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Lee-Yang Zeroes and Logarithmic Corrections in the Φ_4^4 Theory*

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The leading mean-field critical behaviour of ϕ_4^4-theory is modified by multiplicative logarithmic corrections. We analyse these corrections both analytically and numerically. In particular we present a finite-size scaling theory for the Lee-Yang zeroes and temperature zeroes, both of which exhibit logarithmic corrections. On lattices from size 8^4 to 24^4, Monte-Carlo cluster methods and multi-histogram techniques are used to determine the partition function zeroes closest to the critical point. Finite-size scaling behaviour is verified and the logarithmic corrections are found to be in good agreement with our analytical predictions.

1. INTRODUCTION

The single component version of ϕ^4 theory in the d-dimensional Euclidean space-time continuum is defined by the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(\nabla\phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{g_0}{4!}\phi^4 - H(x)\phi(x) \quad (1.1)$$

where $H(x)$ is the source for the fields $\phi(x)$. The lattice parameterization of the theory (in the absence of a source field) is given by the action

$$-\kappa\sum_{x,\mu}\phi_x\phi_{x+\mu} + \sum_x \phi_x^2 + \lambda\sum_x\left(\phi_x^2-1\right)^2. \quad (1.2)$$

Here the hopping parameter κ and the quartic coefficient λ correspond to the bare mass m_0 and bare quartic coupling g_0 respectively. The limit $\lambda\to\infty$ gives the Ising model.

Above one dimension the discretized theory exhibits a phase transition (of second order) near which the continuum theory can be recovered. To remove the cutoff, it turns out that the quartic coupling has to be taken to the infra–red fixed point (IR FP) g_R^*. The theory is believed to be trivial in $d=4$ — although this has never been rigorously proved. This means that it is in the universality class of the theory of free bosonic fields. The leading (mean field) scaling behaviour is modified by logarithmic corrections, which are linked to the triviality of the theory [1]. Their identification provides the primary motivation for this work.

Logarithmic corrections to scaling in the infinite volume system have been studied in [2] and [3]. Here we report on results for finite-size scaling (FSS) [4,5] of the ϕ^4 theory which we have extended to four dimensions. Such finite size theories can be tested using non-perturbative (i.e., numerical) techniques.

The usual statement of FSS is the following [5]: For any thermodynamic quantity $P_L(\kappa)$, measured on a system of linear extent L and near criticality,

$$\frac{P_L(\kappa)}{P_\infty(\kappa)} = f\left(\frac{L}{\xi_\infty(\kappa)}\right), \quad (1.3)$$

where $\xi_\infty(\kappa)$ is the correlation length of the infinite volume system. The usual justification for this formula is that L and ξ_∞ are the only length scales involved and hence their ratio, $x = L/\xi_\infty(\kappa)$, is the scaling variable. Until 1982 this statement had the status of a hypothesis. Then, Brézin [4] succeeded in proving (1.3) from the renormalization group (RG). An essential ingredient in this proof is that the running quartic coupling be approximated by its IR FP value g_R^* in the scaling region. Now, in $d=4$ (in the perturbative formulation at least), the IR FP of the Callan-Symanzik beta function is at the origin. The approximation above then leads to the mean field theory which predicts a phase transition even

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for a finite system. For this reason FSS in the form (1.3) breaks down in $d = 4$. The intuitive justification given above is however a dimension independent argument. It is not clear, then, why it should fail in $d = 4$ while being valid for $d < 4$.

We claim that the usual statement (1.3) is, in fact, flawed and propose a modified FSS formula, valid in any dimension including four.

2. THE PERTURBATIVE RENORMALIZATION GROUP

At some critical value, m^2_{0c}, of the bare mass, the renormalized theory is massless. Writing m^2_0 as $m^2_{0_c} + t$, t becomes a measure of the deviation away from the massless theory. In the Ising version of the model, it is proportional to $\kappa - \kappa_c$, κ_c being the critical hopping parameter. The generating functional $W[H, t]$ is defined by

$$e^{W[H,t]} = C \int \prod_x d\phi(x) e^{-\int d^d x \mathcal{H}}, \tag{2.4}$$

C being a normalization constant. The function conjugate to $H(x)$ is

$$M(x,t) = \frac{\delta W[H,t]}{\delta H(x)} = \langle \phi(x) \rangle_{H,t}. \tag{2.5}$$

If H is independent of x, (which we henceforth assume), then W is a function of its arguments.

The generating functional $\Gamma[M, t]$ of the one particle irreducible vertex functions is defined through the Legendre transformation

$$\Gamma[M,t] + W[H,t] = \int dx H(x) M(x), \tag{2.6}$$

with

$$H(x,t) = \frac{\delta \Gamma[M,t]}{\delta M(x)}. \tag{2.7}$$

After isolating the divergences occurring in the Schwinger functions, one can write down the relationship between the bare and renormalized theories. In order to be able to study the onset of criticality, in both the symmetric and the broken phases, one first considers the massless renormalized theory — renormalized at some arbitrary mass-scale parameter μ. Expanding in the reduced mass t and in the conjugate function M, gives the renormalization group equation (RGE) for the massive theory in the critical region. Because of the local nature of the renormalization group, the renormalization constants of the infinite volume theory render the finite volume theory finite too [4].

For a system of finite volume L^d, with reduced temperature t and magnetization M, the above generating functional becomes the function

$$\Gamma(t, M, g_R, \mu, L)$$

in which g_R represents the renormalized quartic coupling. The RGE expresses the invariance of the physics under a rescaling of μ. I.e., when the mass-scale μ is varied, t,M and g_R respond in a way which is governed by the flow equations [2,3]. In four dimensions these flow equations can be solved perturbatively in g_R. Rescaling μ to μ/L, and using dimensional analysis, gives the following solution of the RGE [6]:

$$\begin{aligned}\Gamma\left(t, M, g_R, 1, L\right) \simeq \\ L^{-4}\Gamma\left(L^2 t \left(\frac{2}{3 g_R \ln L}\right)^{\frac{1}{3}}, LM, \frac{2}{3 g_R \ln L}, 1, 1\right) \\ + \frac{3}{4}\left(\frac{2}{3 g_R}\right)^{\frac{2}{3}} t^2 \left(\ln L\right)^{\frac{1}{3}}.\end{aligned} \tag{2.8}$$

To determine how the running coupling on the right hand side of (2.8) couples to the remaining terms, perturbation theory must be applied to Γ itself. This gives [2,6]

$$\begin{aligned}\Gamma\left(t, M, g_R, 1, L\right) = c_1 t M^2 \left(\ln L\right)^{-\frac{1}{3}} \\ + c_2 M^4 (\ln L)^{-1} + c_3 t^2 (\ln L)^{\frac{1}{3}}\end{aligned} \tag{2.9}$$

where $c_1, \ldots, c_3$ are constants. Applying (2.7) to this yields for the external field

$$\begin{aligned}H\left(t, M, g_R, 1, L\right) \simeq \\ c_4 t M (\ln L)^{-\frac{1}{3}} + c_5 M^3 (\ln L)^{-1},\end{aligned} \tag{2.10}$$

where, again, c_4 and c_5 are constants.

These give for the free energy per unit volume in the presence of an external field

$$\begin{aligned}W_L(t, H) = c_1' t M^2 (\ln L)^{-\frac{1}{3}} \\ + c_2' M^4 (\ln L)^{-1} + c_3 t^2 (\ln L)^{\frac{1}{3}},\end{aligned} \tag{2.11}$$

c_1' and c_2' being constants and M given by (2.10).

If H vanishes, then (2.10) and (2.11) give

$$W_L(t,0) \propto t^2 \, (\ln L)^{\frac{1}{3}} \, . \tag{2.12}$$

One could proceed directly from (2.11) or (2.12) to find the FSS formulae for thermodynamic observables. But it is more complete to study the partition function itself. This is entirely equivalent to the study of its zeroes. For fixed real t the zeroes in the complex h plane are called Lee–Yang zeroes [7], and in the absence of an external field, the zeroes in t are called Fisher zeroes [8]. Their FSS properties below four dimensions was studied in [9]. In this section, the corresponding FSS theory is presented for four dimensions where logarithmic corrections are manifest.

The total free energy at the critical temperature in four dimensions in the presence of an external field is given by (2.11) as

$$L^4 (\ln L)^{\frac{1}{3}} H^{\frac{4}{3}} . \tag{2.13}$$

The partition function is therefore

$$Z_L(t=0,H) = Q\left(L^4 (\ln L)^{\frac{1}{3}} H^{\frac{4}{3}} \right) . \tag{2.14}$$

When the partition function is zero, solving for H gives

$$H_j \propto L^{-3} (\ln L)^{-\frac{1}{4}} \tag{2.15}$$

where the constant of proportionality depends on the index j of the zero. This is the FSS formula for Lee–Yang zeroes in four dimensions.

If H vanishes, (2.12) can be used in a similar way to show that the Fisher zeroes scale as

$$t_j \propto L^{-2} \, (\ln L)^{-\frac{1}{6}} \, . \tag{2.16}$$

Once the FSS behaviour of the partition function zeroes has been found one can easily find the corresponding behaviour for thermodynamic functions by expressing them in terms of the zeroes. These considerations give for the zero field magnetic susceptibility and specific heat

$$\chi_L \, (t=0, H=0) \propto L^2 \, (\ln L)^{\frac{1}{2}} \tag{2.17}$$

and

$$C_L \, (t=0, H=0) \propto (\ln L)^{\frac{1}{3}} \, . \tag{2.18}$$

3. NON-PERTURBATIVE ANALYSIS OF FINITE SIZE SCALING

The Swendsen–Wang cluster algorithm [10] was applied to the Ising version of the theory on lattices of sizes 8^4 to 24^4.

In an external field h ($= \kappa H$), the partition function can be written as

$$Z(\kappa, h) = \sum_{M=-N}^{N} \sum_{S=-4N}^{4N} \rho(S,M) e^{\kappa S + hM} , \tag{3.1}$$

where

$$S = \sum_x \sum_{\mu=1}^{4} \phi_x \phi_{x+\mu} \quad , \quad M = \sum_x \phi_x , \tag{3.2}$$

and the spectral density $\rho(S,M)$ is the relative weight of configurations having given values of S and M. The 'multihistogram' method [11] was used to combine histograms determined at various values of κ. This provides an optimal estimator for the spectral density and allows one to construct $Z(\kappa, h)$ in the complex neighbourhood of the critical point. A Newton–Raphson algorithm was used to determine nearby zeroes.

The leading (power law) FSS behaviour of the Lee–Yang and Fisher zeroes was found to be slightly deviant from the mean field predictions. These deviations find their explanation in the presence of logarithmic corrections. To isolate these corrections, in the case of Fisher zeroes, we plot in fig.1a $\ln (L^2 \mathrm{Im} \kappa_1)$ versus $\ln (\ln L)$. The negative slope is in good agreement with the scaling prediction of $-\frac{1}{6}$. In fact, a fit to all five points gives a slope $-0.217(12)$. Excluding the point corresponding to $L = 8$ gives a slope of $-0.21(4)$. The solid line is the best fit to the remaining points assuming the theoretical prediction $-\frac{1}{6}$ from (2.16).

We may now determine κ_c from $|\kappa_j - \kappa_c| \propto l^{-2} \, (\ln l)^{-1/6}$. Using the first Fisher zeroes, we find $\kappa_c \simeq 0.149703(15)$ in good agreement with the value 0.149668(30) from high temperature expansions [12].

To identify the logarithmic corrections for the Lee–Yang zeroes, we plot in fig.1b $\ln (L^3 \mathrm{Im} h_1)$ against $\ln (\ln L)$. A best fit to all five points gives a slope of $-0.204(9)$ which compares well with

the theoretical prediction of $-\frac{1}{4}$ from (2.15). Excluding the smallest lattice, a fit to the remaining four points gives a slope $-0.22(3)$. The solid line in fig.1b is the best fit to the last four points with given slope $-\frac{1}{4}$.

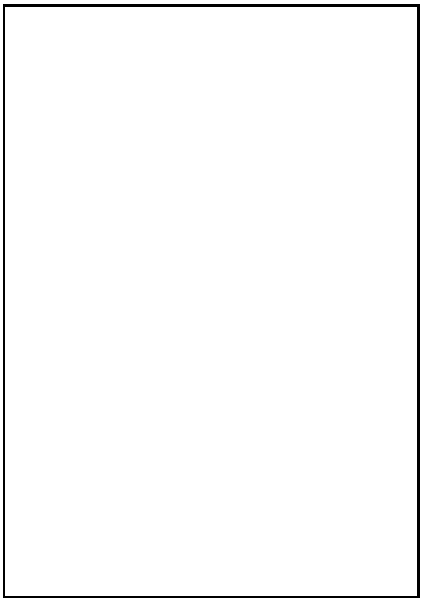

Figure 1. Logarithmic corrections to FSS of (a) Fisher zeroes and (b) Lee–Yang zeroes.

4. CONCLUSIONS

A finite size scaling theory has been developed for the single component ϕ^4 theory in $d = 4$ dimensions. Emphasis has been placed on logarithmic corrections to the mean field predictions. This has been checked non-perturbatively using high precision numerical methods, and good agreement is found.

FSS formulae for other thermodynamic functions are also given. These exhibit logarithmic corrections too. The FSS formula for the correlation length of a four dimensional system also involves logarithmic corrections. This was derived by Brézin [4] for a system of extent L in all directions. At the infinite volume critical point $\kappa = \kappa_c$,

$$\xi_L(\kappa_c) \propto L(\ln L)^{\frac{1}{4}}. \tag{4.1}$$

This suggests that the FSS variable should be

$$\frac{\xi_L(\kappa_c)}{\xi_\infty(\kappa)} = \frac{L(\ln L)^{\frac{1}{4}}}{t^{-\frac{1}{2}} \mid \ln t \mid^{\frac{1}{6}}} \tag{4.2}$$

in four dimensions[13]. Indeed, replacing the scaling variable, $x.$ of the right hand side of (1.3) by the ratio $\xi_L(\kappa_c)/\xi_\infty(\kappa)$ is sufficient to recover all the FSS formulae presented here while still being correct in $d < 4$ dimensions. We suggest that this modified FSS hypothesis is the more appropriate one.

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