Towards Uniform Online Spherical Tessellations

Paul C. Bell
Department of Computer Science, James Parsons Building, Byrom Street, Liverpool John Moores University, Liverpool, L3-3AF, UK
p.c.bell@ljmu.ac.uk
https://orcid.org/0000-0003-2620-635X

Igor Potapov
Department of Computer Science, Ashton Building, Ashton Street, University of Liverpool, Liverpool, L69-3BX, UK
potapov@liverpool.ac.uk

Abstract

The problem of uniformly placing N points onto a sphere finds applications in many areas. An online version of this problem was recently studied with respect to the *gap ratio* as a measure of uniformity. The proposed online algorithm of Chen et al. is upper-bounded by 5.99, which is achieved by considering a circumscribed dodecahedron followed by a recursive decomposition of each face. We analyse a simple tessellation technique based on the regular icosahedron, which decreases the upper-bound for the online version of this problem to around 2.84. Moreover, we show that the lower bound for the gap ratio of placing up to three points is $\frac{1+\sqrt{5}}{2} \approx 1.618$. The uniform distribution of points on a sphere also corresponds to uniform distribution of unit quaternions which represent rotations in 3D space and has numerous applications in many areas.

2012 ACM Subject Classification "Theory of Computation → Online algorithms". "Randomness, geometry and discrete structures → Computational geometry".

Keywords and phrases Online algorithms; Spherical trigonometry; Uniform point placement

1 Introduction

One of the central problems of classical discrepancy theory is to maximize the uniformity of distributing a set of n points into some metric space [5,11]. For example, this includes questions about arranging points over a unit cube in a d-dimensional space, a polyhedral region, a sphere, a torus or even over a hyperbolic plane, etc. Measures of irregularity for a given set can be studied by defining some object, moving the object over the space, and considering the intersection of the object with the point set. Applications of modern day discrepancy theory include those in number theory (Ramsey theory), problems in numerical integration, financial calculations, computer graphics and computational physics [12].

Some motivations and applications of this problem when restricted to the 2-sphere stretch from the classical *Thompson problem* of determining a configuration of N electrons on the surface of a unit sphere that minimizes the electrostatic potential energy [13, 18], to search and rescue/exploration problems (assigning an a-priori unknown number of agents) as well as problems related to extremal energy, crystallography and computational chemistry [14]. There is also a strong connection between discrepancy theory and the study of *tessellations*, important in Computer Science, Mathematics and other areas [9]. In the original offline version of the problem of distributing points over some space, the number of points is predetermined and the goal is to distribute all points as uniformly as possible at the end of the process.

In this paper, we consider the problem of inserting points onto a sphere in the *online* setting. In this case the points should be dynamically inserted one at a time on the surface of a sphere, and the objective is to distribute the points as uniformly as possible at every instance of inserting a point. Once a point has been placed it cannot be later moved. We measure the discrepancy from uniformity following the standard metrics introduced by Teramoto et al. [17] for analysing dynamic discrepancy, and known as the *gap ratio*, i.e. the ratio between the maximum and minimal gap, where the maximum gap is the diameter of the largest empty circle on the sphere and the minimal gap is just the minimum pairwise distance between inserted points.

One might consider defining uniformity of a point set by measuring the closest two points within the given sample, however this does not take into account large gaps that may be present in the point set and in certain scenarios we wish to avoid such gaps, for example in the case of spreading volunteers so that no person can be too far away from a rescuer and in problems related to extremal energy. Alternatively we may use the standard measure from discrepancy theory, where we define some fixed geometric shape R and count the number of inserted points that are contained in R, whilst moving it all over the sphere. This measure has two main disadvantages for our scenario – that of computational hardness of calculating the discrepancy at each stage and also that we must decide upon a given shape R, each of which may give different results [17]. It was also noted in [4] that an advantage of using the gap ratio is that the space requires only a metric, unlike discrepancy that depends on the notion of a range space and its volume. We therefore utilize the gap ratio as our metric in this paper.

In [4], a generalized definition of the gap ratio has been studied considering both discrete and continuous metric spaces, also showing connections to picking uniform samples with clustering, packing and covering problems. The problem of generating a point set on spheres which minimizes criteria such as energy functions, discrepancy, dispersion and mutual distances has been extensively studied in the offline setting [7, 10, 14–16, 20, 21].

The online variant of such problems has begun to attract a lot of attention from the algorithmic community and has been already studied in different settings. The online problem has been studied for inserting integral points on a line [1] or on a grid [22], inserting real points over a unit cube [17] and also recently as a more complex version of inserting real points on a surface of a sphere [6]. In [21], the authors study the problem of generating uniform deterministic samples over the rotation group $SO(3)$, which they point out "is fundamental to computational biology, chemistry, physics and numerous branches of computer science". The online version of inserting points on a spherical surface brings new kinds of geometrical obstacles and constraints for solving the problem, but it also provides new perspectives to a wide range of applications which satisfy the rules of spherical geometry.

A good strategy for online distribution of points on the plane has been found in [2, 17] based on the Voronoi insertion, where the gap ratio is proved to be at most 2. For insertion on a two-dimensional grid, algorithms with a maximal gap ratio $2\sqrt{2} \approx 2.828$ were shown in [22]. The same authors showed that the lower bound for the maximal gap ratio is 2.5 in this context. The other important direction was to solve the problem in a one-dimensional line and an insertion strategy with a uniformity of 2 has been found in [1]. An approach of using *generalised spiral points* was discussed in [13, 14], which performs well for minimizing extremal energy, but this approach is strictly offline (number of points N known in advance).

Recently, the authors of [6] showed that for point insertion on the sphere the simple greedy approach fails and they suggest a *two phase algorithm* with an overall upper bound of 5.99. In the first phase they use an circumscribed dodecahedron to place the first twenty vertices, achieving a maximal gap ratio 2.618. After that, each of the twelve pentagonal faces can be recursively divided according to a defined procedure by having different approaches for isosceles acute triangles and isosceles obtuse triangles. Since each face is identical, this procedure is efficiently described and leads to a gap ratio of no more than 5.99 in the second phase and overall for the whole approach.

One may consider whether such two phase algorithms may perform well for this problem by either modifying the initial shape used in the first phase of the algorithm (such as using initial points derived from other Platonic solids, e.g. a tetrahedron, octahedron, icosahedron, etc.) or else whether the recursive procedure used in phase two to tessellate each regular shape may be improved. We may readily identify an advantage to choosing a Platonic solid for which each face is a triangle (the tetrahedron, octahedron and icosahedron), since in this case at least two procedures for tessellating each triangle immediately spring to mind – namely to recursively place a new point at the centre of *each edge*, denoted *triangular dissection* (creating four subtriangles, see Fig. 2), or else the Delaunay tessellation, which is to place a new point at the *centre of each triangle* (creating three new triangles, see Fig. 1).

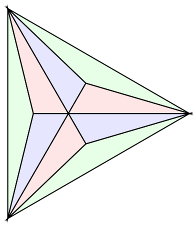

Figure 1 Delauney triangulation applied twice

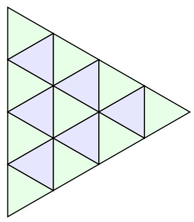

Figure 2 Recursive spherical triangle dissection applied twice

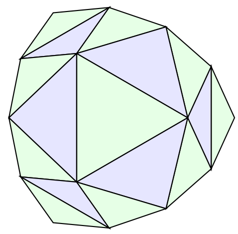

Figure 3 Deformation of tessellation in spherical projection

It can be readily seen that the Delaunay tessellation of each spherical triangle rapidly gives a poor gap ratio, since points start to become dense around the centre of edges of the initial tessellation. Regarding the second recursive tessellation strategy (Fig. 2), it was conjectured in [6], "If we split the icosahedron into four congruent sub-triangles regularly, the gap ratio will be larger since the newly inserted points are on the side of the icosahedron [...] if points are inserted on the side of some configuration, the ratio might be not good". This intuition seems reasonable, since as we recursively decompose each spherical triangle by this strategy the gap ratio increases as for such a triangle this decomposition deforms with each recursive step as can be seen in Fig. 3. It can also be seen that the gap ratio at each level of the triangular dissection is increasing (see Lemma 7). Nevertheless, we show in this paper that as long as the initial tessellation (stage 1) does not create 'large' spherical triangles (with high curvature), then the gap ratio of stage 2 has an upper limit, i.e. can be bounded from above, and performs much better than the tessellation of the regular dodecahedron proposed in [6].

Of the other Platonic solids with triangular faces, one finds that there is a trade-off between the stage 1 and stage 2 effect upon the gap ratio. We performed theoretical analysis and computational simulations of these regular Platonic solids and the results are shown in Table 2 in the conclusion which indicate that of all the Platonic solids, it is the icosahedron which has the best trade off between these two stages. In this paper, we utilise such an circumscribed regular icosahedron and the recursive triangular dissection procedure to reduce the bound of 5.99 derived in [6] to $\frac{\pi}{\arccos\left(\frac{1}{\sqrt{5}}\right)} \approx 2.8376$. Apart from a significantly better upper bound, an advantage of our triangular tessellation procedure is its generalisability and simpler principle of tessellation as we only need to compute the spherical median between two locally introduced points at every step. Note that if we consider the parallel version of the problem i.e. all points at each round of tessellation will be placed simultaneously, the gap ratio of the parallel procedure is $\frac{\pi}{2\arccos\left(\frac{1}{\sqrt{5}}\right)} \approx 1.419$.

The problem of distributing points on a sphere can also be seen as online version of distributing unit quaternions [8] which is important as unit quaternions represent rotations of objects in three dimensions [3] and have various applications in computer graphics, robotics, navigation, and crystallographic texture analysis.

The paper is organized as follows. First we define some notation from spherical trigonometry in Section 2, to make the paper self-contained and as a starting point for continuing research. In Section 3 we provide an overview of the two-stage algorithm, the first of which places points at the vertices of the icosahedron and the second places points following a spherical triangle dissection. Then we present a formal analysis of the tessellation procedure. In Section 4 we show that the gap ratio for the tessellation of an equilateral spherical triangle has a limit and converges to a particular value. This fact is used in Section 5 to prove that the gap ratio of the second stage of our algorithm is bounded by approximately 2.760 and the first stage is bounded by approximately 2.8376. We provide an analysis for inserting the first three points on a sphere which gives the first nontrivial lower bound of $\frac{1+\sqrt{5}}{2} \approx 1.6180$, so no online algorithm can achieve a lower gap ratio. This approach can be extended by considering more points, however we leave this direction for further research. Finally we conclude with some experimental results matching our theoretical bounds.

2 Notations

2.1 Spherical trigonometry

Given a set P, we denote by 2^P the power set of P (the set of all subsets). Let $\mathcal{S}$ denote the 3-dimensional unit sphere. We will deal almost exclusively with unit spheres, since for our purposes the gap ratio (introduced formally later) is not affected by the spherical radius. Let $u_1, u_2, u_3 \in \mathbb{R}^3$ be three unit length vectors, then $T = \langle u_1, u_2, u_3 \rangle$ denotes the spherical triangle on $\mathcal{S}$ with vertices u_1, u_2 and u_3. Given some set of points $\{u_1, u_2, u_3\} \cup \{v_j | 1 \leq j \leq k\}$, a spherical triangle $T = \langle u_1, u_2, u_3 \rangle$ is called *minimal* over that set of points if no v_j for $1 \leq j \leq k$ lies on the interior or boundary of T. As an example, in Fig. 5, triangle $\langle u_1, u_{113}, u_{112} \rangle$ is minimal, but $\langle u_1, u_{13}, u_{12} \rangle$ is not, since points u_{113} and u_{112} lie on the boundary of that triangle.

The edges of a spherical triangle are arcs of great circles. A great circle is the intersection of $\mathcal{S}$ with a central plane, i.e one which goes through the centre of $\mathcal{S}$. We denote the length of a path connecting two points u_1, u_2 on the unit sphere by $\zeta(u_1, u_2)$ (the spherical length).

Given a non-degenerate spherical triangle (i.e. one with positive area, defined later) with two edges e_1 and e_2 which intersect at a point P, then we say that the angle of P is the angle of P measured when projected to the plane tangent at P. We constrain all spherical triangles to have edge lengths strictly between 0 and π, which avoids issues with *antipodal triangles*. Two points on the unit sphere are called antipodal if the angle between them is π (i.e. they lie on opposite sides of the unit sphere) and an antipodal triangle contains two antipodal points. Several results in spherical trigonometry (and in this paper) are derived by working with projections of points and edges to planes tangent to a point on the sphere; in all such cases the projection is from the centre of the sphere.

The following results are all standard from spherical trigonometry, see [19] for proofs and further details. The length of an arc belonging to a great circle corresponds with the angle of the arc, see Fig. 4. Furthermore, given an arc between two points u_1 and u_2 on $\mathcal{S}$, the length of the line connecting u_1 and the projection of u_2 to the plane tangent to u_1 is given by $\tan(\zeta(u_1, u_2))$, see Fig. 4.

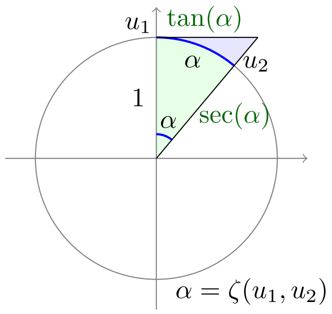

Figure 4 Angular calculations in the plane intersecting the great circle containing (u_1, u_2)

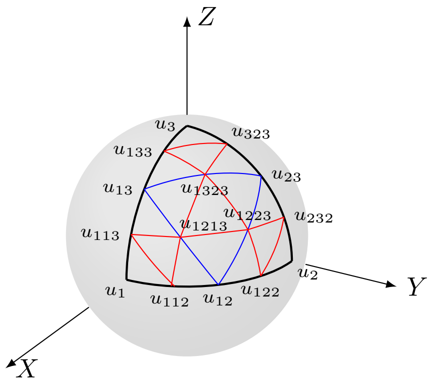

Figure 5 σ tessellations

▶ **Lemma 1** (The Spherical Laws of Cosines)**.** *Given a spherical triangle with sides a, b, c and angles A, B, C opposite to side a, b, c respectively, then:*

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C) \tag{1}$$
$$\cos(C) = -\cos(A)\cos(B) + \sin(A)\sin(B)\cos(c) \tag{2}$$

We shall also require the related *spherical sine law*.

▶ **Lemma 2** (The Spherical Sine Law)**.** *Given a spherical triangle with sides a, b and angles A, B opposite to sides a, b respectively, then:*

$$\frac{\sin(a)}{\sin(A)} = \frac{\sin(b)}{\sin(B)}.$$

It is known that the sum of angles within a spherical triangle is between π (as the volume approaches zero) and 3π (as the triangle fills the whole sphere). We may define a notion of *spherical excess*, which is the sum of its angles minus π radians. Girard's theorem shows a relation between the area and angles of a spherical triangle.

▶ **Theorem 3** (Girard's theorem)**.** *The area of a spherical triangle is equal to its spherical excess.*

2.2 Online point placing on the unit sphere

Our aim is to insert a sequence of 'uniformly distributed' points onto $\mathcal{S}$ in an online manner. After placing a point, it cannot be moved in the future. Let p_i be the i'th point thus inserted and let $S_i = \{p_1, p_2, \ldots, p_i\}$ be the configuration after inserting the i'th point. Teramoto et al. introduced the gap ratio [17], which defines a measure of uniformity for point samples and we use this metric to evaluate equidistant spacing of points on the unit sphere (similarly to [6]). Let $\rho_{\mathsf{min}} : 2^{\mathcal{S}} \to \mathbb{R}$ denote the minimal gap ratio of a set of points, defined by $\rho_{\mathsf{min}}(S_i) = \min_{p,q \in S_i, p \neq q} \zeta(p, q)$. Recall that notation $2^{\mathcal{S}}$ means the set of all points lying on the 2-sphere $\mathcal{S}$. Let $\rho^{\mathcal{S}'}_{\mathsf{max}} : 2^{\mathcal{S}} \to \mathbb{R}$ denote the maximal spherical diameter of the largest empty circle centered at some point of $\mathcal{S}' \subseteq \mathcal{S}$ avoiding a given set of points, defined by $\rho^{\mathcal{S}'}_{\mathsf{max}}(S_i) = \max_{p \in \mathcal{S}'} \min_{q \in S_i} 2 \cdot \zeta(p, q)$. We then define $\rho^{\mathcal{S}'}(S_i) = \frac{\rho^{\mathcal{S}'}_{\mathsf{max}}(S_i)}{\rho_{\mathsf{min}}(S_i)}$ to be the *gap ratio* of S_i over $\mathcal{S}'$. When $\mathcal{S}' = \mathcal{S}$ (i.e. when points can be placed anywhere on the sphere), we define that $\rho(S_i) = \rho^{\mathcal{S}}(S_i)$.

We denote an *equilateral* spherical triangle as one for which each side has the same length. By Lemma 1 (the spherical law of cosines), having three equal length edges implies that an equilateral spherical triangle has the same three angles. By Theorem 3 (Girard's theorem), each such angle is greater than $\frac{\pi}{3}$ (for an equilateral triangle of positive volume). Let $\Delta \subseteq \mathcal{S}$ denote the set of all spherical triangles on the unit sphere.

Consider a spherical triangle $T \in \Delta$. We define a *triangular dissection* function $\sigma : \Delta \to 2^{\Delta}$ in the following way. If $T \in \Delta$ is defined by $T = \langle u_1, u_2, u_3 \rangle$, then $\sigma(T) = \{T_1, T_2, T_3, T_4\} \subset \Delta$, where $T_1 = \langle u_1, u_{12}, u_{13} \rangle$, $T_2 = \langle u_{12}, u_2, u_{23} \rangle$, $T_3 = \langle u_3, u_{13}, u_{23} \rangle$ and $T_4 = \langle u_{12}, u_{13}, u_{23} \rangle$, with u_{ij} being the midpoints (on the unit sphere) of the arc connecting u_i and u_j (see Fig. 5). Define $\sigma_E(T)$ as the set of nine induced edges:

$$\{(u_1, u_{12}), (u_{12}, u_2), (u_2, u_{23}), (u_{23}, u_3), (u_3, u_{13}), (u_{13}, u_1), (u_{13}, u_{23}), (u_{23}, u_{12}), (u_{12}, u_{13})\}$$

We may extend the domain of σ to sets of spherical triangles, such that $\sigma(\{T_1, T_2, \ldots, T_k\}) = \{\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_k)\}$; thus $\sigma : 2^{\Delta} \to 2^{\Delta}$. Given a spherical triangle $T \in \Delta$, we then define that $\sigma^1(T) = \sigma(T)$ and $\sigma^k(T) = \sigma(\sigma^{k-1}(T))$ for $k > 1$. For notational convenience, we also define that $\sigma^0(T) = T$ (the identity tessellation). We may also extend $\sigma_E(T)$ to a set of triangles such that $\sigma_E(\{T_1, T_2, \ldots, T_k\}) = \{\sigma_E(T_1), \sigma_E(T_2), \ldots, \sigma_E(T_k)\}$. See Fig. 5 for an example showing the tessellation of T to depth 2 (e.g. $\sigma^2(T)$) and the set of edges $\sigma_E^2(T)$.

Let $\mu : \Delta \to 2^{\mathcal{S}}$ be a function which, for an input spherical triangle, returns the (unique) set of three points defining that triangle. For example, given a spherical triangle $T = \langle p_1, p_2, p_3 \rangle$, then $\mu(T) = \{p_1, p_2, p_3\}$. Clearly μ may be extended to sets of triangles by defining that $\mu(\{T_1, T_2, \ldots, T_k\}) = \{\mu(T_1), \mu(T_2), \ldots, \mu(T_k)\}$; thus $\mu : 2^{\Delta} \to 2^{\mathcal{S}}$. When there is no danger of confusion, by abuse of notation, we sometimes write T rather than $\mu(T)$. This allows us to write $\rho(T)$ (or $\rho(\sigma^k(T))$) for example, as the gap ratio of the three points defining spherical triangle T (resp. the set of points in the k-fold triangular dissection $\sigma^k(T)$).

We will also require an ordering on the set of points generated by a tessellation $\sigma^k(T)$. Essentially, we wish to order the points as those of $\sigma^0(T) = T$ first (in any order), then those of $\sigma^1(T)$ in any order *but omitting the points of* $\sigma^0(T) = T$, then the points of $\sigma^2(T)$, omitting points in triangles of $\sigma^0(T)$ or $\sigma^1(T)$ etc. To capture this notion, we introduce a function $\tau : \Delta \times \mathbb{Z}^+ \to 2^{\mathcal{S}}$ defined thus:

$$\tau(T, k) = \begin{cases} \mu(\sigma^k(T)) - \mu(\sigma^{k-1}(T)) & ; \text{ if } k \geq 1 \\ \mu(T) & ; \text{ if } k = 0 \end{cases}$$

As an example, consider Fig. 5, then $\tau(T, 0) = \{u_1, u_2, u_3\}$, $\tau(T, 1) = \{u_{12}, u_{13}, u_{23}\}$, $\tau(T, 2) = \{u_{112}, u_{122}, u_{232}, u_{323}, u_{133}, u_{113}, u_{1323}, u_{1213}, u_{1223}\}$. By abuse of notation, we redefine $\sigma^k(T)$ then so that $\sigma^k(T) = \tau(T, 0) \cup \tau(T, 1) \cup \cdots \cup \tau(T, k)$ is an ordered set.

3 Overview of Online Spherical Vertex Insertion Algorithm

As pointed out in [6], a natural strategy for online point insertion is the greedy approach, i.e. to place the next point at the spherical centre of the largest empty spherical surface area. This strategy begins reasonably, with a gap ratio of 2 after placing three points, but the tessellations which are generated after a few iterations become computationally difficult to determine.

Our algorithm is a two stage strategy. In stage one, we project the 12 vertices of the regular icosahedron onto the unit sphere (see Fig. 9). The first two such points inserted should be opposite each other (antipodal points), but the remaining 10 points can be inserted in any order. We show that the gap ratio during stage one of our algorithm is $\frac{\pi}{\arccos\left(\frac{1}{\sqrt{5}}\right)} \approx 2.8376$.

In the second stage, we treat each of the 20 equilateral spherical triangles of the regular icosahedron in isolation. We show in Lemma 7 that the gap ratio for our tessellation is 'local' and depends only on the local configuration of vertices around a given point. This allows us to consider each triangle separately. During stage two, we use the fact that these twenty spherical triangles are equilateral and apply Lemma 7 to independently tesselate each triangle recursively in order to derive an upper bound of the gap ratio in stage two of $\frac{2(3-\sqrt{5})}{\arcsin\left(\frac{1}{2}\sqrt{2-\frac{2}{\sqrt{5}}}\right)} \approx 2.760$.

We note here that the radius of the sphere does not affect the gap ratio of the point insertion problem, and thus we assume a unit sphere throughout.

The algorithmic procedure to generate an infinite set of points is shown in Algorithm 1. To generate a set of k points $\{p_1, p_2, \ldots, p_k\}$, we choose the first k points generated by the algorithm. Recall that a minimal triangle in a set of triangles is essentially the smallest possible triangle, i.e. a triangle containing exactly three vertices and no others.

Stage one: Project 12 vertices of the regular icosahedron to the unit sphere:
    Place two antipodal points on the unit sphere.
    Place the remaining ten points in any order.
    Arbitrarily label the 20 minimal spherical triangles $T = \{T_1, \ldots, T_{20}\}$.
Stage two: Recursively tessellate minimal triangles
    Let $T' \leftarrow T$
    while TRUE **do**
      for all minimal spherical triangles $R \in T$ **do**
          Let $T' \leftarrow (T' \cup \sigma(R)) - R$
      end for
      Let $T \leftarrow T'$
    end while

Algorithm 1: Placing infinitely many points on the unit sphere using our recursive tessellation procedure on the regular icosahedron.

---

4 Gap ratio of equilateral spherical triangles

We will require several lemmata regarding tessellations of spherical triangles.

▶ **Lemma 4.** *Let $T \in \Delta$ be an equilateral triangle. Then the central triangle in the tessellation $\sigma(T)$ is also equilateral.*

Proof. Consider Fig. 5. The lemma claims that if $\langle u_1, u_2, u_3 \rangle$ is equilateral, then so is $\langle u_{12}, u_{13}, u_{23} \rangle$ (and therefore also $\langle u_{1213}, u_{1223}, u_{1323} \rangle$).

As a consequence of the spherical cosine rule, an equilateral spherical triangle will have three equal interior angles, each of which is larger than $\frac{\pi}{3}$ (otherwise, by Girard's theorem, it has zero area). Since the edge lengths of T are identical, then the central triangle of $\sigma(T)$ also has equal length edges, again by the spherical cosine rule. ◀

It is worth again noting in Lemma 4 that the other three triangles in the triangular dissection of an equilateral triangle are *not* equilateral, and indeed they have a strictly smaller area than the central triangle. This *deformation* of the recursive triangular dissection makes the analysis of the algorithm nontrivial. The following lemma equates the distance from the centroid of an equilateral spherical triangle to a vertex of that triangle.

▶ **Lemma 5.** *Let $T = \langle u_1, u_2, u_3 \rangle \in \Delta$ be an equilateral triangle with centroid u_c and edge length $\zeta(u_1, u_2) = \alpha$. Then $\zeta(u_1, u_c) = \zeta(u_2, u_c) = \zeta(u_3, u_c) = \arcsin\left(\frac{2\sin(\frac{\alpha}{2})}{\sqrt{3}}\right)$.*

Proof. Consider Fig. 7. The centroid of T is, as for standard triangles, the unique point u_c of T from which the (spherical) distance satisfies $\zeta(u_c, u_1) = \zeta(u_c, u_2) = \zeta(u_c, u_3)$. Let then $x = \zeta(u_c, u_1)$. By the sine rule of spherical trigonometry:

$$\frac{\sin(x)}{\sin(\pi/2)} = \frac{\sin(\alpha/2)}{\sin(2\pi/6)},$$

and since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, then $x = \arcsin\left(\frac{2\sin(\frac{\alpha}{2})}{\sqrt{3}}\right)$. ◀

Given an equilateral spherical triangle T, we will also need to determine the maximal and minimal edge lengths in $\sigma_E^k(T)$ for $k \geq 1$. The following lemma identifies which edges will have minimal and maximal lengths.

▶ **Lemma 6.** *Let $T = \langle u_1, u_2, u_3 \rangle \in \Delta$ be an equilateral triangle such that $\alpha = \zeta(u_1, u_2) \in (0, \frac{\pi}{2}]$ and $k \geq 1$. Then the minimal length edge in $\sigma_E^k(T)$ is given by any edge lying on the boundary of T. The maximal length edge of $\sigma_E^k(T)$ is any of the edges of the central equilateral triangle of $\sigma^k(T)$.*

Proof. Consider Fig. 4. The lemma states that in $\sigma^2(T)$ shown, the *shortest* length edge of $\sigma_E^2(T)$ is (u_1, u_{113}), or indeed any such edge on the boundary of triangle $\langle u_1, u_2, u_3 \rangle$. The lemma similarly states that the *longest* edge of $\sigma_E^2(T)$ is edge (u_{1213}, u_{1223}), or indeed any edge of the central equilateral triangle $\langle u_{1213}, u_{1223}, u_{1323} \rangle$.

Consider now Fig. 6 illustrating $T = \langle u_1, u_2, u_3 \rangle$. Point u_{12} (resp. u_{13}) is at the midpoint of spherical edge (u_1, u_2) (resp. (u_1, u_3)). Let $\alpha = \zeta(u_1, u_2) = \zeta(u_2, u_3) = \zeta(u_1, u_3)$ be the edge length. The intersection of spherical edges (u_2, u_{13}) and (u_1, u_3) forms a spherical right angle. We denote $y = \zeta(u_{13}, u_{12})$, thus y is the edge length of the central equilateral triangle of $\sigma(T)$ (and $\frac{\alpha}{2} = \zeta(u_1, u_{13})$ is the edge length of the minimal length edge of one of the non-central triangles in $\sigma(T)$; note that this is the same for each such triangle).

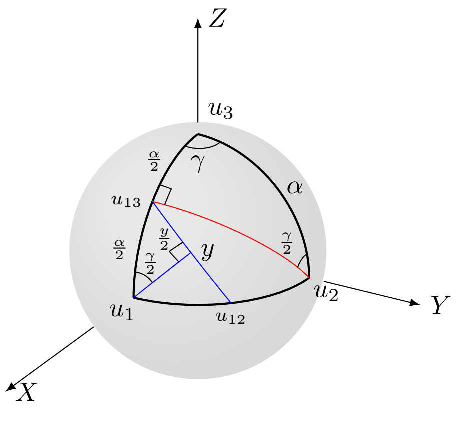

Figure 6 Max and min lengths of σ tessellations of an equilateral triangle.

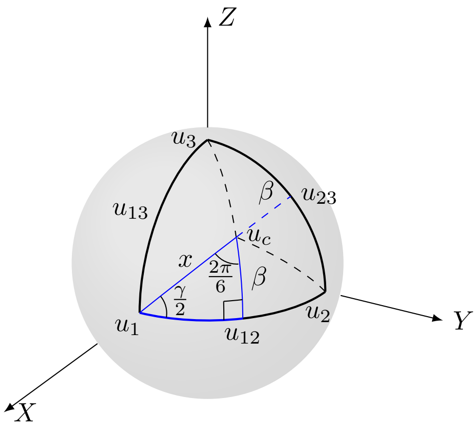

Figure 7 Centroid calculations

By the spherical sine rule, $\sin\left(\frac{\gamma}{2}\right) = \frac{\sin\frac{\alpha}{2}}{\sin\alpha}$, which is illustrated by triangle $\langle u_2, u_{13}, u_3 \rangle$. Therefore $\sin(\frac{\gamma}{2}) = \frac{1}{2}\sec\left(\frac{\alpha}{2}\right)$. Here we used the identity that $\frac{\sin\left(\frac{x}{2}\right)}{\sin x} = \frac{1}{2}\sec\left(\frac{x}{2}\right)$. Furthermore, one can see by the spherical sine rule that

$$\sin\left(\frac{y}{2}\right) = \sin\left(\frac{\alpha}{2}\right)\cdot\sin\left(\frac{\gamma}{2}\right) = \frac{1}{2}\frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)} = \frac{1}{2}\tan\left(\frac{\alpha}{2}\right)$$

This implies that $\zeta(u_{13}, u_{12}) = y = 2\arcsin\left(\frac{1}{2}\tan\frac{\alpha}{2}\right)$ which is larger than $\frac{\alpha}{2}$ for $\alpha \in (0, \frac{\pi}{2}]$. To prove this, let $f(\alpha) = 2\arcsin(\frac{1}{2}\tan\frac{\alpha}{2})$ then $\frac{df}{d\alpha} = \frac{2}{\cos^2\left(\frac{\alpha}{2}\right)\sqrt{4-\tan^2\left(\frac{\alpha}{2}\right)}}$ as is not difficult to prove. Noting that if $\alpha \in (0, \frac{\pi}{2}]$, then $\cos^2\left(\frac{\alpha}{2}\right) \in [\frac{1}{2}, 1]$ and $\sqrt{4-\tan^2(\frac{\alpha}{2})} \in [\sqrt{3}, 2]$, then $\frac{df}{d\alpha} > \frac{1}{2} = \frac{d\frac{\alpha}{2}}{d\alpha}$ and therefore since $f(\alpha) = 0 = \frac{\alpha}{2}$ when $\alpha = 0$, then $y > \frac{\alpha}{2}$ for $\alpha \in (0, \frac{\pi}{2}]$.

For any depth k-tessellation $\sigma^k(T)$, the maximal edge length of $\sigma^k_E(T)$ will thus be given by the length of the edges of the central equilateral triangle and the minimal length edges will be located on the boundary of T as required. ◀

Given an equilateral spherical triangle $T = \langle u_1, u_2, u_3\rangle$, we will now consider the gap ratio implied by the restriction of points to those of T. The first part of this lemma shows that the gap ratio of a depth-k tessellation is lower than the gap ratio of a depth-$k+1$ tessellation (when restricted to points of T), and the second part shows that in the limit, the upper bound converges.

▶ **Lemma 7.** *Let $T = \langle u_1, u_2, u_3\rangle$ be an equilateral spherical triangle with spherical edge length α. Then:*

i) $\rho^T(\mu(\sigma^k(T))) < \rho^T(\mu(\sigma^{k+1}(T)))$;

ii) $\lim_{k\to\infty} \rho^T(\mu(\sigma^k(T))) = \frac{4\sin\left(\frac{\alpha}{2}\right)}{\alpha\sqrt{3-4\sin^2\left(\frac{\alpha}{2}\right)}}$.

Proof. Consider Fig. 7 and let $\alpha = \zeta(u_1, u_2) = \zeta(u_2, u_3) = \zeta(u_1, u_3)$ be the edge length of the equilateral triangle $T = \langle u_1, u_2, u_3\rangle$. Let us calculate $\rho^T(\sigma^0(T)) = \rho^T(T)$. Note by abuse of notation that we write $\rho^T(T)$ rather than the more formal $\rho^T(\mu(T))$, as explained previously. Recall then that $\rho^T(T)$ denotes the gap ratio of point set $\mu(T)$ when the maximal gap ratio calculation is restricted to points of T.

We see that $\rho_{\mathsf{min}}(T) = \alpha$ since all edge lengths of T are identical. Clearly $\rho^T_{\mathsf{max}}(T) = 2x$; in other words the maximal spherical diameter of the largest empty circle centered inside T should be placed at the centroid u_c of T. This follows since if the circle is centered at any other point of T, then it will be closer to at least one vertex of T and therefore the maximal ratio would only decrease. Thus $\rho^T(\sigma^0(T)) = \frac{2x}{\alpha}$.

By Lemma 4, triangle $\langle u_{12}, u_{23}, u_{13}\rangle$ in the decomposition $\sigma(T)$ is also equilateral. It is clear that $\beta > \alpha/2$ in Fig. 7 by the spherical sine rule, since $\gamma > \pi/3$ (by Girard's theorem). Therefore, $\rho_{\mathsf{min}}(\sigma^1(T)) = \frac{\alpha}{2}$, $\rho^T_{\mathsf{max}}(\sigma^1(T)) = 2\beta$ and thus $\rho^T(\sigma^1(T)) = \frac{4\beta}{\alpha}$. We now show that $\frac{2x}{\alpha} < \frac{4\beta}{\alpha}$, which is true if $x < 2\beta$.

Let us consider the projection of equilateral spherical triangle $\langle u_1, u_2, u_3\rangle$ from the center of the unit sphere to a tangent plane at the point u_c. The point u_c is the centroid of spherical triangle $\langle u_1, u_2, u_3\rangle$, as well its projection to the plane P, given by the planar triangle $\langle u'_1, u'_2, u'_3\rangle$.

The median of spherical triangle $\langle u_1, u_2, u_3\rangle$ has length $x + \beta$. The range of x is from β to 2β. This follows from the fact that in the maximal equilateral spherical triangle case (i.e. when each angle is π and the triangle forms a half sphere) $\beta = x = \frac{\pi}{2}$ and when the area of the spherical triangle converges to zero, the median of the spherical triangle $\langle u_1, u_2, u_3\rangle$ converges to the median of the triangle projection $\langle u'_1, u'_2, u'_3\rangle$, and x converges to 2β as the centroid of a Euclidean triangle divides each median in the ratio $2 : 1$.

We thus see that the gap ratio of the (six) points of $\sigma^1(T)$ is greater than the gap ratio of the (three) points of $\sigma^0(T)$, when restricted to points of T. Since the maximal ratio is calculated by using a circle centered at the centroid of the triangle T, this argument applies recursively and for each tessellation $\sigma^k(T)$, the maximal ratio is given by twice the distance of the centroid to the vertices of the central equilateral triangle of the tesselation by Lemma 6, and therefore the gap ratio increases at each depth of the tessellation, which proves statement one of the lemma. We will now determine $\lim_{k\to\infty} \rho^T(\sigma^k(T))$ to prove the second statement.

We may observe that $\rho_{\mathsf{min}}(\sigma^k(T)) = \frac{\alpha}{2^k}$, since the outer edges of triangle T (with length α) are subdivided into two k times under $\sigma^k(T)$ and all interior edges have greater length. As explained above, the maximal diameter circle which may be placed on a point of T which does not intersect points of $\sigma^k(T)$ will be centered at the centroid u_c of T and have a diameter twice the distance from u_c to a vertex of that triangle.

Construct a plane P_{u_c} tangent to the point u_c (the centroid of T). In Fig. 7, note that $x = \zeta(u_1, u_c) = \arcsin\left(\frac{2\sin(\frac{\alpha}{2})}{\sqrt{3}}\right)$ by Lemma 5. The distance[1] from the centroid point to a vertex projected by points u_1, u_2 or u_3 is given by $\tan x$ (see Fig. 4 and Section 2.1), thus the edges of the projection of triangle T have length $y = \sqrt{3}\tan x$ since the projection of an equilateral triangle about its centroid from the origin to P_{u_c} is equilateral, with the same centroid (and the distance from the centroid of a Euclidean triangle to any vertex is of course given by $\frac{e}{\sqrt{3}}$, where e is the edge length of the triangle). The central tessellated triangle thus has edges whose length starts to approximate $\frac{\sqrt{3}\tan x}{2^k}$ in the limit, since as $k \to \infty$, then this triangle lies on the plane tangent at u_c (i.e. the difference between the edge length of the spherical triangle and its projection decreases to zero). This implies that the maximal spherical diameter of the largest empty circle centered at u_c is the distance from u_c to one of these vertices, which approaches $\frac{2\tan x}{2^k}$ in the limit as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} \rho^T(\sigma^k(T)) = \frac{\rho^T_{\mathsf{max}}(\sigma^k(T))}{\rho_{\mathsf{min}}(\sigma^k(T))} = \frac{2\tan x/2^k}{\alpha/2^k} = \frac{2\tan\left(\arcsin\left(\frac{2\sin(\frac{\alpha}{2})}{\sqrt{3}}\right)\right)}{\alpha} \tag{3}$$

$$= \frac{4\sin\left(\frac{\alpha}{2}\right)}{\alpha\sqrt{3-4\sin^2\left(\frac{\alpha}{2}\right)}} \tag{4}$$

Moving from (3) to (4), we used the identity $\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$. ◀

5 Regular icosahedral tessellation

As explained in Section 3 and Algorithm 1, our algorithm consists of two stages. Using the lemmata of the previous section, we are now ready to show that the stage one gap ratio is no more than $\frac{\pi}{\arccos\left(\frac{1}{\sqrt{5}}\right)} \approx 2.8376$ and the second stage gap ratio is no more than $\frac{2(3-\sqrt{5})}{\arcsin\left(\frac{1}{2}\sqrt{2-\frac{2}{\sqrt{5}}}\right)} \approx 2.760$.

▶ **Lemma 8.** *The gap ratio of stage one is no more than* $\frac{\pi}{\arccos\left(\frac{1}{\sqrt{5}}\right)} \approx 2.8376$.

Proof. Consult Fig. 9. The points of a regular icosahedron can be defined by taking circular permutations of $(0, \pm 1, \pm\phi)$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Let V' be the set of the

[1] Note that here we refer to the Euclidean distance between the points, rather than the spherical distance, since the projected points are not on the sphere

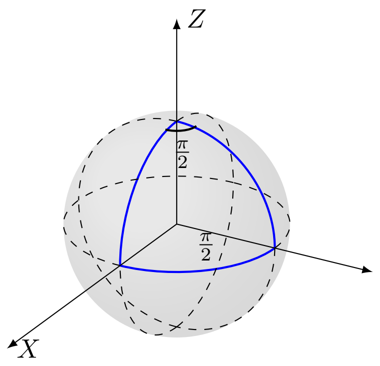

Figure 8 Octahedral tessellation

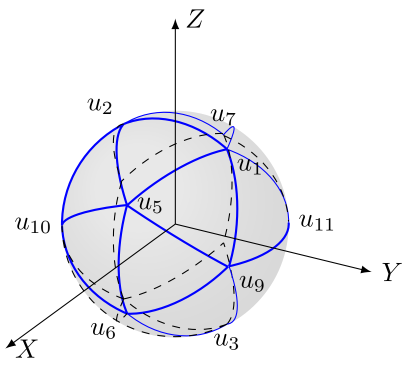

Figure 9 Icosahedral tessellation

twelve such vertices. Normalising each element of V' gives a set V. Note that the area of each spherical triangle is given by $\frac{\pi}{5}$ since we have a unit sphere and twenty identical spherical triangles forming a tessellation. By Girard's theorem (Theorem 3), this implies that $3\gamma - \pi = \frac{\pi}{5}$, where γ is the interior angle of the equilateral triangle and thus $\gamma = \frac{2\pi}{5}$. By the second spherical law of cosines (Lemma 1), this implies that the spherical distance between adjacent vertices, α, is thus given by

$$\cos(\alpha) = \frac{\cos(\frac{2\pi}{5}) + \cos^2(\frac{2\pi}{5})}{\sin^2(\frac{2\pi}{5})} = \frac{\frac{1}{4}(\sqrt{5}-1) + \frac{3}{8} + (\frac{\sqrt{5}}{8})}{\frac{5}{8} + \frac{\sqrt{5}}{8}} = \frac{\frac{1}{8}(1+\sqrt{5})}{\frac{1}{8}(5+\sqrt{5})} = \frac{1}{\sqrt{5}} \tag{5}$$

and therefore $\alpha \approx 1.1071$. The first two points are placed opposite to other, for example $u_2 \approx (0, -0.5257, 0.8507)$ and $u_3 \approx (0, 0.5257, -0.8507)$ in Fig. 9. At this stage, the gap ratio is 1, since the largest circle may be placed on the equator (with u_2 and u_3 at the poles) with a diameter of π, whereas the spherical distance between u_2 and u_3 can be calculated as π. The remaining ten vertices of the normalised regular icosahedron are placed in any order. The minimal distance between them is given by α above, and thus the gap ratio during stage one is no more than $\frac{\pi}{\arccos\left(\frac{1}{\sqrt{5}}\right)} \approx 2.8376$, as required. ◀

As explained in Section 3 and Algorithm 1, we start with the eight equilateral spherical triangles produced in stage one, denoted the depth-0 tessellation of $\mathcal{S}$. We apply σ to each such triangle to generate $8 * 4 = 32$ smaller triangles (note that not all such triangles are equilateral, in fact only eight triangles at each depth tessellation are equilateral). At this stage we have the depth-1 tessellation of $\mathcal{S}$. We recursively apply σ to each spherical triangle at depth-k to generate the depth-$k+1$ tessellation, which contains $8 * 4^{k+1}$ spherical triangles.

▶ **Lemma 9.** *The gap ratio of stage two is no more than* $\frac{12-4\sqrt{5}}{\arccos\left(\frac{1}{\sqrt{5}}\right)} \approx 2.760$.

Proof. At the start of stage two, we have twenty equilateral spherical triangles, $T_1, \ldots, T_{20}$ which are identical (up to rotation). Note that moving from depth-k tessellation to depth-$k+1$, each edge of $\sigma_E^k(T_i)$ will be split at its midpoint, therefore applying σ to any spherical triangle will only 'locally' change the gap ratio of at most two adjacent triangles. Thus the order in which σ is applied to each triangle at depth k is irrelevant.

Assume that we have a (complete) depth-k tessellation with $20 * 4^k$ triangles. Lemma 7 tell us that the gap ratio increases from the depth-k to the depth-$k+1$ tessellations for all

$k \geq 0$ and in the limit, the gap ratio of the depth-k tessellation of each T_i is given by:

$$\lim_{k\to\infty} \rho^T(\sigma^k(T_i)) \quad = \quad \frac{4\sin\left(\frac{\alpha}{2}\right)}{\alpha\sqrt{3-4\sin^2\left(\frac{\alpha}{2}\right)}} \tag{6}$$

where α is the length of the edges of T_i (i.e. the length of those triangles produced by stage 1 via the circumscribed icosahedron). When we start to tessellate the depth-k spherical triangles, until we have a complete depth-$k+1$ tessellation, applying σ to each of the triangles, may decrease the minimal gap ratio at most by a factor up to 2 overall (since we split each edge at its midpoint). The maximal gap ratio cannot increase, but decreases upon completing the depth $k+1$ tessellation. Therefore, we multiply Eq. (6) by 2 to obtain an upper bound of the gap ratio for the entire sequence, not only when some depth-k tessellation is complete.

We must now solve Eq. (6), after multiplication by 2, by substituting $\alpha = \arccos\left(\frac{1}{\sqrt{5}}\right) \approx 1.1071$. This is somewhat laborious, but by noting that $\sin(\frac{\alpha}{2}) = \sqrt{\frac{1}{10}(5-\sqrt{5})}$ and $\sin^2(\frac{\alpha}{2}) = \frac{1}{10}(5-\sqrt{5})$, then:

$$\frac{8\sin\left(\frac{\alpha}{2}\right)}{\alpha\sqrt{3-4\sin^2\left(\frac{\alpha}{2}\right)}} \quad = \quad \frac{8\sqrt{\frac{1}{10}(5-\sqrt{5})}}{\arccos\left(\frac{1}{\sqrt{5}}\right)\sqrt{1+\frac{2}{\sqrt{5}}}} = \frac{12-4\sqrt{5}}{\arccos\left(\frac{1}{\sqrt{5}}\right)} \approx 2.760 \tag{7}$$

Therefore, the gap ratio during stage two is upper bounded by 2.76 as required. ◀

▶ **Theorem 10.** *The gap ratio of the icosahedral triangular dissection is* $\frac{\pi}{\arccos\left(\frac{1}{\sqrt{5}}\right)} \approx 2.8376$.

Proof. This is a corollary of Lemma 8 and Lemma 9. Stage one of the algorithm has a larger gap ratio than stage two in this case. ◀

We can now prove the first nontrivial lower bound when we have only 2 or 3 points on the sphere in this online version of the problem.

▶ **Theorem 11.** *The gap ratio for the problem of placing points on the sphere cannot be less than* $\frac{1+\sqrt{5}}{2} \approx 1.6180$.

Proof. Let us first estimate the ratio with only two points when one point will be located on the north pole of a sphere and another one will be shifted by a distance x from the south pole. The gap ratio in this case will be defined as $\frac{\pi+x}{\pi-x}$, which is increasing from 1 to ∞ when $x \geq 0$.

Let us now consider the case with three points. If we place the third point on the plane P defined by the center of the sphere and the other two points, then the gap ratio will be $\frac{\pi}{\frac{\pi+x}{2}} = \frac{2\pi}{\pi+x}$ as in this case the maximal diameter of an empty circle is π regardless of the position of the third point on P. Note that the diameter of a largest circle will be on the orthogonal plane to P and the smallest function for the gap ratio in terms of x can be defined by positioning the third point at the largest distance from initial two points which is $\frac{\pi+x}{2}$.

If the third point is not on the plane P then the ratio would be equal to some value $\frac{a}{b}$ that is larger than $\frac{2\pi}{\pi+x}$. This follows from the fact that the value a which is the maximal gap would be greater than π and the minimal gap b would be less than $\frac{\pi+x}{2}$. So the minimal gap ratio that can be achieved for the three points will be represented by the expression $\frac{2\pi}{\pi+x}$.

By solving the equation where the left hand side represents the gap ratio in the case of 3 points (decreasing function) and the right hand side representing the case with 2 points (increasing function), we find a positive value of the one unknown x: $\frac{2\pi}{\pi+x} = \frac{\pi+x}{\pi-x}$.

The only positive value x satisfying the above equation has the value $\pi(\sqrt{5}-2)$ and the gap ratio for this value x is equal to $\frac{1+\sqrt{5}}{2}$. ◀

6 Conclusion

In order to illustrate the rate of convergence of the gap ratio for various depths of tessellations starting from a single equilateral triangle of the regular icosahedron T, we wrote a program to perform recursive triangular dissection and to measure the minimum and maximum ratios. The results are shown in Table 1.

The table shows that starting from T, the gap ratio of the complete depth-k tessellation of T quickly approaches 1.38. The next point inserted after reaching a complete depth-k tessellation (with $12 * 4^k$ minimal triangles), requires us to split one of the edges in half according to Algorithm 1. This decreases the minimum gap ratio by a factor of $\frac{1}{2}$ which increases the gap ratio by a factor of 2. Therefore $2 \cdot \rho^T(\sigma^k(T))$ shows the maximal gap ratio at any point, not only restricted to complete depth-k tessellations.

Depth of Tessellation	$\rho^T_{\text{min}}(\sigma^k(T))$	$\rho^T_{\text{max}}(\sigma^k(T))$	$\rho^T(\sigma^k(T))$	$2 \cdot \rho^T(\sigma^k(T))$
0	1.1071	1.3047	1.1784	2.3568
1	0.5536	0.7297	1.3182	2.6364
2	0.2768	0.3774	1.3636	2.7272
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
7	0.0086	0.0119	1.3800	2.7600

Table 1 The gap ratio of the depth-k tessellation of the regular icosahedron when isolated to an equilateral spherical triangle T.

To evaluate the most appropriate initial shape for our algorithm, we derived (both theoretically and with a computational simulation) the gap ratios of the stage 1 and stage 2 tessellations of various Platonic solids, shown in Table 2. The results for the dodecahedron are from [6] using a different tessellation (the dodecahedron has non triangular faces).

	Tetrahedron	Octahedron	Dodecahedron	Icosahedron
Stage 1	2.289	2.0	2.618	*2.8376*
Stage 2	*5.921*	*3.601*	*5.995*	2.760

Table 2 The gap ratio of stage one and two of various regular Platonic solids. Italic elements show which value defines the overall gap ratio in each case.

The results match our intuition, that a finer grained initial tessellation such as that from an icosahedron performs much better in stage 2 than a more coarse grained initial tessellation such as that from a tetrahedron. This is illustrated by Fig. 3 which shows that for a large initial equilateral spherical triangle, the recursive triangular dissection procedure deforms the four triangle by a larger margin. The regular icosahedron thus has the essential criteria that we require; it has a low stage 1 and stage 2 gap ratio, and it is a regular tessellation into equilateral spherical triangles. It would be interesting to consider modifications of the stage 2 procedure which may allow the octahedron to be utilised, given its low stage 1 gap ratio. This may require some a modification of Lemma 7 which works also with non equilateral spherical triangles.

References

1 Tetsuo Asano. Online uniformity of integer points on a line. Inf. Process. Lett. 109(1): 57-60 (2008).

2 T. Asano, S. Teramoto, On-line uniformity of points. In: Book of Abstracts for 8th Hellenic-European Conference on Computer Mathematics and its Applications, Athens, Greece, September, 2007, pp. 21-22.

3 Paul Bell, Igor Potapov: Reachability problems in quaternion matrix and rotation semigroups. Inf. Comput. 206(11): 1353-1361 (2008).

4 Arijit Bishnu, Sameer Desai, Arijit Ghosh, Mayank Goswami, and Subhabrata Paul Uniformity of Point Samples in Metric Spaces Using Gap Ratio SIAM Journal on Discrete Mathematics, (2017), Vol. 31, No. 3 : pp. 2138-2171.

5 B. Chazelle: "The Discrepancy Method: Randomness and Complexity", Cambridge University Press, 2000.

6 C. Chen, F. C. M. Lau, S.-H. Poon, Y. Zhang, and R. Zhou. Online Inserting Points Uniformly on the Sphere. WALCOM: Algorithms and Computation. LNCS 10167 (2017), 243–253.

7 D. P. Hardin, E. B. Saff Discretizing manifolds via minimum energy points. Notices of the American Mathematical Society (2004);51(10):1186–1194.

8 D. Kirk, Graphics Gems III, pp. 124-132, 1992.

9 R. J. Lang. Twists, Tilings, and Tessellations: Mathematical Methods for Geometric Origami, CRC Press, Boca Raton, FL, 2017.

10 A. Lubotsky, R. Phillips, P. Sarnak Hecke operators and distributing points on the sphere I. Comm. Pure Appl. Math (1986);XXXIX:S149–S138.

11 J. Matoušek: "Geometric Discrepancy", Springer, 1991.

12 J. Matoušek: "Geometric Discrepancy: An Illustrated Guide", Algorithms and Combinatorics, Vol 18., Springer, 1999.

13 E. A. Rakhmanov, Minimal discrete energy on the sphere, Mathematical Research Letters (1994), 1, 647–662.

14 E. B. Saff, A. B. J.Kuijlaars Distributing many points on a sphere Mathematical Intelligencer (1997);19(1):5–14.

15 N. J. A. Sloane, T. S. D. R. H Hardin, J. H. Conway Minimal-energy clusters of hard spheres Disc Comp Geom (1995);14:237–259.

16 X. Sun, Z. Chen Spherical basis functions and uniform distribution of points on spheres, J. Approx. Theory (2008);151(2):186–207.

17 S. Teramoto, T. Asano, N. Katoh, and B. Doerr. Inserting Points Uniformly at Every Instance. IEICE (Institute Electronics, Information and Communication Engineers) Transactions on Information and Systems (2006), 89-D(8):2348–2356.

18 J. J. Thompson. On the structure of the Atom: an Investigation of the Stability and Periods of Oscillation of a number of Corpuscles arranged at equal intervals around the Circumference of a Circle; with Application of the Results to the Theory of Atomic Structure, Philosophical Magazine Series 6 (1904), 7(39) 237-265.

19 I. Todhunter. *Spherical Trigonometry*. Macmillan and Co., 1886.

20 G. Wagner. On a new method for constructing good point sets on spheres, Journal of Discrete and Computational Geometry (1993);9(1):119–129.

21 A. Yershova, S. Jain, S. M. Lavalle, J. C. Mitchell. Generating uniform incremental grids on $SO(3)$ using the Hopf fibration, Int. J Rob Res (2010), 29(7), 801–812.

22 Y. Zhang, Z. Chang, F.Y.L. Chin, H-F Ting, and Y.H. Tsin. Uniformly Inserting Points on Square Grid, Inf. Process. Lett., 111(2011), 773-779.