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Scaling and Density of Lee–Yang Zeroes in the four Dimensional Ising Model *

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Abstract

The scaling behaviour of the edge of the Lee–Yang zeroes in the four dimensional Ising model is analyzed. This model is believed to belong to the same universality class as the ϕ_4^4 model which plays a central role in relativistic quantum field theory. While in the thermodynamic limit the scaling of the Yang–Lee edge is not modified by multiplicative logarithmic corrections, such corrections are manifest in the corresponding finite–size formulae. The asymptotic form for the density of zeroes which recovers the scaling behaviour of the susceptibility and the specific heat in the thermodynamic limit is found to exhibit logarithmic corrections too. The density of zeroes for a finite–size system is examined both analytically and numerically.

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1 Zeroes of the Partition Function

The 4D Ising model is believed to belong to the same universality class as the ϕ^4 model which plays a central role in relativistic quantum field theory. The grand canonical partition function for the Ising model (which corresponds to the vacuum to vacuum transition amplitude in ϕ^4 theory) in the presence of an external magnetic field H is

$$Z_N = \frac{1}{\mathcal{N}} \sum_{\{\phi_i\}} e^{-\kappa \left(-J \sum_{\langle i,j \rangle} \phi_i \phi_j - H \sum_i \phi_i \right)} \quad , \quad (1.1)$$

where $\kappa = (k_B T)^{-1}$ is the inverse of the Boltzmann constant times the temperature, J is a coupling constant representing the strength of the intersite interaction (set to unity in the following) and N represents the total number of sites on the lattice. The value, ϕ_i , of the spin at site i , is restricted to ± 1 . In its generic form, only nearest neighbour interactions are considered and such a link is represented by $\langle i, j \rangle$. The sum runs over all \mathcal{N} possible configurations of the spin field, and the normalization ensures $Z_N = 1$ when $\kappa = 0$. The partition function may be expressed as

$$Z_N = \sum_{M=-N}^N \sum_{S=-dN}^{dN} \rho(S, M) e^{\kappa S} e^{hM} \quad , \quad (1.2)$$

in which d is the dimensionality of the system, $h = \kappa H$ is the reduced external magnetic field, and

$$S = \sum_{\langle i,j \rangle} \phi_i \phi_j \quad , \quad M = \sum_{i=1}^N \phi_i \quad , \quad (1.3)$$

are the configuration energy and magnetization. In S the sum is over the dN nearest neighbours or links of a periodic lattice. The spectral density $\rho(S, M)$ denotes the relative weight of configurations having given values of S and M . In the absence of an odd external field a second order phase transition occurs at a critical value κ_c of κ . The reduced temperature

$$t = \frac{\kappa_c - \kappa}{\kappa_c} \quad (1.4)$$

is a measure of the distance away from criticality. The partition function Z_N can be written as an N^{th} degree polynomial in the fugacity z defined by

$$z = e^{-2h} \quad , \quad (1.5)$$

as

$$Z_N(t, z) = z^{-\frac{N}{2}} \sum_{k=0}^N \rho_k(t) z^k \quad , \quad (1.6)$$

in which

$$\rho_k(t) = \sum_{S=-dN}^{dN} \rho(S, N - 2k) e^{\kappa S} \quad (1.7)$$

is an integrated density.

That the partition function (1.6) is analytic for finite N establishes that no phase transition can occur in a finite-size system. However as N is allowed to approach infinity, phase transitions which manifest themselves as points of non-analyticity can and do occur. In 1952 Lee and Yang [1] showed that the study of the onset of criticality is equivalent to that of the scaling behaviour of the zeroes of the partition function. For a finite system, or in the thermodynamic limit but in the symmetric phase ($t > 0$), the zeroes in H are strictly complex and the free energy is analytic in a non-vanishing neighbourhood of the real axis. As criticality is approached ($N \rightarrow \infty, t \rightarrow 0$) the Lee–Yang zeroes pinch the real H axis, precipitating a phase transition. The Lee–Yang theorem states that for the Ising model these zeroes lie on the unit circle in the complex fugacity plane (the imaginary axis in the complex external field plane). This theorem holds independent of the size, dimension and structure of the lattice.

In the forty years since the ideas of Lee and Yang were presented, there has been continual interest in this approach to the problem of phase transitions. Analytical progress has included alternative and modified proofs of the original circle theorem, extensions of the result to other systems, as well as theorems proving that no zeroes can exist in certain regions (see [2] for a review).

It has been shown rigorously that for isotropic nearest neighbour interactions, and for t sufficiently positive (the symmetric phase), there exists a region around $H = 0$ which is free from zeroes [3]. This means there exists a gap $|\text{Im}H| < H_1(t)$ where the density of zeroes is zero. The free energy is

analytic in H in the gap and no phase transition can occur (as a function of H). The point $H = iH_1(t)$, which is a branch point of the partition function, is called the Yang–Lee edge [4]. One expects that this property (the existence of a gap) holds in fact for all $t > 0$.

Early numerical work on Lee–Yang zeroes involved the exact calculation of the density of states $\rho(S, M)$ and was therefore restricted to very small lattice volumes [5]. In the 1980’s, Monte Carlo histogram approximations to the density of states [6], the invention of cluster algorithms [7, 8] and multi-histogram methods [9, 10, 11] all provided boosts to the numerical approach. Nonetheless, numerical studies are necessarily limited to finite volume.

It is, however, the infinite volume limit which is of primary interest. An important analytical breakthrough which related such numerical analyses in finite volume to the thermodynamic limit came when Itzykson, Pearson and Zuber [12] connected the concept of partition function zeroes to the renormalization group and thereby formulated a finite–size scaling theory for these zeroes. Their work applies to dimensions of three or less. This was later extended to dimensions of five or more in [13]. The upper critical dimension of the Ising or ϕ^4 universality class is $d = 4$. Above this the scaling behaviour of the thermodynamic functions simplifies and the critical exponents are exactly those of the mean field theory. Below four dimensions the scaling behaviour is of a power–law type. At $d = 4$ the mean field power–law scaling behaviour is modified by multiplicative logarithmic corrections — a circumstance intimately related to the expected triviality of the theory [14, 15, 16]. Logarithmic corrections to the finite–size scaling of partition function zeroes in four dimensions have recently been identified from a perturbative renormalization group analysis backed up by a high precision numerical study [17].

Recently Salmhofer [18] has proved the existence of a unique density of zeroes in the thermodynamic ($N \rightarrow \infty$) limit. The scaling behaviour of the Yang–Lee edge in the thermodynamic limit was studied by Abe [19] and by Suzuki [20] in 1967 for Ising models below the upper critical dimension. They found asymptotic forms for the density of zeroes and a power–law behaviour for the scaling of the edge in the symmetric phase.

Here we would like to add to this body of knowledge by presenting some results on Lee–yang zeroes in the symmetric phase ($t > 0$) of the four dimensional Ising model. To this end we find an asymptotic form for the density of zeroes in the thermodynamic limit which is sufficient to recover the scaling

forms for the specific heat and susceptibility. The finite-size behaviour of the location and density of the Lee–Yang zeroes is studied both analytically and numerically.

2 The Density of Lee–Yang Zeroes and the Yang–Lee Edge

According to the Lee–Yang theorem [1] the zeroes of the partition function all lie on the unit circle in the complex fugacity plane. Denoting the (t -dependent) position of these zeroes by

$$z_j(t) = e^{i\theta_j(t)} \quad , \quad \theta_j \in \Re \quad , \quad j = 1, \dots, N \quad (2.1)$$

the partition function may be written as

$$Z_N(t, z) = z^{-\frac{N}{2}} \rho_N(t) \prod_{i=1}^N (z - e^{i\theta_j(t)}) \quad . \quad (2.2)$$

The largest coefficient $\rho_N(t)$ plays no rôle in the following and we henceforth set it to unity. The free energy density,

$$f_N(t, z) = \frac{1}{N} \ln Z_N(t, z) \quad , \quad (2.3)$$

can be written as

$$f_N(t, z) = -\frac{1}{2} \ln z + \frac{1}{N} \sum_{j=1}^N \ln (z - e^{i\theta_j(t)}) \quad . \quad (2.4)$$

The discrete measure dG_N is formally given by

$$g_N(\theta, t) = \frac{dG_N(\theta, t)}{d\theta} = \frac{1}{N} \sum_{j=1}^N \delta(\theta - \theta_j(t)) \quad . \quad (2.5)$$

The t -dependent density of Lee–Yang zeroes on the unit circle in the complex z plane is given by g_N and the cumulative density of zeroes G_N is a function monotonically increasing in θ from $G(0, t) = 0$ to $G(2\pi, t) = 1$. The free energy is

$$f_N(t, z) = -\frac{1}{2} \ln z + \int_{\theta=0}^{\theta=2\pi} \ln (z - e^{i\theta}) dG_N(\theta, t) \quad . \quad (2.6)$$

The thermodynamic limit is

$$g(\theta, t) = \lim_{N \rightarrow \infty} g_N(\theta, t) \quad , \quad (2.7)$$

$$G(\theta, t) = \lim_{N \rightarrow \infty} G_N(\theta, t) \quad , \quad (2.8)$$

$$f(t, z) = \lim_{N \rightarrow \infty} f_N(t, z) \quad . \quad (2.9)$$

The coefficients $\rho_k(t)$ of the polynomial (1.6) are real and hence $g(-\theta, t) = g(\theta, t)$. Therefore it is sufficient to consider only the interval $0 \leq \theta \leq \pi$ in the integrals. The Yang–Lee edge $\theta_c(t)$ is defined by

$$g(\theta, t) = 0 \quad \text{for} \quad -\theta_c(t) < \theta < \theta_c(t) \quad . \quad (2.10)$$

Integrating (2.6) by parts gives for the free energy

$$f(t, z) = \frac{1}{2} \ln(2 \cosh(2h) + 2) - \int_{\theta_c(t)}^{\pi} \frac{\sin \theta}{\cosh(2h) - \cos \theta} G(\theta, t) d\theta \quad . \quad (2.11)$$

The magnetization is then

$$\frac{\partial f}{\partial h} = \tanh(h) + 2 \sinh(2h) \int_{\theta_c(t)}^{\pi} \frac{\sin \theta}{(\cosh(2h) - \cos \theta)^2} G(\theta, t) d\theta \quad , \quad (2.12)$$

and the zero field susceptibility

$$\chi(t) = \left(\frac{\partial^2 f}{\partial h^2} \right)_{h=0} = 1 + 4 \int_{\theta_c(t)}^{\pi} \frac{\sin \theta}{(1 - \cos \theta)^2} G(\theta, t) d\theta \quad . \quad (2.13)$$

One expects the contribution of small θ to be dominant [19, 20]. In particular we want to study its contribution singular in t . Expanding the trigonometric functions in (2.13) (and dropping the constant term),

$$\chi(t) = 16 \int_{\theta_c(t)}^{\pi} \frac{G(\theta, t)}{\theta^3} \{1 + O(\theta^2)\} d\theta \quad . \quad (2.14)$$

In four dimensions and in the symmetric phase the perturbative renormalization group gives [21]

$$\chi(t) \sim t^{-1} (-\ln t)^{\frac{1}{3}} \quad . \quad (2.15)$$

A change of variables is introduced via $\theta = \theta_c x$. Then in the critical region where $t > 0$ is sufficiently small

$$t^{-1}(-\ln t)^{1/3} \sim \theta_c(t)^{-2} \int_1^{\pi/\theta_c(t)} \frac{G(x\theta_c, t)}{x^2} dx \quad . \quad (2.16)$$

Following [19, 20], the upper integral limit can be replaced by infinity near criticality. This leads to the requirement that

$$\frac{t(-\ln t)^{-\frac{1}{3}}}{\theta_c(t)^2} \int_1^\infty \frac{G(x\theta_c, t)}{x^2} dx \sim \text{constant}. \quad (2.17)$$

For fixed t the integral is bounded due to the boundedness of G . The constancy leads to a differential equation [19, 20] for G with the general solution

$$G(\theta, t) = t^{-1} (-\ln t)^{\frac{1}{3}} \theta_c(t)^2 \Phi \left(\frac{\theta}{\theta_c(t)} \right) \quad , \quad (2.18)$$

$\Phi(x)$ being an arbitrary function of x with $\Phi(|x| \leq 1) = 0$. Then

$$g(\theta, t) = \frac{dG(\theta, t)}{d\theta} = t^{-1} (-\ln t)^{\frac{1}{3}} \theta_c(t) \Phi' \left(\frac{\theta}{\theta_c(t)} \right) \quad (2.19)$$

where $\Phi'(x) = \frac{d\Phi(x)}{dx}$.

From (2.11) (and using the fact that $G(\theta_c, t) = 0$), one gets the specific heat

$$C_V(t) = \left. \frac{\partial^2 f(t, z)}{\partial t^2} \right|_{h=0} = -2 \int_{\theta_c(t)}^\pi \theta^{-1} \frac{d^2 G(\theta, t)}{dt^2} \{1 + O(\theta^2)\} d\theta \quad . \quad (2.20)$$

Now the cumulative density of zeroes in four dimensions may be found from (2.18). In four dimensions one expects the power law scaling behaviour characteristic of dimensions below the upper critical one to be modified by multiplicative logarithmic corrections. Assume therefore that the Yang–Lee edge has the scaling behaviour

$$\theta_c(t) = At^p (-\ln t)^{-\lambda} \quad (2.21)$$

for small $t > 0$ and with $0 < p < 1$. This gives

$$\begin{aligned} \frac{d^2 G(\theta, t)}{dt^2} &= A^2 t^{2p-3} (-\ln t)^{\frac{1}{3}-2\lambda} \left[1 + O\left(\frac{1}{\ln t}\right) \right] \\ &\times \left\{ 2(1 - 3p + 2p^2)\Phi(x) + p(3 - 4p)x\Phi'(x) + p^2 x^2 \Phi''(x) \right\} \quad (2.22) \end{aligned}$$

where $x = \theta/\theta_c$ and a prime indicates derivative with respect to x . The specific heat is then

$$C_V \propto t^{2p-3} (-\ln t)^{\frac{1}{3}-2\lambda} \left[1 + O\left(\frac{1}{\ln t}\right) \right] \int_1^{\frac{\pi}{\theta_c}} \frac{I(x)}{x} dx \quad , \quad (2.23)$$

where I is some function of x . As $t \rightarrow 0$ ($\theta_c(t) \rightarrow 0$), one has

$$C_V \propto t^{2p-3} (-\ln t)^{\frac{1}{3}-2\lambda} \left[1 + O\left(\frac{1}{\ln t}\right) \right] \quad (2.24)$$

in the symmetric phase ($t > 0$, $H = 0$) and near criticality. From perturbation renormalization group analyses it is known [21] that the zero field specific heat scales as

$$C_V(t) \sim (-\ln t)^{\frac{1}{3}} \quad (2.25)$$

in four dimensions. Therefore $p = \frac{3}{2}$ and $\lambda = 0$. From (2.21) the Yang–Lee edge in four dimensions scales as

$$\theta_c(t) \sim t^{\frac{3}{2}} \quad . \quad (2.26)$$

This is the same formula as that yielded by mean field theory [12, 19, 20].

The density of Lee–Yang zeroes is given by (2.19) as

$$g(\theta, t) = t^{\frac{1}{2}} (-\ln t)^{\frac{1}{3}} \Phi' \left(\frac{\theta}{\theta_c} \right) \quad , \quad (2.27)$$

in which Φ' is an unknown function. This form is sufficient to recover the singular behaviour of the susceptibility and of the specific heat.

The behaviour of the zero at $\theta = x\theta_c$ (for fixed x) as a function of t ($t > 0$) is given by (2.27). At fixed t , $g(\theta, t)$ is an unknown function of θ/θ_c . Kortman and Griffiths emphasized the study of the density of zeroes close to the Yang–Lee edge [22]. Using high temperature and high field series they concluded that below the upper critical dimension and for a fixed (strictly positive) t , the density of zeroes near the edge exhibits a power law behaviour

$$g(\theta, t) \sim (\theta - \theta_c(t))^\sigma \quad . \quad (2.28)$$

In zero dimensions (a single site) σ is known to be -1 [23]. For the exactly solvable one dimensional Ising model $\sigma = -1/2$ for all $t > 0$ [1], i.e.,

the density of zeroes diverges as the edge is approached. The Ising model in the presence of an external field has not been solved in more than one dimension. Nonetheless the value of σ in two dimensions has been found to be $-1/6$ by Dhar [24] by mapping the two dimensional Ising ferromagnet into a solvable model of three dimensional directed animals. Cardy [25] found the same result by using the conformal invariance of two dimensional systems at the critical point. Using high temperature numerical methods, Kurtze and Fisher [4, 26] found $\sigma = 0.086(15)$ in three dimensions. It is believed that this values hold independent of the lattice parametrization used [4]. For the mean field theory $\sigma = 1/2$ [22]. Thus there seems to be a systematic increase of σ with dimensionality.

At criticality $t = 0$, however, the Yang–Lee gap vanishes and one may expect the critical exponent σ to take on a value different than that in the symmetric phase. Now, the density of zeroes is proportional to the discontinuity in the magnetization M crossing the locus of zeroes [1]

$$\lim_{r \rightarrow 1^+} M(t, z = re^{i\theta}) - \lim_{r \rightarrow 1^-} M(t, z = re^{i\theta}) \propto g(\theta, t) \quad . \quad (2.29)$$

The infinite volume behaviour of the magnetization below the upper critical dimension

$$M(t = 0, H) \sim H^{\frac{1}{\delta}} \quad (2.30)$$

should be recovered from (2.28) at $t = 0$ and therefore, for $d < 4$,

$$g(\theta, t = 0) \sim \theta^{\frac{1}{\delta}} \quad . \quad (2.31)$$

In four dimensions where $\delta = 3$, one expects the above formulae to be modified by multiplicative logarithmic corrections. There, (2.30) becomes [21]

$$M(t = 0, H) \sim H^{\frac{1}{3}} (-\ln H)^{\frac{1}{3}} \quad . \quad (2.32)$$

Therefore, in 4D in the thermodynamic limit

$$g(\theta, t = 0) \sim \theta^{\frac{1}{3}} (-\ln \theta)^{\frac{1}{3}} \quad . \quad (2.33)$$

3 Finite–Size Analysis

Non–perturbative means of calculating thermodynamic functions in spin models are provided by stochastic techniques like Monte Carlo integration.

These numerical methods yield exact results subject only to statistical error. They are however limited to finite lattices. One has to rely on finite-size scaling (FSS) extrapolation methods to gain information on the corresponding thermodynamic limit.

Let $P_L(t)$ represent the value of some thermodynamic quantity P at reduced temperature t on a lattice characterized by a linear extent L . Then, if ξ is the correlation length, the FSS hypothesis is that [17, 27]

$$\frac{P_L(t)}{P_\infty(t)} = f\left(\frac{\xi_L(t)}{\xi_\infty(t)}\right) \quad . \quad (3.1)$$

In four dimensions the scaling behaviour of the correlation length is [28]

$$\xi_\infty(t) \sim t^{-\frac{1}{2}}(-\ln t)^{\frac{1}{6}} \quad . \quad (3.2)$$

Its scaling with L is [29]

$$\xi_l(0) \sim L(\ln L)^{\frac{1}{4}} \quad . \quad (3.3)$$

Therefore in $d = 4$ the scaling variable should include logarithmic terms [17],

$$x = \frac{L(\ln L)^{\frac{1}{4}}}{t^{-\frac{1}{2}}(-\ln t)^{\frac{1}{6}}} \quad . \quad (3.4)$$

Let H_1 be the position of the Yang–Lee edge in the complex external magnetic field plane in the thermodynamic limit and let $H_1(L)$ be its finite-size counterpart (i.e., the position of the lowest lying zero for a system of finite linear extent L). From (2.26), the FSS hypothesis applied to the Yang–Lee edge gives

$$H_1(L) \sim t^{\frac{3}{2}} f(x) \quad . \quad (3.5)$$

Fixing x (so that when rescaling L , the temperature is also rescaled in such a way as to keep x constant), we find

$$\begin{aligned} H_1(L) &\sim \left(x^{-1}L^{-2}(\ln L)^{-\frac{1}{6}}\right)^{\frac{3}{2}} f(x) \\ &\sim L^{-3}(\ln L)^{-\frac{1}{4}} \quad . \end{aligned} \quad (3.6)$$

This FSS formula agrees with that derived recently by perturbative renormalization group methods [17].

The perturbative RG analysis of the finite-size ϕ_4^4 model [17] gives the relationship between the magnetization $M_L(t, H)$ and external field H at reduced temperature t

$$H = c_1 t M_L (\ln L)^{-\frac{1}{3}} + c_2 M_L^3 (\ln L)^{-1} \quad , \quad (3.7)$$

where c_1 and c_2 are constants. At $t = 0$, therefore,

$$M_L(t = 0, H) \sim H^{\frac{1}{3}} (\ln L)^{\frac{1}{3}} \quad . \quad (3.8)$$

For a finite-size system the position of the Yang–Lee edge is not zero at $t = 0$ and the origin of non-vanishing density of zeroes has to be correspondingly shifted as in (2.28). One therefore expects the density of zeroes to be

$$g_L(H_j(L)) \sim (H_j(L) - H_1(L))^{\frac{1}{3}} (\ln L)^{\frac{1}{3}} \quad , \quad (3.9)$$

where $H_j(L)$ is the position of the j^{th} Lee–Yang zero. Defining the cumulative density of zeroes at the j^{th} zero by the fractional total of zeroes up to $H_j(L)$,

$$G_L(H_j(L)) = \frac{j-1}{L^4} \quad , \quad (3.10)$$

we find (integrating g_L in (3.9) to G_L)

$$\frac{j-1}{L^4} \sim (H_j(L) - H_1(L))^{\frac{4}{3}} (\ln L)^{\frac{1}{3}} \quad . \quad (3.11)$$

Therefore

$$H_j(L) - H_1(L) \sim \left(\frac{j-1}{L^4} \right)^{\frac{3}{4}} (\ln L)^{-\frac{1}{4}} \quad . \quad (3.12)$$

Eq.(3.12) gives the j dependence of the lowest lying zeroes as well as recovering the FSS prediction of (3.6).

We now compare these FSS results with data obtained for the 4D Ising model in a high statistics Monte Carlo calculation. The simulation was done with the Swendsen–Wang cluster updating algorithm [7] applied to lattices of sizes L^4 for lattice sizes with linear extension $L = 8, 12, 16, 20, 24$ (details of the numerics can be found in [17]).

Histogram approximations to the spectral density $\rho(S, M)$ of (1.2) were determined at zero external field and at various values of κ close to the pseudocritical value (the value of κ at which the zero field specific heat peaks).

For the histogram in the magnetization each of the raw histograms in S and M were firstly binned in a 256×256 array. For each M -bin the corresponding S -subhistograms were then combined to multi-histograms according to [10]. In this way, an optimal histogram in M for arbitrary κ was obtained. From this the partition function may be determined for not too large values of (complex) H .

The critical value of κ in four dimensions has been determined to $\kappa_c = 0.149703(15)$ [17]. Our data yield only three reliable Lee–Yang zeroes for each lattice size. The reason for this is demonstrated in fig.1 where the contours along which $\text{Re}Z = 0$ and $\text{Im}Z = 0$ (for $L = 24$ and $\kappa = 0.149703$) are plotted. Because of the magnification of statistical errors far away from the simulation point $H = 0$ these contours fail to cross the imaginary Z axis when $\text{Im}H$ is large. Thus the zeroes move off the $\text{Im}H$ axis and their positions are unreliable. The remaining lattices give qualitatively similar pictures.

Table 1 lists the positions of the first Lee–Yang zeroes (the Lee–Yang edge) as obtained from the multihistograms for various κ values near κ_c and for all five lattices analyzed. As κ increases one expects the zeroes to approach the real axis in the thermodynamic limit according to (2.26). Fig.2 shows the corresponding behaviour for the finite-size systems considered. At κ_c they should scale according to the FSS formula (3.6).

Table 2 lists the positions of the first three Lee–Yang zeroes as obtained from the multihistogram at our estimated value for the critical coupling in the infinite volume limit, $\kappa_c = 0.149703$. The errors in the quantities calculated from the multihistograms were estimated by the jackknife method. The data for each lattice size were cut to produce 10 subsamples leading to 10 different multihistograms. These 10 different multihistograms give 10 different results for the quantities in question, whence the variance and bias were calculated.

The density of zeroes should behave according to (3.9) or (equivalently) (3.12). The log–log plot of fig.3 gives a slope of $0.778(2)$. The deviation from the exponent 0.75 in (3.12) is presumably due to the presence of logarithmic corrections. This may be seen in fig.4 where we remove the expected leading behaviour: A negative slope is clearly identified. In fact a best fit to all ten points gives a slope $-0.248(17)$. The shaded area is bordered by lines of this slope.

We find that both leading power-law scaling behaviour and multiplicative logarithmic corrections for the density of zeroes (or equivalently for the distance between zeroes) are identified in figs. 3 and 4. This is complementary

to our previous analysis in which the scaling behaviour of the actual positions of these zeroes was analyzed [17]. Both approaches yield quantitative agreement with the (perturbative) theoretical predictions.

4 Conclusions

The scaling behaviour of the Lee–Yang zeroes and in particular of the Yang–Lee edge in four dimensions and in the thermodynamic limit has been examined. The asymptotic form for the density of zeroes in the infinite volume limit is sufficient to recover the scaling formulae for the specific heat, the magnetization and the magnetic susceptibility. This extends the work of Abe and Suzuki to the case of four dimensions where mean field power-law scaling behaviour is modified by multiplicative logarithmic corrections which are linked to the triviality of the theory. An analytical FSS study of the edge and the density of zeroes is in good quantitative agreement with a numerical analysis in the form of Monte Carlo simulations on finite size lattices.

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Tables

Table 1: The positions of the first Lee–Yang zeroes as obtained from the multihistograms for all five lattices and near κ_c . The real part of the zeroes is always zero.

κ	$L = 8$ Im H_1	$L = 12$ Im H_1	$L = 16$ Im H_1	$L = 20$ Im H_1	$L = 24$ Im H_1
0.149600	0.015281	0.004511	0.001958	0.001047	0.000637
0.149650	0.015091	0.004384	0.001860	0.000966	0.000567
0.149703	0.014892	0.004253	0.001761	0.000886	0.000500
0.149750	0.014718	0.004140	0.001677	0.000820	0.000447
0.149800	0.014535	0.004023	0.001593	0.000756	0.000398
0.149850	0.014355	0.003910	0.001512	0.000697	0.000355
0.149900	0.014177	0.003800	0.001437	0.000644	0.000319
0.149950	0.014001	0.003693	0.001366	0.000597	0.000289
0.150000	0.013828	0.003590	0.001299	0.000555	0.000264
0.150050	0.013657	0.003491	0.001237	0.000518	0.000244
0.150100	0.013488	0.003395	0.001180	0.000486	0.000228
0.150150	0.013322	0.003302	0.001127	0.000458	0.000215
0.150200	0.013159	0.003213	0.001077	0.000433	0.000204
0.150250	0.012997	0.003127	0.001032	0.000413	0.000195
0.150300	0.012838	0.003044	0.009909	0.000395	0.000187
0.150350	0.012682	0.002965	0.009530	0.000379	0.000180
0.150400	0.012527	0.002889	0.009185	0.000366	0.000174

Table 2: The positions of the first three Lee–Yang zeroes as obtained from the jackknifed multihistograms at $\kappa = 0.149703$. The real part of the zeroes is always zero.

L	Im(H_1)	Im(H_2)	Im(H_3)
8	0.014892(22)	0.033057(48)	0.047357(174)
12	0.004253(16)	0.009426(15)	0.013349(71)
16	0.001761(6)	0.003905(22)	0.005388(24)
20	0.000886(5)	0.001970(12)	0.002743(29)
24	0.000500(4)	0.001106(5)	0.001541(12)

Figures

Fig 1: Contours along which $\text{Re}Z = 0$ (dotted lines) and $\text{Im}Z = 0$ (full lines) (for $L = 24$ and $\kappa = 0.149703$).

Fig 2: The zeroes approach the real axis as κ increases; at κ_c they should scale with the lattice size L according to (3.6). Here the triangles, circles, diamonds, stars and crosses correspond to lattice sizes 8,12,16,20 and 24 respectively.

Fig 3: The FSS of the density of zeroes is given by (3.12). The leading power-law behaviour is revealed by a log-log plot. Here the open diamonds and triangles correspond to $j = 2$ and $j = 3$ respectively. This gives a slope $0.778(2)$, the deviation away from 0.75 being due to the presence of logarithmic corrections.

Fig 4: Data like in fig.3, but with the leading power-law behaviour removed; we clearly identify the negative exponent in the $\log L$ behaviour. The shaded band indicates the result of a fit giving a slope value of $-0.248(17)$.

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