 0,\ j = 1\ldots 2,\ i = 1\ldots N-2$ for $N \geq 3$:

$$c_N = \frac{1}{N!}\left((-1)^{N-1}\frac{\partial^{N-1}\kappa_1}{\partial {s_1}^{N-1}} + \frac{\partial^{N-1}\kappa_2}{\partial {s_2}^{N-1}}\right) \tag{26}$$

$$c_{N3} = \frac{1}{N!}\left((-1)^{N-1}\frac{\partial^{N}\kappa_1}{\partial {s_1}^{N}} + \left(\frac{\partial \mathbf{c}_2}{\partial \varphi}\cdot \mathbf{t}_1 - 1\right)\frac{\partial^{N}\kappa_2}{\partial {s_2}^{N}}\right) \tag{27}$$

$$c_{N4} = c_{N5} = -\frac{1}{N!}\frac{\partial^{N}\kappa_2}{\partial {s_2}^{N}} \tag{28}$$

Equations (27) and (28) yield that the matrix in Eq. (25) has now only one non-zero entry, as all $c_{ji},\ i = 3\ldots 5$ vanish. An infinite codimension degeneracy occurs, if the two boundaries $\mathbf{c}_1$ and $\mathbf{c}_2$ are locally flat. Because $\frac{\partial^i \kappa_j}{\partial {s_j}^i} = 0,\ j = 1\ldots 2,\ i = 1\ldots\infty$ all $c_N = 0$ due to Eq. (26) and there is an infinitely large number of solutions.

4.3.2 Bifurcations in Concave Regions

Unlike in the case of vanishing curvatures, in the case $d = -\frac{\kappa_1+\kappa_2}{\kappa_1\cdot\kappa_2}$ and $\kappa_1 \neq 0, \kappa_2 \neq 0$ the upper-left 3×3 submatrix of Eq. (25) can have full rank. However, there also exists a concise formulation for c_n and c_{nm} if $\frac{\partial^i \kappa_j}{\partial {s_j}^i} = 0,\ j = 1\ldots 2,\ i = 1\ldots N-2$ for $N \geq 3$ is satisfied:

$$c_N = \frac{{\kappa_2}^{N-1}}{N!\,\kappa_1\left(\kappa_1+{\kappa_2}^2\right)^N}\left({\kappa_2}^{N+1}\frac{\partial^{N-1}\kappa_1}{\partial {s_1}^{N-1}} + {\kappa_1}^{N+1}\frac{\partial^{N-1}\kappa_2}{\partial {s_2}^{N-1}}\right) \tag{29}$$

$$c_{N5} = \frac{{\kappa_2}^{2N}}{N!\,{\kappa_1}^2\left(\kappa_1+{\kappa_2}^2\right)^N}\frac{\partial^{N}\kappa_1}{\partial {s_1}^{N}} \tag{30}$$

It can be recognized from Eq. (30) that the third column vector in matrix (25) is now a zero-vector and hence the rank of this matrix cannot exceed 2. A special case of degeneracy occurs, if the two boundary curves $\mathbf{c}_1$ and $\mathbf{c}_2$ are locally concentric circles. Because all derivatives of their curvatures vanish, the codimension and the number of solutions becomes infinite.

[10]it is assumed, although not proven, that these hold up to arbitrary order.

4.3.3 Other Types of Bifurcations

In general there are bifurcations other than the ones described above that can occur in two-dimensional dynamical systems. These are not relevant for our investigation of the fixed points of Eqs. (3), because

1. Hopf Bifurcations that occur when the Jacobian given by Eq. (9) has two purely complex eigenvalues are associated with the creation and annihilation of limit cycles rather than fixed points.
2. Takens-Bogdanov Bifurcations occur when additionally to $\det(\mathbf{J_{f,s}}) = 0$ the degeneracy condition $\mathrm{tr}(\mathbf{J_{f,s}}) = 0$ is satisfied, but the LHS of Eq. (9) is not a matrix of zeros. The latter condition yields $\kappa_1 = -\kappa_2{}^2$. Even if this condition is satisfied, due to its asymmetric nature swapping the indices of bodies 1 and 2 again yields the case of one zero exponent.
3. The Nilpotent case does not occur, as there is always at least one non-zero element in the matrix in Eq. (9).

4.4 Examples

In the examples below, it is always the case that the Jacobian in Eq. (9) has one zero eigenvalue as this is the only relevant condition for bifurcations of extremal points. All the boundaries[11] $\mathbf{c}_1$ and $\mathbf{c}_2$ of the bodies are described in coordinate systems that yield the simplest form of the expressions for the coefficients c_n and c_{nm} as described above. We adopt the following approach for the further discussion:

1. Compute the relevant Lyapunov values c_n and determine the codimension of the bifurcation.
2. Compute matrix (25) and decide whether the family given by Eq. (3) is in general position with respect to the bifurcation surface.
3. Apply the Tschirnhaus transformation [14] to the $(z+1)$-jet[12] of the system restricted to the center manifold to convert it into a depressed polynomial. This yields either a normal form of the bifurcation or a form that is suitable to investigate the subspace that is reached in the parameter-space of the versal unfolding of the general bifurcation type with the correponding codimension.

4.4.1 Vanishing Curvatures and Codimension 1

$$
\begin{aligned}
\mathbf{c}_1 &= -\begin{bmatrix} s_1 & s_1{}^4 \end{bmatrix}^\mathsf{T} \\
\mathbf{c}_2 &= \begin{bmatrix} s_2 & s_2{}^3 \end{bmatrix}^\mathsf{T}
\end{aligned} \tag{31}
$$

Clearly, at the point $s_1 = 0$, $s_2 = 0$, $\mathbf{r} = \mathbf{0}$ and $\varphi = 0$ the solution of Eq. (3) with Eq. (31) is degenerate as matrix defined by Eq. (9) has one zero eigenvalue. This follows from $\kappa_1 = \kappa_2 = 0$. The codimension of the bifurcation occuring at this point is one since the Lyapunov value $c_2 \neq 0$ as found from Eq. (26). The matrix given by Eq. (25) takes the form (32) and has full

[11] which are not necessarily parametrized by unit arc-length, since we will use only local properties such as curvatures which are independent from the actual parametrization.

[12] where z denotes the codimension

rank.

$$\begin{bmatrix} 1 \; 0 \; 0 \end{bmatrix} \tag{32}$$

Therefore we can find a transverse one parameter family

$$0 = \mu_0 + \tilde{p}_2{}^2 \approx \left(\frac{1}{3}\varphi - \frac{4}{3}\varphi x + \frac{4}{3}x^3\right) + \tilde{p}_2{}^2 \tag{33}$$

that is the normal form of a fold bifurcation [15] by applying the Tschirnhaus transformation to the system restricted to the center manifold and substitution of the original parameters defined by the transformation matrices $\mathbf{P}$ to obtain Jordan canonical form appearing in Eq. (16). From Eq. (33) it is clear that only by a variation of the control parameters φ and x the bifurcation surface is crossed, because y does not occur in the expression. The former case is shown in Fig. 3.

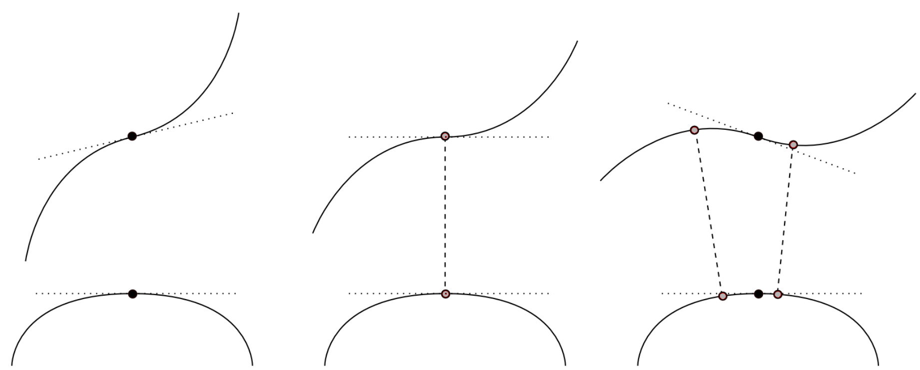

Fig. 3. Fold bifurcation at points of vanishing curvature. Left: no solution, middle: one degenerate solution, right: two regular solutions

4.4.2 Vanishing Curvatures and Codimension 2

$$\begin{aligned} \mathbf{c}_1 &= -\begin{bmatrix} s_1 & s_1{}^4 \end{bmatrix}^\mathsf{T} \\ \mathbf{c}_2 &= \begin{bmatrix} s_2 & s_2{}^4 \end{bmatrix}^\mathsf{T} \end{aligned} \tag{34}$$

Again, because $\kappa_1 = \kappa_2 = 0$, the solution of Eq. (3) with Eq. (34) at the point $s_1 = 0$, $s_2 = 0$, $\mathbf{r} = \mathbf{0}$ and $\varphi = 0$ becomes degenerate. The codimension of the bifurcation is 2 now since the Lyapunov value $c_2 = 0$ and $c_3 \neq 0$ and thus we get one additional degeneracy condition. The matrix in Eq. (25) has the form

$$\begin{bmatrix} 1\ 0\ 0 \\ 0\ 0\ 0 \end{bmatrix} \tag{35}$$

and the rank is only one. Hence, by applying the Tschirnhaus transformation to the system restricted to the center manifold and substitution of the original parameters we find a non-versal 'unfolding' of the bifurcation in a sufficiently small neighborhood of the degenerate point:

$$0 = \mu_0 + \mu_1 \tilde{p}_2 + \tilde{p}_2{}^3 \approx \frac{1}{8}\varphi + \left(\frac{3}{4}x^2 - \frac{3}{2}x\varphi + \frac{3}{4}\varphi^2\right)\tilde{p}_2 + \tilde{p}_2{}^3 \tag{36}$$

In this case there is no bifurcation, as $\mu_1 \geq 0$ and hence the number of solutions to Eq. (36) is always 1. Figure 4 illustrates the situation.

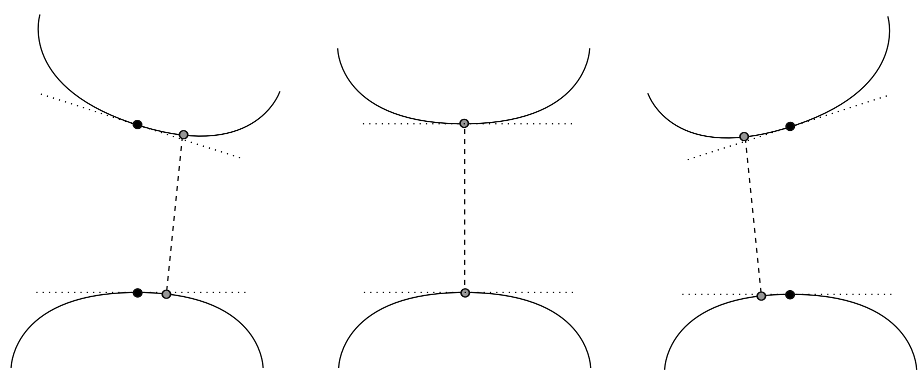

Fig. 4. Codim-2 degeneracy not unfolded at points of vanishing curvature. Left: one regular solution, middle: one degenerate solution, right: one regular solution

4.4.3 Vanishing Curvatures and Codimension 3

$$\begin{aligned} \mathbf{c}_1 &= -\begin{bmatrix} s_1 & s_1{}^3 + 3 \cdot s_1{}^4 \end{bmatrix}^{\mathsf{T}} \\ \mathbf{c}_2 &= \begin{bmatrix} s_2 & s_2{}^3 + 1.5 \cdot s_2{}^4 \end{bmatrix}^{\mathsf{T}} \end{aligned} \tag{37}$$

The solution of Eq. (3) with Eq. (37) is degenerate with codimension 3 at the origin. The additional degeneracy conditions are $c_2 = 0$ and $c_3 = 0$ and the non-degeneracy condition is $c_4 \neq 0$. Since the first derivative of the curvatures do not vanish, the simplified Eq. (26) cannot be used to compute the Lyapunov values. Matrix (25) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & -6 \\ 36 & -54 & -54 \end{bmatrix} \tag{38}$$

and the rank is two. The depressed system restricted to the center manifold with substitution of the original parameters describes a non-versal unfolding of a swallowtail bifurcation [15] described by

$$\begin{aligned} 0 &= \mu_0 + \mu_1 \tilde{p}_2 + \mu_2 \tilde{p}_2{}^2 + \tilde{p}_2{}^4 \\ &\approx -0.41x^2 - 0.0003y^4 + 0.02\varphi - 0.01xy + 1.15x\varphi + 0.01y\varphi + \\ &\quad \left(0.13x - 0.13\varphi - 0.24xy + 28.00x\varphi + 0.40y\varphi + 2.0 \cdot 10^{-9}y^2\right) \tilde{p}_2 + \\ &\quad \left(-1.2x - 0.06y^2 + 2\varphi - 4.32xy + 307.2x\varphi + 8y\varphi\right) \tilde{p}_2{}^2 + \tilde{p}_2{}^4 \end{aligned} \tag{39}$$

in a sufficiently small neighborhood of the degenerate point. The parameters $\mu_i,\ i = 0\ldots 2$ define a 3-space and the bifurcation surface of a swallowtail separates the regions with 0, 2 and 4 solutions. However, due to the nature of the mappings $\mu_i\,(x, y, \varphi)$, that describe a manifold with a 2-dimensional tangent space at the origin neither cutting the region of 4 solutions nor tangent to it, only a subspace in this parameter space can be reached in which Eq. (39) has either 0, 1, or 2 real solution.

4.4.4 Concave Region and Codimension 2

$$\begin{aligned} \mathbf{c}_1 &= -\begin{bmatrix} \sin(s_1) & \cos(s_1) \end{bmatrix}^\mathsf{T} \\ \mathbf{c}_2 &= \begin{bmatrix} s_2 & s_2{}^2 \end{bmatrix}^\mathsf{T} \end{aligned} \tag{40}$$

From Eq. (29) we find that the solution of Eq. (3) with Eq. (40) has a codimension 2 degeneracy at the point $s_1 = 0, s_2 = 0, x = 0, y = -0.5$ and $\varphi = 0$. The matrix (25) has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{4}{3} & -\frac{4}{3} & 0 \end{bmatrix} \tag{41}$$

and rank two. Thus the bifurcation can be versally unfolded and a normal form given by Eq. (42) is found by applying the Tschirnhaus transformation to the 3-jet of the system restricted to the center manifold. Substituting the original parameters yields

$$0 = \mu_0 + \mu_1 \tilde{p}_2 + \tilde{p}_2{}^3 \approx -\frac{27}{16}x + \frac{27}{32}\varphi + \frac{9}{4}y\tilde{p}_2 + \tilde{p}_2{}^3 \; . \tag{42}$$

The bifurcation of one solution into three is shown in Fig. 5.

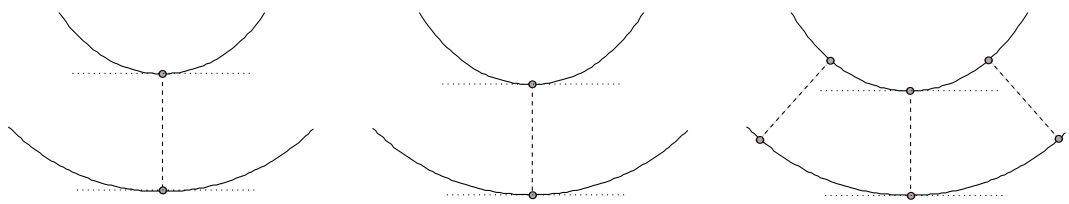

Fig. 5. Cusp bifurcation in concave region of the boundary. Left: one regular solution, middle: one degenerate solution, right: three regular solutions

4.4.5 Concave Region and Codimension 3

$$\begin{aligned} \mathbf{c}_1 &= \begin{bmatrix} -s_1 & 2 \cdot s_1{}^2 + 2 \cdot s_1{}^4 \end{bmatrix}^{\mathsf{T}} \\ \mathbf{c}_2 &= \begin{bmatrix} s_2 & 2 \cdot s_2{}^2 + 24 \cdot s_2{}^4 + 50 \cdot s_2{}^5 \end{bmatrix}^{\mathsf{T}} \end{aligned} \tag{43}$$

In this case the solution of Eq. (3) with Eq. (43) has a codimension 3 degeneracy at $s_1 = 0, s_2 = 0, x = 0, y = 0.25$ and $\varphi = 0$. The matrix (25) takes the form

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{16}{7} & -\frac{16}{7} & 0 \\ -\frac{320}{343} & \frac{192}{49} & \frac{192}{49} \end{bmatrix} \tag{44}$$

and has a rank of three and thus the family given by Eq. (3) is in general position with respect to the bifurcation surface. The normal form of the bifurcation is given by

$$\begin{aligned} 0 &= \mu_0 + \mu_1 \tilde{p}_2 + \mu_2 \tilde{p}_2{}^2 + \tilde{p}_2{}^4 \\ &\approx -\frac{2401}{16000}x + \frac{2401}{64000}\varphi + \frac{343}{1000}y\tilde{p}_2 + \left(\frac{91}{125}x - \frac{77}{100}\varphi\right)\tilde{p}_2{}^2 + \tilde{p}_2{}^4 \end{aligned} \tag{45}$$

and the solutions crossing the swallowtail point from the region with two to the region with four solutions is shown in Fig. 6

where the scale of the vertical to the horizontal axis is ten.

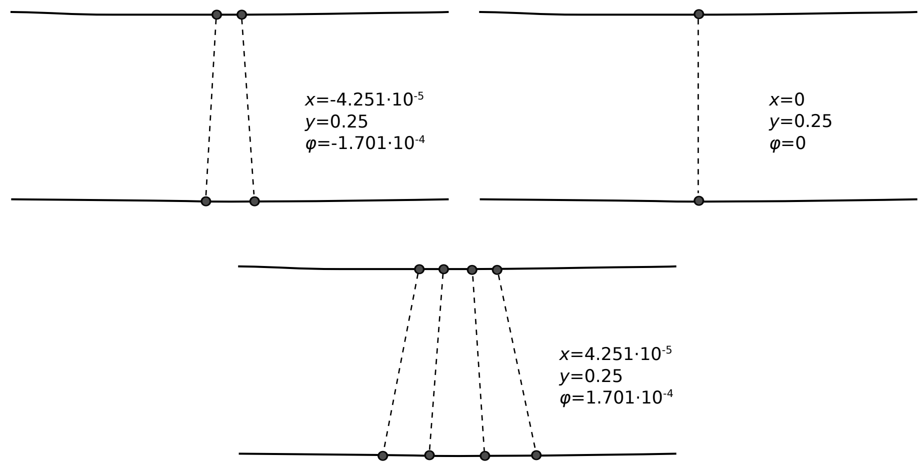

Fig. 6. Swallowtail bifurcation at concave segment. Upper left: two regular solutions, upper right: one degenerate solution, lower: four regular solutions

4.5 Bifurcation Events

When a singularity caused by the degeneration of a solution of the extremal point problem given by Eqs. (3) is encountered during the integration of the equations of motion (5) we are interested in the actual event that occurs. This enables e.g. the reformulation of Eqs. (5) to account for the post-event number of solutions if it is finite and thus restarting the integration just after the occurrence of the degeneracy.

A necessary condition for a bifurcation to occur is $\frac{\mathrm{d}\,\mathrm{Re}(\lambda)}{\mathrm{d}t} \neq 0$ at the degenerate point, where λ is the zero eigenvalue. Alternatively, it can be investigated if the surface defined in Eq. 10 and shown in Fig.1 is crossed transversely with non-zero velocity. The actual bifurcation type is then determined as demonstrated previously.

5 Conclusions

The method of augmenting the equations of motion of a plane multibody system with the extremal point conditions given by Eqs. (3) to keep track of the potential contact points is efficient and straightforward as long as no bifurcations of the solutions occur, i.e. the Jacobian (9) doesn't become singular. However, this can only be guaranteed if one of the bodies under investigation has a purely convex and the other a non-concave boundary. We investigated the more general case of arbitrary smooth boundary curves where degenerate solutions are possible. In order to identify the transition event that occurs when passing the degeneracy we considered the conditions stated in Eqs.(3) as a parameter-dependent dynamical system and restricted it to its invariant center manifold given by Eq. (21). Then we were able to determine the codimension of the degeneracy by computing the Lyapunov values c_n and investigated the transversality of the family defined by Eq. (21) with respect to the bifurcation surface by evaluating the rank of matrix (25).

The coefficients c_n and c_{nm} that appear in Eq. (21) and Eq. (25) depend only on geometrical properties of the boundaries and the distance between the potential contact points. But their determination involves solving a nonlinear system whose size depends on the order n of the corresponding monomial and hence their direct computation is not feasible for higher-order terms. We introduced a robust new method which is computationally less demanding and thus extends the applicability. As the expressions for c_n and c_{nm} do not explicitly depend on the actual boundary functions $\mathbf{c}_1(s_1)$ and $\mathbf{c}_2(s_2)$, they need to be computed exactly once.

Finally we were able to derive general statements about bifurcations of solutions to the plane extremal point problem involving smooth boundary curves. First, we showed that only divergence bifurcations with one zero eigenvalue are of interest in the context of collision detection problems. Furthermore, for extremal points with vanishing curvatures, the family given by Eq. (21) is never in general position with respect to the bifurcation surface if the codimension of the degeneracy is higher than 2. If the curvature at the solution points on the boundary curves differs from zero, degeneracies up to codimension 3 allow for transversal intersection[13]. In the case the derivatives of the curvatures with respect to the curve parameters vanish up to an order equal to the codimension of the degeneracy minus one (see Eqs. (26) through (30)), the family given by Eq. (21) is in general position for degeneracies with codimension 1 for the vanishing curvature case and maximal up to codimension 2 otherwise.

If this family is not in general position with respect to the bifurcation surface the unfolding of a bifurcation and the possible number of solutions after the transition event must be investigated by also inspecting the terms nonlinear in the control parameters. We introduced a method that utilizes the Tschirnhaus transformation for that purpose and we discussed several examples individually. However, the derivation of a generally applicable and simple method to classify the bifurcation types in such a case is still an open problem.

The relevance of the derived results besides the mathematical analysis of the structure of the singularities of the extremal point conditions defined by Eqs. (3) is, that they can be used to predict the complete solution behavior when passing any type of singularity that is related to the extremal point problem during time-integration of the equations of motion[14]. By adding a set of extremal point conditions for each new solution to the problem in the event of a bifurcation, the integration can be restarted with proper initial conditions and the solution of the multibody system can be further computed. As the expressions for the Lyapunov-coefficients c_n and the entries of matrix (25) have been derived in an explicit form, the computational effort to evaluate them is limited and thus they can also be used efficiently as test functions when applying numerical continuation techniques to track the extremal point solutions.

[13] and thus the existence of a versal unfolding of the bifurcation is guaranteed

[14] the creation of new solutions at a singular point where no solution existed before can be analyzed, but is not captured by applying Eq. (5)

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