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In this paper, we address the logarithmic corrections to the leading power laws that govern thermodynamic quantities as a second-order phase transition point is approached. For phase transitions of spin systems on d-dimensional lattices, such corrections appear at some marginal values of the order parameter or space dimension. We present scaling relations for these exponents. We also consider a spin system on a scale-free network which exhibits logarithmic corrections due to the specific network properties. To this end, we analyze the phase behavior of a model with coupled order parameters on a scale-free network and extract leading and logarithmic correction-to-scaling exponents that determine its field and temperature behavior. Although both nontrivial sets of exponents emerge from the network structure rather than from the spin fluctuations they fulfill the respective thermodynamic scaling relations. For the scale-free networks the logarithmic corrections appear at marginal values of the node degree distribution exponent. In addition we calculate scaling functions, which also exhibit nontrivial dependence on intrinsic network properties.

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I. INTRODUCTION

Scaling laws are an intrinsic feature of second-order phase transitions. In their leading asymptotics they are power laws that govern the behavior of the (singular part) of the free energy and of its derivatives in the vicinity of the phase-transition point [1]. For a magnetic phase transition, the Gibbs free energy exhibits universal scaling in terms of its inherent variables, the reduced temperature $\tau = |T-T_c|/T_c$ and the magnetic field h. Beside the critical exponents universality manifests itself in universal amplitude ratios and scaling functions. Moreover, a system defined on a d-dimensional Euclidean space (which we will call a *lattice* hereafter) becomes scale invariant at the critical point. Its correlation length diverges at the transition point $\tau=0$, $h=0$ and the pair correlation function changes from an exponential to a power-law decay. The leading exponents, that govern these scaling laws are related by scaling relations. These form a cornerstone of the modern theory of critical phenomena [1].

Of special interest within this theory of critical phenomena are those situations in which the aforementioned power laws require logarithmic corrections [2,3]. For d-dimensional systems, the most prominent examples are numerous spin models at their upper critical dimension d_c [4–6] and the q-state Potts model in $d=2$ dimensions and $q=q_c=4$ [7]. For spin models the logarithmic corrections appear when the mean-field power-laws observed for $d>d_c$ turn to nontrivial power-law dependencies at $d<d_c$. For the Potts model, the marginal value q_c separates two different phase transition scenarios: for $q>q_c$ the transition is of first order, whereas for $q<q_c$ it is of second order. Another, more subtle example is the $d=2$ Ising model with nonmagnetic impurities (see, e.g., [8], and references therein). Similar to the leading critical exponents, their logarithmic correction counterparts have been shown to obey also a set of scaling relations, as detailed in Ref. [3].

The situations discussed above concern systems with well-defined Euclidean metrics and, as is clearly seen from these examples, the notion of space dimensionality is crucial in defining the situation, where the logarithmic corrections to scaling appear. In this paper we want to attract attention to a different circumstance where critical behavior requires logarithmic corrections to scaling, namely spin models on *networks* or *random graphs* [9]. For Euclidean lattices the space dimension implies a given coordination number ($2d$ for the d-dimensional hypercube). For the networks we will consider here, these coordination numbers (or degrees) are distributed according to a given degree distribution. This amounts to a difference of principal between the origin of logarithmic corrections on regular lattices and on such networks.

The interest to study critical phenomena on complex networks is motivated by a number of reasons [10] both of academic and practical nature. Some models on complex networks may describe exotic phenomena (such as opinion formation in a social network [11]) as well as traditional physical objects (e.g., integrated nanoparticle systems with complex geometry [12]). Real-life complex networks are often characterized by a scale-free behavior: a power-law decay of the node degree distribution

$$P(k) \sim k^{-\lambda}. \tag{1}$$

Here, $P(k)$ is the probability that an arbitrarily chosen node of a network has a degree (the number of links attached to this node) k. The exponent λ is crucial in determining the

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critical behavior of different models on complex networks (see Refs [9,10], and references therein). The general situation is as follows: for small $\lambda<\lambda_s$ the system is always ordered, only an infinite temperature field is able to destroy the order. For large $\lambda>\lambda_c$ the phase transition is described by the usual mean-field critical exponents, whereas systems with intermediate values $\lambda_s<\lambda<\lambda_c$ are generally described by λ-dependent critical exponents. It is the marginal value of $\lambda=\lambda_c$ at which the logarithmic corrections to scaling appear as has been established for a number of classical spin models on scale-free networks [13]. The emergence of these corrections signals the relevance of higher moments of the node degree distribution due to the presence of high-degree nodes (hubs). Here, one observes a certain similarity with the critical behavior on lattices, where the logarithmic corrections appear at the upper critical dimension d_c at which the trivial mean-field exponents turn to the nontrivial ones due to the *correlations* in thermal fluctuations.

In the present work, we pay special attention to the analogy between the role of the upper critical dimension d_c on a regular lattice and the exponent λ_c on a complex network. To this end, we consider the field and temperature dependencies of thermodynamic quantities that characterize the system in the vicinity of the phase transition. The specific example we consider is a system with two coupled order parameters on a scale-free network. This model is widely used to describe ordering phenomena in systems with two possible types of ordering. Physical examples are given by ferromagnetic and antiferromagnetic, ferroelectric and ferromagnetic, structural and magnetic ordering [14]. In sociophysics applications [11], one may think about opinion formation where a coupling exists between the preferences for a candidate and a party in an election. Recently, we have used a Landau-like approach and a mean-field analysis to obtain the phase diagram of this model on a complex scale-free network [15]. In the present paper, we extend this analysis to derive the full set of critical exponents that govern the scaling laws for the thermodynamic quantities in terms of functions of h at $\tau=0$ and of functions of τ at $h=0$. A special focus of our paper is the logarithmic-correction-to-scaling behavior. We check the validity of existing relations for the logarithmic-correction-to-scaling exponents and further derive new scaling relations for exponents of logarithmic corrections, for which these relations were so far unknown.

II. CRITICAL EXPONENTS AND LOGARITHMIC CORRECTIONS TO SCALING

The behavior of a system near a second-order phase transition is described by a number of critical exponents. The magnetization m, susceptibility χ and heat capacity C_h at zero external field, $h=0$, respectively, follow the power laws [16]

$$m \sim \tau^{\beta}, \tag{2}$$

$$\chi \sim \tau^{-\gamma}, \tag{3}$$

$$C_h \sim \tau^{-\alpha}. \tag{4}$$

Spatial characteristics of the system, namely, the correlation length and the correlation function, which are connected with the linear size and the spatial dimension d, scale with their critical exponents ν and η correspondingly. The exponents connected to the spatial structure of the lattice are not well defined for the network. At the phase transition temperature $\tau=0$ the dependencies of the thermodynamic characteristics on the external field are also described by a number of critical exponents [16]

$$m \sim h^{1/\delta}, \tag{5}$$

$$\chi \sim h^{-\gamma_c}, \tag{6}$$

$$C_h \sim h^{-\alpha_c}. \tag{7}$$

The eight critical exponents listed above depend just on a few parameters—spatial dimension, spin dimension and symmetries of the model. Therefore, from a knowledge of just two of the exponents as well as the dimension, any other may be determined. Indeed, the remaining six exponents are related via the following four scaling relations:

$$\alpha + 2\beta + \gamma = 2, \tag{8}$$

$$\beta(\delta - 1) = \gamma, \tag{9}$$

$$\gamma_c = 1 - \frac{1}{\delta}, \tag{10}$$

$$\alpha_c = \frac{2+\gamma}{\beta+\gamma} - 2. \tag{11}$$

For the d-dimensional lattices, the behavior Eqs. (2)–(7) is valid from the lower to the upper critical dimension. Beyond the upper critical dimension, the exponents become those predicted by the mean-field approximation. Just at the upper critical dimension one may see modifications to the dependencies described above: there appear logarithmic corrections [3]. In the absence of an external field ($h=0$) the scaling behavior at the upper critical dimension is

$$m \sim \tau^{\beta}|\ln \tau|^{\hat{\beta}}, \tag{12}$$

$$\chi \sim \tau^{-\gamma}|\ln \tau|^{\hat{\gamma}}, \tag{13}$$

$$C_h \sim \tau^{-\alpha}|\ln \tau|^{\hat{\alpha}}, \tag{14}$$

while at the critical temperature ($\tau=0$) one finds

$$m \sim h^{1/\delta}|\ln h|^{\hat{\delta}}, \tag{15}$$

$$\chi \sim h^{-\gamma_c}|\ln h|^{\hat{\gamma}_c}, \tag{16}$$

$$C_h \sim h^{-\alpha_c}|\ln h|^{\hat{\alpha}_c}. \tag{17}$$

These hatted exponents for the logarithmic corrections are also connected via scaling relations, and in Ref. [3] the following formulas, which are analogous to Eqs. (8) and (9), were derived:

$$\hat{\beta}(\delta-1)=\delta\hat{\delta}-\hat{\gamma}, \tag{18}$$

$$\hat{\alpha}=2\hat{\beta}-\hat{\gamma}. \tag{19}$$

As it was outlined in the Introduction, we are interested in scaling laws for the magnetic phase transition on networks with, generally speaking, undefined Euclidean metrics. Therefore, the exponents we will be interested in are those given by Eqs. (2)–(7) that do not involve the space dimension d. The scaling relations for them are given by Eqs. (8)–(11). However, only two corresponding relations for the hatted exponents, Eqs. (18) and (19), are available in the literature so far [3]. Therefore, before we proceed further, we derive in the next section the scaling relations for the exponents $\hat{\gamma}_c$ Eq. (16) and $\hat{\alpha}_c$ Eq. (17) that characterize logarithmic corrections to the field-strength dependency.

III. NEW SCALING RELATIONS FOR LOGARITHMIC CORRECTIONS

In [3] a Lee-Yang analysis was used to derive relations between the logarithmic-correction exponents, which are analogous to the conventional scaling relations between the leading exponents. Here, these considerations are extended to deal with logarithmic corrections to the field dependency of the susceptibility. The Lee-Yang analysis concerns the zeros of the partition function in the plane of complex magnetic field. The locus of such zeros terminates at the so-called Yang-Lee edge $r_{\rm YL}$, which is temperature dependent. Following [3], we account for the possible existence of logarithmic corrections to the scaling of the edge near the phase transition, and write

$$r_{\rm YL}\sim\tau^{\Delta}|\ln\tau|^{\hat{\Delta}}. \tag{20}$$

The gap exponents Δ and $\hat{\Delta}$ are related to the more conventional exponents through the relations [3]

$$\Delta=\beta+\gamma,\quad \hat{\Delta}=\hat{\beta}-\hat{\gamma}. \tag{21}$$

In [3], the Gibbs free energy is written as a function of τ and h as

$$\Phi(\tau,h)=2\,\mathrm{Re}\int_{r_{\rm YL}}^{\infty}\ln[h-h(r,\tau)]g(r,\tau)dr, \tag{22}$$

in which, following the notation of Ref. [3], $h(r,\tau)$ is the locus of Lee-Yang zeros in the complex h plane and where $g(r,\tau)$ is their density. Integrating by parts yields, for the singular part of the free energy,

$$\Phi(\tau,h)=-2\,\mathrm{Re}\int_{r_{\rm YL}}^{\infty}\frac{G(r,\tau)\exp(i\phi)dr}{h-r\exp(i\phi)}, \tag{23}$$

where $G(r,\tau)=\int_{r_{\rm YL}(\tau)}^{r}g(s,\tau)ds$ is the cumulative distribution function for the zeros, the locus of which is assumed to be $h(r,\tau)=r\exp(i\phi)$ (the Lee-Yang theorem gives $\phi=\pi/2$). In contrast to [3], where h was set to zero in Eq. (23), the external field is now kept as a variable here in order to determine its contribution to scaling near the critical point. From [3], the integrated density is

$$G(r,\tau)=\chi r_{\rm YL}^{2}I\left(\frac{r}{r_{\rm YL}}\right). \tag{24}$$

The functional form of $I(x)$ is undetermined here, but our considerations shall not require such details. Introducing this into Eq. (23), one finds

$$\Phi(\tau,h)=\chi r_{\rm YL}^{2}\mathcal{F}_{\phi}\left(\frac{h}{r_{\rm YL}}\right), \tag{25}$$

where

$$\mathcal{F}_{\phi}(y)=-2\,\mathrm{Re}\int_{1}^{\infty}\frac{I(x)dx}{y\exp(-i\phi)-x}. \tag{26}$$

The specific heat is given by the second derivative of the free energy with respect to τ, and is

$$\mathcal{C}(\tau,h)=\chi r_{\rm YL}^{2}\tau^{-2}\mathcal{F}_{\phi}\left(\frac{h}{r_{\rm YL}}\right). \tag{27}$$

Now, from Eqs. (13) and (20), one may express the scaling of the specific heat in terms of that of the edge:

$$\mathcal{C}(\tau,h)=r_{\rm YL}^{2-\gamma/\Delta-2/\Delta}|\ln r_{\rm YL}|^{(\gamma+2)\hat{\Delta}/\Delta+\hat{\gamma}}\mathcal{F}_{\phi}\left(\frac{h}{r_{\rm YL}}\right), \tag{28}$$

which may in turn be written as

$$\mathcal{C}(\tau,h)=h^{2-\gamma/\Delta-2/\Delta}|\ln h|^{(\gamma+2)\hat{\Delta}/\Delta+\hat{\gamma}}\mathcal{F}'_{\phi}\left(\frac{h}{r_{\rm YL}}\right). \tag{29}$$

Now it is a simple matter to let $\tau\to 0$ so that $r_{\rm YL}\to 0$, and the undetermined function $\mathcal{F}'_{\phi}$ becomes a constant, yielding

$$\mathcal{C}(h)=h^{2-\gamma/\Delta-2/\Delta}|\ln h|^{(\gamma+2)\hat{\Delta}/\Delta+\hat{\gamma}}. \tag{30}$$

From the leading behavior one recovers Eq. (11). The correction exponents lead to the new scaling relation

$$\hat{\alpha}_c=\frac{(\gamma+2)\hat{\Delta}}{\Delta}+\hat{\gamma}, \tag{31}$$

which, from Eq. (21) yields

$$\hat{\alpha}_c=\frac{(\gamma+2)(\hat{\beta}-\hat{\gamma})}{\beta+\gamma}+\hat{\gamma}. \tag{32}$$

Equation (10) and its logarithmic counterpart

$$\hat{\gamma}_c=\hat{\delta} \tag{33}$$

are far more trivial to derive and follow from a single differentiation of Eq. (15) with respect to h. The latter two Eqs. (32) and (33) amount the desired scaling relations for $\hat{\alpha}_c$ and $\hat{\gamma}_c$.

IV. THERMODYNAMICAL FUNCTIONS OF A COUPLED ORDER PARAMETER SYSTEM ON A SCALE-FREE NETWORK

In the previous section we obtained new scaling relations Eqs. (32) and (33) for the logarithmic corrections exponents. Together with the formulas (18) and (19) they form a complete set of scaling relations for the correction to scaling exponents defined in Eqs. (12)–(17). The validity of relations Eqs. (18) and (19) for spin models on *lattices* was subject to a thorough check in Ref. [3]. There, it was shown that the relations hold for all models where the corrections are known explicitly. In particular, these include short- and long-range interacting $O(n)$ models at the upper critical dimension $d_c=4$, spin glasses, percolation and the Yang-Lee edge problem at $d_c=6$, lattice animals at $d_c=8$, regular and structurally disordered Ising model at $d=2$, q-state Potts model at $d=2$ and $q_c=4$ (see [3–8]). Now, we will proceed further to perform a similar check for the case of critical behavior on scale-free *networks*.

A. Temperature dependencies

As a case study, we will consider a rather common situation met in phase transition theory, when a system exhibits several types of ordering. This manifests itself by the appearance of two coupled scalar order parameters denoted by x_1 and x_2. In a microscopic description, such a system may be realized as a coupling between two Ising models, each of them being characterized by its own order parameter x_i, or as an XY model with a single ion anisotropy. Here, we consider the Hamiltonian with a cubic anisotropy term

$$H=-J\sum_{\langle i,j\rangle}\vec{s}_i\cdot\vec{s}_j+u\sum_{i=1}^{N}\sum_{\nu=1}^{2}s_{\nu,i}^4, \tag{34}$$

where $\vec{s}_i$ and $\vec{s}_j$ are spins on nodes i and j correspondingly, J and u are the coupling and anisotropy constants, the index ν numbers the components of the two-component vector, $\vec{s}_i\cdot\vec{s}_j=\Sigma_{\nu=1}^2 s_{\nu,i}s_{\nu,j}$ is a scalar product. The notation $\Sigma_{\langle i,j\rangle}$ denotes the summation over all pairs of connected nodes of the network. Note that the Hamiltonian Eq. (34) is the internal energy of an n-vector anisotropic cubic model in the case $n=2$. The latter is obtained from the $O(n)$ invariant free energy by adding invariants of the symmetry group B_n of the n-dimensional hypercube [17].

In Ref. [15], thermodynamical properties of such a system at $h=0$ were analyzed for a complex scale-free network with a power-law node degree distribution exponent λ, as in Eq. (1). As usual for models on scale-free networks [13], the system remains ordered for any finite temperature for $\lambda\leq 3$, but it possess a second order phase transition with finite T_c for $\lambda>3$. The phase diagram of the system is characterized by two different types of ordering, either along the edges or along the diagonals of the square in the space of the order parameter $\vec{x}=\{x_1,x_2\}$. In the first phase, only one order parameter component is nonzero ($x_1\neq 0$, $x_2=0$ or $x_1=0$, $x_2\neq 0$), whereas $x_1=x_2\neq 0$ in the second phase. The temperature dependencies of the order parameter, the susceptibilities and the heat capacities were obtained and the exponents Eqs. (2)–(4) determined. The marginal value $\lambda_c=5$ was shown to separate two different regimes of the phase transition: for $\lambda>5$ the exponents attain their classical mean field values

$$\beta=1/2,\quad \gamma=1,\quad \alpha=0, \tag{35}$$

whereas for $3<\lambda<5$ two out of the three exponents are λ dependent:

$$\beta=1/(\lambda-3),\quad \gamma=1,\quad \alpha=(\lambda-5)/(\lambda-3). \tag{36}$$

Another prominent feature found at $\lambda=5$ for the temperature dependencies of the order parameter and of the heat capacity is the appearance of logarithmic corrections to scaling that have the form given by Eqs. (12)–(14). The corresponding correction to scaling exponents were found to be [15]

$$\hat{\beta}=-1/2,\quad \hat{\gamma}=0,\quad \hat{\alpha}=-1. \tag{37}$$

To complete the analysis of the phase transition in the above model and to access the leading and correction-to-scaling exponents [Eqs. (5)–(7) and (15)–(17), correspondingly] that govern this transition, it is necessary to analyze the field dependencies of the thermodynamical quantities at $\tau=0$.

B. General relations

The starting point of our analysis is the expressions for the free energy considered for different λ in [15] within a Landau-type analysis [18], which was further supported by the microscopic treatment of the corresponding spin Hamiltonian Eq. (34) in a spirit of the mean-field approach. Such an approach is known to give exact results for tree-like networks, moreover, it gives asymptotically exact results for the networks with a local tree-like structure [10,13]. In the following, it will be more convenient to work within the $(T,\vec{x})$ ensemble and to consider the Helmholtz free energy $F(T,\vec{x})$ related to the Gibbs free energy $\Phi(T,\vec{h})$, Eq. (22), via the Legendre transform

$$F(T,\vec{x})=\Phi(T,\vec{h})+\vec{x}\cdot\vec{h}. \tag{38}$$

For $\lambda>5$, the free energy reads [15]

$$F(T,\vec{x})=\frac{a}{2}(T-T_c)|\vec{x}|^2+\frac{b}{4}|\vec{x}|^4+\frac{c}{4}x_1^2x_2^2, \tag{39}$$

where $|\vec{x}|^2=x_1^2+x_2^2$. Apart from the fact that the parameters a,b,c in Eq. (39) are λ-dependent (see [15] for explicit expressions), the free energy Eq. (38) has the form of a usual Landau-type free energy of a system with coupled scalar-order parameters [14]. Therefore, the network structure does not change the critical exponents for $\lambda>5$. However, for $\lambda\leq 5$ the leading terms of the free energy are modified [15]:

$$F(T,\vec{x})=\frac{a}{2}(T-T_c)|\vec{x}|^2+\frac{b}{4}|\vec{x}|^4\ln\frac{1}{|\vec{x}|}+\frac{c}{4}x_1^2x_2^2\ln\frac{1}{|\vec{x}|},\quad \lambda=5, \tag{40}$$

TABLE I. Critical exponents governing temperature and field dependencies of thermodynamic quantities for different values of λ.

	α	β	γ	δ	α_c	γ_c	Δ
$\lambda \geq 5$	0	1/2	1	3	0	2/3	3/2
$3<\lambda<5$	$\frac{\lambda-5}{\lambda-3}$	$\frac{1}{\lambda-3}$	1	$\lambda-2$	$\frac{\lambda-5}{\lambda-2}$	$\frac{\lambda-3}{\lambda-2}$	$\frac{\lambda-2}{\lambda-3}$

$$F(T,\vec{x}) = \frac{a}{2}(T-T_c)|\vec{x}|^2 + \frac{b}{4}|\vec{x}|^{\lambda-1} + \frac{c}{4}x_1^2x_2^2|\vec{x}|^{\lambda-5}, \quad 3<\lambda<5. \tag{41}$$

Note that the functional form of the coefficients a,b,c in Eqs. (39)–(41) differ, see [15] for detailed formulas. However this explicit form is not important for further calculations therefore we keep the same notation for the coefficients in Eqs. (39)–(41).

To analyze the field dependencies of the thermodynamic quantities we consider that an external magnetic field h is pointing along the order parameter component x_1, $\vec{h}=\{h,0\}$, and write the system of equations of state as

$$\left[\frac{\partial F(T,\vec{x})}{\partial x_1}\right]_T = h, \tag{42}$$

$$\left[\frac{\partial F(T,\vec{x})}{\partial x_2}\right]_T = 0. \tag{43}$$

The stable states are determined from the matrix of second derivatives

$$f_{\mu\eta} = \frac{\partial^2 F(T,\vec{x})}{\partial x_\mu \, \partial x_\eta}. \tag{44}$$

For a given state, the stability condition requires the real parts of the eigenvalues of the matrix Eq. (44) to be positive. Note that these eigenvalues are the inverse susceptibilities, longitudinal $\chi_\parallel^{-1}$ and transverse $\chi_\perp^{-1}$ correspondingly. To derive the heat capacity, one needs to obtain the entropy of the system. Following the definition

$$S(T,\vec{x}) = -\left[\frac{\partial F(T,\vec{x})}{\partial T}\right]_{\vec{x}}, \tag{45}$$

one finds the entropy as a function of the temperature T and order parameter $\vec{x}$. Knowing the dependence of the order parameter $\vec{x}$ on the temperature and external field $\vec{x}=\vec{x}(T,\vec{h})$, obtained from the system of equations of state Eqs. (42) and (43), one finds the entropy as a function of the temperature and external magnetic field $S=S(T,\vec{h})$. Now the heat capacity

$$C_h = T\left[\frac{\partial S(T,\vec{h})}{\partial T}\right]_h, \tag{46}$$

completes the calculations. Below, we sketch the results obtained for the magnetic field dependencies of the order parameter, susceptibilities and of the specific heat for different values of λ at $\tau=0$.

C. Exponents for the magnetic field dependencies at $\tau=0$

Taking the expressions for the free energy Eqs. (39)–(41) in the equation of state Eqs. (42) and (43) one finds stable solutions. It turns out (see the Appendix) that there are always two stable solutions ($x_1 \neq 0$, $x_2=0$) and ($x_1 \neq 0$, $x_2 \neq 0$). In any case the quantities considered have the leading form Eqs. (5)–(7). For $\lambda>5$ one obtains the mean field exponents whereas for $3<\lambda<5$ we arrive at the nontrivial λ-dependent exponents given in Table I.

At the marginal value $\lambda=5$ the logarithmic corrections of the form Eqs. (15)–(17) are obtained as summarized in Table II completed by the gap exponents calculated via Eq. (21).

With the data of Table II at hand, it is straightforward to check the validity of the scaling relations for the logarithmic correction to scaling exponents Eqs. (18), (19), (32), and (33). Moreover, one can see that they constitute a separate family that differs from other logarithmic correction-to-scaling exponents. To this end, we give in the Table II the value of the logarithmic correction-to-scaling exponents that arise for $d=2$ Potts model at marginal number of spin states $q=4$ and for the $O(n)$-symmetrical model at marginal space dimension $d=4$. Table I also gives further evidence of the validity of scaling relations Eqs. (8)–(11) for the leading scaling exponents in particular of those that involve α_c and γ_c. Values of the latter exponents for scale-free networks have not been available before.

V. SCALING FUNCTIONS

Finally, we consider how the underlying structure in the form of a complex network affects the validity of the scaling hypothesis, and if the latter is satisfied, we will find corresponding scaling functions. The hypothesis states that the singular part of a thermodynamic potential of a system in the vicinity of the critical point has the form of a generalized

TABLE II. Exponents for the logarithmic corrections to scaling laws that appear for several models: spin model on a scale-free network at marginal value $\lambda=5$ (our results); $d=2$ Potts model at marginal spin states number $q=4$; and $O(n)$ symmetric model at marginal space dimension $d=4$ (see [3,19], and references therein).

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\alpha}_c$	$\hat{\gamma}_c$	$\hat{\Delta}$
Scale-free network, $\lambda=5$	−1	−1/2	0	−1/3	−1	−1/3	−1/2
$q=4$ Potts model, $d=2$	−1	−1/8	3/4	−1/15	−22/15	−1/15	−7/8
$O(n)$ model, $d=4$	$\frac{4-n}{n+8}$	$\frac{3}{n+8}$	$\frac{n+2}{n+8}$	1/3			$\frac{1-n}{n+8}$

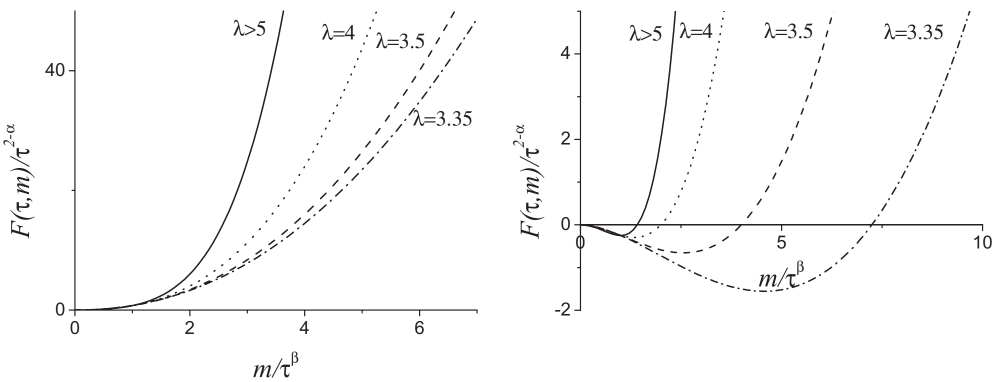

FIG. 1. The dependency of $F(\tau,m)/\tau^{2-\alpha}$ on m/τ^{β} for values of λ between 5 and 3. The left plot applies to temperatures above $T>T_c$ while the right plot applies to $T<T_c$.

homogeneous function [20]. For the Helmholtz potential this statement can be mathematically written as [21]

$$F(\tau,m) = \tau^{2-\alpha} f_{\pm}(m/\tau^{\beta}), \tag{47}$$

where the sign $\pm$ corresponds to $T>T_c$ or $T<T_c$, respectively. Provided the homogeneity hypothesis for the potential Eq. (47) holds, one arrives at the scaling form for the other thermodynamic quantities [23]. In particular, below we will make use of three different equivalent representations for the equation of state [22,24]. The Widom-Griffiths scaling form of the equation of state is [24]

$$h = m^{\delta} h_{\pm}(\tau/m^{1/\beta}), \tag{48}$$

with the alternative representation

$$h = \tau^{\beta\delta} H_{\pm}(m/\tau^{\beta}). \tag{49}$$

The scaling form of the magnetization reads (see also [23])

$$m = \tau^{\beta} \mu_{\pm}(h/\tau^{\beta\delta}), \tag{50}$$

and the isothermal susceptibility may be written as

$$\chi_T = \tau^{-\gamma} \chi_{\pm}(h/\tau^{\beta\delta}). \tag{51}$$

Note that taking temperature derivatives of Eq. (47) one arrives at the scaling functions for the entropy and heat capacity. Since their derivation follows in a similar manner as the above introduced functions Eqs. (48)–(51) we do not give their explicit expressions here.

The formulas given above hold for the single scalar order parameter system, from which we will start our consideration. Then the system of $O(n)$ symmetrical vector order parameter $\vec{m}=\{m_1,m_2\}$ and the system of coupled order parameters will be analyzed, for which one may easily generalize Eqs. (47)–(51).

A. Single order parameter

For $\lambda>5$ the Helmholtz potential $F(\tau,m)$ for the system with a single order parameter (magnetization) m may be obtained from Eq. (39) substituting $x_1=m$, $x_2=0$. Then $F(\tau,m)$ in dimensionless variables is

$$F(\tau,m) = \pm \frac{1}{2}\tau m^2 + \frac{1}{4}m^4, \tag{52}$$

where the energy is measured in units of F_0, and the magnetization in units of m_0:

$$F_0 = a^2 T_c^2/b, \quad m_0 = \sqrt{\frac{aT_c}{b}}. \tag{53}$$

It is easy to see that $F(\tau,m)$ scales as

$$F(\tau,m) = \tau^2 f_{\pm}(m/\tau^{1/2}), \tag{54}$$

where

$$f_{\pm}(\zeta) = \pm \frac{1}{2}\zeta^2 + \frac{1}{4}\zeta^4. \tag{55}$$

For $\lambda=5$ due to the logarithmic corrections in the free energy Eq. (40), the scaling form defined above fails. We refrain from giving a scaling function for this case.

For $3<\lambda<5$, $F(\tau,m)$ as given by Eq. (41) for the single order parameter system reads

$$F(\tau,m) = \pm \frac{1}{2}\tau m^2 + \frac{1}{4}m^{\lambda-1}. \tag{56}$$

Now the dimensionful quantities F_0 and m_0 Eq. (53) become λ dependent:

$$F_0(\lambda) = \frac{(aT_c)^{(\lambda-1)/(\lambda-3)}}{b^{1/(\lambda-3)}}, \quad m_0(\lambda) = \left(\frac{aT_c}{b}\right)^{1/(\lambda-3)}. \tag{57}$$

Again, one can recast Eq. (56) singling out the scaling function as

$$F(\tau,m) = \tau^{(\lambda-1)/(\lambda-3)} f_{\pm}(m/\tau^{1/(\lambda-3)}), \tag{58}$$

where the scaling function $f_{\pm}(\zeta)$ acquires a λ dependence,

$$f_{\pm}(\zeta) = \pm \frac{1}{2}\zeta^2 + \frac{1}{4}\zeta^{\lambda-1}, \tag{59}$$

and we have taken into account the λ dependence of the heat capacity critical exponent $\alpha=(\lambda-5)/(\lambda-3)$, see Table I.

In Fig. 1, we show the dependence of $F(\tau,m)/\tau^{2-\alpha}$ on m/τ^{β} for different values of λ above and below the critical temperature. Here and below we will plot the corresponding

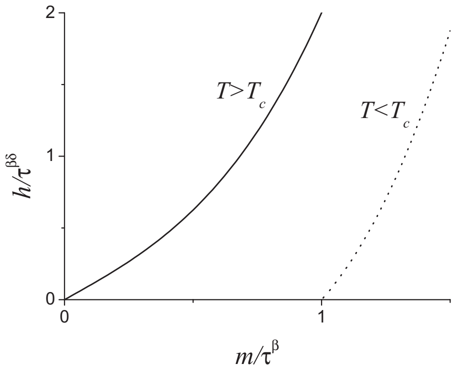

FIG. 2. The dependence of $h/\tau^{\beta\delta}$ on m/τ^{β} given for $\lambda>5$ by the scaling function $H_{\pm}$ (62), above (solid line) and below (dotted line) the critical temperature.

scaling function in the region of positive values of h and m.

Now let us consider the scaling functions $h_{\pm}$, $H_{\pm}$, $\mu_{\pm}$ and $\chi_{\pm}$ defined by Eqs. (48)–(51). For the case $\lambda>5$ one finds from Eq. (52) the equation of state

$$m^3 \pm \tau m - h = 0. \tag{60}$$

Representing Eq. (60) as defined by Eq. (48) we arrive at the scaling function $h_{\pm}(\zeta)$ that describes the equation of state in the Widom-Griffiths form

$$h_{\pm}(\zeta) = 1 \pm \zeta, \tag{61}$$

and the scaling function, defined by Eq. (49) readily follows

$$H_{\pm}(\zeta) = \zeta^3 \pm \zeta. \tag{62}$$

The dependence of $h/\tau^{\beta\delta}$ on m/τ^{β}, given by the scaling function $H_{\pm}$ in Eq. (62) is shown in Fig. 2. To get the magnetization scaling function $\mu_{\pm}$, given by Eq. (50), we first note that $\mu_{\pm}$ express the dependence of m/τ^{β} on $h/\tau^{\beta\delta}$, which is inverse to the dependence, given by the function $H_{\pm}$ in Eq. (49). Correspondingly, the scaling function $\mu_{\pm}$ may be easily plotted by exchanging the axes in Fig. 2 as shown in Fig. 3 by black bold lines. Three analytic solutions of the equation of state Eq. (60) for m give three branches for the

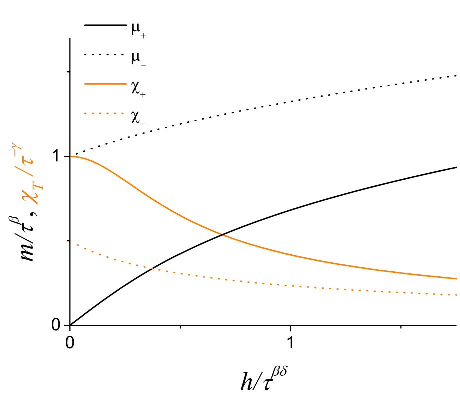

FIG. 3. (Color online) The scaling functions for magnetization and isothermal magnetic susceptibility for $\lambda>5$ above (solid lines) and below (dotted lines) the critical temperature. Black curves, m/τ^{β}; light (orange) curves, isothermal susceptibility scaling function $\chi_{\pm}$.

function $\mu_{\pm}$. Above the critical temperature only one branch is real and is presented by the black solid line in Fig. 3. The black dotted line in Fig. 3 displays the scaling function μ_{-} below the critical temperature $T<T_c$, which corresponds to the real solution of Eq. (60). The above described solutions are given by the following formulas

$$\mu_{\pm}(\zeta) = \frac{\varphi_{\pm}(\zeta)}{6} \mp \frac{2}{\varphi_{\pm}(\zeta)}, \tag{63}$$

where

$$\varphi_{\pm}(\zeta) = (108\zeta + 12\sqrt{81\zeta^2 \pm 12})^{1/3}. \tag{64}$$

The scaling function for the susceptibility Eq. (51) may be easily obtained from Eq. (63) using the relation

$$\chi_{\pm}(\zeta) = \frac{d\mu_{\pm}(\zeta)}{d\zeta} \tag{65}$$

and it reads

$$\chi_{\pm}(\zeta) = \left[\frac{\varphi_{\pm}(\zeta)}{6} \pm \frac{2}{\varphi_{\pm}(\zeta)}\right]\frac{36}{\varphi_{\pm}^3(\zeta)}\left(1 + \frac{9\zeta}{\sqrt{81\zeta^2 \pm 12}}\right). \tag{66}$$

The dependence of $\chi_T/\tau^{-\gamma}$ on $h/\tau^{\beta\delta}$, described by the scaling function Eq. (65) is plotted in Fig. 3 by light (orange online) lines.

As we have observed above, the scaling hypothesis for the Helmholtz free energy holds for $\lambda>5$ and $3<\lambda<5$. To proceed further and to get the scaling functions $h_{\pm}$, $H_{\pm}$, $\mu_{\pm}$, and $\chi_{\pm}$ in the region $3<\lambda<5$ we first derive from Eq. (56) an equation of state, which now has the form

$$\frac{\lambda-1}{4}m^{\lambda-2} \pm \tau m - h = 0. \tag{67}$$

Again, expressing h in terms of m and making use of Eq. (48) we get for the scaling function $h_{\pm}$ that enters the equation of state in Widom-Griffiths form,

$$h_{\pm}(\zeta) = \frac{\lambda-1}{4} \pm \zeta. \tag{68}$$

Note, that in the region $3<\lambda<5$ some of the critical indices acquire λ dependence. In particular, to get Eq. (68) one should take into account that $\beta=1/(\lambda-3)$ and $\delta=\lambda-2$ (Table I). Comparing Eqs. (68) and (61) one can see, that for $3<\lambda<5$ the functional dependence of $h_{\pm}$ on ζ does not change, and the only difference is the λ-dependence of the first term in the right-hand side of Eq. (68). This is not the case for the function $H_{\pm}$. Indeed, from Eqs. (67) and (49) we get for this function

$$H_{\pm}(\zeta) = \frac{\lambda-1}{4}\zeta^{\lambda-2} \pm \zeta. \tag{69}$$

Now not only the coefficient but also the leading power of this function is λ-dependent.

There is one more observation which follows from the comparison of Eqs. (69) and (62) that express function $H_{\pm}$ for $3<\lambda<5$ and $\lambda>5$, correspondingly. Eq. (62) allows for

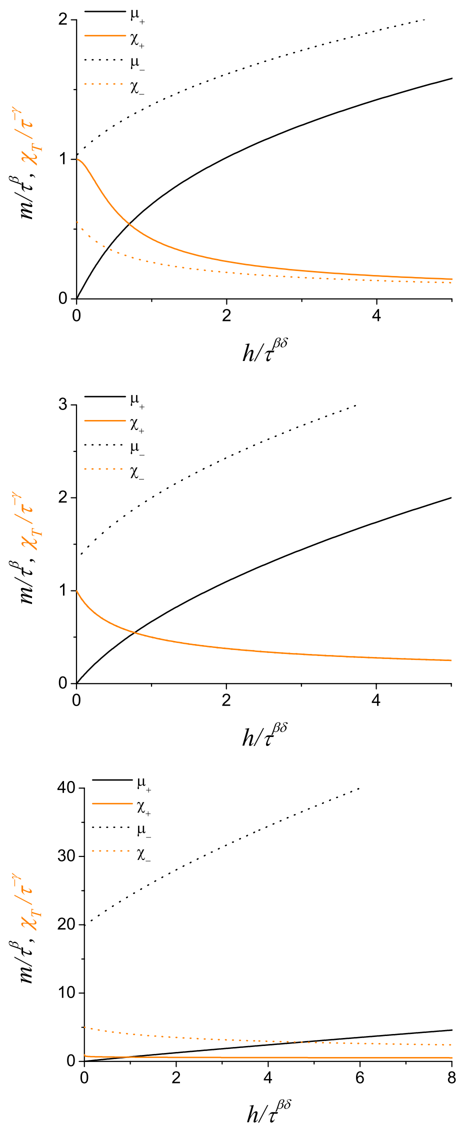

FIG. 4. (Color online) The scaling functions for magnetization and isothermal magnetic susceptibility for (a) $\lambda=4.8$, (b) $\lambda=4.0$, and (c) $\lambda=3.1$ above (solid lines) and below (dotted lines) the critical temperature. Black curves, m/τ^{β}; light (orange) curves, isothermal susceptibility scaling function $\chi_{\pm}$.

an analytic solution, which enables us in particular to find an analytic form for the scaling functions $\mu_{\pm}$, $\chi_{\pm}$ at $\lambda>5$ [see Eq. (63) and (66)]. Whereas a similar analytic treatment is possible for $H_{\pm}$ at integer values of the power $(\lambda-2)$ (i.e., for $\lambda=4$), it is impossible for the general noninteger value of $3<\lambda<5$. Therefore, we make use of the graphic representation to show the behavior of $\mu\pm$, $\chi_{\pm}$ at different λ in Fig. 4.

Comparing the plots in Fig. 4 one observes a particular feature in the behavior of the isothermal susceptibility scaling functions $\chi_{\pm}$. In the region $4<\lambda<5$ the curve χ_{+} is above the corresponding curve χ_{-} for all values of the argument. Qualitatively this resembles the case $\lambda>5$, where the usual Landau theory Eq. (52) holds. In particular, such behavior means that the plot for the zero-field isothermal susceptibility $\chi(\tau)$ has the usual "λ-shape" in the vicinity of $\tau=0$, i.e., the left shoulder of the curve is lower than the right one. This situation changes with a further decrease of λ: first at $\lambda=4$ both curves χ_{+} and χ_{-} coincide, and then, for $3<\lambda<4$ the curve χ_{+} is below χ_{-}. Again, for the zero-field isothermal magnetic susceptibility $\chi(\tau)$ this would mean that its left shoulder $(T<T_c)$ is above its right one $(T>T_c)$: the usual "λ-shape" turns to a "mirror-inversed-λ-shape." The latter is closely related to the universal amplitude ratios of the magnetic susceptibility $\Gamma_{+}/\Gamma_{-}=\lambda-3$, considered in Ref. [15].

B. Vector order parameter

Now let us consider a system with an $O(n)$ symmetric vector order parameter being in particular interested in the $n=2$ case, when $\vec{m}=\{m_1,m_2\}$. The free energy $F(\tau,\vec{m})$ for such a system may be obtained from Eqs. (39)–(41) by excluding the coupling term. So, for $\lambda>5$, $F(\tau,\vec{m})$ in the dimensionless variables is

$$F(\tau,m) = \pm\frac{1}{2}\tau|\vec{m}|^2 + \frac{1}{4}|\vec{m}|^4, \tag{70}$$

where the units of measure are given by Eq. (53). For the vector order parameter the free energy scaling function $f_{\pm}(\vec{\zeta})$ is defined by the equation

$$F(\tau,\vec{m}) = \tau^{2-\alpha}f_{\pm}(\vec{m}/\tau^{\beta}), \tag{71}$$

and reads

$$f_{\pm}(\vec{\zeta}) = \pm\frac{1}{2}|\vec{\zeta}|^2 + \frac{1}{4}|\vec{\zeta}|^4. \tag{72}$$

Taking that the magnetic field points along the m_1 component, one finds that the stable state of the system requires

$$m_1^3 \pm \tau m_1 - h = 0, \tag{73}$$

and

$$m_2 = 0, \tag{74}$$

as follows from the system of equations of state Eqs. (42) and (43). Since the field points only along the first component of the magnetization, we get that $\mu_{2\pm}=0$ and $\mu_{1\pm}\equiv\mu_{\pm}$. The scaling functions $h_{\pm}$, $H_{\pm}$ and $\mu_{1\pm}\equiv\mu_{\pm}$ coincide with the corresponding scaling functions for the single scalar order parameter, and are given by Eqs. (61)–(63) where m is to be replaced by m_1. Nevertheless, one needs to consider two different response functions, that describe the reaction of the system on an external magnetic field: the longitudinal and transverse susceptibilities, which scale as

$$\chi_{\parallel} = \tau^{-\gamma}\chi_{\parallel\pm}(h/\tau^{\beta\delta}), \tag{75}$$

$$\chi_{\perp} = \tau^{-\gamma}\chi_{\perp\pm}(h/\tau^{\beta\delta}). \tag{76}$$

Using the definition for $\chi_{\parallel}$, $\chi_{\perp}$ [see Eq. (44) and below] and substituting solutions Eqs. (73) and (74) into corresponding derivatives of the free energy Eq. (70) we arrive at the scaling functions

$$\chi_{\parallel\pm}(\zeta) = \{\pm 1 + 3[\mu_{\pm}(\zeta)]^2\}^{-1}, \tag{77}$$

$$\chi_{\perp\pm}(\zeta) = \{\pm 1 + [\mu_{\pm}(\zeta)]^2\}^{-1}, \tag{78}$$

with $\mu_{\pm}(\zeta)$ given by Eq. (63). Note that the existence of the continuous $O(n)$ symmetry of the free energy at $h=0$ allows also to present the transverse susceptibility as a ratio of m_1 and h at arbitrary h:

$$\chi_{\perp} = \frac{m_1}{h}. \tag{79}$$

Let us repeat the above calculations for $3<\lambda<5$. Excluding the coupling term from Eq. (41) we get for the $O(n)$ symmetric free energy:

$$F(\tau,\vec{m}) = \pm\frac{1}{2}\tau|\vec{m}|^2 + \frac{1}{4}|\vec{m}|^{\lambda-1}, \tag{80}$$

and the scaling function $f_{\pm}$ follows:

$$f_{\pm}(\vec{\zeta}) = \pm\frac{1}{2}|\vec{\zeta}|^2 + \frac{1}{4}|\vec{\zeta}|^{\lambda-1}. \tag{81}$$

The units of measure are given by Eq. (57). Again, the system of equations of state Eqs. (42) and (43) requires

$$m_2 = 0, \tag{82}$$

and the equation for m_1 coincides with Eq. (67) for a scalar order parameter m:

$$\frac{\lambda-1}{4}m_1^{\lambda-2} \pm \tau m_1 - h = 0. \tag{83}$$

Properly, the scaling functions $h_{\pm}$, $H_{\pm}$, and $\mu_{1\pm}\equiv\mu_{\pm}$ are equal to the corresponding scaling functions for the single scalar order parameter. The susceptibilities scaling functions $\chi_{\parallel\pm}$ and $\chi_{\perp\pm}$ follow

$$\chi_{\parallel\pm}(\zeta) = \left\{\pm 1 + \frac{(\lambda-1)(\lambda-2)}{4}[\mu_{1\pm}(\zeta)]^{\lambda-3}\right\}^{-1}, \tag{84}$$

$$\chi_{\perp\pm}(\zeta) = \left\{\pm 1 + \frac{\lambda-1}{4}[\mu_{1\pm}(\zeta)]^{\lambda-3}\right\}^{-1}. \tag{85}$$

Again, as for the case $\lambda>5$ we note that the transverse susceptibility $\chi_{\perp}$ also for $3<\lambda<5$ can be recast as the Eq. (79).

C. Coupled order parameters

For the model we consider in this paper, the vector $\vec{m}$ has two components which corresponds to two coupled scalar order parameters m_1 and m_2. For the case $\lambda>5$ the free energy is given by Eq. (39) and the corresponding scaling function follows

$$f_{\pm}(\vec{\zeta}) = \pm\frac{1}{2}|\vec{\zeta}|^2 + \frac{1}{4}|\vec{\zeta}|^4 + \frac{c}{4b}\zeta_1^2\zeta_2^2, \tag{86}$$

where the units of measure are given by Eq. (53).

Let us recall that the presence of the coupling between the order parameters m_1 and m_2 leads to two possible stable states of the system. Besides the state

$$[m_1,0], \quad \text{with} \quad m_1\neq 0,\ m_2=0, \tag{87}$$

the system is characterized by an additional stable state

$$[m_1,m_2], \quad \text{with} \quad m_1\neq 0,\ m_2\neq 0. \tag{88}$$

In the ordered state $[m_1,0]$, where only m_1 reflects the system magnetization, the scaling functions $h_{\pm}$, $H_{\pm}$ and $\mu_{1\pm}\equiv\mu_{\pm}$ coincide with the single scalar order parameter given in Eqs. (61)–(63). For the type of ordering considered, $[m_1,0]$, the susceptibility scaling functions read

$$\chi_{\parallel\pm}(\zeta) = \{\pm 1 + 3[\mu_{\pm}(\zeta)]^2\}^{-1}, \tag{89}$$

$$\chi_{\perp\pm}(\zeta) = \left\{\pm 1 + \frac{2b+c}{2b}[\mu_{\pm}(\zeta)]^2\right\}^{-1}, \tag{90}$$

with $\mu_{\pm}(\zeta)$ given by Eq. (63).

For the second type of ordering, $[m_1,m_2]$, the system of equations of state Eqs. (A1) and (A2) may be written as

$$m_1^3 \pm \tilde{\tau}m_1 - \tilde{h} = 0,$$

$$m_2^2 = -\tfrac{2b+c}{2b}m_1^2 - \tau, \tag{91}$$

where

$$\tilde{\tau} = \frac{2b}{4b+c}\tau, \quad \tilde{h} = -\frac{4b^2}{c(4b+c)}h. \tag{92}$$

The coefficient in front of τ in Eq. (92) is positive due to the stability condition in the vicinity of the critical point at zero magnetic field ($h=0$) [15] as well as at the critical temperature ($\tau=0$) with applied field ($h\neq 0$), Eq. (A9).

The Widom-Griffiths form of the equation of state Eq. (48) reads

$$\tilde{h} = m_1^3 h_{\pm}(\tilde{\tau}/m_1^{1/2}), \tag{93}$$

with $h_{\pm}(\zeta)$, given by Eq. (61). The equation of state, given by Eq. (49) is then

$$\tilde{h} = \tilde{\tau}^{3/2}H_{\pm}(m_1/\tilde{\tau}^{1/2}), \tag{94}$$

with $H_{\pm}(\zeta)$ given by Eq. (62).

The order parameters m_1 and m_2, obtained from the system Eq. (91) may be conveniently presented in the scaling form

$$m_1 = \tilde{\tau}^{1/2}\mu_{1\pm}(\tilde{h}/\tilde{\tau}^{3/2}),$$

$$m_2 = \tilde{\tau}^{1/2}\mu_{2\pm}(\tilde{h}/\tilde{\tau}^{3/2}). \tag{95}$$

The scaling function $\mu_{1\pm}(\zeta)$ coincides with the function $\mu_{\pm}(\zeta)$ in Eq. (63) of the single order parameter system, and for $\mu_{2\pm}(\zeta)$ we find

$$\mu_{2\pm}^2(\zeta) = -\frac{2b+c}{2b}\mu_{1\pm}^2(\zeta) \mp \frac{4b+c}{2b}. \tag{96}$$

The susceptibilities follow the scaling form Eqs. (75) and (76), where $\chi_{\parallel\pm}(\zeta)$ and $\chi_{\perp\pm}(\zeta)$ are

$$\chi_{\parallel\pm}(\zeta) = \left[\pm 1 + \frac{8b+c}{4b}(\mu_1^2+\mu_2^2) + \frac{1}{4}\sqrt{\left(4-\frac{c}{b}\right)^2(\mu_1^4+\mu_2^4) + 2\left(16+40\frac{c}{b}+7\frac{c^2}{b^2}\right)\mu_1^2\mu_2^2}\right]^{-1}, \quad (97)$$

$$\chi_{\perp\pm}(\zeta) = \left[\pm 1 + \frac{8b+c}{4b}(\mu_1^2+\mu_2^2) - \frac{1}{4}\sqrt{\left(4-\frac{c}{b}\right)^2(\mu_1^4+\mu_2^4) + 2\left(16+40\frac{c}{b}+7\frac{c^2}{b^2}\right)\mu_1^2\mu_2^2}\right]^{-1}. \quad (98)$$

Here we used the notations $\mu_1 \equiv \mu_{1\pm}(\zeta)$, $\mu_2 \equiv \mu_{2\pm}(\zeta)$.

For the case $3<\lambda<5$ the Helmholtz potential Eq. (41) follows the scaling form Eq. (71) where the scaling function

$$f_\pm(\vec{\zeta}) = \pm \frac{1}{2}|\vec{\zeta}|^2 + \frac{1}{4}|\vec{\zeta}|^{\lambda-1} + \frac{c}{4b}\zeta_1^2\zeta_2^2|\vec{\zeta}|^{\lambda-5} \quad (99)$$

becomes functionally λ dependent.

Similarly as for $\lambda>5$, for $3<\lambda<5$ the system of equations of state Eqs. (A30) and (A31) permits two types of ordering at nonzero external field $h\neq 0$: $[m_1,0]$ [Eq. (87)] and $[m_1,m_2]$ [Eq. (88)].

For the case $[m_1,0]$ where only m_1 depends on the external field, the scaling functions $h_\pm$ and $H_\pm$ coincide with the scaling functions given by Eqs. (68) and (69). Similarly, as for the single order parameter, the analytic treatments for the magnetization and the isothermal susceptibilities are impossible for general noninteger λ, for which $3<\lambda<5$, whereas their shape for different λ is shown in Fig. 4. Note however, that the isothermal susceptibility scaling functions, defined by Eqs. (75) and (76) may be analytically expressed through the magnetization scaling function $\mu_{1\pm}$ as

$$\chi_{\parallel\pm}(\zeta) = \left\{ \pm 1 + \frac{(\lambda-1)(\lambda-2)}{4}[\mu_{1\pm}(\zeta)]^{\lambda-3}\right\}^{-1}, \quad (100)$$

$$\chi_{\perp\pm}(\zeta) = \left\{ \pm 1 + \left[\frac{\lambda-1}{4}+\frac{c}{2b}\right][\mu_{1\pm}(\zeta)]^{\lambda-3}\right\}^{-1}. \quad (101)$$

For the ordering $[m_1,m_2]$, the equations of state Eqs. (A30) and (A31) do not allow to obtain analytical solutions for m_1 and m_2 in the general case for arbitrary noninteger λ. Nevertheless, these equations enable a confirmation of the scaling of the magnetization. Indeed, substituting

$$m_1 = \tau^{1/(\lambda-3)}\mu_{1\pm}(h/\tau^{(\lambda-2)/(\lambda-3)}), \quad (102)$$

$$m_2 = \tau^{1/(\lambda-3)}\mu_{2\pm}(h/\tau^{(\lambda-2)/(\lambda-3)}) \quad (103)$$

into Eqs. (A30) and (A31) one obtains the following equation for the scaling functions $\mu_{1\pm}$ and $\mu_{2\pm}$:

$$\frac{c}{2b}\mu_{1\pm}(\zeta)[\mu_{2\pm}^2(\zeta) - \mu_{1\pm}^2(\zeta)]|\vec{\mu}_\pm(\zeta)|^{\lambda-5} = \zeta, \quad (104)$$

where the relation between $\mu_{1\pm}(\zeta)\equiv\mu_{1\pm}$ and $\mu_{2\pm}(\zeta)\equiv\mu_{2\pm}$ is as follows

$$\pm|\vec{\mu}_\pm|^{\lambda-7} + \frac{\lambda-1}{4}|\vec{\mu}_\pm|^4 + \frac{c}{2b}\mu_{1\pm}^2|\vec{\mu}_\pm|^2 + \frac{(\lambda-5)c}{4b}\mu_{1\pm}^2\mu_{2\pm}^2 = 0. \quad (105)$$

The observed dependence of Eqs. (104) and (105) on a single variable $\zeta = h/\tau^{(\lambda-2)/(\lambda-3)}$ serves as evidence of the validity of the scaling hypothesis. In a similar way, the scaling for h, $\chi_\parallel$, and $\chi_\perp$ may be confirmed.

VI. CONCLUSIONS

Although second order phase transitions take place only under certain conditions, e.g., $\tau=0$, $h=0$ for magnetic systems, the singular part of the free energy and its thermodynamic derivatives are described by functions characterized by scaling properties nearby. Such properties are found experimentally by measurements at various points in temperature and small values of the field. Introducing appropriate scaling fields, data collapse to universal scaling functions is used to determine the exponents. On the other hand, logarithmic corrections present in the temperature and field dependence of quantities described otherwise by power laws may disturb this collapse. Therefore we have on the one hand calculated several scaling functions for ranges of the network parameter λ where power laws are valid and found the characteristic dependence of these functions on λ even when the exponents are independent of λ. On the other hand we have determined the logarithmic corrections at certain borderline values of λ.

It is remarkable that already within mean field theory we find transitions between different behavior at certain values of the exponent λ which plays a role comparable to that of the dimension d for models on regular lattices. In the present case these transitions and the logarithmic corrections that accompany them are due to the fact that certain moments of the degree distributions cease to exist at these transitional values of λ which expresses changes in the average local neighborhood of each node.

In this paper, we gave a comprehensive description of the temperature and field behavior of a system with two coupled scalar order parameters on a scale-free network in the vicinity of the second order phase transition point. Special attention has been paid to the appearance of the logarithmic corrections to scaling. For magnetic systems on d-dimensional lattices, such behavior arises due to the order parameter fluctuations that tend to be strongly correlated in the vicinity of the critical point. It is the space dimension d that is definitive for the relevance of such fluctuations. The scale-free networks we consider here are not characterized by Euclidean metrics or a space dimension d. In the latter case d determines the local environment of any node at different scales, i.e., the number of nearest neighbors, next nearest and higher order neighbors and the way these are interconnected. In the case of equilibrium scale-free networks discussed here the structure of this neighborhood is determined solely by the exponent λ which governs the degree distribution. As λ decreases and the node degree distribution becomes more and more fat tailed, the relative number of the high-degree nodes (hubs) increases and leads to nontrivial critical behavior. This can be related to the fact that below $\lambda=5$ the fourth moment of the degree distribution diverges and below $\lambda=3$ the second moment ceases to exist. First, for $\lambda_c=5$, the nontrivial dependencies appear. With further decrease in λ, for $\lambda<3$, the systems appears to be ordered at any finite temperature.

One observes a certain formal similarity between the behavior of spin systems on d-dimensional lattices and on scale-free networks with exponent λ. Both at $d=d_c$ and at $\lambda=\lambda_c$ the logarithmic corrections to scaling are precursors of the change in the critical behavior. This formal similarity is further pronounced in a more subtle way: the correction exponents found by us for the scale-free networks, although numerically different from those found on the lattices [4–8], obey the same scaling relations. In the present work we derived two new scaling relations for logarithmic correction exponents Eqs. (32) and (33). Together with previously found [3] relations Eqs. (18) and (19) they form a complete set of scaling relations for logarithmic corrections.

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APPENDIX

In this appendix we give the deviation of the exponents governing the field dependencies of thermodynamic quantities at $\tau=0$ for different values of λ.

1. *Case* $\lambda>5$

In this case the free energy follows Eq. (39) and the system of equations of state reads

$$a(T-T_c)x_1 + bx_1|\vec{x}|^2 + \frac{1}{2}cx_1x_2^2 = h, \tag{A1}$$

$$a(T-T_c)x_2 + bx_2|\vec{x}|^2 + \frac{1}{2}cx_1^2x_2 = 0. \tag{A2}$$

At $\tau=0$ one finds two different solutions, which we now detail.

Solution $x_1\neq 0$ and $x_2=0$. In the case of vanishing x_2 this solution is

$$x_1 = \frac{1}{b^{1/3}}h^{1/3}, \quad x_2 = 0. \tag{A3}$$

This solution exists and satisfies the stability conditions if

$$b>0, \quad c>-2b. \tag{A4}$$

The longitudinal and transverse susceptibilities follow

$$\chi_{\parallel} = \frac{1}{3b^{1/3}}h^{-2/3}, \tag{A5}$$

$$\chi_{\perp} = \frac{2b^{2/3}}{2b+c}h^{-2/3}. \tag{A6}$$

The heat capacity at the critical point reads

$$C_h = \frac{a^2}{3b}T_c. \tag{A7}$$

Solution $x_1\neq 0$, $x_2\neq 0$. For x_2 nonvanishing

$$x_1 = \left[-\frac{4b}{c(4b+c)}\right]^{1/3}h^{1/3}, \quad x_2 = \pm\sqrt{-\frac{2b+c}{2b}}x_1. \tag{A8}$$

Note, that the ratio between x_1 and x_2 does not depend on the strength of the field and depends only on the ratio c/b. This solution exists and satisfies stability conditions at

$$b>0, \quad -4b<c<-2b. \tag{A9}$$

The susceptibilities read

$$\chi_{\parallel} = -\frac{8b}{c(8b+c)-\sqrt{\xi}}\left[\frac{c(4b+c)}{4b}\right]^{2/3}h^{-2/3}, \tag{A10}$$

$$\chi_{\perp} = -\frac{8b}{c(8b+c)+\sqrt{\xi}}\left[\frac{c(4b+c)}{4b}\right]^{2/3}h^{-2/3}, \tag{A11}$$

where

$$\xi = c^2(8b+c)^2 - 48bc(2b+c)(4b+c). \tag{A12}$$

The heat capacity follows:

$$C_h = \frac{2}{3}\frac{a^2}{(4b+c)}T_c. \tag{A13}$$

From the above solutions we conclusion that the exponents defined in formulas (5)–(7) are

$$\delta=3, \quad \gamma_c=\frac{2}{3}, \quad \alpha_c=0. \tag{A14}$$

2. *Case* $\lambda=5$

Given the free energy Eq. (40), the system of equations of state reads

$$a(T-T_c)x_1 + bx_1|\vec{x}|^2\ln\frac{1}{|\vec{x}|} - \frac{b}{4}x_1|\vec{x}|^2 + \frac{c}{2}x_1x_2^2\ln\frac{1}{|\vec{x}|} - \frac{c}{4}\frac{x_1^3x_2^2}{|\vec{x}|^2} = h, \quad (A15)$$

$$a(T-T_c)x_2 + bx_2|\vec{x}|^2\ln\frac{1}{|\vec{x}|} - \frac{b}{4}x_2|\vec{x}|^2 + \frac{c}{2}x_1^2x_2\ln\frac{1}{|\vec{x}|} - \frac{c}{4}\frac{x_1^2x_2^3}{|\vec{x}|^2} = 0. \quad (A16)$$

Note that this is a system of transcendent equations and one may estimate the solution at weak external field $h\to 0$. At $\tau=0$ there exist two solutions of the system Eqs. (A15) and (A16).

Solution $x_1\neq 0$, $x_2=0$. The solution for vanishing x_2 is

$$x_1 \approx \left(\frac{3}{b}\right)^{1/3}\frac{h^{1/3}}{(-\ln h)^{1/3}}, \quad x_2=0. \quad (A17)$$

This solution exists and satisfies the stability conditions if

$$b>0, \quad c>-2b. \quad (A18)$$

The susceptibilities follow

$$\chi_{\parallel} = (9b)^{-1/3}h^{-2/3}(-\ln h)^{-1/3}, \quad (A19)$$

$$\chi_{\perp} = \frac{2b}{2b+c}\left(\frac{b}{3}\right)^{-1/3}h^{-2/3}(-\ln h)^{-1/3}. \quad (A20)$$

The heat capacity reads

$$C_h = \frac{a^2}{b}T_c(-\ln h)^{-1}. \quad (A21)$$

Solution $x_1\neq 0$ and $x_2\neq 0$. For nonzero x_2, the solution is of the form

$$x_1 \approx \left(-\frac{c}{2b}\right)^{2/3}\left(\frac{6}{4b+c}\right)^{1/3}h^{1/3}(-\ln h)^{-1/3},$$

$$x_2 = \pm\sqrt{-\frac{2b+c}{2b}}x_1. \quad (A22)$$

This solution exists and satisfies stability conditions if

$$b>0, \quad -4b<c<-2b. \quad (A23)$$

The susceptibilities follow

$$\chi_{\parallel} = \chi_{\parallel}^{II}h^{-2/3}(-\ln h)^{-1/3}, \quad (A24)$$

$$\chi_{\perp} = \chi_{\perp}^{II}h^{-2/3}(-\ln h)^{-1/3}, \quad (A25)$$

where

$$\chi_{\parallel}^{II} = -\frac{6^{1/3}4b}{c(8b+c)-\sqrt{\xi}}\left(-\frac{c}{2b}\right)^{5/3}(4b+c)^{2/3}, \quad (A26)$$

$$\chi_{\perp}^{II} = -\frac{6^{1/3}4b}{c(8b+c)+\sqrt{\xi}}\left(-\frac{c}{2b}\right)^{5/3}(4b+c)^{2/3}, \quad (A27)$$

and ξ is defined by Eq. (A12). The heat capacity is

$$C_h = \frac{2a^2}{4b+c}T_c(-\ln h)^{-1}. \quad (A28)$$

Comparing the obtained solutions with the definition of the logarithmic corrections to scaling exponents Eqs. (12)–(17), we arrive at

$$\hat{\delta} = -\frac{1}{3}, \quad \gamma_c = -\frac{1}{3}, \quad \alpha_c = -1. \quad (A29)$$

3. *Case* $3<\lambda<5$

The system with the free energy Eq. (41) is described by the following equations of state

$$a(T-T_c)x_1 + \frac{\lambda-1}{4}bx_1|\vec{x}|^{\lambda-3} + \frac{1}{2}cx_1x_2^2|\vec{x}|^{\lambda-5} + \frac{\lambda-5}{4}cx_1^3x_2^2|\vec{x}|^{\lambda-7} = h, \quad (A30)$$

$$a(T-T_c)x_2 + \frac{\lambda-1}{4}bx_2|\vec{x}|^{\lambda-3} + \frac{1}{2}cx_1^2x_2|\vec{x}|^{\lambda-5} + \frac{\lambda-5}{4}cx_1^2x_2^3|\vec{x}|^{\lambda-7} = 0. \quad (A31)$$

The system of Eqs. (A30) and (A31) has two solutions.

Solution $x_1\neq 0$ and $x_2=0$. For the vanishing x_2 this solution is

$$x_1 = \left[\frac{4}{(\lambda-1)b}\right]^{1/(\lambda-2)}h^{1/(\lambda-2)}, \quad x_2=0. \quad (A32)$$

This solution exists and satisfies the stability conditions if

$$b>0, \quad c>-\frac{\lambda-1}{2}b. \quad (A33)$$

The longitudinal and transverse susceptibilities read

$$\chi_{\parallel} = \frac{1}{\lambda-2}\left[\frac{4}{(\lambda-1)b}\right]^{1/(\lambda-2)}h^{-(\lambda-3)/(\lambda-2)}, \quad (A34)$$

$$\chi_{\perp} = \frac{4}{(\lambda-1)b+2c}\left[\frac{(\lambda-1)b}{4}\right]^{(\lambda-3)/(\lambda-2)}h^{-(\lambda-3)/(\lambda-2)}. \quad (A35)$$

The heat capacity

$$C_h = \frac{a^2}{\lambda-2}\left[\frac{4}{(\lambda-1)b}\right]^{3/(\lambda-2)}T_ch^{(5-\lambda)/(\lambda-2)}. \quad (A36)$$

Solution $x_1\neq 0$, $x_2\neq 0$. For the nonvanishing x_2 the solution is

$$x_1 = \left[\frac{4(1+\mu^2)^{(7-\lambda)/2}}{(\lambda-1)b(1+\mu^2)^2 + 2c\mu^2(1+\mu^2) - (5-\lambda)c\mu^2} \right]^{1/(\lambda-2)} h^{1/(\lambda-2)}, \quad x_2 = \pm \mu x_1, \tag{A37}$$

where

$$\mu^2 = \frac{-2(\lambda-1)b - (\lambda-3)c + \sqrt{(\lambda-3)^2c^2 - 4(\lambda-1)(5-\lambda)bc}}{2(\lambda-1)b}. \tag{A38}$$

The solution exists and is stable if

$$b > 0, \quad -4b < c < -\frac{\lambda-1}{2}b. \tag{A39}$$

The susceptibilities follow

$$\chi_{\parallel,\perp} = \frac{[(\lambda-1)b(1+\mu^2)^2 + 2c\mu^2(1+\mu^2) + (\lambda-5)c\mu^2]^{(\lambda-3)/(\lambda-2)}}{\{(\lambda-1)^2b + 2c\}(1+\mu^2)^3 - (\lambda+3)(5-\lambda)c\mu^2(1+\mu^2) \pm \sqrt{D}} 4^{1/(\lambda-2)} 2(1+\mu^2)^{(\lambda+3)/2(\lambda-2)} h^{-(\lambda-3)/(\lambda-2)}, \tag{A40}$$

where

$$D = (1-\mu^2)^2\{[(\lambda-1)(\lambda-3)b - 2c](1-\mu^2)^2 + (5-\lambda)(7-\lambda)c\mu^2\}^2 \tag{A41}$$

$$+ 4\mu^2\{[(\lambda-1)b + 2c](\lambda-3)(1+\mu^2)^2 + (5-\lambda)(7-\lambda)c\mu^2\}^2. \tag{A42}$$

Finally, the heat capacity is

$$C_h = \left[\frac{4(1+\mu^2)^{(7-\lambda)/2}}{(\lambda-1)b(1+\mu^2)^2 + 2c\mu^2(1+\mu^2) + (\lambda-5)c\mu^2} \right]^{3/(\lambda-2)} \frac{1+\mu^2}{\lambda-2} a^2 T_c h^{(5-\lambda)/(\lambda-2)}. \tag{A43}$$

The above results give the following values of the critical exponents:

$$\delta = \lambda - 2, \quad \gamma_c = \frac{\lambda-3}{\lambda-2}, \quad \alpha_c = \frac{\lambda-5}{\lambda-2}. \tag{A44}$$

Note that in this case all leading exponents that govern the field dependencies at $\tau=0$ are λ dependent.

Summarizing results of Ref. [15], Eqs. (35)–(37), and those obtained in this section, Eqs. (A14), (A29), and (A44) we give the values of the leading and correction to scaling exponents in Tables I and II completing them by the gap exponents Δ, $\hat{\Delta}$ calculated via Eqs. (21).

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