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Exact finite-size corrections for the spanning-tree model under different boundary conditions

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We express the partition functions of the spanning tree on finite square lattices under five different sets of boundary conditions in terms of a principal partition function with twisted-boundary conditions. Based on these expressions, we derive the exact asymptotic expansions of the logarithm of the partition function for each case. We have also established several groups of identities relating spanning-tree partition functions for the different boundary conditions. We also explain an apparent discrepancy between logarithmic correction terms in the free energy for a two-dimensional spanning-tree model with periodic and free-boundary conditions and conformal field theory predictions. We have obtained corner free energy for the spanning tree under free-boundary conditions in full agreement with conformal field theory predictions.

I. INTRODUCTION

Systems under various boundary conditions have the same per-site free energy, internal energy, specific heat, etc., in the bulk limit. Finite-size corrections, however, are dependent on the boundaries. Theories of finite-size effects have been successful in deriving critical and noncritical properties of infinite systems from their finite or partially finite counterparts. In the quest to improve our understanding of realistic systems of finite extent, two-dimensional models play crucial roles in statistical mechanics as they have long served as a testing ground to explore the general ideas of finite-size scaling under controlled conditions. Of particular importance in such studies are exact results where the analysis can be carried out without numerical errors, the Ising model [1–3], the dimer, and spanning tree model [4–7] being the most prominent examples.

In 2002 Ivashkevich, Izmailian, and Hu [3] proposed a systematic method to compute exact finite-size corrections to the partition functions and their derivatives of free models on the torus, including the Ising model, dimer model, and Gaussian model. They found that the partition functions of all these models can be written in terms of the partition functions with twisted-boundary conditions $Z_{\alpha,\beta}$ with $(\alpha,\beta) = (1/2,0),(0,1/2),$ and $(1/2,1/2)$. Extending this approach, Izmailian, Oganesyan, and Hu [5] computed the finite-size corrections to the free energy for the dimer model on finite square lattices under five different sets of boundary conditions (free, cylindrical, toroidal, Möbius strip, and the Klein bottle). They found that the aspect-ratio dependence of finite-size corrections is sensitive to boundary conditions and the parity of the number of lattice sites along the lattice axis. Quite recently, Izmailian *et al.* [7] have found that the partition functions of the anisotropic dimer model on the rectangular $(2M − 1) × (2N − 1)$ lattice with free and cylindrical-boundary conditions with a single monomer residing on the boundary can be expressed in terms of a partition function with twisted-boundary conditions $Z_{\alpha,\beta}$ with

$(\alpha,\beta) = (0,0)$. Based on these expressions, they derive the exact asymptotic expansions of the free energy.

In this paper we will consider the spanning-tree model. Enumeration of spanning trees on a graph is a classical problem of combinatorial graph theory, first considered by Kirchhoff [8] in his analysis of electrical networks. Let $G = V, E$ denote a connected graph (without loops) with vertex and edge sets V and E. A spanning subgraph of G is a spanning tree T if it has $V − 1$ edges with at least one edge incident at each vertex. The degree of a vertex is the number of edges attached to it (often denoted coordination number). According to the Kirchhoff theorem, the number of spanning-tree subgraphs on a lattice is given by the minors of the discrete Laplacian matrix Δ of this lattice. The Laplacian matrix Δ is defined as

$$\Delta = Q - A, \tag{1}$$

where A is an $\mathcal{N} \times \mathcal{N}$ adjacency matrix, and $\mathcal{N}$ is the number of lattice sites. The elements of matrix A are given by

$$A_{ij} = 1, \quad \text{if sites } i \text{ and } j \text{ are adjacent,}$$
$$A_{ij} = 0, \quad \text{otherwise,}$$

and Q is an $\mathcal{N} \times \mathcal{N}$ degree matrix of G with elements

$$Q_{ij} = k_i \delta_{ij},$$

where k_i is the degree of site i, and δ_{ij} is the Kronecker δ function. In 2000 Tzeng and Wu [9] obtained the closed-form expressions for the spanning tree generating function for a hypercubic lattice in d dimensions under free, periodic, and a combination of free and periodic-boundary conditions. They also obtained the spanning-tree-generating function for a simple quartic net embedded on two nonorientable surfaces, a Möbius strip and the Klein bottle.

In this paper we will express the partition functions of the spanning tree on finite square lattices under five different sets of boundary conditions (free, cylindrical, toroidal, Möbius strip, and Klein bottle) in terms of a principal partition function with twisted-boundary conditions. Based on these expressions, we derive the exact asymptotic expansions of the logarithm of the partition function for all boundary conditions mentioned above. We will show that the exact asymptotic expansion of

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the free energy for all boundary conditions can be written as

$$f = f_{\text{bulk}} + \frac{2f_{1s}}{M} + \frac{2f_{2s}}{N} + \frac{f_0(z\xi)}{S} + \sum_{p=1}^{\infty} \frac{f_p(z\xi)}{S^{p+1}}, \quad (2)$$

where $S = M \times N$ is the area of the lattice, $\xi = N/M$ is the aspect ratio, f_{bulk} is the bulk free energy, f_{1s} and f_{2s} are the surface free energies in the horizontal and vertical directions, respectively, along which x_1 and x_2 are the weights assigned to the edges of the spanning tree with

$$z = \sqrt{\frac{x_1}{x_2}},$$

$f_0(z\xi)$ is the leading finite-size correction term, and $f_p(z\xi)$ for $p = 1, 2, 3, \ldots$ are subleading correction terms.

In general, the bulk free energy f_{bulk}, the surface free energies f_{1s} and f_{2s} and subleading correction terms $f_p(z\xi)$ ($p = 1, 2, 3, \ldots$) are nonuniversal, but the coefficient f_0 is supposed to be universal [10]. The value of f_0 is known [11] to be related to the conformal anomaly number c and conformal weights of the underlying conformal theory. Cardy and Peschel [12] have shown that corners on the boundary induce a trace anomaly in the stress tensor. They predicted that corner contribution to free energy f_{corner} gives rise to a term in f_0 equal to

$$f_{\text{corner}} = -\frac{c}{8} \ln S, \quad (3)$$

where c is the central charge defining the universality class of the system and S is the area of the domain. Later, Kleban and Vassileva [13] extended the study of the free energy on a rectangle. They further derived a geometry-dependent universal part of the free energy in the rectangular geometry and showed that in addition to corner contribution predicted by Cardy and Peschel [12], the term f_0 contains also another universal part f_u depending on the aspect ratio

$$f_u = \frac{c}{4} \ln[\eta(q)\eta(q')], \quad (4)$$

where $\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function and $q = \exp(-2\pi\xi)$, $q' = \exp(-2\pi/\xi)$. Moreover, the term f_0 contains also nonuniversal, geometry-independent constant f_{nonuniv}. Thus, the term f_0 can be written as

$$f_0 = f_{\text{univ}} + f_{\text{nonuniv}}, \quad (5)$$

where the universal part f_{univ} of the free energy in the rectangular geometry can be calculated by conformal field theory methods [13] and given by

$$f_{\text{univ}} = f_{\text{corner}} + f_u, \quad (6)$$

while nonuniversal part f_{nonuniv} of the free energy is not calculable via the conformal field theory methods.

Until now there is little evidence for these predictions from exact solutions or numerical calculations. Quite recently, an efficient bond propagation algorithm was applied for computing the partition function of the Ising model with free edges and corners in two dimensions on square and triangular lattices [14–16]. They verify the conformal field theory prediction given by Eq. (6) with central charge $c = 1/2$. Later the conformal field theory prediction Eq. (6)

was confirmed [7] for the dimer model on odd-odd square lattices with one monomer on the boundary, for which the central charge is $c = -2$. However, for another model in the $c = -2$ universality class, namely the spanning-tree model, Duplantier and David [17] found logarithmic correction terms in the free energy for periodic and free-boundary conditions, which appeared to contradict the conformal field theory prediction. In what follows we will explain such discrepancy.

In this paper we will show that for the spanning tree on finite square lattices under free-boundary conditions f_0 contains the universal part f_{univ}, given by Eq. (6), and does not contain the term coming from corners of the lattice for periodic, cylindrical, Mobius, and Klein bottle-boundary conditions. This also confirms the conformal field theory prediction for the corner free energy in models for which the central charge is $c = -2$. Moreover, the nonuniversal, geometry-independent constant f_{nonuniv}, which is not calculable via the method of Kleban and Vassileva [13], is determined:

$$f_{\text{nonuniv}} = -\ln 2 - \frac{1}{4}\ln(1 + z^2) + \frac{5}{4}\ln z. \quad (7)$$

We have also established several groups of identities relating spanning-tree partition functions for the different boundary conditions.

Our objective in this paper is to study the finite-size properties of a spanning tree on the plane square lattice under five different sets of boundary conditions using the same techniques developed in Refs. [3,5] and [7]. The paper is organized as follows. In Sec. II we introduce the principal partition functions with twisted-boundary conditions $Z_{\frac{1}{2},\frac{1}{2}}(z, M, N)$ and $Z_{0,0}(z, M, N)$. In Sec. III we show how the partition functions of the spanning tree under different boundary conditions can be expressed in terms of the principal partition functions with twisted-boundary conditions. In Sec. IV we derive several groups of identities relating spanning-tree partition functions for the different boundary conditions. In Sec. V we discuss the finite-size corrections of the spanning-tree model and derive the exact asymptotic expansions of the logarithm of the partition functions for all five sets of boundary conditions and write down the expansion coefficients up to arbitrary order. Our results are summarized and discussed in Sec. VI.

II. PARTITION FUNCTION WITH TWISTED BOUNDARY CONDITION

We will show that the exact partition functions of the anisotropic spanning-tree model on finite rectangular lattices with free, cylindrical, toroidal, Möbius-strip, and Klein-bottle boundary conditions can be expressed in terms of the principal partition functions with twisted-boundary conditions $Z_{\frac{1}{2},\frac{1}{2}}(z, M, N)$ and $Z_{0,0}(z, M, N)$, where

$$Z_{\alpha,\beta}^2(z, M, N) = \prod_{m=0}^{M-1} \prod_{n=0}^{N-1} f\left(\frac{(m+\alpha)\pi}{M}, \frac{(n+\beta)\pi}{N}\right), \quad (8)$$

with $(\alpha, \beta) \neq (0,0)$. Here $f(x, y)$ is given by

$$f(x, y) = 4(z^2 \sin^2 x + \sin^2 y). \quad (9)$$

Since $Z_{0,0}(z, M, N)$ vanishes due to the zero mode at $(m, n) = (0,0)$, therefore, when $\alpha = \beta = 0$ we remove the zero mode

and denote remaining product in Eq. (8) by $Z_{0,0}^2(z,M,N)$,

$$Z_{0,0}^2(z,M,N) = \prod_{m=0}^{M-1} \prod_{n=0}^{N-1}{}' f\left(\frac{m\pi}{M}, \frac{n\pi}{N}\right), \qquad (10)$$

where the prime on the product denotes the restriction $(m,n) \neq (0,0)$.

The general theory for the asymptotic expansion of $Z_{0,0}(z,M,N)$ has been given in Ref. [7] and the asymptotic expansion of $Z_{\alpha,\beta}(z,M,N)$ for $(\alpha,\beta) \neq (0,0)$ has been given in Refs. [3,5].

III. THE SPANNING-TREE MODEL

The enumeration of spanning trees on a graph or lattice is a problem of longstanding interest in mathematics and physics

and it has been first considered by Kirchhoff in his analysis of electrical networks [8]. The enumeration of the weighted spanning trees on the $M \times N$ rectangular lattice concerns with the evaluation of the tree-generating function

$$Z_{M,N} = \sum x_1^{n_h} x_2^{n_v}, \qquad (11)$$

where x_1 and x_2 are the edge weights along directions M and N, respectively, and n_h and n_v are the numbers of edges in the horizontal and vertical directions and summation runs over all possible spanning-tree configurations.

In what follows, we will show that the partition function of the spanning tree on an $M \times N$ lattice is expressed in terms of the principal partition function with twisted-boundary conditions $Z_{0,0}(z,M,N)$ and $Z_{\frac{1}{2},\frac{1}{2}}(z,M,N)$ only, namely,

$$Z_{M,N}^{\text{Torus}} = x_2^{MN-1} \frac{1}{MN} Z_{0,0}^2(z,M,N), \qquad (12)$$

$$Z_{M,N}^{\text{Cyl}} = x_2^{MN-1} \frac{1}{2Nz \sinh\left(M \operatorname{arcsinh} 1/z\right)} Z_{0,0}(z,M,2N), \qquad (13)$$

$$Z_{M,N}^{\text{Free}} = x_2^{MN-1} \frac{(1+z^2)^{1/4}}{z\sqrt{2MN \sinh\left(2N \operatorname{arcsinh} z\right) \sinh\left(2M \operatorname{arcsinh} 1/z\right)}} Z_{0,0}^{1/2}(z,2M,2N), \qquad (14)$$

$$Z_{M,N}^{\text{Mob}} = x_2^{MN-1} \frac{1}{2M\left|\sinh\left(N \operatorname{arcsinh} z + \frac{i\pi M}{2}\right)\right|} Z_{0,0}(z,M,N) Z_{\frac{1}{2},\frac{1}{2}}(z,M,N), \qquad (15)$$

$$Z_{M,N}^{\text{Klein}} = x_2^{MN-1} \frac{\coth\left(N \operatorname{arcsinh} z/a\right)}{2M} Z_{0,0}(z,M,2N), \qquad (16)$$

where $a = 1$ for even M and $a = 0$ for odd M.

A. Partition function of the spanning tree model with toroidal-boundary condition

For a rectangular $M \times N$ lattice with toroidal-boundary conditions, the exact partition function for the weighted spanning trees $Z_{M,N}^{\text{Torus}}$ is given by [9]

$$Z_{M,N}^{\text{Torus}} = \frac{1}{MN} \prod_{m=0}^{M-1} \prod_{n=0}^{N-1}{}' 2\left[x_1\left(1 - \cos\frac{2m\pi}{M}\right) + x_2\left(1 - \cos\frac{2n\pi}{N}\right)\right], \qquad (17)$$

where the prime on the product denotes the restriction $(m,n) \neq (0,0)$. The partition function can be transformed as

$$Z_{M,N}^{\text{Torus}} = \frac{x_2^{MN-1}}{MN} \prod_{m=0}^{M-1} \prod_{n=0}^{N-1}{}' f\left(\frac{m\pi}{M}, \frac{n\pi}{N}\right), \qquad (18)$$

and $f(x,y)$ is given by Eq. (9). Now it is easy to see from Eqs. (8) and (18) that the partition function of the spanning-tree model on an $M \times N$ rectangular lattice with toroidal-boundary conditions can be expressed in terms of $Z_{0,0}(z,M,N)$ and written in the form given by Eq. (12).

B. Partition function of the spanning-tree model with cylindrical-boundary condition

For a rectangular $M \times N$ lattice with cylindrical-boundary condition (periodic-boundary conditions in the M direction and free-boundary conditions in the N direction), the exact partition function for the weighted spanning trees $Z_{M,N}^{\text{Cyl}}$ is given by [9]

$$Z_{M,N}^{\text{Cyl}} = \frac{1}{MN} \prod_{m=0}^{M-1} \prod_{n=0}^{N-1}{}' 2\left[x_1\left(1 - \cos\frac{2m\pi}{M}\right) + x_2\left(1 - \cos\frac{n\pi}{N}\right)\right], \qquad (19)$$

which can be transformed as

$$Z_{M,N}^{\text{Cyl}} = \frac{x_2^{MN-1}}{MN} \prod_{m=0}^{M-1} \prod_{n=0}^{N-1}{}' f\left(\frac{m\pi}{M}, \frac{n\pi}{2N}\right), \qquad (20)$$

where $f(x,y)$ is given by Eq. (9). Since

$$f\left(\frac{m\pi}{M}, \pi - \frac{n\pi}{2N}\right) = f\left(\frac{m\pi}{M}, \frac{n\pi}{2N}\right), \qquad (21)$$

the product over n in Eq. (20) can be extended up to $2N-1$ and the double product $\prod_{n=0}^{2N-1} \prod_{m=0}^{M-1}{}' f(\frac{m\pi}{M}, \frac{n\pi}{2N})$ can be expressed

in terms of the $\prod_{n=0}^{N-1} \prod_{m=0}^{'M-1} f(\frac{m\pi}{M}, \frac{n\pi}{2N})$ as

$$\prod_{n=0}^{2N-1} \prod_{m=0}^{M-1} {}' f\left(\frac{m\pi}{M}, \frac{n\pi}{2N}\right)$$

$$= \frac{\prod_{m=0}^{M-1} f\left(\frac{m\pi}{M}, \frac{\pi}{2}\right)}{\prod_{m=1}^{M-1} f\left(\frac{m\pi}{M}, 0\right)} \left[\prod_{n=0}^{N-1} \prod_{m=0}^{M-1} {}' f\left(\frac{m\pi}{M}, \frac{n\pi}{2N}\right) \right]^2. \quad (22)$$

Using the identity [18]

$$\prod_{m=0}^{M-1} 4\left[\sinh^2 \omega + \sin^2 \frac{(m+\alpha)\pi}{M} \right]$$

$$= 4|\sinh(M\omega + i\pi\alpha)|^2, \quad (23)$$

for $\alpha = 0$, the products $\prod_{m=0}^{M-1} f(\frac{m\pi}{M}, \frac{\pi}{2})$ and $\prod_{m=1}^{M-1} f(\frac{m\pi}{M}, 0)$ can be written as

$$\prod_{m=0}^{M-1} f\left(\frac{m\pi}{M}, \frac{\pi}{2}\right) = \prod_{m=0}^{M-1} 4\left[1 + z^2 \sin^2 \frac{m\pi}{M} \right]$$

$$= z^{2M} \prod_{m=0}^{M-1} 4\left[z^{-2} + \sin^2 \frac{m\pi}{M} \right]$$

$$= 4z^{2M} \sinh^2(M \operatorname{arcsinh} 1/z), \quad (24)$$

$$\prod_{m=1}^{M-1} f\left(\frac{m\pi}{M}, 0\right) = \prod_{m=1}^{M-1} 4z^2 \sin^2 \frac{m\pi}{M} = z^{2M-2} M^2, \quad (25)$$

respectively. From Eqs. (20)–(22), (24), and (25) the partition function of the spanning tree on cylinder can be expressed as

$$Z_{M,N}^{\mathrm{Cyl}} = \frac{x_2^{MN-1}}{2Nz \sinh(M \operatorname{arcsinh} 1/z)} \sqrt{\prod_{m=0}^{M-1} \prod_{n=0}^{2N-1} {}' f\left(\frac{m\pi}{M}, \frac{n\pi}{2N}\right)}. \quad (26)$$

Now it is easy to see from Eqs. (8) and (26) that the partition function of the spanning-tree model on an $M \times N$ rectangular lattice with cylindrical-boundary conditions can be expressed in terms of $Z_{0,0}(z, M, 2N)$ and written in the form given by Eq. (13).

C. Partition function of the spanning-tree model with free-boundary conditions

Let us now consider a rectangular $M \times N$ lattice with free boundaries. The exact partition function for the weighted spanning trees $Z_{M,N}^{\mathrm{Free}}$ is given by [9]

$$Z_{M,N}^{\mathrm{Free}} = \frac{1}{MN} \prod_{m=0}^{M-1} \prod_{n=0}^{N-1} {}' 2\left[x_1\left(1 - \cos\frac{m\pi}{M}\right) \right.$$

$$\left. + x_2\left(1 - \cos\frac{n\pi}{N}\right) \right], \quad (27)$$

which can be transformed as

$$Z_{M,N}^{\mathrm{Free}} = \frac{x_2^{MN-1}}{MN} \prod_{m=0}^{M-1} \prod_{n=0}^{N-1} {}' f\left(\frac{m\pi}{2M}, \frac{n\pi}{2N}\right), \quad (28)$$

where $f(x, y)$ is given by Eq. (9). It is easy to show that $f(\pi - \frac{m\pi}{2M}, \frac{n\pi}{2N}) = f(\frac{m\pi}{2M}, \pi - \frac{n\pi}{2N}) = f(\frac{m\pi}{2M}, \frac{n\pi}{2N})$. This allows us to express the double product $\prod_{n=0}^{2N-1} \prod_{m=0}^{'2M-1} f(\frac{m\pi}{2M}, \frac{n\pi}{2N})$, which is

$$\prod_{n=0}^{2N-1} \prod_{m=0}^{2M-1} {}' f\left(\frac{m\pi}{2M}, \frac{n\pi}{2N}\right)$$

$$= \prod_{n=0}^{2N-1} \prod_{m=0}^{2M-1} {}' 4\left[z^2 \sin^2 \frac{m\pi}{2M} + \sin^2 \frac{n\pi}{2N} \right]$$

$$= Z_{0,0}^2(z, 2M, 2N) \quad (29)$$

in terms of $\prod_{n=0}^{N-1} \prod_{m=0}^{'M-1} f(\frac{m\pi}{2M}, \frac{n\pi}{2N})$ through

$$\prod_{n=0}^{2N-1} \prod_{m=0}^{2M-1} {}' f\left(\frac{m\pi}{2M}, \frac{n\pi}{2N}\right) = \frac{f\left(0, \frac{\pi}{2}\right) f\left(\frac{\pi}{2}, 0\right)}{f\left(\frac{\pi}{2}, \frac{\pi}{2}\right)} \frac{\prod_{m=0}^{2M-1} f\left(\frac{m\pi}{2M}, \frac{\pi}{2}\right) \prod_{n=0}^{2N-1} f\left(\frac{\pi}{2}, \frac{n\pi}{2N}\right)}{\prod_{m=1}^{2M-1} f\left(\frac{m\pi}{2M}, 0\right) \prod_{n=1}^{2N-1} f\left(0, \frac{n\pi}{2N}\right)} \left[\prod_{n=0}^{N-1} \prod_{m=0}^{M-1} {}' f\left(\frac{m\pi}{2M}, \frac{n\pi}{2N}\right) \right]^4. \quad (30)$$

Now with the help of identity given by Eq. (23) we can find that

$$\prod_{m=0}^{2M-1} f\left(\frac{m\pi}{2M}, \frac{\pi}{2}\right) = z^{4M} \prod_{m=0}^{2M-1} 4\left[z^{-2} + \sin^2 \frac{m\pi}{2M} \right] = 4z^{4M} \sinh^2(2M \operatorname{arcsinh} 1/z), \quad (31)$$

$$\prod_{n=0}^{2N-1} f\left(\frac{\pi}{2}, \frac{n\pi}{2N}\right) = \prod_{n=0}^{2N-1} 4\left[z^2 + \sin^2 \frac{n\pi}{2N} \right] = 4\sinh^2(2N \operatorname{arcsinh} z), \quad (32)$$

$$\prod_{m=1}^{2M-1} f\left(\frac{m\pi}{2M}, 0\right) = \prod_{m=1}^{2M-1} 4z^2 \sin^2 \frac{m\pi}{2M} = 4z^{4M-2} M^2, \quad (33)$$

$$\prod_{n=1}^{2N-1} f\left(0, \frac{n\pi}{2N}\right) = \prod_{n=1}^{2N-1} 4\sin^2 \frac{n\pi}{2N} = 4N^2. \quad (34)$$

It is easy to show that

$$\frac{f\left(0,\frac{\pi}{2}\right)f\left(\frac{\pi}{2},0\right)}{f\left(\frac{\pi}{2},\frac{\pi}{2}\right)} = \frac{4z^2}{1+z^2}. \tag{35}$$

Now, plugging Eqs. (31)–(35) back to Eq. (30) and using Eq. (29) we obtain

$$Z_{0,0}^2(z,2M,2N) = \frac{4z^4\sinh^2\left(2N\operatorname{arcsinh}z\right)\sinh^2\left(2M\operatorname{arcsinh}1/z\right)}{(1+z^2)M^2N^2}\left[\prod_{n=0}^{N-1}\prod_{m=0}^{M-1}{}'f\left(\frac{m\pi}{2M},\frac{n\pi}{2N}\right)\right]^4. \tag{36}$$

Finally, from Eqs. (28) and (36), the partition function of the spanning-tree model on an $M\times N$ rectangular lattice with free-boundary conditions can be written in the form given by Eq. (14).

D. Partition function of the spanning-tree model with Möbius-strip-boundary condition

For a rectangular $M\times N$ lattice with Möbius-strip-boundary conditions (with free-boundary conditions in the M direction and twisted boundaries in the N direction), the exact partition function for the weighted spanning trees $Z_{M,N}^{\text{Mob}}$ is given by [9]

$$Z_{M,N}^{\text{Mob}} = \frac{1}{MN}\prod_{m=0}^{M-1}\prod_{n=0}^{N-1}{}'2\left[x_1\left(1-\cos\frac{m\pi}{M}\right)+x_2\left(1-\cos\frac{4n+1-(-1)^m}{2N}\pi\right)\right] \tag{37}$$

$$= \frac{x_2^{MN-1}}{MN}\prod_{m=0}^{M-1}\prod_{n=0}^{N-1}{}'4\left[z^2\sin^2\frac{m\pi}{2M}+\sin^2\frac{4n+1-(-1)^m}{4N}\pi\right]. \tag{38}$$

The double product in Eq. (38) can be split into two part by considering even m and odd m separately,

$$Z_{M,N}^{\text{Mob}} = \frac{x_2^{MN-1}}{MN}\prod_{m=0}^{\lfloor\frac{M-1}{2}\rfloor}\prod_{n=0}^{N-1}{}'f\left(\frac{m\pi}{M},\frac{n\pi}{N}\right)\prod_{m=0}^{\lfloor\frac{M}{2}\rfloor-1}\prod_{n=0}^{N-1}f\left[\frac{(m+1/2)\pi}{M},\frac{(n+1/2)\pi}{N}\right], \tag{39}$$

where $\lfloor a\rfloor$ is the integer part of a and $f(x,y)$ is given by Eq. (9).

For even M

$$Z_{M,N}^{\text{Mob}} = \frac{x_2^{MN-1}}{MN}\prod_{m=0}^{\frac{M}{2}-1}\prod_{n=0}^{N-1}{}'f\left(\frac{m\pi}{M},\frac{n\pi}{N}\right)\prod_{m=0}^{\frac{M}{2}-1}\prod_{n=0}^{N-1}f\left[\frac{(m+1/2)\pi}{M},\frac{(n+1/2)\pi}{N}\right] \tag{40}$$

$$= \frac{x_2^{MN-1}}{2M\sinh\left(N\operatorname{arcsinh}z\right)}Z_{0,0}(z,M,N)Z_{\frac{1}{2},\frac{1}{2}}(z,M,N). \tag{41}$$

Here we extend the product over m in the double products of Eq. (40) up to $M-1$. For odd M,

$$Z_{M,N}^{\text{Mob}} = \frac{x_2^{MN-1}}{MN}\prod_{m=0}^{\frac{M-1}{2}}\prod_{n=0}^{N-1}{}'f\left(\frac{m\pi}{M},\frac{n\pi}{N}\right)\prod_{m=0}^{\frac{M-1}{2}-1}\prod_{n=0}^{N-1}f\left[\frac{(m+1/2)\pi}{M},\frac{(n+1/2)\pi}{N}\right] \tag{42}$$

$$= \frac{x_2^{MN-1}}{2M\cosh\left(N\operatorname{arcsinh}z\right)}Z_{0,0}(z,M,N)Z_{\frac{1}{2},\frac{1}{2}}(z,M,N). \tag{43}$$

Here again we extend the product over m in the double products of Eq. (42) up to $M-1$.

Thus, we have shown that the partition function of the spanning tree on the $M\times N$ lattice with Möbius-boundary condition is expressed in terms of the principal partition function with twisted-boundary conditions $Z_{0,0}(z,M,N)$ and $Z_{\frac{1}{2},\frac{1}{2}}(z,M,N)$ only, and can be written in the form given by Eq. (15).

E. Partition function of the spanning-tree model with Klein-bottle-boundary condition

For a rectangular $M\times N$ lattice with Klein-bottle-boundary condition (periodic-boundary conditions in the M direction and twisted boundaries in the N direction), the exact partition function for the weighted spanning trees $Z_{M,N}^{\text{Cyl}}$ depends on the parity of the M and given by [9]

$$Z_{M,N}^{\text{Klein}} = \frac{2^{MN-1}}{MN}\prod_{k=1}^{N-1}x_2\left(1-\cos\frac{2\pi k}{N}\right)\prod_{m=1}^{\frac{M-1}{2}}\prod_{n=0}^{2N-1}\left[x_1\left(1-\cos\frac{2m\pi}{M}\right)+x_2\left(1-\cos\frac{n\pi}{N}\right)\right], \tag{44}$$

for odd M and

$$Z_{M,N}^{\text{Klein}} = \frac{2^{MN-1}}{MN} \prod_{n=1}^{N-1} x_2 \left(1 - \cos\frac{2\pi n}{N}\right) \prod_{n=0}^{N-1} \left[2x_1 + x_2\left(1 - \cos\frac{(2n+1)\pi}{N}\right)\right]$$

$$\times \prod_{m=1}^{\frac{M}{2}-1} \prod_{n=0}^{2N-1} \left[x_1\left(1 - \cos\frac{2m\pi}{M}\right) + x_2\left(1 - \cos\frac{n\pi}{N}\right)\right], \tag{45}$$

for even M. The partition function can be transformed as

$$Z_{M,N}^{\text{Klein}} = \frac{x_2^{MN-1}}{MN} \prod_{k=1}^{N-1} 4\sin^2\frac{\pi n}{N} \prod_{m=1}^{\frac{M-1}{2}} \prod_{n=0}^{2N-1} f\left(\frac{m\pi}{M}, \frac{n\pi}{2N}\right) \tag{46}$$

$$= x_2^{MN-1} \frac{N}{M} \prod_{m=1}^{\frac{M-1}{2}} \prod_{n=0}^{2N-1} f\left(\frac{m\pi}{M}, \frac{n\pi}{2N}\right), \tag{47}$$

for odd M and

$$Z_{M,N}^{\text{Klein}} = \frac{x_2^{MN-1}}{MN} \prod_{n=1}^{N-1} 4\sin^2\frac{\pi n}{N} \prod_{n=0}^{N-1} 4\left[z^2 + \sin^2\frac{(n+1/2)\pi}{N}\right] \prod_{m=1}^{\frac{M}{2}-1} \prod_{n=0}^{2N-1} f\left(\frac{m\pi}{M}, \frac{n\pi}{2N}\right) \tag{48}$$

$$= \frac{4x_2^{MN-1} \cosh^2(N\operatorname{arcsinh} z)}{M} \prod_{m=1}^{\frac{M}{2}-1} \prod_{n=0}^{2N-1} f\left(\frac{m\pi}{M}, \frac{n\pi}{2N}\right), \tag{49}$$

for even M. Here we have used the identity given by Eq. (23) for $\alpha = 0$ and $1/2$.

Now extending product over m in the double product of Eqs. (47) and (49) up to $M-1$, we obtain

$$Z_{M,N}^{\text{Klein}} = \frac{x_2^{MN-1}}{2M} Z_{0,0}(z, M, 2N) \quad \text{for odd } M, \tag{50}$$

$$Z_{M,N}^{\text{Klein}} = x_2^{MN-1} \frac{\coth(N\operatorname{arcsinh} z)}{2M} Z_{0,0}(z, M, 2N)$$
$$\text{for even } M. \tag{51}$$

Thus, we have shown that the partition function of the spanning-tree model on an $M \times N$ rectangular lattice with Klein-bottle-boundary conditions is expressed in terms of the principal partition function with twisted-boundary conditions $Z_{0,0}(z, M, 2N)$ and can be written in the form given by Eq. (16).

Equations (12)–(16) give how the partition functions of the spanning-tree model on an $M \times N$ rectangular lattice with different boundary conditions can be expressed in terms of the principal objects $Z_{0,0}(z, M, N)$ and $Z_{\frac{1}{2},\frac{1}{2}}(z, M, N)$. In the next section, based on such results, we will established a group of identities relating spanning-tree partition functions for the different boundary conditions.

IV. IDENTITIES FOR THE SPANNING-TREE MODEL

From Eqs. (12)–(14) and (16) one can see that the partition functions of the spanning tree on $M \times N$ lattices with toroidal, cylindrical, free, and Klein-bottle boundary conditions are all expressed in terms of the principal objects $Z_{0,0}(z, \mathcal{M}, \mathcal{N})$ only. Based on such results, it is easy to establish the following group of identities relating spanning-tree partition functions

for the different boundary conditions

$$Z_{M,2N}^{\text{Torus}} = \frac{2Mx_2}{N} \left(Z_{M,N}^{\text{Klein}}\right)^2 \quad \text{for odd } M, \tag{52}$$

$$Z_{M,2N}^{\text{Torus}} = A_1 \left(Z_{M,N}^{\text{Klein}}\right)^2 \quad \text{for even } M, \tag{53}$$

$$Z_{M,2N}^{\text{Torus}} = A_2 \left(Z_{M,N}^{\text{Cyl}}\right)^2, \tag{54}$$

$$Z_{2M,2N}^{\text{Torus}} = A_3 \left(Z_{M,N}^{\text{Free}}\right)^4, \tag{55}$$

where the coefficients A_1 and A_2 are given by

$$A_1 = \frac{2Mx_2}{N\coth^2(N\operatorname{arcsinh} z)}, \tag{56}$$

$$A_2 = \frac{2Nx_1 \sinh^2(M\operatorname{arcsinh} 1/z)}{M}, \tag{57}$$

$$A_3 = \frac{MNx_2^3 z^4 \sinh^2(2M\operatorname{arcsinh} 1/z)\sinh^2(2N\operatorname{arcsinh} z)}{1 + z^2}. \tag{58}$$

Thus, we have established a group of identities relating spanning-tree partition functions for the toroidal, cylindrical, free, and Klein-bottle boundary conditions [see Eqs. (52)–(55)].

V. ASYMPTOTIC EXPANSION OF FREE ENERGY

In Sec. II we have expressed the partition functions of the spanning tree on finite square lattices under five different boundary conditions (free, cylindrical, toroidal, Möbius strip, and Klein bottle) in terms of a principal partition function with

twisted-boundary conditions $Z_{0,0}(z,M,N)$ and $Z_{\frac{1}{2},\frac{1}{2}}(z,M,N)$ only. Based on such results, one can use the exact asymptotic expansions of $Z_{0,0}(z,M,N)$ and $Z_{\frac{1}{2},\frac{1}{2}}(z,M,N)$ given in Refs. [7] and [3] to derive the exact asymptotic expansions of the free energy of the spanning tree $F = -\ln Z$ for all boundary conditions mentioned above in terms of the Kronecker's double series [3,19], which are directly related to elliptic θ functions.

Now we can easily write down all the terms of the exact asymptotic expansion Eq. (2) of the free energy, $F = -\ln Z$, for all models under consideration by using Eqs. (A1) and (A2).

The bulk free energy f_{bulk} in Eq. (2) for the weighted spanning tree on finite $M \times N$ lattices for all boundary conditions is given by

$$f_{\text{bulk}} = -\frac{2}{\pi}\int_0^\pi \omega_z(x)dx = -\frac{1}{\pi}\sum_{n=0}^\infty (-1)^n (n+1/2)^{-2}z^{2n}$$
$$= -\frac{\Phi\left(-z^2,2,\tfrac{1}{2}\right)}{\pi}, \tag{59}$$

where $\Phi(-z^2,2,1/2)$ is the Lerch transcendent. In particular, for isotropic spanning tree ($z=1$), the Lerch transcendent is now $\Phi(-1,2,1/2) = 4G$, where $G = 0.915965594\ldots$ is the Catalan constant. In what follows we can set $x_2 = 1$ without loss of generality.

A. Spanning tree on the torus

Using Eqs. (12) and (A1), the exact asymptotic expansions of the free energy for the spanning tree on torus, $F = -\ln Z_{M,N}^{\text{Torus}}$ can be written as

$$F = -\ln Z_{M,N}^{\text{Torus}} = \ln S - 2\ln Z_{0,0}(z,M,N)$$
$$= Sf_{\text{bulk}} - \ln\xi - 4\ln\eta(iz\xi)$$
$$+ 4\pi\xi\sum_{p=1}^\infty \left(\frac{\pi^2\xi}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(iz\xi)}{2p+2}, \tag{60}$$

where f_{bulk} is given by Eq. (59). Thus, the exact asymptotic expansion of the free energy for the periodic-boundary conditions can be written in the form given by Eq. (2). The bulk free energy is given by Eq. (59). The surface free energy for the spanning tree f_{1s} and f_{2s} in Eq. (2) are equal to zero. For the leading correction terms $f_0(z\xi)$ we obtain

$$f_0(z\xi) = -\ln\xi - 4\ln\eta(iz\xi)$$
$$= -2\ln\eta(iz\xi)\eta(i/(z\xi)) + \ln z, \tag{61}$$

in which $\xi = \frac{N}{M}$. Here we use the behavior of the Dedekind η function $\eta(\tau') = \sqrt{-i\tau}\,\eta(\tau)$ under the Jacobi transformation $\tau \to \tau' = -1/\tau$, for $\tau = iz\xi$.

For subleading correction terms $f_p(z\xi)$ for $p = 1,2,3,\ldots,$ we obtain

$$f_p(z\xi) = 4\pi^{2p+1}\xi^{p+1}\frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(iz\xi)}{2p+2}.$$

The coefficients Λ_{2p} are listed in Ref. [7] and Kronecker's double series $K_{2p+2}^{0,0}(iz\xi)$ in terms of the elliptic θ functions are given in Refs. [3,5,7] for $p = 1, 2, 3,$ and 4.

It is easy to see from Eq. (61) that for the spanning tree on finite square lattices under periodic-boundary conditions, $f_0(z\xi)$ does not contain the corner free energy f_{corner} given by Eq. (3), which confirm both conformal theory [12] and finite-size scaling [20] predictions that logarithmic corner corrections to the free-energy density should be absent for periodic-boundary conditions. However, such terms have been found by Duplantier and David [17] in the two-dimensional spanning tree (ST) model under periodic-boundary conditions:

$$f_{\text{corner}} = -\ln S. \tag{62}$$

This discrepancy comes from the fact that Eq. (62) has been obtained for the rooted spanning-tree model. The logarithmic correction to the free energy obtained by Duplantier and David (62) is connected with the fact that the number of rooted spanning trees is S times larger than that of the unrooted spanning trees (see Eq. (1.3) of [17]). It is not related to the contribution to free energy from the corner. Taking into account that the result for the free energy for unrooted spanning trees (considering in the present paper) differs from the rooted spanning trees by a factor $\ln S$, we can obtain the correct version for the corner free energy $f_{\text{corner}} = 0$ by adding to Eq. (62) the term $\ln S$.

B. Spanning tree on the cylinder

Using Eqs. (13) and (A1), the exact asymptotic expansions of the free energy for the spanning tree on cylinder, $F = -\ln Z_{M,N}^{\text{Cyl}}$ can be written as

$$F = -\ln Z_{M,N}^{Cyl}$$
$$= M\operatorname{arcsinh} 1/z + \ln N + \ln z - \ln Z_{0,0}(z,M,2N)$$
$$= Sf_{\text{bulk}} - 2\ln\eta(iz\xi) - \frac{1}{2}\ln 2 + \ln z$$
$$- 2\pi\xi\sum_{p=1}^\infty \left(\frac{\pi^2\xi}{2S}\right)^p \frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(iz\xi)}{2p+2}. \tag{63}$$

Thus, the exact asymptotic expansions of the free energy for the cylindrical-boundary conditions can be written in the form given by Eq. (2). The bulk free energy is given by Eq. (59). The surface free energies for the spanning trees f_{1s} and f_{2s} in Eq. (2) are

$$f_{1s} = \tfrac{1}{2}\ln(z + \sqrt{1+z^2}), \tag{64}$$
$$f_{2s} = 0. \tag{65}$$

For the leading correction terms $f_0(z\xi)$ we obtain

$$f_0(z\xi) = -2\ln\eta(iz\xi) - \tfrac{1}{2}\ln 2 + \ln z$$
$$= -\ln\eta(iz\xi)\eta[i/(z\xi)] - \tfrac{1}{2}\ln 2 + \tfrac{1}{2}\ln z - \tfrac{1}{2}\ln\xi, \tag{66}$$

in which $\xi = \frac{N}{M}$. For subleading correction terms $f_p(z\xi)$ for $p = 1,2,3,\ldots,$ we obtain

$$f_p(z\xi) = \frac{\pi^{2p+1}\xi^{p+1}}{2^{p-1}}\frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(iz\xi)}{2p+2}.$$

The coefficients Λ_{2p} are listed in Ref. [7] and Kronecker's double series $K_{2p+2}^{0,0}(iz\xi)$ in terms of the elliptic θ functions are given in Refs. [3,5,7] for $p = 1, 2, 3$, and 4.

C. Spanning tree on the plane

Using Eqs. (14) and (A1), the exact asymptotic expansions of the free energy for the spanning tree on plane, $F = -\ln Z_{M,N}^{\text{Free}}$, can be written as

$$
\begin{aligned}
F &= -\ln Z_{M,N}^{\text{Free}} \\
&= \frac{1}{2}\ln S + N\operatorname{arcsinh} z + M\operatorname{arcsinh} 1/z + \ln z \\
&\quad - \frac{1}{4}\ln(1+z^2) - \frac{1}{2}\ln 2 - \frac{1}{2}\ln Z_{0,0}(z,2M,2N) \\
&= Sf_{\text{bulk}} + N\operatorname{arcsinh} z + M\operatorname{arcsinh} 1/z + \frac{1}{4}\ln S - \frac{1}{4}\ln\xi \\
&\quad - \ln\eta(iz\xi) + \ln z - \frac{1}{4}\ln(1+z^2) - \ln 2 \\
&\quad + \pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{4S}\right)^p \frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(iz\xi)}{2p+2}.
\end{aligned}
\tag{67}
$$

Thus, the exact asymptotic expansions of the free energy for the free-boundary conditions can be written in the form given by Eq. (2). The bulk free energy is given by Eq. (59). The surface free energy for the spanning tree f_{1s} is given by Eq. (64) and $f_{2s} = -\frac{1}{2}\ln z + \frac{1}{2}\ln(1+\sqrt{1+z^2})$. For the leading correction terms $f_0(z\xi)$ we obtain

$$
\begin{aligned}
f_0(z\xi) &= \frac{1}{4}\ln S - \frac{1}{4}\ln\xi - \ln\eta(iz\xi) - \ln 2 \\
&\quad - \frac{1}{4}\ln(1+z^2) + \ln z \\
&= \frac{1}{4}\ln S - \frac{1}{2}\ln\eta(iz\xi)\eta[i/(z\xi)] - \ln 2 \\
&\quad - \frac{1}{4}\ln(1+z^2) + \frac{5}{4}\ln z,
\end{aligned}
\tag{68}
$$

in which $\xi = \frac{N}{M}$. For subleading correction terms $f_p(z\xi)$ for $p = 1,2,3,\ldots$, we obtain

$$
f_p(z\xi) = \frac{\pi^{2p+1}\xi^{p+1}}{2^{2p}}\frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(iz\xi)}{2p+2}.
$$

The coefficients Λ_{2p} are listed in Ref. [7] and Kronecker's double series $K_{2p+2}^{0,0}(iz\xi)$ in terms of the elliptic θ functions are given in Ref. [7] for $p = 1, 2, 3$, and 4.

It is easy to see from Eq. (68) that for the spanning tree on finite square lattices under free-boundary condition $f_0(z\xi)$ contains the universal part f_{univ} given by Eq. (6). This confirms the conformal field theory prediction for the corner free energy in models for which the central charge is $c = -2$. Moreover, $f_0(z\xi)$ contains the nonuniversal, geometry-independent constant f_{nonuniv} given by Eq. (7). Again, as in the case of periodic-boundary conditions, there is discrepancy with the results of Duplantier and David [17] for the corner free energy in the spanning tree on finite square lattices under free-boundary condition. They obtained for the corner free energy the expression

$$
f_{\text{corner}} = -\frac{3}{4}\ln S, \tag{69}
$$

which is different from the conformal field theory prediction Eq. (3). Noting that the result for the unrooted spanning tree differs from that of the rooted spanning tree by a factor $\ln S$, we obtain the correct version for the corner free energy given by Eq. (3) with $c = -2$ by adding to Eq. (69) the term $\ln S$.

D. Spanning tree on the Möbius strip

Using Eqs. (12) and (A1), the exact asymptotic expansions of the free energy for the spanning tree on Möbius strip, $F = -\ln Z_{M,N}^{\text{Mob}}$ can be written as

$$
\begin{aligned}
F &= -\ln Z_{M,N}^{\text{Mob}} = \ln M + N\operatorname{arcsinh} z - \ln Z_{0,0}(z,M,N) \\
&\quad - \ln Z_{\frac{1}{2},\frac{1}{2}}(z,M,N) \\
&= Sf_{\text{bulk}} + N\operatorname{arcsinh} z - \ln\xi - \ln\theta_{\frac{1}{2},\frac{1}{2}}(iz\xi)\eta(iz\xi) \\
&\quad + 2\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(iz\xi) + K_{2p+2}^{\frac{1}{2},\frac{1}{2}}(iz\xi)}{2p+2}.
\end{aligned}
\tag{70}
$$

Thus, the exact asymptotic expansions of the free energy for Möbius-strip-boundary conditions can be written in the form given by Eq. (2). The bulk free energy is given by Eq. (59), and the surface free energy f_{1s} is given by Eq. (64) and f_{2s} in equal to zero.

For the leading correction terms $f_0(z\xi)$ we obtain

$$
f_0(z\xi) = -\ln\xi - \ln\theta_{\frac{1}{2},\frac{1}{2}}(iz\xi)\eta(iz\xi), \tag{71}
$$

in which $\xi = \frac{N}{M}$. For subleading correction terms $f_p(z\xi)$ for $p = 1,2,3,\ldots$, we obtain

$$
f_p(z\xi) = 2\pi^{2p+1}\xi^{p+1}\frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(iz\xi) + K_{2p+2}^{\frac{1}{2},\frac{1}{2}}(iz\xi)}{2p+2}.
$$

The coefficients Λ_{2p} are listed in Ref. [7] and Kronecker's double series $K_{2p+2}^{0,0}(iz\xi)$ and $K_{2p+2}^{\frac{1}{2},\frac{1}{2}}(iz\xi)$ in terms of the elliptic θ functions are given in Refs. [3,5,7].

E. Spanning tree on the Klein bottle

Using Eqs. (12) and (A1), the exact asymptotic expansions of the free energy for the spanning tree on the Klein bottle, $F = -\ln Z_{M,N}^{\text{Klein}}$ can be written as

$$
\begin{aligned}
F &= -\ln Z_{M,N}^{\text{Klein}} = \ln 2M - \ln Z_{0,0}(z,M,2N) \\
&= Sf_{\text{bulk}} - \ln 2\xi - 2\ln\eta(2iz\xi) \\
&\quad + 4\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(2iz\xi)}{2p+2}.
\end{aligned}
\tag{72}
$$

Thus, the exact asymptotic expansions of the free energy for the Klein-bottle-boundary conditions can be written in the form given by Eq. (2). The bulk free energy is given by Eq. (59). The surface free energy f_{1s} and f_{2s} in Eq. (2) are equal to zero. For the leading correction terms $f_0(z\xi)$ we obtain

$$
f_0(z\xi) = -\ln 2\xi - 2\ln\eta(2iz\xi), \tag{73}
$$

in which $\xi = \frac{N}{M}$. For subleading correction terms $f_p(z\xi)$ for $p = 1, 2, 3, \ldots$, we obtain

$$f_p(z\xi) = 4\pi^{2p+1}\xi^{p+1}\frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(2iz\xi)}{2p+2}.$$

The coefficients Λ_{2p} are listed in Ref. [7] and Kronecker's double series $K_{2p+2}^{0,0}(2iz\xi)$ in terms of the elliptic θ functions are given in Ref. [7] for $p = 1, 2, 3,$ and 4.

VI. CONCLUSION

In this paper, we have used the method of Refs. [3] and [7] to derive exact finite-size corrections for the logarithm of the partition function of the spanning-tree model on the $M \times N$ square lattice with five different sets of boundary conditions. We have found that the exact asymptotic expansion of the free energy of the spanning-tree model can be written in the form given by Eq. (2). Except for the bulk free energy f_{bulk}, all other coefficients in this expansion are sensitive to the boundary conditions. We have established several groups of new identities relating to the spanning-tree partition functions for different boundary conditions. We explain an apparent discrepancy between conformal field theory predictions and a two-dimensional spanning-tree model with periodic and free-boundary conditions [17,21]. We have also obtained the corner free energy for free-boundary conditions. We proved the conformal field theory prediction about the corner free energy and have shown that the corner free energy, which is proportional to the central charge c, is indeed universal. We find the central charge in the framework of the conformal field theory to be $c = -2$.

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APPENDIX: ASYMPTOTIC EXPANSION OF $Z_{0,0}(z, M, N)$ AND $Z_{\frac{1}{2}, \frac{1}{2}}(z, M, N)$

For the convenience of the reader, in this appendix we present the exact asymptotic expansions of the logarithm of $Z_{0,0}(z, M, N)$ and $Z_{\frac{1}{2}, \frac{1}{2}}(z, M, N)$ given, respectively, in Refs. [7] and [5].

The exact asymptotic expansion of the logarithm of $Z_{0,0}(z, M, N)$ and $Z_{\frac{1}{2}, \frac{1}{2}}(z, M, N)$ in terms of the Kronecker's double series [3,19] can be written as

$$\ln Z_{0,0}(z, M, N) = \frac{S}{\pi}\int_0^\pi \omega_z(x)dx + \ln\sqrt{S\xi} + 2\ln\eta(iz\xi)$$
$$- 2\pi\xi\sum_{p=1}^\infty \left(\frac{\pi^2\xi}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{0,0}(iz\xi)}{2p+2},$$
$$\text{(A1)}$$

$$\ln Z_{\frac{1}{2}, \frac{1}{2}}(z, M, N) = \frac{S}{\pi}\int_0^\pi \omega_z(x)dx + \ln\frac{\theta_{\frac{1}{2}, \frac{1}{2}}(iz\xi)}{\eta(iz\xi)}$$
$$- 2\pi\xi\sum_{p=1}^\infty \left(\frac{\pi^2\xi}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!}\frac{K_{2p+2}^{\frac{1}{2}, \frac{1}{2}}(iz\xi)}{2p+2},$$
$$\text{(A2)}$$

where $S = MN$, $\xi = N/M$, $\eta(\tau)$ is the Dedekind-η function, $K_{2p}^{0,0}(\tau)$ and $K_{2p}^{\frac{1}{2}, \frac{1}{2}}(\tau)$ is Kronecker's double series [3,19], and function $\theta_{\frac{1}{2}, \frac{1}{2}}(i\tau) = \theta_3(\tau)$ is elliptic θ function. Λ_{2p} is the differential operators that have appeared here and can be expressed via coefficients z_{2p} of the Taylor expansion of the lattice dispersion relation $\omega_z(k)$ (see, for example, Ref. [7])

$$\omega_z(k) = \text{arcsinh}\,(z\sin k) = k\left(z + \sum_{p=1}^\infty \frac{z_{2p}}{(2p)!}k^{2p}\right), \quad \text{(A3)}$$

The Kronecker's double series $K_{2p}^{0,0}(\tau)$ and $K_{2p}^{\frac{1}{2}, \frac{1}{2}}(\tau)$ can all be expressed in terms of the elliptic θ functions only [3,5,7].

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