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RESEARCH ARTICLE

Parity Equations-based Unknown Input Reconstruction for MIMO Stochastic Systems with an Application

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In this paper an approach for reconstructing an unknown input for multiple-input multiple-output systems is presented. It is assumed that the system is affected by process noise, whereas the available system inputs are contaminated by measurement noise. The novel approach is based on a parity equations concept and forms an extension of the idea developed previously by the authors. A modification of the algorithm is also provided, which allows the approach to deal with systems whose zero is close to unity. The order of the parity space can be treated as a tuning parameter allowing for a trade-off between the smoothness of the reconstructed unknown input and a phase lag. An analytical solution of the overall problem is obtained by making use of a Lagrange multiplier method. The utility of the scheme is demonstrated when applied to a practical hydrological system.

Keywords: filtering; parity equations; unknown input reconstruction

1 Introduction

In the literature the problem of the unknown (unmeasurable) input estimation is solved either by a system inversion or by a joint state and input estimation. Early contributions to the inversion of multiple-input multiple-output (MIMO) deterministic systems have been presented by Dorato (1969) and Sain and Massey (1969); however, their approaches do not ensure stability of the inverted systems. Moylan (1977) provided a stable inversion algorithm for minimum-phase systems, whilst Antsaklis (1978) developed a straightforward state feedback-based method, which allows the poles of the inverted system to be assigned. This latter method is, however, limited to systems with stable zeros.

Over the last decade a geometric approach to the unknown input reconstruction problem has gained interest, see e.g. Kirtikar et al. (2009) where an unknown input reconstruction scheme for minimum phase systems is proposed. An exhaustive solution to the unknown-state, unknown-input reconstruction problem for both minimum-phase and nonminimum-phase systems has relatively recently been developed by Marro and Zattoni (2010). Nevertheless, this approach does not consider the effects of measurement noise.

Another approach to the unknown input estimation problem for deterministic systems is based on the well-known state observer. The Luenberger state observer, see Luenberger (1964), has been extended to the class of systems with both known and unknown system inputs, see, for example Hou and Müller (1992), Darouach and Zasadzinski (1997). The work of Fernando and Trinh (2006) presents a joint input and state observer based on a descriptor approach.

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When dealing with stochastic systems Kalman filter-based approaches have gained wide interest, see, for example, Hsieh (2000), Floquet and Barbot (2006). Gillijns and De Moor (2007a) combined the state observer proposed by Darouach and Zasadzinski (1997) and the unknown input estimator of Hsieh (2000) creating a joint state and unknown input observer, which is optimal in the minimum variance sense. This approach has subsequently been extended to the case of a linear system with a direct feedthrough term in Gillijns and De Moor (2007b). Palanthandalam-Madapusi and Bernstein (2007) introduced a concept of state and input observability, i.e. they provided a scheme, which allows one to determine if both unknown input and state can be derived from the output measurements. Keller and Sauter (2010) proposed a variable geometric Kalman filter, where the statistical effect of each unknown input is tested before deriving the state estimate. In the recent work of Ghahremani and Kamwa (2011) an extended Kalman filter with unknown inputs has been developed and applied to the state estimation of a synchronous machine in a power system.

The area where the unknown input reconstruction is commonly used is fault detection and diagnosis. A fault, which can be modelled as an additional input to the system, needs to be firstly detected and then isolated and reconstructed (identified). Modelling of uncertainties for robust fault diagnosis is described in Patton et al. (1992). Hou and Patton (1998) discussed input observability and reconstruction using a matrix pencil approach. Edwards et al. (2000) propose sliding mode observers as a powerful tool for fault diagnosis.

In this paper a novel approach for the unknown input reconstruction of MIMO discrete-time stochastic systems is described. The parity equation-based unknown input observer (PE-UIO) utilises a parity equation (PE) concept for disturbance-decoupled unknown input reconstruction. The design freedom is used to optimise the filter parameters in order to minimise the effect of stochastic disturbances on the unknown input estimate. For this purpose a Lagrange multiplier method is utilised. The use of the Lagrange multiplier method produces an analytical solution to the optimisation problem, which, unlike heuristic approaches, see, for example, Chen et al. (1996), provides the optimal filter parameters in a relatively short time. The proposed method is suitable for both minimum and nonminimum-phase systems, which is an important result, because unstable zeros may result from the discretisation of a continuous-time system. The PE-UIO was originally developed for single-input single-output (SISO) output error (OE) systems in Sumińska et al. (2010a). The algorithm has been subsequently extended to the errors-in-variables (EIV) framework in Sumińska et al. (2010b). In Sumińska et al. (2011) the analysis of the PE-UIO in the frequency domain has been provided and it has been demonstrated that, unlike the Kalman filter-based approaches, see, for example, Gillijns and De Moor (2007b), Hou and Patton (1998), one can reduce the filter bandwidth by increasing the parity space order. This leads to attenuation of the impact of noise on the unknown input estimate; however, the accompanying phase lag causes an estimation delay to be increased at the same time. In Sumińska et al. (2010) the scheme has been extended to the MIMO case. In this paper the PE-UIO is extended to the case where the process noise is coloured. A generalised form of the algorithm is provided, where the output is subjected to coloured noise (accounting for measurement and process noise), whilst the input is affected by white measurement noise. Furthermore, this paper also provides an extension of the PE-UIO algorithm for the cases when a system zero is close or equal to unity, see Sumińska (2013).

The paper is organised as follows: in Section 2 the proposed approach is presented. Then, in Section 3, a limitation of the scheme in the case when a system has a zero close or equal to unity is discussed and an extension which deals with this problem is provided. The applicability of the scheme is demonstrated on a practical example of a hydrological system in Section 4. Conclusions are provided in Section 5.

2 Description of approach

In this section the PE-UIO algorithm is presented. Firstly, for completeness in Subsection 2.1, a definition of a general linear system representation according to the stochastic properties of the noise is provided. Then, in Subsection 2.2, the concept of PE is explained, and, in Subsection 2.3, the design of the PE-UIO is presented. An extension of the scheme is presented in Subsection 2.4 which is developed to allow for disturbance decoupling.

2.1 Linear system representation

A linear dynamic discrete-time time-invariant MIMO system is defined as:

$$\begin{aligned} \mathbf{y}(t) &= \frac{\mathcal{B}_u(z^{-1})}{\mathcal{A}(z^{-1})} \mathbf{u}_0(t) + \frac{\mathcal{B}_v(z^{-1})}{\mathcal{A}(z^{-1})} v(t) + \boldsymbol{\xi}(t) \\ \mathbf{u}(t) &= \mathbf{u}_0(t) + \tilde{\mathbf{u}}(t) \end{aligned} \quad (1)$$

where $\mathbf{u}(t) \in \mathbb{R}^p$ and $\mathbf{y}(t) \in \mathbb{R}^m$ denote, respectively, measured input and output vectors, whereas $v(t)$ denotes the scalar unknown (unmeasurable) input. The term $\mathbf{u}_0(t) \in \mathbb{R}^p$ denotes the noise-free input to the system, whilst $\tilde{\mathbf{u}}(t) \in \mathbb{R}^p$ denotes a vector of white zero-mean noise sequences representing the input measurement noise. The terms $\mathcal{A}(z^{-1})$, $\mathcal{B}_u(z^{-1})$, and $\mathcal{B}_v(z^{-1})$ are polynomial matrices in the backward shift operator z^{-1} .

The term $\boldsymbol{\xi}(t) \in \mathbb{R}^m$ denotes a vector of coloured random sequences which represent both the process and measurement noise. A widely used representation for coloured noise is the following autoregressive moving average process, see, for example, Young (2011) and Box and Jenkins (1976)

$$\boldsymbol{\xi}(t) = \frac{\mathcal{C}(z^{-1})}{\mathcal{D}(z^{-1})} \mathbf{e}(t) \quad (2)$$

where $\mathbf{e}(t) \in \mathbb{R}^m$ is a vector of zero-mean, white Gaussian, independent and identically distributed (i.i.d.) random sequences uncorrelated with $\mathbf{u}_0(t)$. Terms $\mathcal{C}(z^{-1})$ and $\mathcal{D}(z^{-1})$ are polynomial matrices which describe properties of the coloured noise $\boldsymbol{\xi}(t)$. Equation (2) is suitable for a wide range of real-world applications where the process noise is strongly correlated, see, Young (2011), Box and Jenkins (1976). The scheme presented in this paper minimises the impact of $\tilde{\mathbf{u}}(t)$ and $\boldsymbol{\xi}(t)$ on the unknown input estimate taking into consideration the properties of the coloured noise $\boldsymbol{\xi}(t)$. The following state-space representation of the system (1) is adopted

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}_0(t) + \mathbf{G}v(t) + \mathbf{\Pi}\mathbf{e}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}_0(t) + \mathbf{H}v(t) + \mathbf{\Omega}\mathbf{e}(t) \\ \mathbf{u}(t) &= \mathbf{u}_0(t) + \tilde{\mathbf{u}}(t) \end{aligned} \quad (3)$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times p}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$, $\mathbf{D} \in \mathbb{R}^{m \times p}$, $\mathbf{G} \in \mathbb{R}^{n \times 1}$, $\mathbf{H} \in \mathbb{R}^{m \times 1}$, $\mathbf{\Pi} \in \mathbb{R}^{n \times m}$, $\mathbf{\Omega} \in \mathbb{R}^{m \times m}$. Equation (3) is a generalised representation of a linear system and can be simplified to provide more specific cases, some of which and their relations to Equation (2) are given below.

2.1.1 Auto-regressive with moving average and exogenous input model

A MIMO auto-regressive with moving average and exogenous input (ARMAX) model is given by, see Ljung (1999), Yiu and Wang (2007):

$$\mathbf{y}(t) = \sum_{i=1}^{n_a} \mathbf{a}_i \mathbf{y}(t-i) + \sum_{j=0}^{n_b} \mathbf{b}_j \mathbf{u}_0(t-j) + \sum_{k=0}^{n_c} \mathbf{c}_k \mathbf{e}(t-k) \quad (4)$$

where $\mathbf{u}_0(t)$ and $\mathbf{y}(t)$ are, respectively, input and output vectors of the system and $\mathbf{e}(t)$ is a vector of white, zero-mean, i.i.d. Gaussian noise sequences. The terms n_a , n_b , n_c , with $n_a \geq n_b$ and $n_a \geq n_c$, are the orders of the auto-regressive, exogenous and moving average polynomials (including the number of coefficient matrices), respectively, and \mathbf{a}_i , \mathbf{b}_j and \mathbf{c}_k are the respective coefficient matrices. The third term of the right-hand side of (4) refers to the moving average (coloured) process noise, noting that the noise $\boldsymbol{\xi}(t)$ in the ARMAX case is defined by the following polynomials, cf. (2)

$$\begin{aligned} \mathcal{C}(z^{-1}) &= \mathbf{c}_0 + \mathbf{c}_1 z^{-1} + \cdots + \mathbf{c}_{n_c} \\ \mathcal{D}(z^{-1}) &= \mathbf{a}_0 + \mathbf{a}_1 z^{-1} + \cdots + \mathbf{a}_{n_a} \end{aligned} \quad (5)$$

The state space system matrices (3) for the ARMAX model (4) in the observable canonical form are given by:

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} -\mathbf{a}_1 & \mathbf{I}_m & 0 & \cdots & 0 \\ -\mathbf{a}_2 & 0 & \mathbf{I}_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mathbf{a}_{n_a-1} & 0 & 0 & \cdots & \mathbf{I}_m \\ -\mathbf{a}_{n_a} & 0 & 0 & \cdots & 0 \end{bmatrix} & \mathbf{B} &= \begin{bmatrix} \mathbf{b}_1 - \mathbf{a}_1 \mathbf{b}_0 \\ \mathbf{b}_2 - \mathbf{a}_2 \mathbf{b}_0 \\ \vdots \\ \mathbf{b}_{n_b} - \mathbf{a}_{n_b} \mathbf{b}_0 \\ -\mathbf{a}_{n_b+1} \mathbf{b}_0 \\ \vdots \\ -\mathbf{a}_{n_a} \mathbf{b}_0 \end{bmatrix} & \mathbf{\Pi} &= \begin{bmatrix} \mathbf{c}_1 - \mathbf{a}_1 \mathbf{c}_0 \\ \mathbf{c}_2 - \mathbf{a}_2 \mathbf{c}_0 \\ \vdots \\ \mathbf{c}_{n_c} - \mathbf{a}_{n_c} \mathbf{c}_0 \\ -\mathbf{a}_{n_c+1} \mathbf{c}_0 \\ \vdots \\ -\mathbf{a}_{n_a} \mathbf{c}_0 \end{bmatrix} \\ \mathbf{C} &= [\mathbf{I}_m \ 0 \ \cdots \ 0] & \mathbf{D} &= \mathbf{b}_0 & \mathbf{\Omega} &= \mathbf{c}_0 \end{aligned} \quad (6)$$

where \mathbf{I}_m is an identity matrix of dimension m . Matrices \mathbf{G} and \mathbf{H} are constructed by replacing \mathbf{b}_i in, respectively, matrices \mathbf{B} and \mathbf{D} with exogenous matrix parameters related to the unknown input $v(t)$. Note that the ARMAX model assumes that the input is known exactly (there is no noise present on the input variable), hence $\tilde{\mathbf{u}}(t) = 0$. An autoregressive model with exogenous input (ARX) is obtained from the ARMAX model by setting \mathbf{c}_i , $i = 1, \dots, n_c$, to zero.

2.1.2 Output error model

An OE model assumes, that there is no process noise present in the system, however, the noise-free output, denoted $\mathbf{y}_0(t)$, is subjected to a zero-mean, white, Gaussian, i.i.d. measurement noise $\mathbf{e}(t)$, see Ljung (1999):

$$\begin{aligned} \mathbf{y}_0(t) &= \sum_{i=1}^{n_a} \mathbf{a}_i \mathbf{y}_0(t-i) + \sum_{j=0}^{n_b} \mathbf{b}_j \mathbf{u}_0(t-j) \\ \mathbf{y}(t) &= \mathbf{y}_0(t) + \mathbf{e}(t) \end{aligned} \quad (7)$$

This case can be modelled by the system representation (3), where matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} and also \mathbf{G} and \mathbf{H} are all given as in the ARMAX case. The matrix $\mathbf{\Pi}$ is null, and $\mathbf{\Omega}$ is diagonal. (Note that different values of diagonal elements of matrix $\mathbf{\Omega}$ assume that the variances of the measurement noise sequences affecting different outputs are in general also different, which is

often the case in practice.) Also there is no noise present on the input variable, hence $\tilde{\mathbf{u}}(t) = 0$. Note that the OE model assumes that $\boldsymbol{\xi}(t) = \mathbf{e}(t)$, cf. (2). The PE-UIO algorithm for a SISO OE case has been developed in Sumińska et al. (2010a).

2.1.3 Errors-in-variables framework

In the EIV framework, see, for example, Söderström (2007), all measured variables, i.e. inputs and outputs of the system, are affected by zero-mean, white, Gaussian, i.i.d. measurement noise sequences. This can be represented by (3), where $\tilde{\mathbf{u}}(t) \neq 0$, $\mathbf{\Pi}$ is null, and $\mathbf{\Omega}$ is diagonal. The PE-UIO algorithm for a SISO EIV case has been presented in Sumińska et al. (2010b, 2011).

2.2 Parity equations

PE are widely used for the purpose of fault detection and isolation, see, for example, Chow and Willsky (1984), Gertler and Singer (1990), Li and Shah (2002). The approach described in this paper utilises the PE to design an unknown input reconstructor.

The stacked vector of the system output $\mathbf{y}(t)$ is considered:

$$\mathbf{Y} = [\mathbf{y}^T(t-s) \mathbf{y}^T(t-s+1) \cdots \mathbf{y}^T(t)]^T \in \mathbb{R}^{(s+1)m} \quad (8)$$

where the term s denotes the order of the parity space. Analogously, one can construct stacked vectors of $v(t)$, $\mathbf{u}(t)$, $\mathbf{u}_0(t)$, $\tilde{\mathbf{u}}(t)$ and $\mathbf{e}(t)$ which are denoted, respectively, as \mathbf{V} , \mathbf{U} , \mathbf{U}_0 , $\tilde{\mathbf{U}}$ and \mathbf{E} . Using this notation the system defined by (3) can be expressed in the form:

$$\mathbf{Y} = \mathbf{\Gamma}\mathbf{x}(t-s) + \mathbf{Q}\mathbf{U}_0 + \mathbf{T}\mathbf{V} + \mathbf{\Xi}\mathbf{E} \quad (9)$$

where $\mathbf{\Gamma}$ is an extended observability matrix:

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^s \end{bmatrix} \in \mathbb{R}^{(s+1)m \times n} \quad (10)$$

and \mathbf{Q} is the following block Toeplitz matrix:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{D} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{C}\mathbf{B} & \mathbf{D} & \cdots & \mathbf{0} \\ \mathbf{C}\mathbf{A}\mathbf{B} & \mathbf{C}\mathbf{B} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}\mathbf{A}^{s-1}\mathbf{B} & \mathbf{C}\mathbf{A}^{s-2}\mathbf{B} & \cdots & \mathbf{D} \end{bmatrix} \in \mathbb{R}^{(s+1)m \times (s+1)p} \quad (11)$$

Analogously, one can construct the matrix $\mathbf{T} \in \mathbb{R}^{(s+1)m \times (s+1)p}$ by replacing \mathbf{B} and \mathbf{D} in \mathbf{Q} by, respectively, \mathbf{G} and \mathbf{H} , and the matrix $\mathbf{\Xi} \in \mathbb{R}^{(s+1)m \times (s+1)m}$ is obtained by replacing \mathbf{B} and \mathbf{D} , respectively, by $\mathbf{\Pi}$ and $\mathbf{\Omega}$. To eliminate the unknown state vector from (9), a row vector $\mathbf{w}^T \in \mathbb{R}^{1 \times (s+1)m}$ is defined, which belongs to the left nullspace of $\mathbf{\Gamma}$, i.e.

$$\mathbf{w}^T \mathbf{\Gamma} = 0 \quad (12)$$

Note that the row vector \mathbf{w}^T can always be found by choosing s to be sufficiently large. Therefore, (9) can be reformulated as:

$$\mathbf{w}^T \mathbf{Y} = \mathbf{w}^T \mathbf{Q}\mathbf{U} - \mathbf{w}^T \mathbf{Q}\tilde{\mathbf{U}} + \mathbf{w}^T \mathbf{T}\mathbf{V} + \mathbf{w}^T \mathbf{\Xi}\mathbf{E} \quad (13)$$

By rearranging the measured (known) variables to the right-hand side of (13) and the unknowns to the left-hand side, the following PE is obtained, cf. Li and Shah (2002):

$$\mathbf{w}^T \mathbf{T} \mathbf{V} + \mathbf{w}^T \mathbf{\Xi} \mathbf{E} - \mathbf{w}^T \mathbf{Q} \tilde{\mathbf{U}} = \mathbf{w}^T \mathbf{Y} - \mathbf{w}^T \mathbf{Q} \mathbf{U} \quad (14)$$

In the following subsection the PE is used to derive a novel algorithm for the unknown input estimation.

2.3 Proposed input estimation procedure

Denote the matrix spanning the left nullspace of $\mathbf{\Gamma}$ as $\mathbf{\Gamma}^\perp$. Consequently, the row vector \mathbf{w}^T is a linear combination of rows of $\mathbf{\Gamma}^\perp$. In the disturbance free case, i.e. when $\mathbf{U} = \mathbf{U}_0$ and $\mathbf{E} = 0$, the following equation holds (cf. (14)):

$$\mathbf{\Gamma}^\perp \mathbf{T} \mathbf{V} = \mathbf{b} \quad (15)$$

where \mathbf{b} is a column vector given by:

$$\mathbf{b} = \mathbf{\Gamma}^\perp \mathbf{Y} - \mathbf{\Gamma}^\perp \mathbf{Q} \mathbf{U} \quad (16)$$

Selection of a sufficiently large s would result in (15) being a set of equations with an explicit solution or an overdetermined set of equations. Nevertheless, in practice, precision of the solution to (15) can still be seriously affected by noise. The algorithm proposed here provides an on-line approximation of the unknown input, simultaneously minimising the unwanted effects of noise.

It is proposed to calculate the value of the unknown input as:

$$\hat{v}(t - \tau) = \mathbf{w}^T \mathbf{Y} - \mathbf{w}^T \mathbf{Q} \mathbf{U} \quad (17)$$

where $\hat{v}(t)$ denotes the estimate of $v(t)$. The term τ is an estimation lag (estimation delay) and it accounts for the fact that the unknown input may not be reconstructed instantaneously. Therefore, at the time instance t the estimate of $v(t - \tau)$ is obtained. The estimation delay τ is defined further in this section. In the noise-free case, $\hat{v}(t - \tau)$ is simply:

$$\hat{v}(t - \tau) = \mathbf{w}^T \mathbf{T} \mathbf{V} \quad (18)$$

Therefore, based on the assumption that the unknown input is varying relatively slowly (see Subsection 3.1), it can be estimated as a linear combination of the sequence $v(t - s)$, $v(t - s + 1)$, \dots , $v(t)$, i.e.

$$\hat{v}(t - \tau) = \alpha_0 v(t) + \alpha_1 v(t - 1) + \dots + \alpha_s v(t - s) \quad (19)$$

where the α_i parameters are dependent on the choice of the vector \mathbf{w}^T , such that:

$$\mathbf{w}^T \mathbf{T} = [\alpha_s \ \alpha_{s-1} \ \dots \ \alpha_0] \quad (20)$$

One can note that (19) represents a moving average finite impulse response filter with the gain being given by the sum of the α_i parameters, i.e. the sum of elements of the vector $\mathbf{w}^T \mathbf{T}$. Thus, it is suggested that \mathbf{w}^T should be selected in such a way, that the sum of the elements of the vector $\mathbf{w}^T \mathbf{T}$ is equal unity. Furthermore, it is anticipated that the choice of both, i.e. the order of the parity space s , as well as the vector \mathbf{w}^T , influences the estimation lag τ in the estimate of the unknown input (due to the moving average filtering property of the unknown input estimator).

The estimation lag is defined as the centre of gravity of the moving average filter rounded to the nearest natural number and is calculated via:

$$\tau = \text{round} \left(\frac{\sum_i \alpha_i i}{\sum_i \alpha_i} \right) \quad (21)$$

In the following subsection an algorithm for the selection of the optimal row vector \mathbf{w}^T is derived based on the Lagrange multiplier method.

2.3.1 Selection of optimal row vector \mathbf{w}^T

In the case of noisy input and output measurements, equation (18) becomes:

$$\hat{v}(t - \tau) = \mathbf{w}^T \mathbf{T} \mathbf{V} + \mathbf{w}^T \mathbf{\Xi} \mathbf{E} - \mathbf{w}^T \mathbf{Q} \tilde{\mathbf{U}} \quad (22)$$

which can be expanded to give:

$$\begin{aligned} \hat{v}(t - \tau) = & \alpha_0 v(t) + \alpha_1 v(t - 1) + \cdots + \alpha_s v(t - s) + \mathbf{w}_{\xi_{s+1}} \mathbf{e}(t) + \mathbf{w}_{\xi_s} \mathbf{e}(t - 1) + \cdots + \\ & \mathbf{w}_{\xi_1} \mathbf{e}(t - s) - \mathbf{w}_{q_{s+1}} \tilde{\mathbf{u}}(t) - \mathbf{w}_{q_s} \tilde{\mathbf{u}}(t - 1) - \cdots - \mathbf{w}_{q_1} \tilde{\mathbf{u}}(t - s) \end{aligned} \quad (23)$$

where the vector coefficients \mathbf{w}_{ξ_i} and \mathbf{w}_{q_i} are constructed from the appropriate elements of the vectors $\mathbf{w}^T \mathbf{\Xi}$ and $\mathbf{w}^T \mathbf{Q}$, respectively. (In the case when $p = m = 1$, i.e. $\mathbf{u}(t)$ and $\mathbf{y}(t)$ are scalars, \mathbf{w}_{ξ_i} and \mathbf{w}_{q_i} refer to the i^{th} elements of the vectors $\mathbf{w}^T \mathbf{\Xi}$ and $\mathbf{w}^T \mathbf{Q}$, respectively.) Note, that in (23) the estimate of the unknown input is affected by two coloured noise sequences. However, by a careful choice of \mathbf{w}^T , the degrading effect of these disturbances can be minimised. Furthermore, the influence of measurement noise on the unknown input estimate can be reduced by minimising the variance of the term $\mathbf{w}^T \mathbf{\Xi} \mathbf{E} - \mathbf{w}^T \mathbf{Q} \tilde{\mathbf{U}}$, i.e.:

$$\begin{aligned} \mathbb{E}\{(\mathbf{w}^T \mathbf{\Xi} \mathbf{E} - \mathbf{w}^T \mathbf{Q} \tilde{\mathbf{U}})(\mathbf{w}^T \mathbf{\Xi} \mathbf{E} - \mathbf{w}^T \mathbf{Q} \tilde{\mathbf{U}})^T\} = \\ = \mathbf{w}^T \mathbf{\Xi} \mathbf{\Sigma}_e \mathbf{\Xi}^T \mathbf{w} + \mathbf{w}^T \mathbf{Q} \mathbf{\Sigma}_{\tilde{\mathbf{u}}} \mathbf{Q}^T \mathbf{w} - \mathbf{w}^T \mathbf{\Xi} \mathbf{\Sigma}_{\tilde{\mathbf{u}}e}^T \mathbf{Q}^T \mathbf{w} - \mathbf{w}^T \mathbf{Q} \mathbf{\Sigma}_{\tilde{\mathbf{u}}e} \mathbf{\Xi}^T \mathbf{w} \end{aligned} \quad (24)$$

where $\mathbf{\Sigma}_{\tilde{\mathbf{u}}} = \mathbb{E}\{\tilde{\mathbf{U}} \tilde{\mathbf{U}}^T\}$, $\mathbf{\Sigma}_e = \mathbb{E}\{\mathbf{E} \mathbf{E}^T\}$, and $\mathbf{\Sigma}_{\tilde{\mathbf{u}}e} = \mathbb{E}\{\tilde{\mathbf{U}} \mathbf{E}^T\} = 0$. Consequently, the vector \mathbf{w}^T should be selected to minimise the cost function $f(\mathbf{w}^T)$:

$$f(\mathbf{w}^T) = \mathbf{w}^T \mathbf{\Xi} \mathbf{\Sigma}_e \mathbf{\Xi}^T \mathbf{w} + \mathbf{w}^T \mathbf{Q} \mathbf{\Sigma}_{\tilde{\mathbf{u}}} \mathbf{Q}^T \mathbf{w} \quad (25)$$

subject to the following constraints:

- (1) Sum of elements of $\mathbf{w}^T \mathbf{T}$ is equal to 1.
- (2) $\mathbf{w}^T \mathbf{\Gamma} = 0$.

Note that constraint (1) is sufficient to ensure unity gain, because $\mathbb{E}\{\mathbf{e}(t)\} = 0$ and $\mathbb{E}\{\tilde{\mathbf{u}}(t)\} = 0$.

The cost function (25) can be minimised by making use of the Lagrange multiplier method, see, for example, Bertsekas (1982). Denote the rows of $\mathbf{\Gamma}^\perp$ by $\gamma_1^T, \gamma_2^T, \dots, \gamma_k^T$:

$$\mathbf{\Gamma}^\perp = [\gamma_1 \ \gamma_2 \ \cdots \ \gamma_k]^T \quad (26)$$

The vector \mathbf{w}^T is a linear combination of the rows of $\mathbf{\Gamma}^\perp$, which ensures that constraint (2) is satisfied, i.e.

$$\mathbf{w}^T = \sum_{i=1}^k p_i \gamma_i^T \quad (27)$$

Hence, the cost function (25) can be reformulated as a function of the parameter vector $\mathbf{P} = [p_1 \ p_2 \ \cdots \ p_k]^T$:

$$f(\mathbf{P}) = \left(\sum_{i=1}^k p_i \gamma_i^T \right) \Sigma \left(\sum_{j=1}^k p_j \gamma_j \right) \quad (28)$$

where

$$\Sigma = \Xi \Sigma_e \Xi^T + \mathbf{Q} \Sigma_{\tilde{u}} \mathbf{Q}^T \quad (29)$$

The cost function $f(\mathbf{P})$ is required to be minimised subject to the constraint:

$$g(\mathbf{P}) = \text{sum}_{\text{row}}(\mathbf{w}^T \mathbf{T}) - 1 = 0 \quad (30)$$

where the operator $\text{sum}_{\text{row}}(\mathbf{A})$ denotes a column vector whose elements are sums of the appropriate rows of an arbitrary matrix \mathbf{A} . (In the case of a row vector \mathbf{q} , the term $\text{sum}_{\text{row}}(\mathbf{q})$ is simply a scalar being sum of elements of the vector \mathbf{q} , whilst, if \mathbf{q} is a column vector, $\text{sum}_{\text{row}}(\mathbf{q}) = \mathbf{q}$.) The solution to the Lagrange minimisation problem is given by, see Bertsekas (1982):

$$\nabla f(\mathbf{P}) = \lambda \nabla g(\mathbf{P}) \quad (31)$$

which, after some manipulations, may be reformulated as (for details see Appendix A.):

$$\mathbf{S} \mathbf{P} = \lambda \boldsymbol{\psi} \quad (32)$$

where

$$\mathbf{S} = \mathbf{\Gamma}^\perp \Sigma (\mathbf{\Gamma}^\perp)^T + (\mathbf{\Gamma}^\perp \Sigma (\mathbf{\Gamma}^\perp)^T)^T \quad (33)$$

and

$$\boldsymbol{\psi} = \text{sum}_{\text{row}}(\mathbf{\Gamma}^\perp \mathbf{T}) \quad (34)$$

Therefore, the parameter vector \mathbf{P} is given by:

$$\mathbf{P} = \lambda \mathbf{S}^{-1} \boldsymbol{\psi} \quad (35)$$

where the Lagrange multiplier λ is calculated via:

$$\lambda = \left((\mathbf{S}^{-1} \boldsymbol{\psi})^T \boldsymbol{\psi} \right)^{-1} \quad (36)$$

The algorithm for calculating the optimal row vector \mathbf{w}^T is summarised as follows:

Algorithm 1:

- (1) Select the order of the parity space s and construct matrices $\mathbf{\Gamma}$, \mathbf{Q} , \mathbf{T} , and Ξ .
- (2) Obtain $\mathbf{\Gamma}^\perp$.
- (3) Compute Σ using (29).
- (4) Calculate the column vector \mathbf{S} and the matrix $\boldsymbol{\psi}$ by making use of (33) and (34), respectively.
- (5) Obtain the Lagrange multiplier λ using (36).
- (6) Calculate the parameter vector \mathbf{P} by (35).

- (7) Compute the vector \mathbf{w}^T using (27).
- (8) Calculate estimation delay τ using (20) and (21).
- (9) Use designed estimator to calculate $v(t - \tau)$ as in equation (17).

2.4 Robust unknown input reconstruction

Nonlinearities in system dynamics, parameter variations, linearisation and model order reduction errors etc., cause model mis-match which can seriously affect the input reconstruction process. It has been presented in Patton et al. (1992), Patton and Chen (1992) that these discrepancies can be modelled as an additional input to the system, denoted $\mathbf{d}(t) \in \mathbb{R}^r$.

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}_0(t) + \mathbf{G}v(t) + \mathbf{R}_x\mathbf{d}(t) + \mathbf{\Pi}\mathbf{e}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}_0(t) + \mathbf{H}v(t) + \mathbf{R}_y\mathbf{d}(t) + \mathbf{\Omega}\mathbf{e}(t) \\ \mathbf{u}(t) &= \mathbf{u}_0(t) + \tilde{\mathbf{u}}(t) \end{aligned} \quad (37)$$

where $\mathbf{R}_x \in \mathcal{R}^{n \times r}$ and $\mathbf{R}_y \in \mathcal{R}^{m \times r}$ are disturbance distribution matrices. Methods for determining the structure of matrices \mathbf{R}_x and \mathbf{R}_y are presented in Patton and Chen (1992) and Chen and Patton (1999). The scheme proposed in this paper can be easily extended to facilitate disturbance decoupling. For this purpose define matrix $\mathbf{O} \in \mathbb{R}^{(s+1)m \times (s+1)r}$ by replacing \mathbf{B} and \mathbf{D} in \mathbf{Q} by \mathbf{R}_x and \mathbf{R}_y , respectively. Then (9) can be reformulated as

$$\mathbf{Y} = \mathbf{\Gamma}\mathbf{x}(t-s) + \mathbf{Q}\mathbf{U}_0 + \mathbf{TV} + \mathbf{OK} + \mathbf{\Xi}\mathbf{E} \quad (38)$$

where \mathbf{K} is the stacked vector of disturbances $\mathbf{d}(t)$, cf. (8). In order to eliminate both the unknown state vector and the disturbance vector from (38), the vector \mathbf{w}^T should fulfil the constraint

$$\mathbf{w}^T [\mathbf{\Gamma} \mathbf{O}] = 0 \quad (39)$$

Thus the matrix $\mathbf{\Gamma}^\perp$ should be selected is such a way that it spans the left nullspace of the matrix $[\mathbf{\Gamma} \mathbf{O}] \in \mathcal{R}^{(s+1)m \times (n+(s+1)r)}$. Disturbance decoupling is possible if

$$(s+1)m > \text{rank} [\mathbf{\Gamma} \mathbf{O}] \quad (40)$$

It has been shown in Massoumnia (1986) that disturbance decoupling is possible for system (37) if $r < m$. Note that if this condition is satisfied then the matrix $\mathbf{\Gamma}^\perp$ and therefore the row vector \mathbf{w}^T can always be found by choosing s to be sufficiently large.

3 Extension to systems with zero close to unity

The scheme described in Section 2 cannot be applied to single output systems, whose transfer function between the unknown input and the output contains a derivative term. Furthermore, performance of the PE-UIO is seriously impaired when the system zero lies close to unity. The background of this problem is explained in Subsection 3.1. Furthermore, in Subsection 3.2 a modified version of the PE-UIO is proposed, where this limitation is alleviated.

3.1 Zero assignment filter

Consider the system (3) where $p = m = 1$. The two relationships between each of the system inputs (both known and unknown) and the output can be described by discrete-time transfer

functions in the z -domain of the following form, cf. (3):

$$\begin{aligned} G_u(z) &= \frac{Y_0(z)}{U_0(z)} = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\ G_v(z) &= \frac{Y_0(z)}{V(z)} = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{G} + \mathbf{H} \end{aligned} \quad (41)$$

where the terms $Y_0(z)$, $U_0(z)$, and $V(z)$ denote, respectively, $\mathbf{y}_0(t)$, $\mathbf{u}_0(t)$, and $v(t)$ in the z -domain. Equation (13) in the noise-free case can be reformulated as the following relation:

$$W(z)Y_0(z) = W_Q(z)U_0(z) + W_T(z)V(z) \quad (42)$$

where terms $W(z)$, $W_Q(z)$ and $W_T(z)$ are polynomials in the z -variable with parameters defined by vectors \mathbf{w}^T , $\mathbf{w}^T\mathbf{Q}$, and $\mathbf{w}^T\mathbf{T}$, respectively. The transfer functions corresponding to $\mathbf{u}_0(t)$ and $v(t)$ are given, respectively, by:

$$\begin{aligned} G_u(z) &= \frac{W_Q(z)}{W(z)} \\ G_v(z) &= \frac{W_T(z)}{W(z)} \end{aligned} \quad (43)$$

where $G_u(z)$ defines the relationship between $U_0(z)$ and the output $Y_0(z)$, whereas $G_v(z)$ describes the relationship between the unknown input $V(z)$ and $Y_0(z)$, cf. (41). In the case when $s = n$, the left nullspace of $\mathbf{\Gamma}$ is a row vector $\mathbf{\Gamma}^\perp = \mathbf{w}^T$ (it is assumed here that the system (3) is observable) and the order of the polynomial $W(z)$ is equal to the order of the system. Hence, one can deduce from (43) that the roots of the polynomial $W(z)$ are eigenvalues of the matrix \mathbf{A} (i.e. poles of both $G_v(z)$ and $G_u(z)$). Denote the set of poles and zeros of $G_v(z)$ by P_v and Z_v , respectively. Analogously, denote P_u and Z_u as, respectively, poles and zeros of $G_u(z)$. Then, it is true that the roots of $W(z)$ are $P_v \cup P_u$, the roots of $W_Q(z)$ are defined by the set $Z_u \cup (P_v \setminus P_u)$, whilst roots of $W_T(z)$ are $Z_v \cup (P_u \setminus P_v)$.

If the order of the parity space is higher than that of the system, i.e. $s > n$, then the set of equations (43) must still be fulfilled. This means, that $W(z)$, $W_Q(z)$ and $W_T(z)$ have common $s - n$ roots (a zero-pole cancellation occurs, hence both $\frac{W_Q(z)}{W(z)}$ and $\frac{W_T(z)}{W(z)}$ remain unaltered). The choice of those additional $s - n$ zeros influences the properties of the noise filtration of the filter (22). Hence, the problem of finding the optimal row vector \mathbf{w}^T can be reformulated as a filter zeros assignment problem. The unknown input reconstruction is possible when the bandwidth of the unknown input is narrower than that of $W_T(z)$, whilst the ability of the PE-UIO to filter $\tilde{\mathbf{u}}(t)$ and $\mathbf{e}(t)$ depends on the frequency response of both, i.e. $W_Q(z)$ and $W(z)$. Note that the relationship between the unknown input and its estimate in the z -domain is given by, cf. (19) and (20):

$$\hat{V}(z) = W_T(z)V(z) \quad (44)$$

This also explains, why the PE-UIO cannot be used when $G_v(z)$ contains a derivative term, i.e. a zero equal to unity. In such a case the polynomial $W_T(z)$ also contains the derivative term and its steady state gain is zero. Therefore, use of the standard PE-UIO for the purpose of unknown input estimation is infeasible. Furthermore, if $G_v(z)$ contains a zero close to unity, the step response of $W_T(z)$ is characterised by a large overshoot (characteristic of systems whose zero lies close to unity), hence the unknown input estimation problem becomes seriously affected. The overshoot of $W_T(z)$ can be minimised by a significant increase of the order of parity space, however, this results in a reduction of the filter bandwidth. Therefore, a modification of the

PE-UIO is required, and this is provided in the next subsection. Note that this problem will occur also for multiple input systems as long as the system has a single output. Thus, during the derivation of the modified PE-UIO filter in Subsection 3.2 it is assumed that the single output system may have an arbitrary number of measured inputs, i.e. $m = 1$ and $p \in \mathbb{N}$. The algorithm developed in the following subsection is applicable to systems, whose ‘problematic’ zero lies on the real axis and is less than or equal to unity. However, this result can also be extended to cases with multiple zeros which lie on or within the unit circle and that are relatively close to unity.

3.2 Two stage filter design

Consider a single output system, whose transfer function between the unknown input and the output, denoted as $G_v(z)$, contains a zero, denoted z_0 , which is close or equal to unity. Such a transfer function can be represented by:

$$G_v(z) = G'_v(z) \frac{z - z_0}{z} \quad (45)$$

Therefore, the input-output relationship in the z -domain can be represented as:

$$\begin{aligned} Y(z) &= G_v(z)V(z) + G_u(z)U_0(z) = G'_v(z) \frac{z - z_0}{z} V(z) + G_u(z)U_0(z) = \\ &= G'_v(z)V'(z) + G_u(z)U_0(z) \end{aligned} \quad (46)$$

where $V'(z) = \frac{z - z_0}{z} V(z)$ is the z -domain representation of the variable $v'(t)$, whose relation with the unknown input is defined as:

$$v'(t) = v(t) - z_0 v(t - 1) \quad (47)$$

The transfer function $G'_v(z)$ can be represented by:

$$G'_v(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1} \mathbf{G}' + \mathbf{H}' \quad (48)$$

where \mathbf{H}' and \mathbf{G}' are the appropriately modified matrices \mathbf{H} and \mathbf{G} , respectively. Define the matrix \mathbf{T}' by replacing \mathbf{G} and \mathbf{H} in \mathbf{T} by, respectively, \mathbf{G}' and \mathbf{H}' . Subsequently the PE (14) can be reformulated as:

$$\mathbf{w}^T \mathbf{T}' \mathbf{V}' + \mathbf{w}^T \mathbf{\Xi} \mathbf{E} - \mathbf{w}^T \mathbf{Q} \tilde{\mathbf{U}} = \mathbf{w}^T \mathbf{Y} - \mathbf{w}^T \mathbf{Q} \mathbf{U} \quad (49)$$

Analogously to the algorithm described in Section 2, it is proposed to estimate the variable $v'(t - \tau)$ as:

$$\hat{v}'(t - \tau) = \mathbf{w}^T \mathbf{Y} - \mathbf{w}^T \mathbf{Q} \mathbf{U} \quad (50)$$

which, in the noise-free case, is equal to:

$$\hat{v}'(t - \tau) = \mathbf{w}^T \mathbf{T}' \mathbf{V}' \quad (51)$$

Subsequently, the unknown input estimate can be calculated via, cf. (47):

$$\hat{v}(t) = z_0 \hat{v}(t - 1) + \hat{v}'(t) \quad (52)$$

Note that this scheme is applicable only to systems with $|z_0| \leq 1$. Otherwise, (52) becomes unstable.

In the noisy case the term $\hat{v}'(t - \tau)$ is given by:

$$\hat{v}'(t - \tau) = \mathbf{w}^T \mathbf{T}' \mathbf{V}' + \epsilon(t) \quad (53)$$

where $\epsilon(t)$ represents the disturbance introduced by $\mathbf{e}(t)$ and $\tilde{\mathbf{u}}(t)$, i.e.:

$$\epsilon(t) = \mathbf{w}^T \Xi \mathbf{E} - \mathbf{w}^T \mathbf{Q} \tilde{\mathbf{U}} \quad (54)$$

Hence, it follows from equations (52) and (53), that the estimate of $v(t - \tau)$ is affected by the error term $\epsilon^*(t)$, whose relation to $\epsilon(t)$ is given by:

$$\epsilon^*(t) = \epsilon(t) + z_0 \epsilon^*(t - 1) \quad (55)$$

For convenience, the following notation is introduced:

$$\begin{aligned} \tilde{\mathbf{u}}^*(t) &= \tilde{\mathbf{u}}(t) + z_0 \tilde{\mathbf{u}}^*(t - 1) \\ \mathbf{e}^*(t) &= \mathbf{e}(t) + z_0 \mathbf{e}^*(t - 1) \end{aligned} \quad (56)$$

Thus, the term $\epsilon^*(t)$ is given by:

$$\epsilon^*(t) = \mathbf{w}^T \Xi \mathbf{E}^* - \mathbf{w}^T \mathbf{Q} \tilde{\mathbf{U}}^* \quad (57)$$

where \mathbf{E}^* and $\tilde{\mathbf{U}}^*$ are constructed from the current and previous values of $\mathbf{e}^*(t)$ and $\tilde{\mathbf{u}}^*(t)$, respectively, cf. (8). It is now required to minimise the variance of the term $\epsilon^*(t)$, which is given by, cf. (24):

$$\begin{aligned} E\{(\mathbf{w}^T \Xi \mathbf{E}^* - \mathbf{w}^T \mathbf{Q} \tilde{\mathbf{U}}^*)(\mathbf{w}^T \Xi \mathbf{E}^* - \mathbf{w}^T \mathbf{Q} \tilde{\mathbf{U}}^*)^T\} = \\ = \mathbf{w}^T \Xi \Sigma_{e^*} \Xi^T \mathbf{w} + \mathbf{w}^T \mathbf{Q} \Sigma_{\tilde{u}^*} \mathbf{Q}^T \mathbf{w} - \mathbf{w}^T \Xi (\Sigma_{\tilde{u}^* e^*})^T \mathbf{Q}^T \mathbf{w} - \mathbf{w}^T \mathbf{Q} \Sigma_{\tilde{u}^* e^*} \Xi^T \mathbf{w} \end{aligned} \quad (58)$$

where $\Sigma_{\tilde{u}^*} = E\{\tilde{\mathbf{U}}^* \tilde{\mathbf{U}}^{*\top}\}$, $\Sigma_{e^*} = E\{\mathbf{E}^* \mathbf{E}^{*\top}\}$ and $\Sigma_{\tilde{u}^* e^*} = E\{\tilde{\mathbf{U}}^* \mathbf{E}^{*\top}\} = 0$. Hence, the function to be minimised is given by:

$$f(\mathbf{w}^T) = \mathbf{w}^T \Xi \Sigma_{e^*} \Xi^T \mathbf{w} + \mathbf{w}^T \mathbf{Q} \Sigma_{\tilde{u}^*} \mathbf{Q}^T \mathbf{w} \quad (59)$$

In order to calculate (59), first, the terms Σ_{e^*} and $\Sigma_{\tilde{u}^*}$ are required. The signal $\mathbf{e}^*(t)$ can be described by a function of its previous values, cf. (56):

$$\mathbf{e}^*(t) = \mathbf{e}(t) + z_0 \mathbf{e}^*(t - 1) + z_0^2 \mathbf{e}^*(t - 2) + \dots \quad (60)$$

Therefore, by recalling that $\mathbf{e}(t)$ is assumed to be white, the expected value of $\mathbf{e}^*(t) \mathbf{e}^*(t - i)$ is calculated as:

$$\begin{aligned} E\{\mathbf{e}^*(t) \mathbf{e}^*(t - i)\} &= E\{z_0^i \mathbf{e}^2(t - i) + z_0^{i+2} \mathbf{e}^2(t - i - 1) + z_0^{i+4} \mathbf{e}^2(t - i - 2) + \dots\} \\ &= E\{\mathbf{e}^2(t)\} z_0^i (1 + z_0^2 + z_0^4 + \dots) \end{aligned} \quad (61)$$

which is a sum of a geometric series and, in the case when $|z_0| < 1$, can be simplified to:

$$E\{\mathbf{e}^*(t) \mathbf{e}^*(t - i)\} = E\{\mathbf{e}^2(t)\} \frac{z_0^i}{1 - z_0^2} \quad (62)$$

Analogously, by recalling that $\tilde{\mathbf{u}}(t)$ is assumed to be white, the expected value of $\tilde{\mathbf{u}}^*(t)\tilde{\mathbf{u}}^*(t-i)$ can be derived as:

$$\mathbb{E}\{\tilde{\mathbf{u}}^*(t)(\tilde{\mathbf{u}}^*(t-i))^T\} = \mathbb{E}\{\tilde{\mathbf{u}}(t)\tilde{\mathbf{u}}^T(t)\} \frac{z_0^i}{1-z_0^2} \quad (63)$$

In the cases when $z = 1$ the sum of the geometric series (61) is infinite. Therefore, to cope with such a case it is proposed to replace z_0 in (62) and (63) by a value smaller than unity in order to indicate that $\mathbf{e}(t)$ and $\tilde{\mathbf{u}}(t)$ are not white.

The matrices Σ_{e^*} and $\Sigma_{\tilde{u}^*}$ are constructed by making use of appropriate values of, respectively, $\mathbb{E}\{\mathbf{e}^*(t)\mathbf{e}^*(t-i)\}$ and $\mathbb{E}\{\tilde{\mathbf{u}}^*(t)\tilde{\mathbf{u}}^*(t-i)\}$. For convenience, a new term is introduced, cf. (29):

$$\Sigma^* = \Xi \Sigma_{e^*} \Xi^T + \mathbf{Q} \Sigma_{\tilde{u}^*} \mathbf{Q}^T \quad (64)$$

Hence the cost function (59) becomes:

$$f(\mathbf{P}) = \left(\sum_{i=1}^k p_i \gamma_i^T \right) \Sigma^* \left(\sum_{j=1}^k p_j \gamma_j \right) \quad (65)$$

which is required to be minimised subject to the constraint:

$$g(\mathbf{P}) = \text{sum}_{\text{row}}(\mathbf{w}^T \mathbf{T}') - 1 = 0 \quad (66)$$

Consequently, the algorithm for calculating the unknown input is summarised as follows:

Algorithm 2:

- (1) Select the order of the parity space s and construct matrices Γ , \mathbf{Q} , \mathbf{T}' , and Ξ .
- (2) Obtain Γ^\perp .
- (3) Calculate Σ_{e^*} and $\Sigma_{\tilde{u}^*}$ using (62) and (63), respectively.
- (4) Compute Σ^* using (64).
- (5) Calculate the matrix \mathbf{S} by:

$$\mathbf{S} = \Gamma^\perp \Sigma^* (\Gamma^\perp)^T + (\Gamma^\perp \Sigma^* (\Gamma^\perp)^T)^T \quad (67)$$

- (6) Calculate the column vector $\boldsymbol{\psi}$ via:

$$\boldsymbol{\psi} = \text{sum}_{\text{row}}(\Gamma^\perp \mathbf{T}') \quad (68)$$

- (7) Obtain the Lagrange multiplier λ using (36).
- (8) Calculate the parameter vector \mathbf{P} by (35).
- (9) Compute the vector \mathbf{w}^T , cf. (27), as:

$$\mathbf{w}^T = \mathbf{P}^T \Gamma^\perp \quad (69)$$

- (10) Calculate estimation delay τ using (20) and (21).
- (11) Use the designed estimator to calculate $\hat{v}'(t-\tau)$ via (51).
- (12) Obtain the estimate of $v(t-\tau)$ using (52).

Note that the above algorithm can be extended to facilitate disturbance decoupling. This can be achieved by selecting such matrix Γ^\perp that it spans the left nullspace of the matrix $[\Gamma \ \mathbf{O}]$, where \mathbf{O} is defined in Subsection 2.4.

4 Application to hydrological system

This example is based on the data collected during a potassium bromide (KBr) tracer experiment carried out in a wetland area by Martinez and Wise (2003). A schematic illustration of the experiment is depicted in Figure 1. A tracer has been poured into the river at the point (1). Two tracer concentration sensors have been placed downstream, at points (2) and (3), whose readings are denoted, respectively, as $v(t)$ and $y(t)$, see Figure 2. The data used for this experiment are available on http://captaintoolbox.co.uk/Data_Sets.html. A linear model of the relation between the input, denoted $v(t)$, and the output, denoted $y(t)$ has been developed in Young and Sumińska (2012) and is given by

$$G_v(z) = \frac{0.017591(z + 4.302)(z - 0.9735)}{(z - 0.9764)(z - 0.8916)} \quad (70)$$

The aim is to use the PE-UIO scheme to estimate the input $v(t)$ based on the output measurements and knowledge of the system model (70). It should be noted that the model is nonminimum-phase, thus naive inversion is not feasible. Although the PE-UIO can cope with the zero at -4.302 , the zero at 0.9735 requires the two stage PE-UIO, i.e Algorithm 2, to be used. Hence, after removing the zero at 0.9735 from $G_v(z)$, the transfer function $G'_v(z)$ is given by, cf. (45)

$$G'_v(z) = \frac{0.017591(z + 4.302)}{(z - 0.9764)(z - 0.8916)} \quad (71)$$

and corresponding state space matrices are

$$\mathbf{A} = \begin{bmatrix} 1.8680 & 1 \\ -0.8706 & 0 \end{bmatrix} \quad \mathbf{G}' = \begin{bmatrix} 0.0176 \\ 0.0757 \end{bmatrix} \quad \mathbf{C} = [1 \ 0] \quad \mathbf{H}' = 0 \quad (72)$$

It has been assumed that the measurements are affected by white zero-mean noise ($\mathbf{\Pi}$ is null, $\mathbf{\Pi} = 1$). Algorithm 2 has been used to generate the input reconstruction scheme. The parity space order has been set to 12, which leads to $\tau = 5$ samples.

$$\begin{aligned} \hat{v}'(t-5) &= 0.4202y(t) - 0.0281y(t-1) - 0.0337y(t-2) - 0.0400y(t-3) - 0.0470y(t-4) \\ &\quad - 0.0548y(t-5) - 0.0636y(t-6) - 0.0734y(t-7) - 0.0845y(t-8) \\ &\quad - 0.0968y(t-9) - 0.1106y(t-10) - 0.1261y(t-11) + 0.3658y(t-12) \\ \hat{v}(t) &= 0.9735\hat{v}(t-1) + \hat{v}'(t) \end{aligned} \quad (73)$$

The result of the unknown input estimation is compared with the original input in Figure 3, which indicates a satisfactory performance.

It is believed that the input reconstruction discrepancies are caused mainly by the modelling inaccuracy presumably due to unmodelled system nonlinearities. This hypothesis is supported in Figure 4, which compares the measured output with the model output.

5 Conclusions

A new approach to the unknown input reconstruction problem has been proposed. The scheme is applicable to multiple-input multiple-output systems with a single unmeasurable input, whereas the number of outputs and known inputs may be arbitrary. The method allows for disturbance decoupling which is often found to be requirement encountered in practice.

The main advantage of this new parity equations-based unknown input observer (PE-UIO) is its simplicity; the filter parameters are calculated once at the beginning of the reconstruction process. The method is fast as it utilises two moving average filters. The only tuning parameter of the PE-UIO is the order of the parity space. By changing this, the bandwidth of the input reconstruction filter is shaped. This property allows the designer to tune the algorithm for different levels of noise. It should be noted, however, that reduction of the filter bandwidth (hence improvement of the noise filtering properties of the scheme) introduces an estimation lag into the scheme. Furthermore, the PE-UIO is applicable for non-minimum phase systems.

Potential practical application of the proposed scheme has also been demonstrated. The two stage PE-UIO has successfully estimated the tracer concentration in a hydrological system which exhibited non-minimum phase behaviour and had a zero close to unity.

Further work is aimed towards an extension of the developed algorithms to systems with multiple unmeasurable inputs. Although the proposed algorithms are generally applicable for nonminimum-phase systems, a solution for systems, whose nonminimum-phase zero is close to unity, still remains an open question. Other applications of the PE-UIO of an engineering nature are on-going, with promising results being obtained from a simulated model of a hot finishing steel rolling mill.

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References

- Antsaklis, P. (1978), "Stable Proper nth-order Inverses," *IEEE Transactions on Automatic Control*, 23, 1104–1106.
- Bertsekas, D.P., *Constrained Optimisation and Lagrange Multiplier Methods*, London: Academic press, Inc. (1982).
- Box, G., and Jenkins, G., *Time Series Analysis - Forecasting and Control*, Oakland, CA, USA: Holden-Day (1976).
- Chen, J., Patton, R., and Liu, G. (1996), "Optimal Residual Design for Fault Diagnosis Using Multiobjective Optimisation and Genetic Algorithms," *International Journal of Systems Science*, 27, 567–576.
- Chen, J., and Patton, R.J., *Robust Model-Based Fault Diagnosis for Dynamic Systems*, Boston: Kluwer Academic Publishers (1999).
- Chow, E., and Willsky, A. (1984), "Analytical Redundancy and the Design of Robust Failure Detection Systems," *IEEE Transactions on Automatic Control*, 29, 603–614.
- Darouach, M., and Zasadzinski, M. (1997), "Unbiased Minimum Variance Estimation for Systems with Unknown Exogenous Inputs," *Automatica*, 33, 717–719.
- Dorato, P. (1969), "On the Inverse of Linear Dynamical Systems," *IEEE Transactions on Systems Science and Cybernetics*, 5, 43–48.
- Edwards, C., Spurgeon, S.K., and Patton, R. (2000), "Sliding Model Observers for Fault Detection and Isolation," *Automatica*, 36, 541–553.
- Fernando, T., and Trinh, H. (2006), "Design of Reduced-order State/Unknown Input Observers: A Descriptor System Approach," in *Proceedings of the 2006 IEEE International Conference on Control Applications*, October 4–6, Munich, Germany.
- Floquet, T., and Barbot, J. (2006), "State and Unknown Input Estimation for Linear Discrete-time Systems," *Automatica*, 42, 1883–1889.
- Gertler, J., and Singer, D. (1990), "A New Structural Framework for Parity Equation-based Failure Detection and Isolation," *Automatica*, 26, 381–388.

- Ghahremani, E., and Kamwa, I. (2011), “Simultaneous State and Input Estimation of a Synchronous Machine Using the Extended Kalman Filter with Unknown Inputs,” in *Proceedings of the 2011 IEEE International Electric Machines & Drives Conference (IEMDC)*, 15–18 May, pp. 1468–473.
- Gillijns, S., and De Moor, B. (2007a), “Unbiased Minimum Variance Input and State Estimation for Linear Discrete-time Systems,” *Automatica*, 43, 111–116.
- Gillijns, S., and De Moor, B. (2007b), “Unbiased Minimum Variance Input and State Estimation for Linear Discrete-time Systems with Direct Feedthrough,” *Automatica*, 43, 934–937.
- Hou, M., and Müller, P. (1992), “Design of Observers for Linear Systems with Unknown Inputs,” *IEEE Transactions on Automatic Control*, 37, 871–875.
- Hou, M., and Patton, R. (1998), “Input Observability and Input Reconstruction,” *Automatica*, 34, 789–794.
- Hou, M., and Patton, R. (1998), “Optimal Filtering for Systems with Unknown Inputs,” *IEEE Transactions on Automatic Control*, AC-43, 445–449.
- Hsieh, C. (2000), “Robust Two-stage Kalman Filters for Systems with Unknown Inputs,” *IEEE Transactions on Automatic Control*, 45, 2374–2378.
- Keller, J., and Sauter, D. (2010), “A Variable Geometric State Filtering for Stochastic Linear Systems Subject to Intermittent Unknown Inputs,” in *Proceedings of the Conference on Control and Fault-Tolerant Systems (SysTol)*, 6–8 October, pp. 558–563.
- Kirtikar, S., Palanthandalam-Madapusi, H., Zattoni, E., and Bernstein, D.S. (2009), “ l -Delay Input Reconstruction for Discrete-Time Linear Systems,” in *Proc. of the Conference on Decision and Control*, Shanghai, China, pp. 1848–1853.
- Li, W., and Shah, S. (2002), “Structured Residual Vector-based Approach to Sensor Fault Detection and Isolation,” *Journal of Process Control*, 12, 429–443.
- Ljung, L., *System Identification - Theory for the User*, 2nd ed., PTR Prentice Hall Information and System Sciences Series, New Jersey: Prentice Hall (1999).
- Luenberger, D.G. (1964), “Observing the State of Linear Systems,” *IEEE Trans. Mil. Electr.*, MIL-8, 70–80.
- Marro, G., and Zattoni, E. (2010), “Unknown-state, Unknown-input Reconstruction in Discrete-time Nonminimum-phase Systems: Geometric Approach,” *Automatica*, 46, 815–822.
- Martinez, C.J., and Wise, W.R. (2003), “Analysis of Constructed Treatment Wetland Hydraulics with the Transient Storage Model OTIS,” *Ecological Engineering*, 20, 211–222.
- Massoumnia, M.A. (1986), “A Geometric Approach to the Synthesis of Failure Detection Filters,” *IEEE Transaction on Automatic Control*, AC-31, 839–846.
- Moylan, P. (1977), “Stable Inversion of Linear Systems,” *IEEE Transactions on Automatic Control*, 22, 74–78.
- Palanthandalam-Madapusi, H., and Bernstein, D. (2007), “Unbiased Minimum-variance Filtering for Input Reconstruction,” in *Proceedings of the American Control Conference (ACC'07)*, 9–13 July, pp. New York City, USA.
- Patton, R., Zhang, H.Y., and Chen, J. (1992), “Modelling of Uncertainties for Robust Fault Diagnosis,” in *Proceedings of the 31st Conference on Decision and Control*, Tuscon, AR, USA, pp. 921–926.
- Patton, R.J., and Chen, J. (1992), “Robust Fault Detection of Jet Engine Sensor Systems Using Eigenstructure Assignment,” *Journal of Guidance, Control, and Dynamics*, 15, 1491–1497.
- Sain, M., and Massey, J. (1969), “Invertibility of Linear Time-Invariant Dynamical Systems,” *IEEE Transactions on Automatic Control*, 14, 141–149.
- Söderström, (2007), “Errors-in-variables Methods In System Identification,” *Automatica*, 43, 939–958.
- Sumińska, M. (2013), “Fault detection and diagnosis and unknown input reconstruction based on parity equations concept,” Coventry University.
- Sumińska, M., Burnham, K.J., and Larkowski, T.M. (2010a), “Design of Unknown Input Estimator of a Linear System Based on Parity Equations,” in *Proceedings of the XVII International*

- Conference on Systems Science*, Wrocław, Poland, pp. 81–90.
- Sumisławska, M., Larkowski, T., and Burnham, K.J. (2010), “Parity Equations-based Unknown Input Estimator for Multiple-input Multiple-output Linear Systems,” *Systems Science*, 36, 49–56.
- Sumisławska, M., Larkowski, T., and Burnham, K.J. (2011), “Design of Unknown Input Reconstruction Filter Based on Parity Equations for Errors-in-Variables Case,” in *Proceedings of the 18th IFAC World Congress*, Milan, Italy, pp. 4272–4277.
- Sumisławska, M., Larkowski, T.M., and Burnham, K.J. (2010b), “Design of Unknown Input Reconstruction Algorithm in Presence of Measurement Noise,” in *Proceedings of the 8th European ACD2010 Workshop on Advanced Control and Diagnosis*, 18-19 November, Ferrara, Italy, pp. 213–216.
- Yiua, J.C., and Wang, S. (2007), “Multiple ARMAX Modeling Scheme for Forecasting Air Conditioning System Performance,” *Energy Conversion and Management*, 48, 2276–2285.
- Young, P.C., *Recursive Estimation and Time Series Analysis: An Introduction for the Student and Practitioner*, Springer (2011).
- Young, P.C., and Sumisławska, M. (2012), “A Control Systems Approach to Input Estimation with Hydrological Applications,” in *Proceedings of the 16th IFAC Symposium on System Identification*, Brussels, Belgium, pp. 1043–1048.

Appendix A: Derivation of Algorithm 1.

The solution to the Lagrange optimisation problem is calculated via:

$$\nabla f(\mathbf{P}) = \lambda \nabla g(\mathbf{P}) \quad (\text{A1})$$

The cost function $f(\mathbf{P})$ is given by:

$$f(\mathbf{P}) = \left(\sum_{i=1}^k p_i \gamma_i \right) \Sigma \left(\sum_{i=1}^k p_i \gamma_i^T \right) \quad (\text{A2})$$

where $k = s - n + 1$, and can be expanded as:

$$\begin{aligned} f(\mathbf{P}) = & p_1^2 \gamma_1 \Sigma \gamma_1^T + p_1 p_2 \gamma_1 \Sigma \gamma_2^T + \cdots + p_1 p_k \gamma_1 \Sigma \gamma_k^T + \\ & p_1 p_2 \gamma_2 \Sigma \gamma_1^T + p_2^2 \gamma_2 \Sigma \gamma_2^T + \cdots + p_2 p_k \gamma_2 \Sigma \gamma_k^T + \\ & \vdots \\ & p_1 p_k \gamma_k \Sigma \gamma_1^T + p_2 p_k \gamma_k \Sigma \gamma_2^T + \cdots + p_k^2 \gamma_k \Sigma \gamma_k^T \end{aligned} \quad (\text{A3})$$

Hence, the partial derivative of $f(\mathbf{P})$ with respect to the i^{th} element of the \mathbf{P} vector (denoted as p_i) is given by:

$$\begin{aligned} \frac{\partial f(\mathbf{P})}{\partial p_i} = & p_1 \gamma_i \Sigma \gamma_1^T + p_2 \gamma_i \Sigma \gamma_2^T + \cdots + p_i \gamma_i \Sigma \gamma_i^T + \cdots + p_k \gamma_i \Sigma \gamma_k^T + \\ & p_1 \gamma_1 \Sigma \gamma_i^T + p_2 \gamma_2 \Sigma \gamma_i^T + \cdots + p_i \gamma_i \Sigma \gamma_i^T + \cdots + p_k \gamma_k \Sigma \gamma_i^T \end{aligned} \quad (\text{A4})$$

Consequently, the gradient of $f(\mathbf{P})$ can be written as:

$$\begin{aligned} \begin{bmatrix} \frac{\partial f(\mathbf{P})}{\partial p_1} \\ \frac{\partial f(\mathbf{P})}{\partial p_2} \\ \vdots \\ \frac{\partial f(\mathbf{P})}{\partial p_k} \end{bmatrix} &= \begin{bmatrix} \gamma_1 \boldsymbol{\Sigma} \gamma_1^T & \gamma_1 \boldsymbol{\Sigma} \gamma_2^T & \cdots & \gamma_1 \boldsymbol{\Sigma} \gamma_k^T \\ \gamma_2 \boldsymbol{\Sigma} \gamma_1^T & \gamma_2 \boldsymbol{\Sigma} \gamma_2^T & \cdots & \gamma_2 \boldsymbol{\Sigma} \gamma_k^T \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_k \boldsymbol{\Sigma} \gamma_1^T & \gamma_k \boldsymbol{\Sigma} \gamma_2^T & \cdots & \gamma_k \boldsymbol{\Sigma} \gamma_k^T \end{bmatrix} \mathbf{P} + \begin{bmatrix} \gamma_1 \boldsymbol{\Sigma} \gamma_1^T & \gamma_2 \boldsymbol{\Sigma} \gamma_1^T & \cdots & \gamma_k \boldsymbol{\Sigma} \gamma_1^T \\ \gamma_1 \boldsymbol{\Sigma} \gamma_2^T & \gamma_2 \boldsymbol{\Sigma} \gamma_2^T & \cdots & \gamma_k \boldsymbol{\Sigma} \gamma_2^T \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1 \boldsymbol{\Sigma} \gamma_k^T & \gamma_2 \boldsymbol{\Sigma} \gamma_k^T & \cdots & \gamma_k \boldsymbol{\Sigma} \gamma_k^T \end{bmatrix} \mathbf{P} \\ &= (\nabla f(\mathbf{P}))^T \end{aligned} \quad (\text{A5})$$

Thus, recalling that $\boldsymbol{\Sigma}$ is symmetric, expression (A5) can be reformulated as:

$$(\nabla f(\mathbf{P}))^T = \left(\boldsymbol{\Gamma}^\perp \boldsymbol{\Sigma} (\boldsymbol{\Gamma}^\perp)^T + \left(\boldsymbol{\Gamma}^\perp \boldsymbol{\Sigma} (\boldsymbol{\Gamma}^\perp)^T \right)^T \right) \mathbf{P} \quad (\text{A6})$$

The constraint function $g(\mathbf{P})$ is:

$$g(\mathbf{P}) = \text{sum}_{\text{row}}(\mathbf{w}^T \mathbf{T}) - 1 = \sum_{i=1}^k \text{sum}_{\text{row}}(p_i \gamma_i \mathbf{T}) - 1 \quad (\text{A7})$$

Hence the partial derivative of $g(\mathbf{P})$ with respect to p_i is calculated via:

$$\frac{\partial g(\mathbf{P})}{\partial p_i} = \text{sum}_{\text{row}}(\gamma_i \mathbf{T}) \quad (\text{A8})$$

Thus, the gradient of $g(\mathbf{P})$ can be reformulated as:

$$(\nabla g(\mathbf{P}))^T = \text{sum}_{\text{row}}(\boldsymbol{\Gamma}^\perp \mathbf{T}) \quad (\text{A9})$$

Using the notation:

$$\mathbf{S} = \boldsymbol{\Gamma}^\perp \boldsymbol{\Sigma} (\boldsymbol{\Gamma}^\perp)^T + \left(\boldsymbol{\Gamma}^\perp \boldsymbol{\Sigma} (\boldsymbol{\Gamma}^\perp)^T \right)^T \quad (\text{A10})$$

and

$$\boldsymbol{\psi} = \text{sum}_{\text{row}}(\boldsymbol{\Gamma}^\perp \mathbf{T}) \quad (\text{A11})$$

the solution to the Lagrange optimisation problem (A1) can be rewritten as:

$$\mathbf{S} \mathbf{P} = \lambda \boldsymbol{\psi} \quad (\text{A12})$$

Hence, the optimal parameter vector \mathbf{P} is given by:

$$\mathbf{P} = \lambda \mathbf{S}^{-1} \boldsymbol{\psi} \quad (\text{A13})$$

The constraint function $g(\mathbf{P}) = 0$ can be rewritten as:

$$\mathbf{P}^T \boldsymbol{\psi} - 1 = 0 \quad (\text{A14})$$

Incorporating (A13) into (A14) yields:

$$\lambda (\mathbf{S}^{-1}\boldsymbol{\psi})^T \boldsymbol{\psi} - 1 = 0 \quad (\text{A15})$$

Finally, the Lagrange multiplier is given by:

$$\lambda = \left((\mathbf{S}^{-1}\boldsymbol{\psi})^T \boldsymbol{\psi} \right)^{-1} \quad (\text{A16})$$

Figure captions

- Figure 1. Schematic illustration of tracer experiment.
- Figure 2. Input and output signals in tracer experiment.
- Figure 3. Result of unknown input estimation.
- Figure 4. Model output vs. measured output.