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Elastic buckling of columns with end restraint effects

R. Adman[a] and M. Saidani[b]
[a]Faculty of Civil Engineering, U.S.T.H.B., Algiers, Algeria
[b]Faculty of Engineering and Computing, Coventry University, Priory Street, Coventry CV1 5FB, UK

Abstract

It is a well established fact that the behaviour of columns as part of a structure is affected by the end restraints. The main aim of the current study is to develop a criterion of stability capable of predicting an impending failure by elastic buckling of a column of a structure. The rigidities at the ends of a column element are modelled using rotational and translational springs, which have been considered by taking into account their coupling effects. The role of the springs is to model the nodal restraints of any column of a given structure. This formulation offers significant practical advantages in the elastic buckling analysis of such structures. This approach is performed through a relationship to several parameters, such as the non-dimensional rotational and translational restraint indices and the effective length factor K. The approach was applied in analysing the elastic buckling of a number of structures and good results were obtained, thus justifying its reliability. In determining the effective length factor K, a marked difference was noted between the results obtained using the Eurocode approach and that proposed by the current study, particularly in the case of non-braced structures.

Keywords: Stability, Elastic buckling, Effective length factor, Column, Second order.

1. Introduction and context of the current work

A substantial body of research has been carried out by a number of researchers on the stability of frames and the concept of effective length and effective length factors. In the 70's, Wood [1, 2, 3] investigated the effective length of columns in multi-storey buildings. In doing so, he defined the rigidity of a joint in a multi-storey setting in terms of the effective length factor K. The Eurocode later adopted this approach. This approach, although limited in practice, remains a powerful analytical tool for the engineer [4, 5]. Cheong-Siat-Moy [6] and Chen and Lui [7] have developed expressions for the effective length factor for single columns with partial lateral end restraints. However, the effect of coupling between rotational and translational rigidities is not taken into account. Also the relative stiffness G is simply adopted as given in the codes.

The K factor represents an important parameter vis-à-vis the elastic buckling analysis. It can easily accommodate the elastic critical load by using a single formula covering all situations of boundary condition, expressed by the following equation:

$$N = \pi^2 EI/(KL)^2 \tag{1}$$

Where, K represents the ratio between the effective length "l_f" and the actual length "l" of the column:

$$K = l_f/l \tag{2}$$

From a physical point of view, "the effective length is the length of the equivalent pin-ended column that would have the same elastic critical load as the actual end-restrained column". [8]

This study offers a simple and yet a global approach that allows a very rigorous assessment of the effective length factor K. The criterion of analysis derives from the solution of the deflection y(x) of a column element where four springs are introduced at the ends to model the rotational and translational flexibilities. This formulation offers significant practical advantages in the elastic buckling analysis of such structures. Furthermore, the values of the effective length factor K for some situations of conventional boundary condition are known. Figure 1 below provides an overview regarding the values of K and their respective buckling modes. Obviously, in practice there are an infinite number of situations relating to the boundary conditions, which do not correspond to conventional situations especially when the column element of interest is considered in relation to partially braced structures.

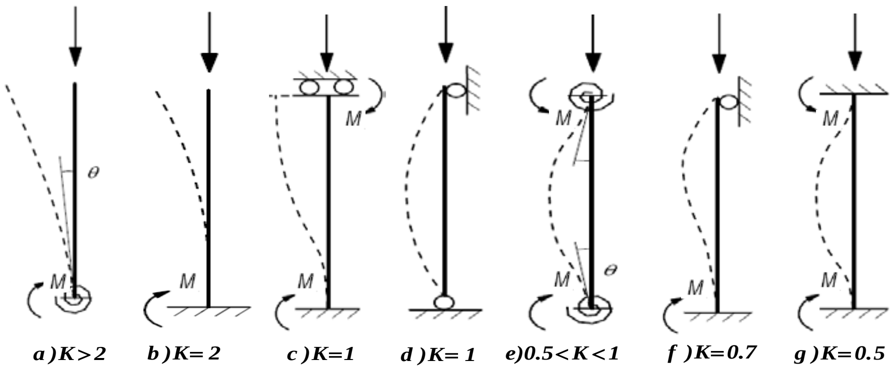

Fig. 1 Values of K and respective buckling modes for some conventional boundary conditions

Hellesland and Bjorhovde [9] used what they described the method of means to determine the effective length factors for continuous columns and beams. The method uses effective length factors from isolated columns as input. Comparison between actual results and those using the method were found to be in order of 5%.

The method was limited when dealing with unbraced frames or when columns were laterally very flexible. Also, coupling effects were not considered.

Previous researchers such as Aristizabal-Ochoa [10, 11] and Hellesland [12,13] used the so-called fixity factors or as also known degree of rotational fixity factor in solving the stability equations. However, such factors do not take into account the coupling effects between rotational and translational flexibilities.

The novelty of the current study is that efficient and more general parameters, equivalent to those previously cited, were obtained by solving the problem of global stability of the column with end restraints with coupling effects fully taken into consideration.

The purpose of the current study is to investigate the effect of end restraints on the column elastic buckling. This is achieved by proposing a general criterion resulting from the solution of the stability equations, taking into account the end conditions of the column in terms of non-dimensional translational and rotational end restraint indices and their coupling effects.

2. Formulation

Consider a column segment (ij) given in Figure 2.

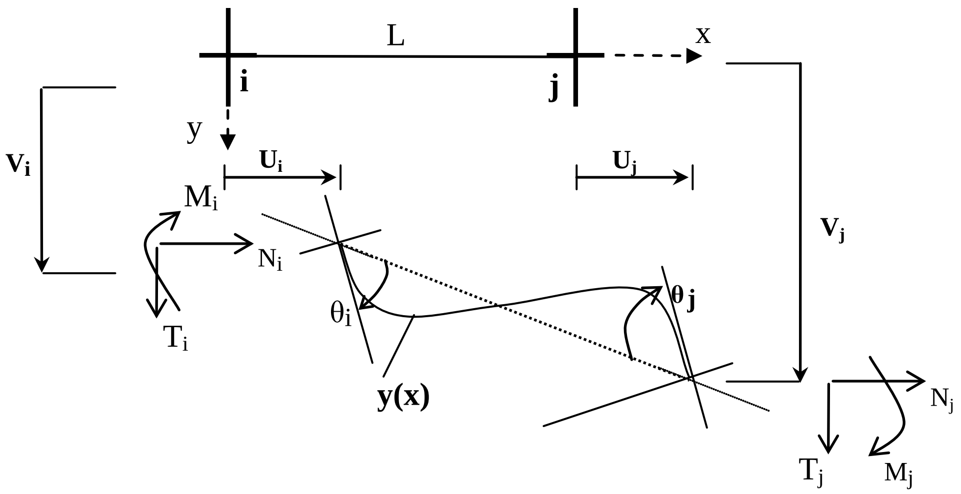

Fig.2 Column model (End moments, forces, rotations and deflections)

The equilibrium equation along the element length can be expressed in compressive case as:

$$y(x)_{xx} + (\beta/L)^2\, y(x) = -\, M(x)\,/EI$$

(3)

where the non-dimensional parameter β is given as:

$$\beta = \sqrt{\frac{|N|L^2}{EI}} \qquad (4)$$

and (xx) indicate the second derivative operator.

$$M(x) = M_i + T_{i.x} \qquad (5)$$

$$T_i = (M_j - M_i)/L \qquad (6)$$

N is the axial force, M_i and M_j are the nodal moments, T_i and T_j are the nodal shear forces (shown as positives in Figure 1.), E=Young modulus and I= second moment of area of the section about the axis of bending.

The solution of Equation (3) is obtained as:

$$y(x) = C_1 \cos(\beta \frac{x}{L}) + C_2 \sin(\beta \frac{x}{L}) + C_3(\frac{x}{L}) + C_4 \qquad (7)$$

C1, C2, C3, C4, are constants depending on the boundary conditions of the element (ij). Four degrees of freedom (d.o.f.) are taken into account: two displacements Vi and Vj according to OY axis, and two rotations θi and θj around OZ axis, (⊥x0y).

To undertake this task, it is essential to consider the exact boundary conditions on the ends of the nodal element. Indeed, the nodal displacements and rotations described by the variables (vi θi, vj, θj), depend directly of requirements for the ends of the element. In this regard, the physical model illustrated in Figure 3 is adopted, which is distinguished by an unconventional behaviour at each node of the element.

The particular solution for eqn. (3), which refers to the loading applied to the column has no effect on the formulation of the stiffness matrix, and therefore has no influence on the parameter K being the main objective of this study.

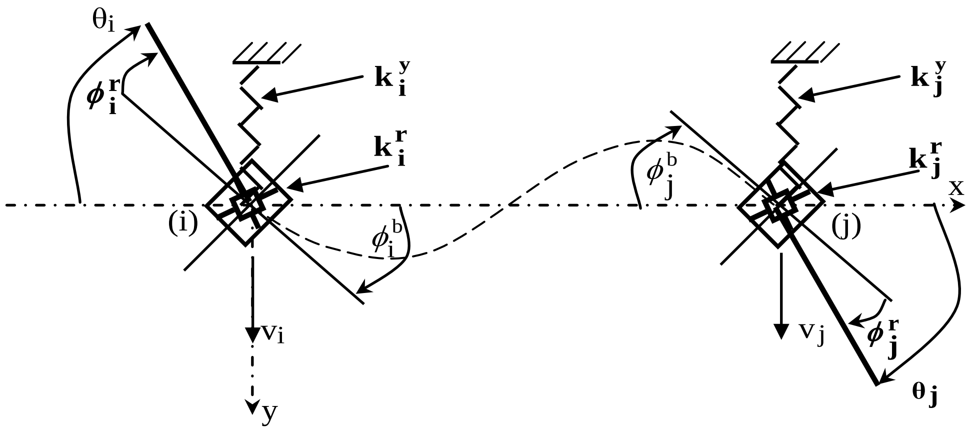

Fig.3 Structural model involving the presence of end restraints relating to rotations and translations

By using the appropriate boundary conditions,

$$\begin{cases} v_i = y(0) + v_i^y & \text{(a)} \\ \theta_i = y_{,x}(0) + \varphi_i^r & \text{(b)} \\ v_j = y(L) + v_j^y & \text{(c)} \\ \theta_j = y_{,x}(L) + \varphi_j^r & \text{(d)} \end{cases} \tag{8}$$

where, (φ_i^r , φ_j^r, v_i^y , v_j^y) refers to equations (8(a), (b), (c), and (d)), the rotations and displacements, respectively associated with the springs of rotation and translation located at nodes (i) and (j) of the column element. The following relations express these quantities:

$$\begin{aligned} V_i^y &= S_i^y\, T_i & \text{(a)} \\ \varphi_i^r &= S_i^r\, M_i & \text{(b)} \\ V_j^y &= S_j^y\, T_j & \text{(c)} \\ \varphi_j^r &= S_j^r\, M_j & \text{(d)} \end{aligned} \tag{9}$$

where ($S_i^r = \frac{1}{k_i^r}$, $S_j^r = \frac{1}{k_j^r}$, $S_i^y = \frac{1}{k_i^y}$, $S_j^y = \frac{1}{k_j^y}$) respectively designate the flexibilities of the springs located at the ends of the column element. These flexibilities correspond to the inverse of the rigidity $\left(k_i^r , k_i^y , k_j^r , k_j^y\right)$ of the springs located at the ends of the column element. Also, Bending moments and shear forces (Ti, Mi, Tj, Mj)., relative to the nodes (i) and (j) of the element, respectively, are deduced from the general relations of bending moment and shear force obtained from the equilibrium of an infinitesimal elementary section.

Quantities y(0) and y(L) in equations 8(a) and (c) refer to the lateral displacements associated with nodes (i) and (j) respectively, of the column element. These quantities are obtained from equation (7).

Lastly, (x) indicate the first derivative operator and the quantities ($y_{,x}(0)$, $y_{,x}(L)$) refers to relations 8(b) and 8(d), the rotations due to column bending only, respectively associated with the nodes (i) and (j). These quantities are obtained from equation (7).

For convenience, the following four parameters $\left(f_i^r, f_i^y, f_j^r, f_j^y\right)$ respectively designate the non-dimensional rotational and translational restraint indices of the springs located at the ends of the element. These indices vary from zero for fully restrained connection to infinity for simple connection.

$$\begin{cases} f_i^r = \dfrac{S_i^r}{S_b^r} & \text{(a)} \\ f_j^r = \dfrac{S_i^r}{S_b^r} & \text{(b)} \\ f_i^y = \dfrac{S_i^y}{S_b^y} & \text{(c)} \\ f_j^y = \dfrac{S_j^y}{S_b^y} & \text{(d)} \end{cases} \qquad (11)$$

Note that the flexibility characterizing the column itself is represented by analogy as two springs: one rotational and one translational, denoted by $\left(S_b^r, S_b^y\right)$. The following relations express these flexibilities:

$$\begin{cases} Sb^{r} = \frac{L}{EI} & \text{(a)} \\ Sb^{y} = \frac{L^{3}}{EI} & \text{(b)} \end{cases} \tag{12}$$

Therefore, we can observe that when y(x) is infinitely large, the denominator of y(x) will approach zero, leading to a non-dimensional quantity denoted dL expressed by the following relationships (13).

$$\begin{aligned} dL = {} & \beta\left(-2+2\cos(\beta)+\beta\sin(\beta)\right)-\left(f_i^r+f_j^r\right)\left(\beta^3\cos(\beta)-\beta^2\sin(\beta i)\right)-f_i^r f_j^r\beta^4\sin(\beta) \\ & -\left(f_i^y+f_j^y\right)\left(\beta^4\sin(\beta)+\left(f_i^r+f_j^r\right)\beta^4\cos(\beta)-f_i^r f_j^r\beta^6\sin(\beta)\right) \end{aligned} \tag{13}$$

3. Numerical analysis

The next step is intended to highlight the second order effects associated with the phenomenon of instability by buckling. This task is carried out based on the expression of the displacement, which mathematically would tend to infinity when the axial load approaches the critical buckling load. This extreme limit corresponds physically to the condition of instability of the column, which from a mathematical point of view is reached when a quantity dL becomes zero for the first time after the initial equilibrium (axial load equal to zero). This condition is obtained for specific boundary conditions described by the non-dimensional translational and rotational restraint indices. Thus, it is sufficient to monitor changes in the quantity dL compared to the K factor, which gives the value of K that makes dL equal to zero. Finally, the critical axial load of the element is deduced from the value of K obtained. This is obtained from the equation (1) given above.

The values of K that nullify the expression dL are compared with those obtained from the relationships (14) proposed in the Eurocode 2 [14]. Thus, it may be observed that for braced structures the equation of K as given by Eurocode 2 is expressed in relation to indexes of flexibility on both rotational springs located at each end node of the element.

$$K = 0.5\sqrt{\left(1+\frac{f_i^r}{0.45+f_i^r}\right)\left(1+\frac{f_j^r}{0.45+f_j^r}\right)} \quad (14)$$

Figure 4 shows the variation of the effective length factor K of a beam column element belonging to braced structures against the indexes of flexibility. It may be seen that the curves (blue) obtained numerically in this study and those drawn from the relationship of K by Eurocode 2 (red) are virtually identical.

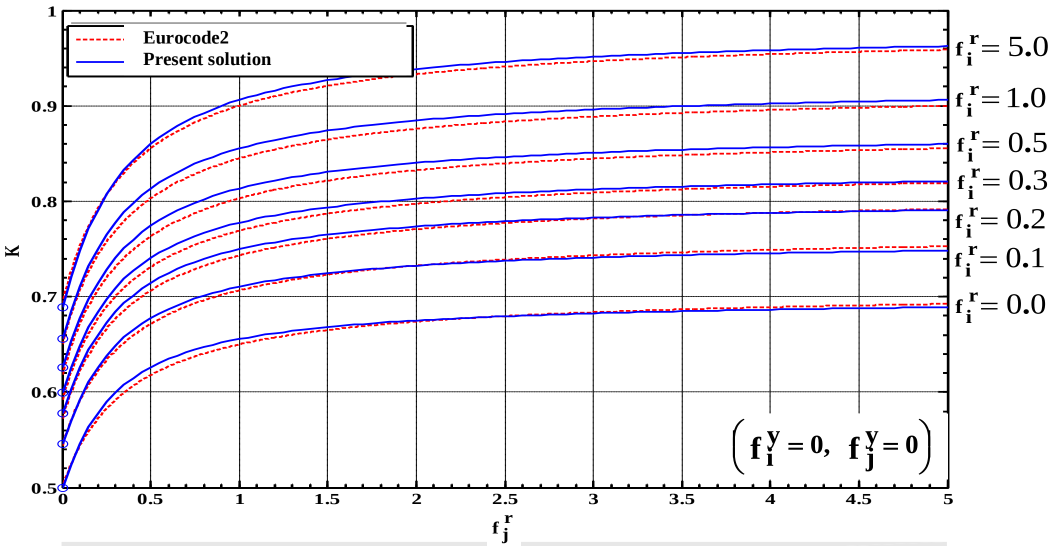

Figure 4: Curves of the effective length factor K
(Case of braced structures)

Obviously, in the case of elements belonging to braced structures, in the expression of dL, $f_i^y = 0$ and $f_j^y = 0$. Thus, it can be seen from Figure 4 above, that a quasiperfect similarity between the results of the current study and those proposed by Eurocode2 exists. Moreover, both approaches lead to classical results: K = 0.5 when $\left(f_i^r = 0, f_j^r = 5\right)$ and K approaching unity when $\left(f_i^r = 5, f_j^r = 5\right)$ corresponding, respectively, to the two extreme cases of a built-in-ended member and a pin-ended member.

For elements belonging to partially braced structures, it is necessary to emphasize the complexity of the relationship expressing the K factor which depends on a number of parameters related to the boundary conditions reflected by the indexes of flexibilities, particularly those relating to translational springs. In this context, the same reference [14] suggests for non-braced structures, a relationship of K based on two limits as shown by the relations (15) below (see also Westerberg [15]). It is worth noting in this formulation the lack of parameters reflecting the translational springs.

$$K = \max\left\{ \sqrt{\left(1 + 10\frac{f_i^r f_j^r}{f_i^r + f_j^r}\right)}\ ;\ \left(1 + \frac{f_i^r}{1 + f_i^r}\right)\left(1 + \frac{f_j^r}{1 + f_j^r}\right) \right\} \tag{15}$$

Let us now consider an element whose node (i) is completely fixed $\left(f_i^r = 0, f_i^y = 0\right)$. The node (j) may in turn change from a completely fixed situation $\left(f_j^r = 0, f_j^y = 0\right)$ (case of braced element) to a quasi-free situation in rotation and translation $\left(f_j^r = 5, f_j^y = 5\right)$, which corresponds to the cases of a non-braced element. All situations between these two extreme cases correspond to partially braced structures. Figure 5 shows the variation of the effective length factor K of a beam column element belonging to partially braced structures versus the indexes of flexibility.

It may be seen that factor K tends to value equal to 2 for a quasi-free situation on node (j) $\left(f_j^r = 5, f_j^y = 5\right)$, for both approaches (current study and Eurocode 2). Moreover, this approach allows to evaluate the effective length factor K corresponding to different levels of fixation relating to transverse displacement, which reflects cases of partially braced structures. As an indication, for $f_j^r = 5$ (i.e. node (j) quasi-articulated), the value of K increases gradually from 0.7 when $f_j^y = 0$, to 1.8 when $f_j^y = 5$, which approximates the situation when node (j) becomes quasi-free.

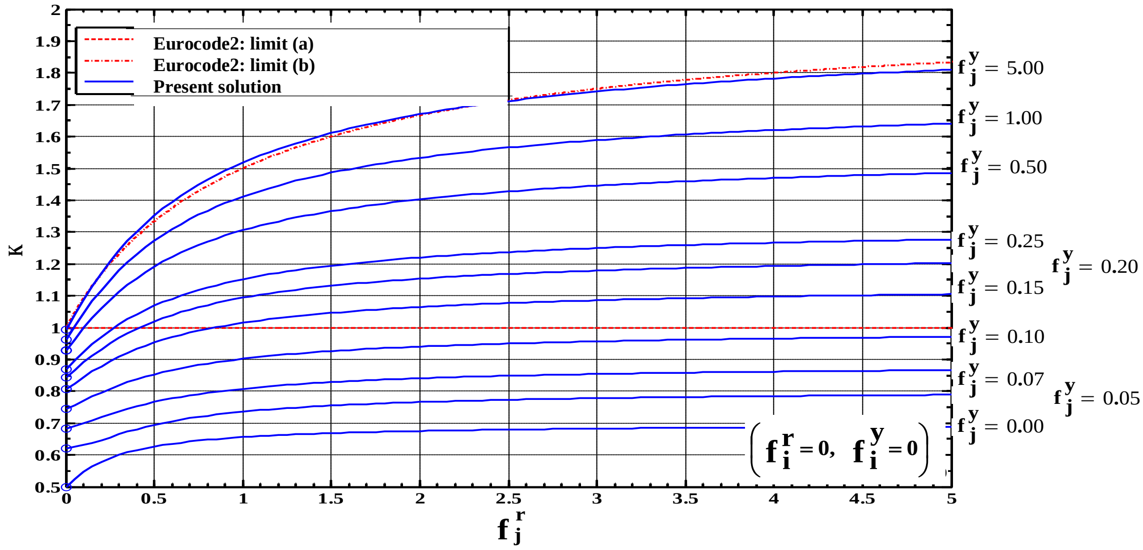

Figure 5: Effective length factor K for partially braced structures
(end i built-in)

As shown in Figure 6, the analysis conducted by considering node (i) quasi-articulated $\left(f_i^r = 5, f_i^y = 0\right)$ leads to a significant difference between the results obtained from both approaches (current study and Eurocode 2). It is worth noting that for $\left(f_j^r = 5, f_j^y = 5\right)$ (node (j) quasi-free), Eurocode 2 [14] leads to K= 5.1 which is approximately 20% higher than the result obtained by the current study (K= 4.25), which will result in an additional cost in design.

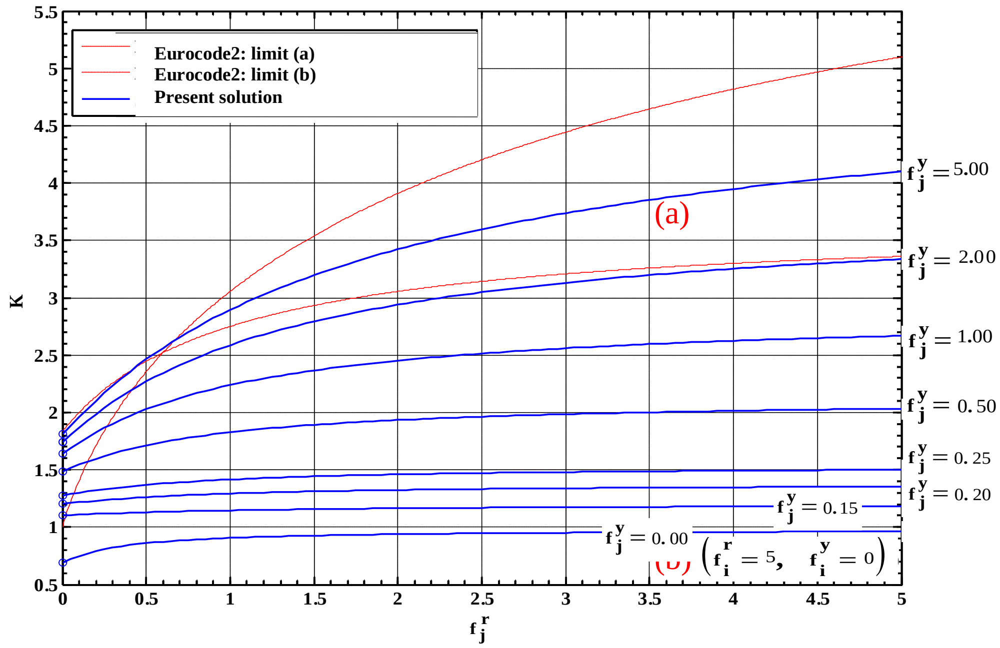

Figure 6: Effective length factor K for partially braced structures (end i being pinned)

4. Conclusions

The stability of a beam-column element has been investigated by the current study by considering general end conditions with varying flexibilities. A criterion of stability is obtained analytically from the resolution of the differential equilibrium equation.

Using the proposed criterion, reliable results are obtained for the effective length factor K of a beam column element belonging to braced, non-braced or partially braced structures. This is highly desirable for the numerical convergence, and for predicting the buckling load accurately.

The results of the effective length factor K obtained from the current study and those using the Eurocode 2 are compared. The following observations may be made:

- In the case of braced structures, a perfect correlation in the results exists.
- In the case of non-braced structures, a relative difference of 20% was observed, making Eurocode 2 a more expensive way to design.

Additionally, using the current approach, the value of factor K of any element belonging to braced, non-braced or partially braced structures may be obtained.

This performance is achieved through the formulation of the stability criterion, which is based on indexes of flexibility that reflect its actual original nodal boundaries.

Finally, this result offers to the finite element approach the needed tool of accuracy required besides the simplicity of the method.

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