THE UNIVERSITY *of* LIVERPOOL

Ultra-Parallel Complex Hyperbolic Triangle Groups

Sam Povall

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For my nephew Tom,
the brightest star in the sky;
and my sister Sarah,
the bravest person I know.

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Contents

Chapter 1

Introduction

In this thesis we study the discreteness criteria for complex hyperbolic ultra-parallel triangle groups of type $[m, m, 0; n_1, n_2, 2]$ (see the explanation of this notation later), generated by three complex reflections ι_1, ι_2 and ι_3 of orders n_1, n_2 and 2 respectively in complex hyperbolic 2-space $\mathbb{H}^2_{\mathbb{C}}$.

A complex hyperbolic triangle group is a group generated by complex reflections of arbitrary orders in the sides of a complex hyperbolic triangle. We define a triangle in $\mathbb{H}^2_{\mathbb{C}}$ as three distinct complex slices. The type of triangle formed depends on how the pairs of slices intersect.

One example is if each pair of complex slices intersect in $\mathbb{H}^2_{\mathbb{C}} \cup \partial\mathbb{H}^2_{\mathbb{C}}$. The angle at this vertex is defined as the infimum of the angles between the real geodesics contained in the complex geodesics and passing through the intersection point. In this case, a complex hyperbolic triangle group is a group generated by three complex reflections, denoted ι_1, ι_2 and ι_3, in the three complex geodesics connecting the pairs of vertices of the triangle.

Another example is if each pair of complex slices are disjoint in $\mathbb{H}^2_{\mathbb{C}}$, then we measure the distance between the real geodesics contained in the complex geodesics. In this case, a complex hyperbolic *ultra-parallel* triangle group is a group generated by three complex reflections, denoted ι_1, ι_2 and ι_3, in the three complex geodesics of distances m_1, m_2 and m_3 apart.

In principle, a mixed case is also possible, where one or two pairs of complex slices intersect inside $\mathbb{H}^2_{\mathbb{C}}$, while the remaining pairs of complex slices are disjoint in $\mathbb{H}^2_{\mathbb{C}}$.

A complex hyperbolic triangle is uniquely determined up to isometry by the angles or distances between the complex geodesics and the angular invariant $\alpha \in [0, 2\pi]$. For more details see section 2.1.4. Note that unlike real reflections, complex reflections can be of arbitrary order. Work on higher order reflections has been discussed by Parker and Paupert [19] and Pratoussevitch [21].

One of the main areas of study in complex hyperbolic geometry is lattices in $\mathrm{PU}(n,1)$ - the holomorphic isometry group of the n-dimensional complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. Complex hyperbolic space is an example of a non-compact symmetric space of rank 1. Discrete subgroups of finite covolume, i.e. lattices, in $\mathbb{H}^2_{\mathbb{C}}$ are isolated points in a large space. A fundamental problem in the study of symmetric spaces is the existence of non-arithmetic lattices. Complex hyperbolic space is the only class of symmetric spaces of non-compact type where this question is still open in higher dimensions.

In 1980, Mostow [13] constructed examples of lattices in $\mathbb{H}^2_{\mathbb{C}}$ from triangle groups and gave the first examples of non-arithmetic lattices in $\mathrm{PU}(2,1)$. Following this, Deligne and Mostow [3] constructed further examples of non-arithmetic lattices in $\mathrm{PU}(2,1)$ and $\mathrm{PU}(3,1)$. Since then, Deraux, Parker and Paupert [7], [6], [5] have produced examples of new non-arithmetic lattices in $\mathrm{PU}(2,1)$ which are not commensurable to each other.

More recently, Deraux [4] has given a new non-arithmetic lattice in $\mathrm{PU}(3,1)$ (new in a sense that they are not commensurable to any Deligne-Mostow examples). The question still remains open of whether non-arithmetic lattices in $\mathrm{PU}(n,1)$ for $n \geq 4$ exist. The question of classifying all such lattices is still open even for complex hyperbolic 2-space $\mathbb{H}^2_{\mathbb{C}}$.

This is the motivation for studying complex hyperbolic triangle groups in $\mathbb{H}^2_{\mathbb{C}}$.

In 1992, Goldman and Parker [18] were the first to study the ideal triangle group (triangles with vertices on the boundary). They proved that an ideal triangle group representation is a discrete embedding for an interval in the parameter space. They also conjectured that the ideal triangle group representations were discrete and faithful if and only if the product of the three generators, $\iota_1\iota_2\iota_3$, was not elliptic. A stronger version of this conjecture was then proved by Schwartz [22]. He proved that ideal triangle group representations are discrete and faithful if and only if $\iota_1\iota_2\iota_3$ is not elliptic and are not discrete otherwise.

Much work followed that of Goldman and Parker, but the work we are mainly interested in is the ultra-parallel case. In his thesis [24] in 2000, Justin Wyss-Gallifent studied ultra-parallel triangle groups. He used a 'ping-pong' method to show discreteness, similar to that used by Schwartz [22]. Wyss-Gallifent used this approach in studying ultra-parallel triangle groups with distances between the complex slices $l, l, 2l$ and $l, l, 0$ and obtained partial results similar to that of Goldman and Parker [18] for the ideal case.

More recently Monaghan [11] and Monaghan, Parker and Pratoussevitch [12] considered the case of ultra-parallel triangle groups with distances $m_1, m_2, 0$ and obtained more general discreteness results. In these studies, the complex reflections were all of order 2. As stated earlier, complex reflections can be of arbitrary order. In this thesis, we will set out the discreteness conditions for the case of complex hyperbolic ultra-parallel triangle groups with distances between the complex slices $m, m, 0$ with complex reflections of orders n_1, n_2 and 2 respectively. We denote these groups $[m, m, 0; n_1, n_2, 2]$-complex hyperbolic triangle groups.

This thesis is organised as follows. In chapter 2 we discuss the background on complex hyperbolic geometry and the main tools needed for the proofs. We begin by introducing the complex hyperbolic 2-space and a model of the complex hyperbolic plane. We describe the isometries and how to classify them and then move on to define complex slices and complex hyperbolic triangle groups. We finish by introducing chains and Heisenberg space, which is where most of the calculations and proofs will be performed.

Chapter 3 gives a parametrisation of $[m_1, m_2, 0; n_1, n_2, n_3]$-triangle groups. We reintroduce criteria for discreteness called the *compressing* method used in [22] and [24], which uses a 'ping-pong' method to show discreteness of a group. We then discuss the possible orders for the complex reflections ι_1 and ι_2 and use the work of Hersonsky and Paulin [9] and Parker [17] to prove what orders can occur in a discrete group:

Theorem. (3.3.2) *A complex hyperbolic ultra-parallel $[m_1, m_2, 0; n_1, n_2, 2]$-triangle group can only be discrete if the unordered pair of orders of the complex reflections ι_1 and ι_2 is one of*

$$\{2,2\},\ \{2,3\},\ \{2,4\},\ \{2,6\},\ \{3,3\},\ \{3,6\}\ \textit{or}\ \{4,4\}.$$

In chapter 4 we find discreteness conditions on the angular invariant α and on the distance m for the complex hyperbolic ultra-parallel $[m, m, 0; n_1, n_2, 0]$-triangle group for all possible orders of the complex reflections ι_1 and ι_2 (except for the $\{2, 2\}$ case as this has been considered previously). We use the 'ping-pong' method on the boundary of the complex hyperbolic 2-space $\partial\mathbb{H}^2_{\mathbb{C}}$ and identify a domain of discreteness in the parameter space for each case:

Proposition. (4.1.2) & (4.2.2). *A complex hyperbolic ultra-parallel $[m, m, 0; n, 3, 2]$-triangle group with $n \in \{2, 3\}$ is discrete if the following conditions on the angular invariant α and on m are satisfied:*

$$\cos(\alpha) \leq -\frac{1}{2} \quad \textit{and} \quad m \geq \log(3).$$

Proposition. (4.3.2) & (4.4.2). *A complex hyperbolic ultra-parallel* $[m, m, 0; n, 4, 2]$-*triangle group with* $n \in \{2, 4\}$ *is discrete if the following conditions on the angular invariant* α *and on* m *are satisfied:*

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \textit{and} \quad m \geq \log\left(3 + 2\sqrt{2}\right).$$

Proposition. (4.5.2) & (4.6.4). *A complex hyperbolic ultra-parallel* $[m, m, 0; n, 6, 2]$-*triangle group with* $n \in \{2, 3\}$ *is discrete if the following conditions on the angular invariant* α *and on* m *are satisfied:*

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \textit{and} \quad m \geq \log\left(7 + 4\sqrt{3}\right).$$

Chapter 5 contrasts the discreteness results with non-discreteness results. Here we use the complex hyperbolic version of Shimizu's Lemma introduced in [16] to find the conditions on the angular invariant α and the distance m for which the complex hyperbolic ultra-parallel $[m, m, 0; n_1, n_2, 2]$-triangle groups are not discrete:

Proposition. (5.2.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 3, 2]$-*triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{12\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Proposition. (5.3.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 3, 2]$-*triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{48\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Proposition. (5.4.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 4, 2]$-*triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{32 \cdot \cosh^2\left(\frac{m}{2}\right)}.$$

Proposition. (5.5.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 4, 4, 2]$-*triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{8 \cdot \cosh^2\left(\frac{m}{2}\right)}.$$

Proposition. (5.6.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 6, 2]$-*triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{8\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Proposition. (5.7.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 6, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

We then further these non-discreteness results by considering the results of Parker [15]. Here we obtain further conditions on the angular invariant α and the distance m for which the complex hyperbolic ultra-parallel $[m, m, 0; n_1, n_2, 2]$-triangle groups are not discrete:

Proposition. (5.8.2). *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 3, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{6\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{6\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proposition. (5.8.3). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 3, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{24\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{24\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proposition. (5.8.4). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 4, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{16\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{16\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proposition. (5.8.5). *A complex hyperbolic ultra-parallel* $[m, m, 0; 4, 4, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{4\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{4\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proposition. (5.8.6). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 6, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proposition. (5.8.7). *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 6, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{2\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{2\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Finally chapter 6 gives a brief summary of the results obtained throughout the thesis and suggestions on how to improve these results. Comparing the values of α and m for discreteness against non-discreteness for each case, we notice that there is a gap. This is illustrated for the case of ultra-parallel $[m, m, 0; 3, 3, 2]$-triangle groups in the figure below.

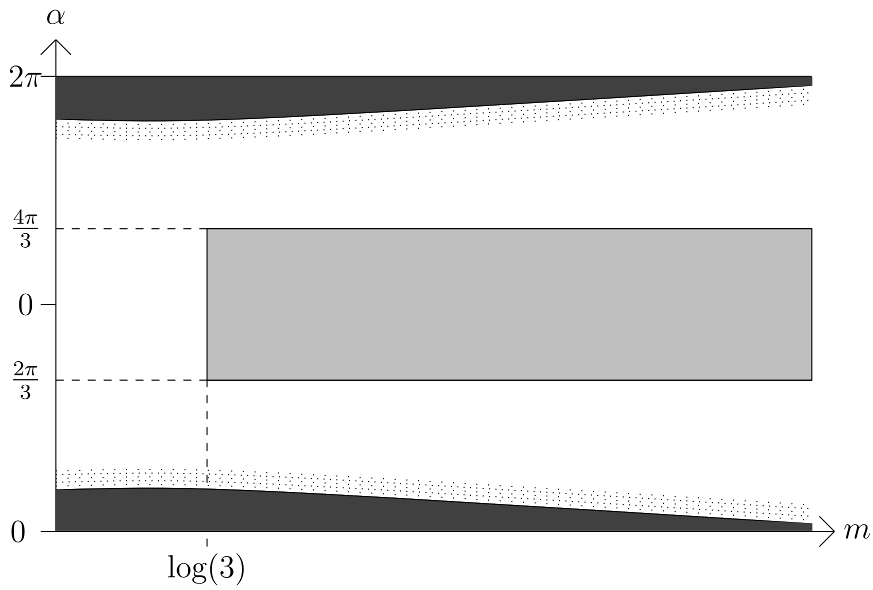

Figure 1.1: Gap between discreteness (light grey) and non-discreteness (dark grey) results.

We suggest several ways of attempting to close this gap which would improve the known results and also give ideas on how to further this work.

Chapter 2

Preliminaries

2.1 Complex Hyperbolic 2-Space

In order to introduce complex hyperbolic 2-space, $\mathbb{H}^2_{\mathbb{C}}$, let $\mathbb{C}^{2,1}$ be the complex vector space of dimension 3 equipped with a Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(2,1)$. We can describe the Hermitian form with a 3×3 Hermitian matrix J, with 2 positive eigenvalues and 1 negative eigenvalue. The choice of a particular Hermitian form is equivalent to a choice of a basis in $\mathbb{C}^{2,1}$. The Hermitian form we will consider is defined as

$$\langle \mathbf{z}, \mathbf{w} \rangle = w^* J z = z_1 \bar{w}_1 + z_2 \bar{w}_2 - z_3 \bar{w}_3,$$

where $z = (z_1, z_2, z_3)^t$, $w = (w_1, w_2, w_3)^t$, and the Hermitian matrix is given by

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

2.1.1 The Projective Model of the Complex Hyperbolic Plane

If $z \in \mathbb{C}^{2,1}$ then $\langle z, z \rangle$ is real. Thus we define subsets V_-, V_0 and V_+ of $\mathbb{C}^{2,1}$ as follows

1. $V_- = \{z \in \mathbb{C}^{2,1} \mid \langle z, z \rangle < 0\}$;
2. $V_0 = \{z \in \mathbb{C}^{2,1} - \{0\} \mid \langle z, z \rangle = 0\}$;
3. $V_+ = \{z \in \mathbb{C}^{2,1} \mid \langle z, z \rangle > 0\}$.

We say that $z \in \mathbb{C}^{2,1}$ is *negative*, *null* or *positive* if z is in V_-, V_0 or V_+ respectively. For any non-zero complex scalar λ, the point λz is negative, null or positive if and only if z is. This is because $\langle \lambda z, \lambda z \rangle = |\lambda|^2 \langle z, z \rangle$. We define a projection map $\mathbb{P}$ on the points of $\mathbb{C}^{2,1}$ with $z_3 \neq 0$. This projection map is defined by

$$\mathbb{P} : z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{pmatrix} z_1/z_3 \\ z_2/z_3 \end{pmatrix} \in \mathbb{P}\left(\mathbb{C}^{2,1}\right).$$

That is, provided $z_3 \neq 0$,

$$z = (z_1, z_2, z_3) \mapsto [z] = [z_1 : z_2 : z_3] = \left[\frac{z_1}{z_3} : \frac{z_2}{z_3} : 1\right].$$

Definition 2.1.1. The *projective model* of the complex hyperbolic plane is defined to be the collection of negative lines in $\mathbb{C}^{2,1}$ and its boundary is defined to be the collection of null lines. That is

$$\mathbb{H}^2_{\mathbb{C}} = \mathbb{P}\left(V_-\right) \text{ and } \partial\mathbb{H}^2_{\mathbb{C}} = \mathbb{P}\left(V_0\right).$$

Definition 2.1.2. The metric on $\mathbb{H}^2_{\mathbb{C}}$, called the *Bergman metric*, is given by the distance function ρ defined by the formula

$$\cosh^2\left(\frac{\rho([z],[w])}{2}\right) = \frac{\langle z, w\rangle\langle w, z\rangle}{\langle z, z\rangle\langle w, w\rangle},$$

where $[z]$ and $[w]$ are the images of z and w in $\mathbb{C}^{2,1}$ under the projectivisation map $\mathbb{P}$.

2.1.2 Isometries

Let A be a 3×3 complex matrix that preserves the Hermitian form above, i.e. $A \in \mathrm{U}(2,1)$ is a unitary matrix. That is, for z and w in $\mathbb{C}^{2,1}$, we have

$$\langle Az, Aw\rangle = (Aw)^* JAz = w^* A^* JAz = w^*(A^* JA)z.$$

We want this to be equal to $\langle z, w\rangle = w^* Jz$. Therefore

$$\langle Az, Aw\rangle = \langle z, w\rangle \Leftrightarrow w^*(A^* JA)z = w^* Jz.$$

If we let z and w run through a basis of $\mathbb{C}^{2,1}$ this implies that the matrices coincide, so for all z and w we have

$$\langle Az, Aw\rangle = \langle z, w\rangle \Leftrightarrow A^* JA = J \Leftrightarrow A^{-1} = J^{-1} A^* J.$$

Any matrix in $\mathrm{U}(2,1)$ which is a non-zero complex scalar multiple of the identity matrix, Id, maps each line in $\mathbb{C}^{2,1}$ to itself and so acts trivially on the complex hyperbolic space.

Definition 2.1.3. The *projective unitary group* $\mathrm{PU}(2,1)$ is the quotient of $\mathrm{U}(2,1)$ by the right multiplication of $\mathrm{U}(1)$, where $\mathrm{U}(1)$ is identified with the set $\{e^{i\theta}\,\mathrm{Id} \,:\, 0 \leq \theta \leq 2\pi\}$. That is

$$\mathrm{PU}(2,1) = \mathrm{U}(2,1)/\,\mathrm{U}(1).$$

Sometimes we will consider $\mathrm{SU}(2,1)$, the group of matrices with determinant 1 which are unitary with respect to the Hermitian form. The group $\mathrm{SU}(2,1)$ is a 3-fold covering of $\mathrm{PU}(2,1)$:

$$\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/\{\mathrm{Id}, w\,\mathrm{Id}, w^2\,\mathrm{Id}\},$$

where $w = \frac{-1+i\sqrt{3}}{2}$ is a cube root of unity.

Since the Bergman metric is given in terms of the Hermitian form, we see that if A is unitary with respect to the Hermitian form then A acts isometrically on the projective model of the complex hyperbolic space. Thus $\mathrm{PU}(2,1)$ is a subgroup of the complex hyperbolic isometry group. There are isometries of $\mathbb{H}^2_{\mathbb{C}}$ not in $\mathrm{PU}(2,1)$. For example, consider the complex conjugation $z \mapsto \bar{z}$:

$$\cosh^2\left(\frac{\rho(\bar{z},\bar{w})}{2}\right) = \frac{\overline{\langle z,w\rangle}\;\overline{\langle w,z\rangle}}{\langle z,z\rangle\langle w,w\rangle} = \frac{\langle w,z\rangle\langle z,w\rangle}{\langle z,z\rangle\langle w,w\rangle} = \cosh^2\left(\frac{\rho(z,w)}{2}\right).$$

Therefore, complex conjugation is also an isometry of the complex hyperbolic space. The following theorem will describe the full isometry group of $\mathbb{H}^2_{\mathbb{C}}$:

Theorem 2.1.4. *Every isometry of $\mathbb{H}^2_{\mathbb{C}}$ is either holomorphic or anti-holomorphic. Moreover, each holomorphic isometry of $\mathbb{H}^2_{\mathbb{C}}$ is given by a matrix in $\mathrm{PU}(2,1)$ and each anti-holomorphic isometry is given by complex conjugation followed by a matrix in $\mathrm{PU}(2,1)$.*

Proof. For proof, see [14], Theorem 3.5 and [8], section 6.2. □

In a similar way to real hyperbolic geometry, we are able to classify the holomorphic isometries by their fixed point behaviour. A holomorphic complex hyperbolic isometry A is said to be

1. *Loxodromic* if it has exactly two fixed points in $\partial\mathbb{H}^2_{\mathbb{C}}$;
2. *Parabolic* if it has exactly one fixed point in $\partial\mathbb{H}^2_{\mathbb{C}}$;
3. *Elliptic* if it has at least one fixed point in $\mathbb{H}^2_{\mathbb{C}}$;

We are able to further refine this classification by considering eigenvalues and eigenvectors of matrices in $\mathrm{SU}(2,1)$. An elliptic element is called *regular elliptic* if all of its eigenvalues are distinct. An element whose eigenvalues are all 1 is called *unipotent*. With the exception of the identity, all unipotent elements are parabolic.

To determine the type of isometry by the trace of the corresponding matrix, we can use the *discriminant function* introduced in [8] (Theorem 6.2.4):

$$f(z) = |z|^4 - 8\,\mathrm{Re}(z^3) + 18|z|^2 - 27.$$

Suppose $A \in \mathrm{SU}(2,1)$, then

1. A is regular elliptic if and only if $f(\mathrm{tr}(A)) < 0$;
2. A is loxodromic if and only if $f(\mathrm{tr}(A)) > 0$;
3. A is either a complex reflection in a complex geodesic, or a complex reflection about a point, or is parabolic if and only if $f(\mathrm{tr}(A)) = 0$.

Remark 2.1.5. The type of isometry can be determined from the position of the trace of the corresponding matrix in the complex plane. The deltoid curve, $f(z) = 0$, has the property that an isometry $A \in \mathrm{SU}(2,1)$ is regular elliptic if and only if $\mathrm{tr}(A)$ is inside the deltoid, and is loxodromic if and only if $\mathrm{tr}(A)$ is outside the deltoid.

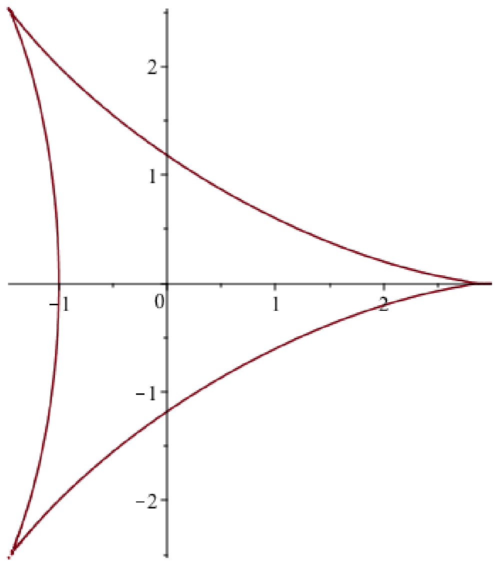

Figure 2.1: The deltoid given by $f(z) = 0$.

2.1.3 Complex Geodesics

Definition 2.1.6. A *complex geodesic* is a projectivisation of a 2-dimensional complex subspace of $\mathbb{C}^{2,1}$. In fact, complex geodesics are totally geodesic subspaces of real dimension 2. Any complex geodesic is isometric to

$$\{[z : 0 : 1] \mid z \in \mathbb{C}\}$$

in the projective model. Given any two points in $\mathbb{H}^2_{\mathbb{C}}$ there is a unique complex geodesic containing them. Any positive vector $c \in \mathbb{C}^{2,1}$ determines a 2-dimensional complex subspace

$$\{z \in \mathbb{C}^{2,1} \mid \langle c, z \rangle = 0\}.$$

Projecting this subspace we obtain a complex geodesic

$$\mathbb{P}\left(\{z \in \mathbb{C}^{2,1} \mid \langle c, z \rangle = 0\}\right).$$

Conversely, any complex geodesic is represented by a positive vector $c \in \mathbb{C}^{2,1}$ called a *polar vector* of the complex geodesic. To describe how two complex geodesics intersect, we first need the following definition.

Definition 2.1.7. The *Hermitian cross-product* is defined to be

$$\boxtimes : \mathbb{C}^{2,1} \times \mathbb{C}^{2,1} \to \mathbb{C}^{2,1},$$

explicitly given as

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \boxtimes \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{vmatrix} \bar{z}_1 & \bar{w}_1 & \overrightarrow{e_1} \\ \bar{z}_2 & \bar{w}_2 & \overrightarrow{e_2} \\ -\bar{z}_3 & -\bar{w}_3 & \overrightarrow{e_3} \end{vmatrix} = \begin{bmatrix} \overline{z_3 w_2} - \overline{z_2 w_3} \\ \overline{z_1 w_3} - \overline{z_3 w_1} \\ \overline{z_1 w_2} - \overline{z_2 w_1} \end{bmatrix}.$$

If we have two complex geodesics, C_1 and C_2, with respective polar vectors, c_1 and c_2, we can normalise the polar vectors so that

$$\langle c_1, c_1 \rangle = \langle c_2, c_2 \rangle = 1.$$

Assuming that the complex geodesics do not coincide, then we have three possible types of intersection:

1. The complex geodesics C_1 and C_2 intersect at a single point in $\mathbb{H}^2_{\mathbb{C}}$ if and only if
$$\left|\langle c_1, c_2 \rangle\right| < 1,$$
in which case $c_1 \boxtimes c_2$ is a negative vector that corresponds to the intersection point and
$$\left|\langle c_1, c_2 \rangle\right| = \cos(\theta),$$
where $\theta \in [0, \pi/2]$ is the angle of intersection between C_1 and C_2.
2. The complex geodesics C_1 and C_2 intersect at a single point in $\partial\mathbb{H}^2_{\mathbb{C}}$ if and only if
$$\left|\langle c_1, c_2 \rangle\right| = 1,$$
in which case $c_1 \boxtimes c_2$ is a null vector and corresponds to the intersection point.

3. The complex geodesics C_1 and C_2 are disjoint in $\mathbb{H}^2_{\mathbb{C}} \cup \partial\mathbb{H}^2_{\mathbb{C}}$ if and only if

$$\big|\langle c_1, c_2\rangle\big| > 1,$$

in which case $c_1 \boxtimes c_2$ is a positive vector that corresponds to the intersection point of C_1 and C_2 outside $\mathbb{H}^2_{\mathbb{C}} \cup \partial\mathbb{H}^2_{\mathbb{C}}$ and

$$\big|\langle c_1, c_2\rangle\big| = \cosh(m/2),$$

where m is the distance between C_1 and C_2, i.e. $\inf\{d(w,z) \mid w \in C_1, z \in C_2\}$.

Definition 2.1.8. For a given complex geodesic C, a *minimal complex hyperbolic reflection of order* n in C is the isometry $\iota_C^{(n)}$ $(= \iota_C)$ in $\mathrm{PU}(2,1)$ of order n with fixed point set C given by

$$\iota_C^{(n)}(z) = -z + (1-\mu)\frac{\langle z, c\rangle}{\langle c, c\rangle}c,$$

where c is a polar vector of C and $\mu = \exp(2\pi i/n)$. To show that $\iota_C^{(n)}$ is of order n see [21].

2.1.4 Complex Hyperbolic Triangle Groups

Definition 2.1.9. A *complex hyperbolic triangle* is a triple (C_1, C_2, C_3) of complex geodesics in $\mathbb{H}^2_{\mathbb{C}}$.

The different types of triangle groups depend on how the complex geodesics intersect each other. The type we will consider is when the complex geodesics do not intersect in $\mathbb{H}^2_{\mathbb{C}}$. That is, the case when each pair of complex geodesics is *ultra-parallel.*

Definition 2.1.10. A triangle (C_1, C_2, C_3) is a *complex hyperbolic ultra-parallel* $[m_1, m_2, m_3]$*-triangle* if the complex geodesics C_{k-1} and C_{k+1} are ultra-parallel at distance $m_k \geq 0$ for $k = 1, 2, 3$ (subscripts are modulo 3).

Remark 2.1.11. The case $m_k = 0$ refers to the case when C_{k-1} and C_{k+1} are asymptotic and so intersect on the boundary with distance equal to 0.

Definition 2.1.12. A *complex hyperbolic ultra-parallel* $[m_1, m_2, m_3; n_1, n_2, n_3]$*-triangle group* is a subgroup of $\mathrm{PU}(2,1)$ generated by $\iota_1, \iota_2, \iota_3$ where ι_k is the minimal complex reflection of order n_k in C_k and C_1, C_2, C_3 is a complex hyperbolic ultra-parallel $[m_1, m_2, m_3]$-triangle (C_1, C_2, C_3).

One of the main questions we can ask is, when is a given complex hyperbolic ultra-parallel $[m_1, m_2, m_3; n_1, n_2, n_3]$-triangle group discrete?

For each fixed triple $[m_1, m_2, m_3]$ in $\mathbb{H}^2_{\mathbb{C}}$, the space of $[m_1, m_2, m_3]$-triangles up to isometry is of real dimension one. We can describe a parametrisation of the space of complex hyperbolic triangles in $\mathbb{H}^2_{\mathbb{C}}$ by means of an angular invariant α (for more details see [21]).

Definition 2.1.13. The *angular invariant* α of the triangle (C_1, C_2, C_3) is defined as

$$\alpha = \arg\left(\prod_{k=1}^{3}\langle c_{k-1}, c_{k+1}\rangle\right),$$

where c_k is the normalised polar vector of the complex geodesic C_k.

We use the following proposition, given in [21], which gives criteria for the existence of a triangle group in terms of the angular invariant.

Proposition 2.1.14. *An $[m_1, m_2, m_3]$-triangle in $\mathbb{H}^2_{\mathbb{C}}$ is determined uniquely up to isometry by the three distances between the pairs of complex geodesics and the angular invariant α. For any $\alpha \in [0, 2\pi]$, an $[m_1, m_2, m_3]$-triangle with angular invariant α exists if and only if*

$$\cos(\alpha) < \frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1r_2r_3},$$

where $r_k = \cosh(m_k/2)$.

In the case $m_3 = 0$ we have $r_3 = 1$ and

$$\frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1r_2r_3} = \frac{r_1^2 + r_2^2}{2r_1r_2} \geq 1.$$

So Proposition 2.1.14 states that for any $\alpha \in (0, 2\pi)$ there exists a unique (up to isometry) $[m_1, m_2, 0]$-triangle in $\mathbb{H}^2_{\mathbb{C}}$.

2.1.5 Heisenberg Group

In the same way that the boundary of the real hyperbolic space is identified with the one point compactification of the Euclidean space of one dimension lower, we can identify the boundary of the complex hyperbolic space $\partial\mathbb{H}^2_{\mathbb{C}}$ with a one point compactification of the Heisenberg group.

Definition 2.1.15. The *Heisenberg space* is defined as

$$\mathcal{N} = \mathbb{C} \times \mathbb{R} \cup \{\infty\} = \{(\zeta, \nu) \mid \zeta \in \mathbb{C}, \nu \in \mathbb{R}\} \cup \{\infty\}.$$

One homeomorphism taking $\partial\mathbb{H}^2_{\mathbb{C}}$ to $\mathcal{N}$ is given by the projection:

$$[z_1 : z_2 : z_3] \mapsto \left(\frac{z_1}{z_2 + z_3}, \operatorname{Im}\left(\frac{z_2 - z_3}{z_2 + z_3}\right)\right) \text{ if } z_2 + z_3 \neq 0, \;\; [0 : z : -z] \mapsto \infty.$$

We can visualise this projection as a stereographic projection from the 3-dimensional sphere $S^3 \cong \partial\mathbb{H}^2_{\mathbb{C}}$ in $\mathbb{R}^4$ to $\mathbb{R}^3 \cong \mathcal{N}$.

Definition 2.1.16. The *Heisenberg group* is the set of all pairs $(\zeta, \nu) \in \mathbb{C} \times \mathbb{R}$ with the group law

$$(\zeta_1, \nu_1) * (\zeta_2, \nu_2) = \Big(\zeta_1 + \zeta_2, \nu_1 + \nu_2 + 2\,\mathrm{Im}\left(\zeta_1 \bar{\zeta_2}\right)\Big).$$

Note that $(0, 0)$ is the identity element and $(-\zeta, -\nu)$ is the inverse element of (ζ, ν).

The centre of $\mathfrak{N}$ consists of elements of the form $(0, \nu)$ for $\nu \in \mathbb{R}$. The Heisenberg group is not abelian but is 2-step nilpotent. To see this, observe that

$$(\zeta_1, \nu_1) * (\zeta_2, \nu_2) * (-\zeta_1, -\nu_1) * (-\zeta_2, -\nu_2) = \Big(0, 4\,\mathrm{Im}\left(\zeta_1 \bar{\zeta_2}\right)\Big).$$

Therefore, the commutator of any two elements lies in the centre.

Recall an alternaive description of the Heisenberg group $\mathfrak{N}$:

Definition 2.1.17. The *three dimensional Heisenberg group* is given as

$$\mathfrak{N} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\},$$

the group of 3×3 upper triangular matrices. The correspondence with $\mathbb{C} \times \mathbb{R}$ is given as $(x, y, z) \mapsto \left(\frac{iz-x}{2}, \frac{2y-xz}{2}\right)$.

Definition 2.1.18. For any integer $k \neq 0$, we obtain a uniform lattice N_k in $\mathfrak{N}$ which is the subgroup generated by

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 1 & 0 & \frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The group N_k has the presentation

$$N_k \;=\; \langle a, b, c \mid [b, a] = c^k, [c, a] = [c, b] = 1 \rangle.$$

Remark 2.1.19. Any uniform lattice in $\mathfrak{N}$ is isomorphic to N_k for some $k \neq 0$. For more details see section 6.1 in [1].

Definition 2.1.20. The group generated by translations and $\mathrm{U}(n-1)$ act isometrically with respect to several left-invariant metrics. This is the group of Heisenberg isometries. The metric we will consider is the *Cygan metric* given as

$$\rho_0\left((\zeta_1, \nu_2), (\zeta_2, \nu_2)\right) = \Big| |\zeta_1 - \zeta_2|^2 - i(\nu_1 - \nu_2) - 2i\,\mathrm{Im}(\zeta_1 \bar{\zeta_2}) \Big|^{1/2}.$$

Definition 2.1.21. A *Heisenberg translation* $T_{(\xi,\nu)}$ by $(\xi,\nu) \in \mathfrak{N}$ is given by

$$(\zeta,\omega) \mapsto (\xi,\nu) * (\zeta,\omega) = (\xi+\zeta, \nu+\omega+2\,\mathrm{Im}(\xi\bar{\zeta})).$$

$T_{(\xi,\nu)}$ is an isometry of $\mathfrak{N}$ with respect to the Cygan metric and corresponds to the following matrix in $\mathrm{PU}(2,1)$:

$$\begin{pmatrix} 1 & \xi & \xi \\ -\bar{\xi} & 1-\frac{|\xi|^2-i\nu}{2} & -\frac{|\xi|^2-i\nu}{2} \\ \bar{\xi} & \frac{|\xi|^2-i\nu}{2} & 1+\frac{|\xi|^2-i\nu}{2} \end{pmatrix}.$$

A *vertical Heisenberg translation* is a Heisenberg translation of the form

$$(\zeta,\omega) \mapsto (0,\nu) + (\zeta,\omega) = (\zeta,\omega+\nu), \ \text{ for } \ \nu \in \mathbb{R},$$

which is just a vertical Euclidean translation.

Definition 2.1.22. A *Heisenberg rotation* R_μ by $\mu \in \mathbb{C}$, with $|\mu| = 1$, is given by

$$(\zeta,\omega) \mapsto (\mu \cdot \zeta, \omega).$$

R_μ is an isometry of $\mathfrak{N}$ with respect to the Cygan metric and corresponds to the following matrix in $\mathrm{PU}(2,1)$:

$$\begin{pmatrix} \mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Following Goldman [8], section 4.2.2, we refer to the group generated by Heisenberg translations and Heisenberg rotations as the group $\mathrm{Isom}(\mathfrak{N})$ of Heisenberg isometries of $\mathfrak{N}$. The group of Heisenberg translations can be identified with $\mathfrak{N}$, and the group of Heisenberg rotations can be identified with $\mathrm{U}(1) = \{\mu \in \mathbb{C} \mid |\mu| = 1\}$. The group $\mathrm{Isom}(\mathfrak{N})$ has the structure of a semi-direct product $\mathfrak{N} \rtimes \mathrm{U}(1)$.

2.1.6 Chains

Definition 2.1.23. A complex geodesic in $\mathbb{H}^2_{\mathbb{C}}$ is homeomorphic to a disc, its intersection with the boundary of the complex hyperbolic plane is homeomorphic to a circle. Such circles that arise as the boundaries of complex geodesics are called *chains*.

There is a bijection between chains and complex geodesics. We can therefore, without loss of generality, talk about reflections in chains instead of reflections in complex geodesics.

The description of chains in the Heisenberg space $\mathfrak{N}$ is as follows, for more details see [8]. Chains passing through ∞ are represented as vertical straight lines defined by $\zeta = \zeta_0$. Such chains are called *vertical*. The vertical chain C_{ζ_0} defined by $\zeta = \zeta_0$ has a polar vector

$$c_{\zeta_0} = \begin{bmatrix} 1 \\ -\bar{\zeta}_0 \\ \bar{\zeta}_0 \end{bmatrix}.$$

A chain not containing ∞ is called *finite*. A finite chain is represented by an ellipse whose vertical projection $\mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is a circle in $\mathbb{C}$. There is a unique finite chain for each point $(\zeta_0, \nu_0) \in \mathbb{C} \times \mathbb{R}$ and $r_0 > 0$. The projection to $\mathbb{C}$ is the circle of radius of r_0 centred at ζ_0. The finite chain with centre $(\zeta_0, \nu_0) \in \mathfrak{N}$ and radius $r_0 > 0$ has a polar vector

$$\begin{bmatrix} 2\zeta_0 \\ 1 + r_0^2 - \zeta_0\bar{\zeta}_0 + i\nu_0 \\ 1 - r_0^2 + \zeta_0\bar{\zeta}_0 - i\nu_0 \end{bmatrix}$$

and consists of all $(\zeta, \nu) \in \mathfrak{N}$ satisfying the equations

$$|\zeta - \zeta_0| = r_0, \quad \nu = \nu_0 - 2\,\mathrm{Im}(\zeta\bar{\zeta}_0).$$

For example, the finite chain with centre $(0, 0)$ and radius 1 is the unit circle in the $\mathbb{C} \times \{0\}$ plane. It has a normalised polar vector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

A complex reflection of order 2 in this chain is given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{SU}(2, 1).$$

This complex reflection acts as an inversion on the set

$$U = \{(\zeta, \nu) \in \mathfrak{N} \,:\, |\zeta|^4 + \nu^2 = 1\},$$

known as the *unit spinal sphere*, switching the inside of the spinal sphere with the outside. For more details, see section 2.1.7.

Using the formula for the minimal complex reflection ι_{C_ζ} of order n in a vertical chain C_ζ with polar vector

$$c_\zeta = \begin{bmatrix} 1 \\ -\bar{\zeta} \\ \bar{\zeta} \end{bmatrix},$$

we are able to find the corresponding element in $\mathrm{PU}(2,1)$. We have that

$$\iota_{C_\zeta}(z) = -z + (1-\mu)\frac{\langle z, c_\zeta\rangle}{\langle c_\zeta, c_\zeta\rangle}c_\zeta, \quad \text{where} \quad \mu = \exp(2\pi i/n).$$

Recall that the hermitian form we are using is $\langle z, w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2 - z_3\bar{w}_3$. Therefore

$$\langle z, c_\zeta\rangle = z_1 + z_2\overline{(-\bar{\zeta})} - z_3\overline{(\bar{\zeta})} = z_1 - z_2\zeta - z_3\zeta \ \text{ and } \ \langle c_\zeta, c_\zeta\rangle = 1.$$

So we have that

$$\begin{aligned}
\iota_{C_\zeta}(z) &= -z + (1-\mu)(z_1 - z_2\zeta - z_3\zeta)\begin{pmatrix} 1 \\ -\bar{\zeta} \\ \bar{\zeta} \end{pmatrix} \\
&= \begin{pmatrix} -z_1 \\ -z_2 \\ -z_3 \end{pmatrix} + (1-\mu)\begin{pmatrix} z_1 - z_2\zeta - z_3\zeta \\ -\bar{\zeta}z_1 + |\zeta|^2 z_2 + |\zeta|^2 z_3 \\ \bar{\zeta}z_1 - |\zeta|^2 z_2 - |\zeta|^2 z_3 \end{pmatrix} \\
&= \begin{pmatrix} (-\mu)z_1 - (1-\mu)\zeta z_2 - (1-\mu)\zeta z_3 \\ -(1-\mu)\bar{\zeta}z_1 + ((1-\mu)|\zeta|^2 - 1)z_2 + (1-\mu)|\zeta|^2 z_3 \\ (1-\mu)\bar{\zeta}z_1 - (1-\mu)|\zeta|^2 z_2 - ((1-\mu)|\zeta|^2 + 1)z_3 \end{pmatrix} \\
&= \begin{bmatrix} -\mu & -(1-\mu)\zeta & -(1-\mu)\zeta \\ -(1-\mu)\bar{\zeta} & (1-\mu)|\zeta|^2 - 1 & (1-\mu)|\zeta|^2 \\ (1-\mu)\bar{\zeta} & -(1-\mu)|\zeta|^2 & -(1-\mu)|\zeta|^2 - 1 \end{bmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.
\end{aligned}$$

This is the corresponding element in $\mathrm{PU}(2,1)$ of a minimal complex reflection of order n in the vertical chain C_ζ.

What effect does the minimal complex reflection of order n in the vertical chain C_ζ have on another vertical chain, C_ξ, which intersects $\mathbb{C}\times\{0\}$ at ξ?

We calculate

$$\begin{bmatrix} -\mu & -(1-\mu)\zeta & -(1-\mu)\zeta \\ -(1-\mu)\bar{\zeta} & (1-\mu)|\zeta|^2 - 1 & (1-\mu)|\zeta|^2 \\ (1-\mu)\bar{\zeta} & -(1-\mu)|\zeta|^2 & -(1-\mu)|\zeta|^2 - 1 \end{bmatrix}\begin{bmatrix} 1 \\ -\bar{\xi} \\ \bar{\xi} \end{bmatrix} = \begin{bmatrix} -\mu \\ -(1-\mu)\bar{\zeta} + \bar{\xi} \\ (1-\mu)\bar{\zeta} - \bar{\xi} \end{bmatrix}.$$

This vector is a multiple of

$$\begin{pmatrix} 1 \\ (1-\mu)\bar{\mu}\bar{\zeta} - \bar{\mu}\bar{\xi} \\ -(1-\mu)\bar{\mu}\bar{\zeta} + \bar{\mu}\bar{\xi} \end{pmatrix} = \begin{pmatrix} 1 \\ -\overline{(\mu\xi - (\mu-1)\zeta)} \\ \overline{(\mu\xi - (\mu-1)\zeta)} \end{pmatrix},$$

which is the polar vector of the vertical chain that intersects $\mathbb{C} \times \{0\}$ at $\mu\xi - (\mu - 1)\zeta$. This corresponds to rotating ξ around ζ through $\frac{2\pi}{n}$. So if we have a vertical chain C_ξ, the minimal complex reflection of order n in another vertical chain C_ζ rotates C_ξ as a set around C_ζ through $\frac{2\pi}{n}$. (But not point-wise as there may also be a vertical translation on the chain).

On the boundary of complex hyperbolic 2-space, $\partial\mathbb{H}^2_{\mathbb{C}}$, any complex reflection ι_{C_ζ} can be described as a composition of a Heisenberg translation and a Heisenberg rotation:

$$\iota_{C_\zeta} = R_\mu \circ T_{(\xi,\nu)},$$

where

$$\xi = (\bar{\mu} - 1)\,\zeta \;\text{ and }\; \nu = 2|\zeta|^2 \cdot \operatorname{Im}(1 - \mu) = -2|\zeta|^2 \sin\left(\frac{2\pi}{n}\right).$$

The order of the Heisenberg rotation is equal to the order of the complex reflection.

Alternatively, any complex reflection ι_{C_ζ} can be described as a composition of a Heisenberg rotation and a Heisenberg translation given that

$$T_{(\mu\xi,\nu)} \circ R_\mu = R_\mu \circ T_{(\xi,\nu)}.$$

To see this, observe that

$$\begin{aligned}
\left(T_{(\mu\xi,\nu)} \circ R_\mu\right)(\zeta,\omega). &= T_{(\mu\xi,\nu)}\left(R_\mu(\zeta,\omega)\right) = T_{(\mu\xi,\nu)}(\mu\zeta,\omega) \\
&= \left(\mu\xi + \mu\zeta, \nu + \omega + 2\operatorname{Im}\left((\mu\xi)\overline{(\mu\zeta)}\right)\right) \\
&= \left(\mu(\xi + \zeta), \nu + \omega + 2\operatorname{Im}\left(\xi\bar{\zeta}\right)\right) \\
&= R_\mu\left(\xi + \zeta, \nu + \omega + 2\operatorname{Im}\left(\xi\bar{\zeta}\right)\right) \\
&= \left(R_\mu \circ T_{(\xi,\nu)}\right)(\zeta,\omega).
\end{aligned}$$

2.1.7 Bisectors and Spinal Spheres

Unlike in the real hyperbolic space, there are no totally geodesic real hypersurfaces in $\mathbb{H}^2_{\mathbb{C}}$. An acceptable substitute are the metric bisectors.

Definition 2.1.24. Let $z_1, z_2 \in \mathbb{H}^2_{\mathbb{C}}$ be two distinct points. The *bisector equidistant* from z_1 and z_2 is defined as

$$\mathcal{C}\{z_1, z_2\} = \{z \in \mathbb{H}^2_{\mathbb{C}} \mid \rho(z_1, z) = \rho(z_2, z)\}.$$

Definition 2.1.25. The intersection of a bisector with the boundary of $\mathbb{H}^2_{\mathbb{C}}$ is a smooth hypersurface in $\partial\mathbb{H}^2_{\mathbb{C}}$ called a *spinal sphere*, which is diffeomorphic to a sphere.

Definition 2.1.26. Let $z_1, z_2 \in \mathbb{H}^2_{\mathbb{C}}$ be two distinct points. Let $\Sigma \subset \mathbb{H}^2_{\mathbb{C}}$ be the complex geodesic spanned by z_1 and z_2. We call Σ the *complex spine* of the bisector $\mathcal{C} = \mathcal{C}\{z_1, z_2\}$. The *spine* of $\mathcal{C}$ equals

$$\sigma\{z_1, z_2\} = \mathcal{C}\{z_1, z_2\} \cap \Sigma = \{z \in \Sigma \mid \rho(z_1, z) = \rho(z_2, z)\}.$$

That is, σ is the orthogonal bisector of the geodesic segment joining z_1 and z_2 in Σ.

Theorem 2.1.27. *Let $\mathcal{C}, \Sigma$ and σ be as above. Let $\Pi_\Sigma : \mathbb{H}^2_{\mathbb{C}} \to \Sigma$ be the orthogonal projection onto Σ. Then*

$$\mathcal{C} = \Pi_\Sigma^{-1}(\sigma) = \bigcup_{s \in \sigma} \Pi_\Sigma^{-1}(s).$$

Proof. For proof, see [8], Theorem 5.1.1. □

Definition 2.1.28. The complex hyperplanes $\Pi_\Sigma^{-1}(s)$, for $s \in \sigma$, are called the *slices* of $\mathcal{C}$.

From [8], section 5.1.4, we have that the spine $\sigma \subset \mathbb{H}^2_{\mathbb{C}}$ is completely determined by the hypersurface $\mathcal{C}$, and not by the pair $\{z_1, z_2\}$ used to define $\mathcal{C}$. Associated to every bisector is a geodesic , i.e. its spine. Conversely, if $\sigma \subset \mathbb{H}^2_{\mathbb{C}}$ is a geodesic, there exists a unique bisector $\mathcal{C} = \mathcal{C}_\sigma$ that has spine σ.

A fixed point set of an isometry of $\mathbb{H}^2_{\mathbb{C}}$ is a totally geodesic submanifold, see [10], Theorem 1.10.15. The only proper totally geodesic submanifolds in $\mathbb{H}^2_{\mathbb{C}}$ are complex slices, real slices and real geodesics, see [8], section 5.1.4. In particular, the real dimension of any proper totally geodesic submanifold of $\mathbb{H}^2_{\mathbb{C}}$ is at most 2, hence bisectors are not totally geodesic. It follows that there exists no isometry of $\mathbb{H}^2_{\mathbb{C}}$ whose fixed point set is a bisector $\mathcal{C}$.

Instead, for each slice S of a bisector $\mathcal{C}$, inversion ι_S in S leaves $\mathcal{C}$ invariant, but pointwise fixes only the slice S. Inversion ι_S acts by reflection on the spine σ, interchanging the two endpoints and fixing the point $S \cap \sigma$.

Definition 2.1.29. The endpoints of the spine of $\mathcal{C}$ are called the *vertices* of the bisector $\mathcal{C}$.

Theorem 2.1.30. *Let $S \subset \mathbb{H}^2_{\mathbb{C}}$ be a complex hyperplane and let ι_S denote the inversion in S. Suppose that $u_1, u_2 \in \partial\mathbb{H}^2_{\mathbb{C}}$. Then S is a slice of the bisector $\mathcal{C}$ having vertices u_1, u_2 if and only if ι_S interchanges u_1 and u_2.*

Proof. For proof, see [8], Theorem 5.2.1. □

Chapter 3

A parametrisation of $[m_1, m_2, 0; n_1, n_2, n_3]$-triangle groups

3.1 A parametrisation of $[m_1, m_2, 0; n_1, n_2, n_3]$ -triangle groups

As mentioned in the previous chapter, we are interested in the question: When is a complex hyperbolic ultra-parallel triangle group discrete? We will focus on the case of complex hyperbolic ultra-parallel $[m, m, 0]$-triangle groups, i.e. where two of the complex geodesics intersect on the boundary $\partial\mathbb{H}^2_{\mathbb{C}}$. First, we give a parametrisation of $[m_1, m_2, 0]$-triangle groups.

For $r_1, r_2 \geq 1$ and $\alpha \in (0, 2\pi)$, let C_1, C_2 and C_3 be the complex geodesics with respective polar vectors

$$c_1 = \begin{pmatrix} 1 \\ -r_2e^{-i\theta} \\ r_2e^{-i\theta} \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 \\ r_1e^{i\theta} \\ -r_1e^{i\theta} \end{pmatrix} \quad \text{and} \quad c_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where $\theta = (\pi - \alpha)/2 \in (-\pi/2, \pi/2)$. The type of triangle formed by C_1, C_2 and C_3 is an ultra-parallel $[m_1, m_2, 0]$-triangle with angular invariant α, where $r_k = \cosh(m_k/2)$ for $k = 1, 2$.

Let ι_k be the minimal complex reflection of order n_k in the chain C_k for $k = 1, 2, 3$. The group $\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle$ generated by these three complex reflections is an ultra-parallel complex hyperbolic triangle group of type $[m_1, m_2, 0; n_1, n_2, n_3]$. Looking at the arrangement of the chains C_1, C_2 and C_3 in the Heisenberg space $\mathfrak{N}$, the finite chain C_3 is the (Euclidean) unit circle in $\mathbb{C} \times \{0\}$, whereas C_1 and C_2 are vertical lines through $\varphi_1 = r_2e^{i\theta}$ and $\varphi_2 = -r_1e^{-i\theta}$ respectively, see Figure 3.1.

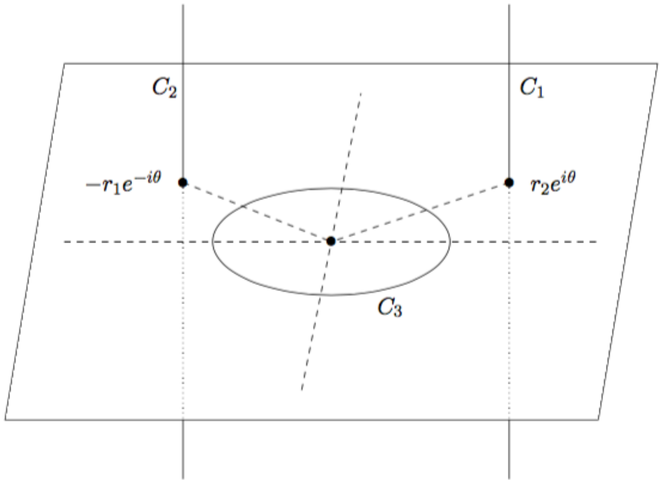

Figure 3.1: Chains C_1, C_2 and C_3.

3.2 Discreteness Criterion

Let C_1, C_2 and C_3 be chains in $\mathfrak{N}$ as in the previous section. Let ι_k be the minimal complex reflection of order n_k in the chain C_k for $k = 1, 2, 3$. We will only consider the case when $n_3 = 2$. Let

$$\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle \quad \text{and} \quad \Gamma' = \langle \iota_1, \iota_2 \rangle.$$

Definition 3.2.1. If there exist open subsets U_1, U_2 and V in $\mathfrak{N}$ with $U_1 \cap U_2 = \varnothing$ and $V \subsetneq U_1$ such that

1. $\iota_3(U_1) = U_2$;
2. $g(U_2) \subsetneq V$, $\forall\, g \in \Gamma' \backslash \{\mathrm{Id}\}$

then the group Γ is *compressing*.

Remark 3.2.2. Γ is said to be *semi-compressing* if only the second of these conditions is met.

To prove the discreteness of the group Γ we will use the following discreteness criterion discussed in [24]:

Proposition 3.2.3. *If Γ is compressing, then Γ is a discrete subgroup of* $\mathrm{PU}(2, 1)$.

Proof. Consider an element $g \in \Gamma$. Each $g \in \Gamma, g \neq \mathrm{Id}$, has an action on either U_1 or U_2 that is isolated from the identity. To see this, notice that any element g can be written in one of the following 4 forms:

1. $\kappa_1 \iota_3 \kappa_2 \iota_3 ... \iota_3 \kappa_n$,

2. $\kappa_1\iota_3\kappa_2\iota_3...\iota_3$,
3. $\iota_3\kappa_2\iota_3...\iota_3\kappa_n$,
4. $\iota_3\kappa_2\iota_3...\iota_3$,

where all the κ_j are elements in Γ' (and assume that the leading and trailing κ_j's are non-identity). For case (1), let $x \in U_2$, then $g(x) \in V$ and

$$U_2 \cap V \subset U_2 \cap U_1 = \varnothing.$$

Therefore g is isolated from the identity. For case (2), let $x \in U_1$ with $x \notin \overline{V}$, then $x \in U_1 \backslash \overline{V}$. Then $g(x) \in V$ and

$$U_1 \backslash \overline{V} \cap V = \varnothing.$$

Therefore g is isolated from the identity. Similarly for case (3), $g(U_2) \subsetneq \iota_3(V)$ and for case (4), $g(U_1) \subsetneq \iota_3(V)$. We can conclude that in all four cases g is non-trivial and is isolated from the identity. Therefore Γ is discrete. □

As discussed in the previous chapter, the minimal complex reflection ι_k of order n_k in the vertical chain C_k rotates any other vertical chain as a set around C_k through $\frac{2\pi}{n_k}$. Projecting the action of the complex reflections ι_1 and ι_2 to $\mathbb{C} \times \{0\}$, let j_1 and j_2 be the rotations of $\mathbb{C}$ around $\varphi_1 = r_2 e^{i\theta}$ and $\varphi_2 = -r_1 e^{-i\theta}$ through $\frac{2\pi}{n_1}$ and $\frac{2\pi}{n_2}$ respectively. Let $\Lambda = \langle j_1, j_2 \rangle$ be the group of isometries of $\mathbb{C} \times \{0\}$ generated by j_1 and j_2. We will need the following Lemma:

Lemma 3.2.4. *If $\left|f(0)\right| \geq 2$ for all $f \in \Lambda \backslash \{\mathrm{Id}\}$ and $\left|h(0)\right| \geq 2$ for all vertical translations $h \in \Gamma' \backslash \{\mathrm{Id}\}$, then the group Γ is discrete.*

Proof. We will use Proposition 3.2.3. Consider the unit spinal sphere

$$U = \{(\zeta, \nu) \in \mathfrak{N} : |\zeta|^4 + \nu^2 = 1\}.$$

The complex reflection ι_3 in C_3 is given by

$$\iota_3([z_1 : z_2 : z_3]) = [-z_1 : z_2 : -z_3] = [z_1 : -z_2 : z_3].$$

The complex reflection ι_3 preserves the bisector

$$\mathcal{C} = \{[z : it : 1] \in \mathbb{H}^2_{\mathbb{C}} \mid |z|^2 < 1 - t^2, z \in \mathbb{C}, t \in \mathbb{R}\}$$

and hence preserves the unit spinal sphere U which is the boundary of the bisector $\mathcal{C}$. The complex reflection ι_3 interchanges the points $[0 : 1 : 1]$ and $[0 : -1 : 1]$ in $\mathbb{H}^2_{\mathbb{C}}$, which correspond to the points $(0, 0)$ and $\infty \in \mathfrak{N}$. Therefore, ι_3 leaves U invariant and switches the inside of U with the outside.

Let U_1 be the part of $\mathcal{N} \setminus U$ outside U, containing ∞, and let U_2 be the part inside U, containing the origin. Clearly

$$U_1 \cap U_2 = \varnothing \quad \text{and} \quad \iota_3(U_1) = U_2.$$

Therefore, if we find a subset $V \subsetneq U_1$ such that $g(U_2) \subsetneq V$ for all elements $g \in \Gamma' \setminus \{\mathrm{Id}\}$, then this will show that Γ is compressing and hence discrete. Let

$$W = \{(\zeta, \nu) \in \mathcal{N} \mid |\zeta| = 1\}$$

be the cylinder consisting of all vertical chains through $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Let

$$W_1 = \{(\zeta, \nu) \in \mathcal{N} : |\zeta| > 1\} \quad \text{and} \quad W_2 = \{(\zeta, \nu) \in \mathcal{N} : |\zeta| < 1\}$$

be the parts of $\mathcal{N} \setminus W$ outside and inside the cylinder W respectively. We have that $U_2 \subset W_2$ and so $g(U_2) \subset g(W_2)$ for all elements $g \in \Gamma' \setminus \{\mathrm{Id}\}$. The set W_2 is a union of vertical chains. We know that elements of Γ' map vertical chains to vertical chains. There may also be a vertical translation on the chain itself. Therefore, we look at both the intersection of the images of W_2 with $\mathbb{C} \times \{0\}$, and the vertical displacement of W_2.

Elements of Γ' move the intersection with $\mathbb{C} \times \{0\}$ by rotations j_1 and j_2 around $r_2 e^{i\theta}$ and $-r_1 e^{-i\theta}$ through $\frac{2\pi}{n_1}$ and $\frac{2\pi}{n_2}$ respectively. Provided that the interior of the unit circle is mapped completely off itself under all elements in $\Lambda \setminus \{\mathrm{Id}\}$, then the same is true for W_2 and hence for U_2 under all elements in Γ' that are not vertical translations.

Vertical Heisenberg translations are vertical Euclidean translations. Such translations will shift W_2 and U_2 and their images $g(W_2)$ and $g(U_2)$ vertically by the same distance.

We choose V to be the union of all the images of U_2 under all elements of $\Gamma' \setminus \{\mathrm{Id}\}$. This subset will satisfy the compressing conditions assuming that the interior of the unit circle is mapped off itself by any element in $\Lambda \setminus \{\mathrm{Id}\}$, and that the interior of the spinal sphere is mapped off itself by any vertical translation in $\Gamma' \setminus \{\mathrm{Id}\}$. Since the radius of a circle is preserved under rotations, we need to show that the origin is moved to a distance of at least twice the radius of the circle by any element in $\Lambda \setminus \{\mathrm{Id}\}$. That is

$$\left| f(0) \right| \geq 2 \text{ for all } f \in \Lambda \setminus \{\mathrm{Id}\}.$$

Since vertical translations shift the spinal spheres vertically, we need to show that they shift by at least the height of the spinal sphere. That is

$$\left| h(0) \right| \geq 2 \text{ for all vertical translations } h \in \Gamma' \setminus \{\mathrm{Id}\}.$$

$\square$

3.3 Orders of Reflection

We are considering the case of complex hyperbolic ultra-parallel $[m_1, m_2, 0; n_1, n_2, n_3]$-triangle groups. We now want to find the possible orders for the complex reflections ι_1 and ι_2 for the triangle group to be discrete. To find these orders, we will use the work of Hersonsky and Paulin [9] and Parker [17].

Recall the Heisenberg group $\mathfrak{N}$ endowed with the group law

$$(\zeta_1, \nu_1) * (\zeta_2, \nu_2) = \left(\zeta_1 + \zeta_2, \nu_1 + \nu_2 + 2\,\mathrm{Im}\left(\zeta_1 \bar{\zeta}_2\right)\right)$$

introduced in the previous chapter. We will first use Proposition 5.4 (specifically when $n = 2$) of [9]:

Proposition 3.3.1. *Let Γ be a discrete cocompact subgroup in $\mathfrak{N}$. Let $\pi : \mathfrak{N} \to \mathbb{C}$ be the canonical projection defined by $\pi(\zeta, \nu) = \zeta$. Then $\pi(\Gamma)$ is a cocompact lattice in $\mathbb{C}$.*

Proof. From the Heisenberg group $\mathfrak{N}$ there exists a central extension

$$0 \to \mathbb{R} \to \mathfrak{N} \to \mathbb{C} \to 0.$$

By this, we have that

$$\mathrm{Ker}(\pi) = \mathbb{R} \quad \text{and} \quad \mathfrak{N}/\mathbb{R} = \mathbb{C}.$$

Note that $\Gamma \cap \mathbb{R}$ is a normal subgroup of Γ, since $\mathbb{R}$ is in the centre of $\mathfrak{N}$. Therefore, the group

$$G = \Gamma / \left(\Gamma \cap \mathbb{R}\right)$$

which identifies to $\pi(\Gamma)$ acts on $\mathbb{C}$. We can see that G acts with bounded quotient on $\mathfrak{N}/\mathbb{R}$ and therefore $\pi(\Gamma)$ acts cocompactly on $\mathbb{C}$.

We next need to show that G acts discretely on $\mathbb{C}$. For a contradiction, suppose not. For a sequence to converge on the plane $\mathbb{C}$, we are able to bound their corresponding elements in Γ by applying a vertical Heisenberg translation H in Γ, which exists due to Γ being non-abelian. This implies that there is a convergent subsequence to an element $(0, t), t \in \mathbb{R}$. This is the required contradiction since Γ is discrete, and hence discrete on $\mathbb{C}$, which completes the proof. $\square$

With this proposition, we now refer to section 5 of [17] (page 454). For the group of Heisenberg isometries with a lattice subgroup of index greater than or equal to 3, Parker states that the canonical projection π, with $\pi : \mathfrak{N} \to \mathbb{C}$, of this isometry group is a $(3, 3, 3)-$, $(2, 4, 4)-$, or $(2, 3, 6)-$triangle group. This gives rise to the following theorem:

Theorem 3.3.2. *A complex hyperbolic ultra-parallel $[m_1, m_2, 0; n_1, n_2, n_3]$-triangle group can only be discrete if the unordered pair of orders of the complex reflections ι_1 and ι_2 is one of*

$$\{2,2\},\ \{2,3\},\ \{2,4\},\ \{2,6\},\ \{3,3\},\ \{3,6\}\ \text{ or } \{4,4\}.$$

Note that the subgroup $E = \langle \iota_1, \iota_2 \rangle$ in an ultra-parallel $[m_1, m_2, 0; n_1, n_2, n_3]$-triangle group $\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle$ is an almost crystallographic group in the sense of Dekimpe [1, 2]:

Definition 3.3.3. Let G be a connected, simply connected nilpotent Lie group, and C a maximal compact subgroup of $\mathrm{Aut}(G)$. A uniform discrete subgroup E of $G \rtimes C$ is called an *almost-crystallographic* group.

For the group $E = \langle \iota_1, \iota_2 \rangle$, we consider almost-crystallographic groups in the setting where

$$G = \mathfrak{N}, \quad \mathrm{U}(1) \subset C \quad \text{and} \quad E \subset \mathfrak{N} \rtimes \mathrm{U}(1),$$

so the theory developed by Dekimpe applies.

Chapter 4

Discreteness Results

We will now derive the discreteness results for the complex hyperbolic ultra-parallel $[m, m, 0; n_1, n_2, 2]$-triangle groups for all orders of complex reflections $\{n_1, n_2\}$.

Remark 4.0.1. The case $\{n_1, n_2\} = \{2, 2\}$ has already been discussed in [11] and [12].

Recall that we can assume without loss of generality that the corresponding chains C_1 and C_2 are vertical chains through $\varphi_1 = re^{i\theta}$ and $\varphi_2 = -re^{-i\theta}$ respectively. Consider the group $\Gamma' = \langle \iota_1, \iota_2 \rangle$. We are able to write every $h \in \Gamma' \backslash \{\mathrm{Id}\}$ as a reduced word in the generators ι_1 and ι_2 of order n_1 and n_2 respectively. Throughout the thesis we mean that any word is reduced i.e. has at most $(n_1 - 1)$ ι_1 and at most $(n_2 - 1)$ ι_2 in a row. The complex reflections ι_1 and ι_2 correspond to the following elements in $\mathrm{PU}(2, 1)$:

$$\iota_1 = \begin{bmatrix} \delta - 1 & -\delta\varphi_1 & -\delta\varphi_1 \\ -\delta\bar{\varphi}_1 & \delta|\varphi_1|^2 - 1 & \delta|\varphi_1|^2 \\ \delta\bar{\varphi}_1 & -\delta|\varphi_1|^2 & -\delta|\varphi_1|^2 - 1 \end{bmatrix}, \ \iota_2 = \begin{bmatrix} \phi - 1 & -\phi\varphi_2 & -\phi\varphi_2 \\ -\phi\bar{\varphi}_2 & \phi|\varphi_2|^2 - 1 & \phi|\varphi_2|^2 \\ \phi\bar{\varphi}_2 & -\phi|\varphi_2|^2 & -\phi|\varphi_2|^2 - 1 \end{bmatrix}, \tag{4.1}$$

where

$$r = \cosh(m/2), \quad \varphi_1 = re^{i\theta}, \quad \varphi_2 = -re^{-i\theta}, \quad \delta = 1 - \mu, \quad \phi = 1 - \lambda,$$

$$\mu = \exp(2\pi i/n_1) \quad \text{and} \quad \lambda = \exp(2\pi i/n_2).$$

For the purpose of the following calculations, we will use the notation

$$\iota_{a_1 a_2 \dots a_n} = \iota_{a_1}\iota_{a_2}\dots\iota_{a_n}.$$

Projecting ι_1 and ι_2 to $\mathbb{C}$ we obtain rotations j_1 and j_2 of $\mathbb{C} \times \{0\}$ through $\frac{2\pi}{n_1}$ and $\frac{2\pi}{n_2}$ around φ_1 and φ_2 respectively. We will use the notation

$$j_{a_1 a_2 \dots a_n} = j_{a_1} j_{a_2} \dots j_{a_n}.$$

The method for the proof of each case is as follows. We will use Lemma 3.2.4. First we will find the Heisenberg translations in the group $\Gamma' = \langle \iota_1, \iota_2 \rangle$, and then find the generators for the group that contains all Heisenberg translations. The vertical Heisenberg translations are used to satisfy the conditions for the second part of Lemma 3.2.4.

To satisfy the first part of Lemma 3.2.4, we will project ι_1 and ι_2 to $\mathbb{C}$ to obtain rotations j_1 and j_2 of $\mathbb{C} \times \{0\}$. We will find the translation maps in the group $\Lambda = \langle j_1, j_2 \rangle$, and find the generators of this group. We will then obtain a generating system so that every element $f \in \Lambda$ can be written as a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word $p = w(0)$ for some remainder word w. We will then consider each p in turn to confirm that the conditions on the angular invariant α and on the distance m satisfy the first part of Lemma 3.2.4.

All of the calculations throughout the proofs were solved using Maple. The Maple document for the case $[m, m, 0; 3, 6, 2]$ can be downloaded and viewed in [20]. This document can be used for the other cases by altering the orders of the complex reflections, ι_1 and ι_2, and the translation maps.

4.1 The case $[m, m, 0; 3, 3, 2]$

Proposition 4.1.1. *Every Heisenberg translation in $\Gamma' = \langle \iota_1, \iota_2 \rangle$ is of the form $T_1^x T_2^y H^z$, where T_1 and T_2 are Heisenberg translations, H is a vertical Heisenberg translation and $x, y, z \in \mathbb{Z}$. Every vertical Heisenberg translation in Γ' is of the form H^z, $z \in \mathbb{Z}$. In particular, the shortest non-trivial vertical translations in Γ' are*

$$H^{\pm 1} = [T_1, T_2]^{\pm 1} = (\iota_{12})^{\pm 3}.$$

Proof. The complex reflections ι_1 and ι_2 are of the form (4.1) with

$$r = \cosh(m/2), \quad \varphi_1 = re^{i\theta}, \quad \varphi_2 = -re^{-i\theta}, \quad \delta = \phi = 1 - \mu \quad \text{and} \quad \mu = \exp(2\pi i/3).$$

As μ is a third root of unity, we will obtain Heisenberg translations from words in ι_1 and ι_2 with length divisible by 3. Straightforward computation shows that the elements $\iota_{112}, \iota_{121}, \iota_{122}, \iota_{211}, \iota_{212}$ and ι_{221} are Heisenberg translations. Let $\mathcal{T}$ be the group generated by these 6 Heisenberg translations. The group $\mathcal{T}$ is generated by $T_1 = \iota_{212}$ and $T_2 = \iota_{112}$ since all other generators can be expressed in terms of T_1 and T_2 :

$$\iota_{121} = T_2 T_1^{-1} T_2^{-1}, \iota_{122} = T_2 T_1^{-1}, \iota_{211} = T_1 T_2^{-1} \quad \text{and} \quad \iota_{221} = T_2^{-1}.$$

Therefore, given any reduced word in ι_1 and ι_2, we are able to break it down into a sequence of Heisenberg translations T_1 and T_2 and their inverses, followed by a word of length at most 2. Hence $\mathcal{T}$ contains all Heisenberg translations in $\Gamma' = \langle \iota_1, \iota_2 \rangle$. The group $\mathcal{T}$ has the presentation

$$\langle T_1, T_2, H \mid [T_1, T_2] = H, [T_1, H] = [T_2, H] = 1 \rangle.$$

Note that $\mathcal{T}$ is isomorphic to N_1 (see Definition 2.1.18). Any vertical translation in Γ' belongs to the subgroup $\mathcal{T}$. Computing the commutator $H = [T_1, T_2] = T_1^{-1} T_2^{-1} T_1 T_2$ we obtain the Heisenberg translation

$$(\zeta, \omega) \mapsto \big(\zeta, \omega + 4\,\mathrm{Im}(\xi_1 \bar{\xi_2})\big)\,. \tag{4.2}$$

So H is the vertical Heisenberg translation in $\mathcal{N}$ by $\big(0, 4\,\mathrm{Im}(\xi_1 \bar{\xi_2})\big)$. Recall that all elements of the form $(0, \star)$ are central in the group $\mathcal{N}$, hence the vertical translation H commutes with any other Heisenberg translation. Using the identities

$$T_1 H = H T_1, \quad T_2 H = H T_2 \quad \text{and} \quad T_1 T_2 = T_2 T_1 H$$

every element of N_1 can be written in the form $T_1^x T_2^y H^z$ for some $x, y, z \in \mathbb{Z}$. If we project to $\mathbb{C} \times \{0\}$, the element $T_1^x T_2^y H^z$ acts as a translation by $x v_1 + y v_2$. That is, this element is a vertical translation if and only if it is a power of H. Direct computation should that $H^{\pm 1} = (\iota_{12})^{\pm 3}$. Note that this is inline with the results of Dekimpe [1] (Chapter 7, Case 13). $\square$

Proposition 4.1.2. *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 3, 2]$*-triangle group is discrete if the following conditions on the angular invariant* α *and on* m *are satisfied:*

$$\cos(\alpha) \le -\frac{1}{2} \quad \textit{and} \quad m \ge \log(3).$$

Proof. We will use Lemma 3.2.4. Direct computation shows that the Heisenberg translations T_k for $k = 1, 2$ by (ξ_k, ν_k) are given as

$$\begin{aligned} \xi_1 &= i \cdot 2r\sqrt{3}\cos(\theta) \ \text{ and } \ \nu_1 = 12\sqrt{3} r^2 \cos^2(\theta), \\ \xi_2 &= r\left(3 + i\sqrt{3}\right)\cos(\theta) \ \text{ and } \ \nu_2 = 12 r^2 \sin(\theta)\cos(\theta). \end{aligned}$$

Substituting ξ_1 and ξ_2 into (4.2) we have

$$\begin{aligned} &\left(\zeta, \omega + 4\,\mathrm{Im}\left(i \cdot 2r\sqrt{3}\cos(\theta)\left(3r\cos(\theta) - i \cdot r\sqrt{3}\cos(\theta)\right)\right)\right) \\ =\ &\left(\zeta, \omega + 4\,\mathrm{Im}\left(i \cdot 6\sqrt{3} r^2 \cos^2(\theta) + 6r^2\cos^2(\theta)\right)\right) \\ =\ &\left(\zeta, \omega + 24\sqrt{3} r^2 \cos^2(\theta)\right). \end{aligned}$$

So the vertical Heisenberg translation H is given as $\left(0, 24\sqrt{3}r^2\cos^2(\theta)\right)$. To satisfy the second part of Lemma 3.2.4, we need the displacement of every vertical translation $H^z, z \neq 0$, to be at least the height of the spinal sphere, i.e.

$$24\sqrt{3}r^2\cos^2(\theta) \geq 2 \Leftrightarrow r^2\cos^2(\theta) \geq \frac{\sqrt{3}}{36}.$$

Under our assumptions, $\cos(\alpha) \leq -\frac{1}{2}$, hence $\frac{2\pi}{3} \leq \alpha \leq \frac{4\pi}{3}$. Using $\alpha = \pi - 2\theta$, we have that $|\theta| \leq \frac{\pi}{6}$. We also have that $m = 2\cosh^{-1}(r) \geq \log(3)$, hence $r \geq \frac{2}{\sqrt{3}}$. So we have

$$r^2\cos^2(\theta) \geq \left(\frac{2}{\sqrt{3}}\right)^2 \cdot \left(\frac{\sqrt{3}}{2}\right)^2 = 1 > \frac{\sqrt{3}}{36},$$

hence the condition $\left|h(0)\right| \geq 2$ is satisfied for all vertical translations $h \in \Gamma' \backslash \{\mathrm{Id}\}$.

To satisfy the first part of Lemma 3.2.4, we project ι_1 and ι_2 to $\mathbb{C}$ to obtain rotations j_1 and j_2 of $\mathbb{C} \times \{0\}$ through $\frac{2\pi}{3}$ around φ_1 and φ_2 respectively. We can write every element $f \in \Lambda$ as a word in the generators $j_1^{\pm 1}$ and $j_2^{\pm 1}$. Using the relations $j_k^{-1} = j_k^2$, for $k = 1, 2$, we can rewrite every element f as a word in terms of j_1 and j_2.

Figure 4.1 shows the points $f(0)$ for all reduced words f of length up to 6 in the case $r = 1$ and $\theta = 0$.

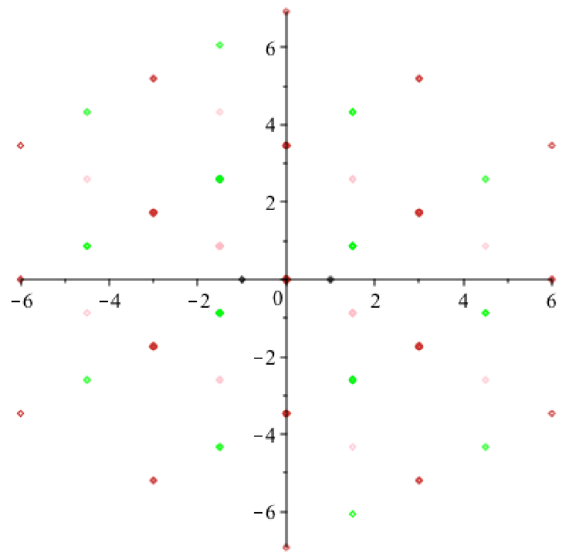

Figure 4.1: Points $f(0)$ for all words f up to length 6.

Straightforward computation gives that

$$j_{a_1 a_2 a_3}(z) = z + (1 - \mu)(\mu^2\varphi_{a_3} + \mu\varphi_{a_2} + \varphi_{a_1}),$$

where

$$\mu = e^{\frac{2\pi i}{3}}, \quad \varphi_1 = re^{i\theta} \quad \text{and} \quad \varphi_2 = -re^{-i\theta}.$$

The explicit formulas for the translations are as follows

$$\begin{aligned}
j_{212}(z) &= z + i\sqrt{3}(\varphi_1 - \varphi_2), \\
j_{121}(z) &= z - i\sqrt{3}(\varphi_1 - \varphi_2), \\
j_{122}(z) &= z + \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2), \\
j_{211}(z) &= z - \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2), \\
j_{112}(z) &= z + \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2), \\
j_{221}(z) &= z - \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2).
\end{aligned}$$

Remark 4.1.3. The remaining maps $j_{111}(z)$ and $j_{222}(z)$ are equal to the identity map. These six translations generate the subgroup of all translations in the group Λ. This subgroup can be generated by two translations. Notice that the six translations are in fact three pairs of inverse translations. We can pair these translations to obtain three generators, the translations by v_1, v_2 and v_3, where

$$\begin{aligned}
v_1 &:= j_{212}(0), \\
v_2 &:= j_{112}(0), \\
v_3 &:= j_{122}(0).
\end{aligned}$$

One of these generators can be written as a linear combination of the other two i.e. $v_3 = -v_1 + v_2$. Therefore, we have a generating system of two translations

$$j_{212}(z) = z + v_1 \quad \text{and} \quad j_{112}(z) = z + v_2.$$

Using the translations $j_{a_1 a_2 a_3}$, we will be able to break down any element of Λ, written as a word in the generators j_1 and j_2, into a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word of length at most 2, so that every point in the orbit of 0 under Λ is of the form

$$f_p(x, y) := p + xv_1 + yv_2,$$

where $p = w(0)$ for some word w of length at most 2 and $x, y \in \mathbb{Z}$. We can further reduce the choices of p. Notice that

$$j_2(z) = j_{211}\left(j_1(z)\right), \quad j_{22}(z) = j_{221}\left(j_{11}(z)\right), \quad j_{12}(z) = j_{121}\left(j_{11}(z)\right)$$

$$\text{and} \quad j_{21}(z) = j_{211}\left(j_{11}(z)\right).$$

Therefore, we can write any element $f \in \Lambda$ as a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word $p = w(0)$ for some word $w \in \{\mathrm{Id}, j_1, j_{11}\}$.

To apply lemma 3.2.4, we want to show that $\left|f(0)\right| \geq 2$ for all $f \in \Lambda \backslash \{\mathrm{Id}\}$. This is equivalent to showing that

$$\left|f_p(x,y)\right|^2 = |p + xv_1 + yv_2|^2 \geq 4$$

for all possible choices of p and for all $x, y \in \mathbb{Z}$, except for $p = x = y = 0$. We have

$$\begin{aligned}\left|f_p(x,y)\right|^2 = &|p + xv_1 + yv_2|^2 = (p + xv_1 + yv_2)\left(\bar{p} + x\bar{v_1} + y\bar{v_2}\right) \\ = &|p|^2 + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2}).\end{aligned} \tag{4.3}$$

Calculating the terms that do not depend on p, we have

$$\begin{aligned} v_1 &= j_{212}(0) = i\sqrt{3}(\varphi_1 - \varphi_2) = i\sqrt{3}\left(re^{i\theta} + re^{-i\theta}\right) = i \cdot 2r\sqrt{3}\cos(\theta), \\ v_2 &= j_{112}(0) = \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2) = \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)\left(re^{i\theta} + re^{-i\theta}\right) \\ &= \left(3 + i\sqrt{3}\right) r\cos(\theta). \end{aligned}$$

We then obtain

$$\begin{aligned} |v_1|^2 = |v_2|^2 = 12r^2\cos^2(\theta), \quad v_1\bar{v_2} = i \cdot 2\sqrt{3}\left(3 - i\sqrt{3}\right) r^2\cos^2(\theta) \\ \text{and} \quad \mathrm{Re}(v_1\bar{v_2}) = 6r^2\cos^2(\theta). \end{aligned}$$

Hence, $x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2})$ is equal to

$$\begin{aligned} &x^2\left(12r^2\cos^2(\theta)\right) + y^2\left(12r^2\cos^2(\theta)\right) + 2xy\left(6r^2\cos^2(\theta)\right) \\ = &12r^2\cos^2(\theta)\left(x^2 + xy + y^2\right). \end{aligned}$$

So we have

$$\begin{aligned} \left|f_p(x,y)\right|^2 = &|p|^2 + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2}) \\ = &12r^2\cos^2(\theta)\left(x^2 + xy + y^2\right) + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + |p|^2. \end{aligned}$$

We want to minimise this expression. In order to do so, we make a coordinate change. Let $u = y - x$ and $v = x + y$, that is,

$$x = \frac{v - u}{2} \quad \text{and} \quad y = \frac{u + v}{2}. \tag{4.4}$$

Under this change of coordinates, points $(x,y)\in\mathbb{Z}^2$ are mapped to points $(u,v)\in\mathbb{Z}^2$ with $u\equiv v \mod 2$. Applying the change of coordinates, we have

$$\begin{aligned}
\left|f_p(u,v)\right|^2 &= 12r^2\cos^2(\theta)\left(\left(\frac{v-u}{2}\right)^2+\left(\frac{v-u}{2}\right)\left(\frac{u+v}{2}\right)+\left(\frac{u+v}{2}\right)^2\right)\\
&\quad+2\left(\frac{v-u}{2}\right)\operatorname{Re}(p\bar{v}_1)+2\left(\frac{u+v}{2}\right)\operatorname{Re}(p\bar{v}_2)+|p|^2\\
&=3r^2\cos^2(\theta)\left(v^2-2uv+u^2+v^2-u^2+u^2+2uv+v^2\right)\\
&\quad+v\left(\operatorname{Re}(p\bar{v}_1)+\operatorname{Re}(p\bar{v}_2)\right)+u\left(\operatorname{Re}(p\bar{v}_2)-\operatorname{Re}(p\bar{v}_1)\right)+|p|^2\\
&=3r^2\cos^2(\theta)\left(u^2+3v^2\right)+v\left(\operatorname{Re}(p(\bar{v}_1+\bar{v}_2))\right)+u\left(\operatorname{Re}(p(\bar{v}_2-\bar{v}_1))\right)+|p|^2.
\end{aligned}$$

This can be rewritten as

$$\left|f_p(u,v)\right|^2=3r^2\cos^2(\theta)\left((u-a)^2+3(v-b)^2\right),$$

where

$$\begin{aligned}
a&=\frac{\operatorname{Re}(p(\bar{v}_1-\bar{v}_2))}{6r^2\cos^2(\theta)}=-\frac{\operatorname{Re}(p(3+i\sqrt{3}))}{6r\cos(\theta)},\\
b&=-\frac{\operatorname{Re}(p(\bar{v}_1+\bar{v}_2))}{18r^2\cos^2(\theta)}=-\frac{\operatorname{Re}(p(1-i\sqrt{3}))}{6r\cos(\theta)},\\
a^2+3b^2&=\frac{|p|^2}{3r^2\cos^2(\theta)}.
\end{aligned}$$

Our aim is to show that $|p+xv_1+yv_2|^2\geq 3r^2\geq 4$ for all $(x,y)\in\mathbb{Z}^2$ excluding the case $p=x=y=0$ which corresponds to the identity case. That is,

$$3r^2\cos^2(\theta)\left((u-a)^2+3(v-b)^2\right)\geq 3r^2\Leftrightarrow (u-a)^2+3(v-b)^2\geq\sec^2(\theta)$$

for all $(u,v)\in\mathbb{Z}^2$ with $u\equiv v \mod 2$, excluding the case $a=b=u=v=0$. Notice that this inequality is always satisfied if $|u-a|\geq\sec(\theta)$ or $|v-b|\geq\frac{\sec\theta}{\sqrt{3}}$ and so we only need to check that

$$g_p^{3,3}(u,v)=(u-a)^2+3(v-b)^2-\sec^2(\theta)\geq 0$$

for all $(u,v)\in\mathbb{Z}^2$ with $u\equiv v \mod 2$ inside the bounding box

$$(a-\sec(\theta),a+\sec(\theta))\times\left(b-\frac{\sec(\theta)}{\sqrt{3}},b+\frac{\sec(\theta)}{\sqrt{3}}\right).$$

For the choices of p, we look at the words $w\in\{\mathrm{Id},j_1,j_{11}\}$. We have 3 possibilities: the identity Id and

$$\begin{aligned}
j_1(z)&=\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)z+\left(\frac{3}{2}-\frac{i\sqrt{3}}{2}\right)\varphi_1,\\
j_{11}(z)&=\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)z+\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)\varphi_1.
\end{aligned}$$

Figure 4.2 shows the points $w(0)$ for all words $w \in \{\mathrm{Id}, j_1, j_{11}\}$ in the case $r = 1$ and $\theta = 0$.

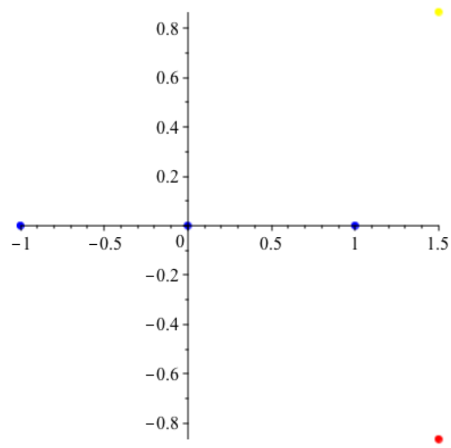

Figure 4.2: Points $w(0)$ for all words $w \in \{\mathrm{Id}, j_1, j_{11}\}$.

Evaluating these words at $z = 0$, we have three possible choices for p:

$$p = 0 \quad \text{and} \quad p = \left(\frac{3}{2} \pm \frac{i\sqrt{3}}{2}\right)\varphi_1.$$

For each choice of p, the following table shows the values of a, b and $a^2 + 3b^2$ in terms of $t = \tan(\theta)$:

$p = w(0)$	a	b	$a^2 + 3b^2$
Id	0	0	0
$j_1(0)$	-1	$-\frac{t}{\sqrt{3}}$	$t^2 + 1$
$j_{11}(0)$	$-\frac{1}{2}\left(1 - t\sqrt{3}\right)$	$-\frac{1}{6}\left(3 + t\sqrt{3}\right)$	$t^2 + 1$

Under the assumption $\lvert\theta\rvert \leq \frac{\pi}{6}$ we have that

$$t = \tan(\theta) \in \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \quad \text{and} \quad \sec(\theta) \in \left[1, \frac{2}{\sqrt{3}}\right].$$

So for each p, we need to calculate the bounds on a, b and the size of the bounding box

$$\left(\min(a) - \frac{2}{\sqrt{3}}, \max(a) + \frac{2}{\sqrt{3}}\right) \times \left(\min(b) - \frac{2}{3}, \max(b) + \frac{2}{3}\right).$$

We then need to show that

$$\begin{aligned} g_p^{3,3}(u, v) &= (u - a)^2 + 3(v - b)^2 - \sec^2(\theta) \geq 0 \\ &= u^2 - 2au + 3v^2 - 6bv + (a^2 + 3b^2) - (t^2 + 1) \geq 0 \end{aligned}$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$ inside the bounding box.

For $p = \mathrm{Id}$, we have $a = 0$ and $b = 0$. The bounding box

$$\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \times \left(-\frac{2}{3}, \frac{2}{3}\right) \subset (-2, 2) \times (-1, 1)$$

contains the point $(0, 0)$ which corresponds to the excluded case.

For $p = j_1(0)$, we have $a = -1$ and $b = -\frac{t}{\sqrt{3}} \in \left[-\frac{1}{3}, \frac{1}{3}\right]$. The bounding box

$$\left(-1 - \frac{2}{\sqrt{3}}, -1 + \frac{2}{\sqrt{3}}\right) \times \left(-\frac{1}{3} - \frac{2}{3}, \frac{1}{3} + \frac{2}{3}\right) \subset (-3, 1) \times (-1, 1)$$

contains the points $(-2, 0)$ and $(0, 0)$. The function

$$g_1^{3,3}(u, v) = u^2 + 2u + 3v^2 + 2tv\sqrt{3}$$

evaluated at these points is non-negative: $g_1^{3,3}(-2, 0) = g_1^{3,3}(0, 0) = 0$.

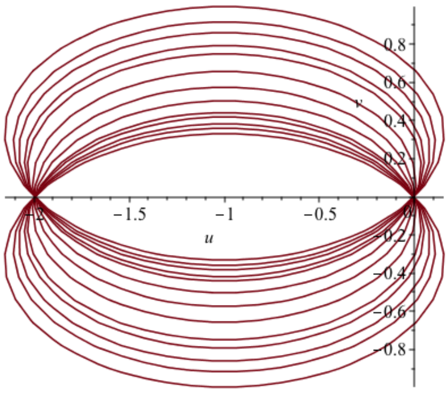

Figure 4.3: The level curves $g_1^{3,3}(u, v) = 0$ for several $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$.

For $p = j_{11}(0)$, we have $a = -\frac{1}{2}\left(1 - t\sqrt{3}\right) \in [-1, 0]$ and $b = -\frac{1}{6}\left(3 + t\sqrt{3}\right)$ $\in \left[-\frac{2}{3}, -\frac{1}{3}\right]$. The bounding box

$$\left(-1 - \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \times \left(-\frac{2}{3} - \frac{2}{3}, -\frac{1}{3} + \frac{2}{3}\right) \subset (-3, 2) \times (-2, 1)$$

contains the points $(-2, 0), (-1, -1), (0, 0)$ and $(1, -1)$. The function

$$g_{11}^{3,3}(u, v) = u^2 + \left(1 - t\sqrt{3}\right) u + 3v^2 + \left(3 + t\sqrt{3}\right) v$$

evaluated at these points is non-negative:

$$g_{11}^{3,3}(-2,0) = 2\left(1 + t\sqrt{3}\right) \geq 0, \quad g_{11}^{3,3}(-1,-1) = g_{11}^{3,3}(0,0) = 0$$

$$\text{and} \quad g_{11}^{3,3}(1,-1) = 2\left(1 - t\sqrt{3}\right) \geq 0.$$

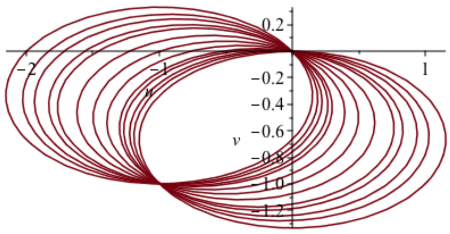

Figure 4.4: The level curves $g_{11}^{3,3}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$.

Therefore, as $g_p^{3,3}(u,v) \geq 0$ for all p, we have that

$$\left|f_p(x,y)\right|^2 = |p + xv_1 + yv_2|^2 \geq 3r^2 \geq 4$$

under the assumption that $r \geq \frac{2}{\sqrt{3}}$. That is, $\left|f_p(x,y)\right| \geq 2$ for all $f \in \Lambda \backslash \{\mathrm{Id}\}$. Hence the conditions of Lemma 3.2.4 are satisfied, and we can conclude that the complex hyperbolic ultra-parallel $[m,m,0;3,3,2]$-triangle group is discrete for

$$\cos(\alpha) \leq -\frac{1}{2} \quad \text{and} \quad m \geq \log(3).$$

□

4.2 The case $[m,m,0;2,3,2]$

Proposition 4.2.1. *Every Heisenberg translation in $\Gamma' = \langle \iota_1, \iota_2 \rangle$ is of the form $T_1^x T_2^y H^z$, where T_1 and T_2 are Heisenberg translations, H is a vertical Heisenberg translation and $x, y, z \in \mathbb{Z}$. Every vertical Heisenberg translation in Γ' is of the form H^z, $z \in \mathbb{Z}$. In particular, the shortest non-trivial vertical translations in Γ' are*

$$H^{\pm 1} = [T_1, T_2]^{\pm 1} = (\iota_{12})^{\pm 6}.$$

Proof. The complex reflections ι_1 and ι_2 are of the form (4.1) with

$$r = \cosh(m/2), \quad \varphi_1 = re^{i\theta}, \quad \varphi_2 = -re^{-i\theta}, \quad \delta = 1 - \mu,$$

$$\phi = 1 - \lambda, \quad \mu = \exp(i \cdot \pi) \quad \text{and} \quad \lambda = \exp(2\pi i/3).$$

As μ is a second root of unity, and λ is a third root of unity, we will obtain Heisenberg translations with words containing an even number of ι_1 and a multiple of 3 of ι_2. Straightforward computation shows that the elements $\iota_{21212}, \iota_{12212}, \iota_{22121}, \iota_{21221}$ and ι_{12122} are Heisenberg translations. Let $\mathcal{T}$ be the group generated by these 5 Heisenberg translations. The group $\mathcal{T}$ is generated by $T_1 = \iota_{21212}$ and $T_2 = \iota_{12212}$ since all other generators can be expressed in terms of T_1 and T_2 :

$$\iota_{12122} = T_2 T_1^{-1}, \iota_{21221} = T_1 T_2^{-1} \quad \text{and} \quad \iota_{22121} = T_2^{-1}.$$

The reduced length 5 words which are not Heisenberg translations can be expressed in terms of the generators T_1, T_2 and a remainder word of length at most 4:

$$\iota_{12121} = T_2 T_1^{-1} \iota_{221} \quad \text{and} \quad \iota_{22122} = T_1^{-1} \iota_{21}.$$

Therefore, given any reduced word in ι_1 and ι_2, we are able to break it down into a sequence of Heisenberg translations T_1 and T_2 and their inverses, followed by a word of length at most 4. Hence $\mathcal{T}$ contains all Heisenberg translations in $\Gamma' = \langle \iota_1, \iota_2 \rangle$.

The group $\mathcal{T}$ has the presentation

$$\langle T_1, T_2, H \mid [T_1, T_2] = H, [T_1, H] = [T_2, H] = 1 \rangle.$$

Note that $\mathcal{T}$ is isomorphic to N_1 (see Definition 2.1.18). Any vertical translation in Γ' belongs to the subgroup $\mathcal{T}$. Computing the commutator $H = [T_1, T_2] = T_1^{-1} T_2^{-1} T_1 T_2$ we obtain the Heisenberg translation of the form (4.2) which is a vertical translation in $\mathfrak{N}$. Using the identities

$$T_1 H = H T_1, \quad T_2 H = H T_2 \quad \text{and} \quad T_1 T_2 = T_2 T_1 H,$$

every element of N_1 can be written in the form $T_1^x T_2^y H^z$ for some $x, y, z \in \mathbb{Z}$. If we project to $\mathbb{C} \times \{0\}$, the element $T_1^x T_2^y H^z$ acts as a translation by $xv_1 + yv_2$. That is, this element is a vertical translation if and only if it is a power of H. Direct computation shows that $H^{\pm 1} = (\iota_{12})^{\pm 6}$. Note that this is inline with the results of Dekimpe [1] (Chapter 7, Case 16). $\square$

Proposition 4.2.2. *A complex hyperbolic ultra-parallel $[m, m, 0; 2, 3, 2]$-triangle group is discrete if the following conditions on the angular invariant α and on m are satisfied:*

$$\cos(\alpha) \leq -\frac{1}{2} \quad \textit{and} \quad m \geq \log(3).$$

Proof. We will use Lemma 3.2.4. Direct computation shows that the Heisenberg translations T_k for $k = 1, 2$ by (ξ_k, ν_k) are given as

$$\begin{aligned}
&\xi_1 = i \cdot 4r\sqrt{3}\cos(\theta) \text{ and } \nu_1 = 32\sqrt{3}r^2\cos^2(\theta), \\
&\xi_2 = 2r\left(3 + i\sqrt{3}\right)\cos(\theta) \text{ and } \nu_2 = 24r^2\sin(\theta)\cos(\theta) - 8\sqrt{3}r^2\cos^2(\theta).
\end{aligned}$$

Substituting ξ_1 and ξ_2 into (4.2) we have that the vertical Heisenberg translation H is given as $\left(0, 96\sqrt{3}r^2\cos^2(\theta)\right)$. To satisfy the second part of Lemma 3.2.4, we need the displacement of every vertical translation $H^z, z \neq 0$, to be at least the height of the spinal sphere, i.e.

$$96\sqrt{3}r^2\cos^2(\theta) \geq 2 \Leftrightarrow r^2\cos^2(\theta) \geq \frac{\sqrt{3}}{144}.$$

By our assumption, $\cos(\alpha) \leq -\frac{1}{2}$ and $m \geq \log(3)$, hence as in the previous section we have that $|\theta| \leq \frac{\pi}{6}$ and $r \geq \frac{2}{\sqrt{3}}$. So we have

$$r^2\cos^2(\theta) \geq \left(\frac{2}{\sqrt{3}}\right)^2 \cdot \left(\frac{\sqrt{3}}{2}\right)^2 = 1 > \frac{\sqrt{3}}{144},$$

hence the condition $\left|h(0)\right| \geq 2$ is satisfied for all vertical translations $h \in \Gamma' \backslash \{\mathrm{Id}\}$.

To satisfy the first part of Lemma 3.2.4, we project ι_1 and ι_2 to $\mathbb{C}$ to obtain rotations j_1 and j_2 of $\mathbb{C} \times \{0\}$ through π and $\frac{2\pi}{3}$ around φ_1 and φ_2 respectively. We can write every element $f \in \Lambda$ as a word in the generators j_1 and $j_2^{\pm 1}$. Using the relation $j_2^{-1} = j_2^2$, we can rewrite every element f as a word in terms of j_1 and j_2.

Figure 4.5 shows the points $f(0)$ for all reduced words f of length up to 6 in the case $r = 1$ and $\theta = 0$.

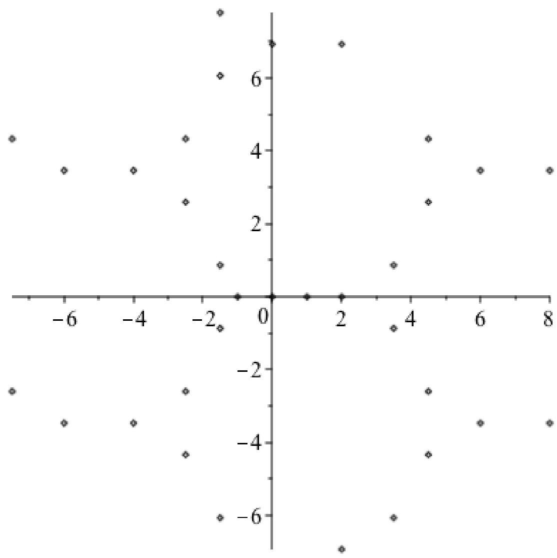

Figure 4.5: Points $f(0)$ for all words f up to length 6.

Projecting the Heisenberg translations to $\mathbb{C}$ we obtain Euclidean translations. The explicit formulas for these Euclidean translations are as follows

$$\begin{aligned}
j_{22121}(z) &= z - (3 + i\sqrt{3})\,(\varphi_1 - \varphi_2)\,, \\
j_{21221}(z) &= z - (3 - i\sqrt{3})\,(\varphi_1 - \varphi_2)\,, \\
j_{21212}(z) &= z + i \cdot 2\sqrt{3}\,(\varphi_1 - \varphi_2)\,, \\
j_{12212}(z) &= z + (3 + i\sqrt{3})\,(\varphi_1 - \varphi_2)\,, \\
j_{12122}(z) &= z + (3 - i\sqrt{3})\,(\varphi_1 - \varphi_2)\,.
\end{aligned}$$

These translations generate the subgroup of all translations in the group Λ. This subgroup is generated by two translations

$$j_{21212}(z) = z + v_1 \quad \text{and} \quad j_{12212}(z) = z + v_2,$$

where $v_1 = i \cdot 2\sqrt{3}\,(\varphi_1 - \varphi_2)$ and $v_2 = (3 + i\sqrt{3})\,(\varphi_1 - \varphi_2)$.

Using these translations, we are able to break down any element of Λ, written as a word in the generators j_1 and j_2, into a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word of length at most 4, so that every point in the orbit of 0 under Λ is of the form

$$f_p(x, y) := p + xv_1 + yv_2,$$

where $p = w(0)$ for some word w of length at most 4 and $x, y \in \mathbb{Z}$.

Remark 4.2.3. For words of length 5 which are not translation maps, notice that these maps are equal to maps of greater length which can be broken down into a sequence of translations followed by a word of length at most 4:

$$\begin{aligned} j_{22122}(z) &= j_{12212}^{-1}\left(j_{12}(z)\right); \\ j_{12121}(z) &= j_{21212}^{-1}\left(j_{122}(z)\right). \end{aligned}$$

Therefore the form of every point in the orbit of 0 under Λ is still valid.
We can further reduce the choices of p. Notice that

$$\begin{aligned} j_{21}(z) &= j_{21212}\left(j_{12212}^{-1}\left(j_{12}(z)\right)\right), \\ j_{121}(z) &= j_{12212}\left(j_{21212}^{-1}\left(j_{2}(z)\right)\right), \\ j_{212}(z) &= j_{21212}\left(j_{12212}^{-1}\left(j_{122}(z)\right)\right), \\ j_{221}(z) &= j_{12212}^{-1}\left(j_{122}(z)\right), \\ j_{1212}(z) &= j_{12212}\left(j_{21212}^{-1}\left(j_{22}(z)\right)\right), \\ j_{1221}(z) &= j_{12212}\left(j_{22}(z)\right), \\ j_{2121}(z) &= j_{21212}\left(j_{22}(z)\right), \\ j_{2122}(z) &= j_{21212}\left(j_{12212}^{-1}\left(j_{1}(z)\right)\right), \\ j_{2212}(z) &= j_{12212}^{-1}\left(j_{1}(z)\right). \end{aligned}$$

Therefore, we can write any element $f \in \Lambda$ as a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word $p = w(0)$ for some word $w = \{\mathrm{Id}, j_1, j_2, j_{12}, j_{22}, j_{122}\}$.

To apply lemma 3.2.4, we want to show that

$$\left|f_p(x,y)\right|^2 = |p + xv_1 + yv_2|^2 \geq 3r^2 \geq 4$$

for all possible choices of p and for all $x, y \in \mathbb{Z}$. Using (4.3) we have

$$\left|f_p(x,y)\right|^2 = |p|^2 + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2}).$$

Calculating the terms that do not depend on p, we have

$$\begin{aligned} v_1 &= j_{21212}(0) = i \cdot 2\sqrt{3}(\varphi_1 - \varphi_2) = i \cdot 2\sqrt{3}\left(re^{i\theta} + re^{-i\theta}\right) = i \cdot 4r\sqrt{3}\cos(\theta), \\ v_2 &= j_{12212}(0) = \left(3 + i\sqrt{3}\right)(\varphi_1 - \varphi_2) = \left(3 + i\sqrt{3}\right)\left(re^{i\theta} + re^{-i\theta}\right) \\ &= \left(6 + i2\sqrt{3}\right) r\cos(\theta). \end{aligned}$$

We then obtain

$$|v_1|^2 = |v_2|^2 = 48r^2\cos^2(\theta), \quad v_1\bar{v_2} = i \cdot 4\sqrt{3}\left(6 - i2\sqrt{3}\right) r^2\cos^2(\theta)$$
$$\text{and} \quad \mathrm{Re}(v_1\bar{v_2}) = 24r^2\cos^2(\theta).$$

Hence, $x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2})$ is equal to

$$\begin{aligned} &x^2\left(48r^2\cos^2(\theta)\right) + y^2\left(48r^2\cos^2(\theta)\right) + 2xy\left(24r^2\cos^2(\theta)\right) \\ &= 48r^2\cos^2(\theta)\left(x^2 + xy + y^2\right). \end{aligned}$$

So we have

$$\left|f_p(x,y)\right|^2 = 48r^2\cos^2(\theta)\left(x^2 + xy + y^2\right) + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + |p|^2.$$

We want to minimise this expression. In order to do so, we apply the coordinate change (4.4). Under this coordinate change, we have

$$\begin{aligned} \left|f_p(u,v)\right|^2 &= 48r^2\cos^2(\theta)\left(\left(\frac{v-u}{2}\right)^2 + \left(\frac{v-u}{2}\right)\left(\frac{u+v}{2}\right) + \left(\frac{u+v}{2}\right)^2\right) \\ &\quad + 2\left(\frac{v-u}{2}\right)\mathrm{Re}(p\bar{v_1}) + 2\left(\frac{u+v}{2}\right)\mathrm{Re}(p\bar{v_2}) + |p|^2 \\ &= 12r^2\cos^2(\theta)\left(v^2 - 2uv + u^2 + v^2 - u^2 + u^2 + 2uv + v^2\right) \\ &\quad + v\left(\mathrm{Re}(p\bar{v_1}) + \mathrm{Re}\left(p\bar{v_2}\right)\right) + u\left(\mathrm{Re}(p\bar{v_2}) - \mathrm{Re}(p\bar{v_1})\right) + |p|^2 \\ &= 12r^2\cos^2(\theta)\left(u^2 + 3v^2\right) + v\left(\mathrm{Re}(p(\bar{v_1} + \bar{v_2}))\right) + u\left(\mathrm{Re}(p(\bar{v_2} - \bar{v_1}))\right) + |p|^2. \end{aligned}$$

This can be rewritten as

$$\left|f_p(u,v)\right|^2 = 12r^2\cos^2(\theta)\left((u-a)^2 + 3(v-b)^2\right),$$

where

$$\begin{aligned} a &= -\frac{\mathrm{Re}(p(\bar{v_2} - \bar{v_1}))}{24r^2\cos^2(\theta)} = -\frac{\mathrm{Re}(p(3 + i\sqrt{3}))}{12r\cos(\theta)}, \\ b &= -\frac{\mathrm{Re}(p(\bar{v_1} + \bar{v_2}))}{72r^2\cos^2(\theta)} = -\frac{\mathrm{Re}(p(1 - i\sqrt{3}))}{12r\cos(\theta)}, \\ a^2 + 3b^2 &= \frac{|p|^2}{12r^2\cos^2(\theta)}. \end{aligned}$$

Our aim is to show that $|p + xv_1 + yv_2|^2 \geq 3r^2$ for all $(x,y) \in \mathbb{Z}^2$ excluding the case $p = x = y = 0$ which corresponds to the identity case. That is,

$$12r^2\cos^2(\theta)\left((u-a)^2 + 3(v-b)^2\right) \geq 3r^2 \Leftrightarrow (u-a)^2 + 3(v-b)^2 \geq \frac{\sec^2(\theta)}{4}$$

for all $(u,v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$, excluding the case $a = b = u = v = 0$. Notice that this inequality is always satisfied if $|u-a| \geq \frac{\sec(\theta)}{2}$ or $|v-b| \geq \frac{\sec\theta}{2\sqrt{3}}$ and so we only need to check that

$$g_p^{2;3}(u,v) = (u-a)^2 + 3(v-b)^2 - \frac{\sec^2(\theta)}{4} \geq 0$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$ inside the bounding box

$$\left(a - \frac{\sec(\theta)}{2}, a + \frac{\sec(\theta)}{2}\right) \times \left(b - \frac{\sec(\theta)}{2\sqrt{3}}, b + \frac{\sec(\theta)}{2\sqrt{3}}\right).$$

For the choices of p, we look at the words $w = \{\mathrm{Id}, j_1, j_2, j_{12}, j_{22}, j_{122}\}$. We have 6 possibilities: the identity Id and

$$\begin{aligned}
j_1(z) &= 2\varphi_1 - z, \\
j_2(z) &= -\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) z + \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right) \varphi_2, \\
j_{12}(z) &= \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) z + 2\varphi_1 - \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right) \varphi_2, \\
j_{22}(z) &= -\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) z + \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right) \varphi_2, \\
j_{122}(z) &= \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) z + 2\varphi_1 - \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right) \varphi_2.
\end{aligned}$$

Figure 4.6 shows the points $w(0)$ for all words $w = \{\mathrm{Id}, j_1, j_2, j_{12}, j_{22}, j_{122}\}$ in the case $r = 1$ and $\theta = 0$.

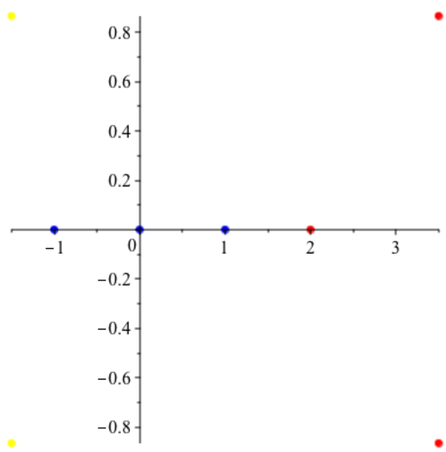

Figure 4.6: Points $w(0)$ for all words $w = \{\mathrm{Id}, j_1, j_2, j_{12}, j_{22}, j_{122}\}$.

Evaluating these words at $z = 0$, we have six possible choices for p:

$$p = 0, \quad p = 2\varphi_1, \quad p = \left(\frac{3}{2} \pm \frac{i\sqrt{3}}{2}\right) \varphi_2,$$

$$\text{and} \quad p = 2\varphi_1 - \left(\frac{3}{2} \pm \frac{i\sqrt{3}}{2}\right) \varphi_2.$$

For each choice of p, the following table shows the values of a, b and a^2+3b^2 in terms of $t = \tan(\theta)$:

$p = w(0)$	a	b	$a^2 + 3b^2$
0	0	0	0
$j_1(0)$	$\frac{1}{2\sqrt{3}}\left(t - \sqrt{3}\right)$	$-\frac{1}{2\sqrt{3}}\left(t + \frac{1}{\sqrt{3}}\right)$	$\frac{1}{3}\left(t^2+1\right)$
$j_2(0)$	$\frac{1}{2}$	$-\frac{t}{2\sqrt{3}}$	$\frac{1}{4}\left(t^2+1\right)$
$j_{12}(0)$	$\frac{1}{2\sqrt{3}}\left(t - 2\sqrt{3}\right)$	$-\frac{1}{6}$	$\frac{1}{12}\left(t^2 - 4t\sqrt{3} + 13\right)$
$j_{22}(0)$	$\frac{1}{4}\left(1 + t\sqrt{3}\right)$	$\frac{1}{4\sqrt{3}}\left(\sqrt{3} - t\right)$	$\frac{1}{4}\left(t^2+1\right)$
$j_{122}(0)$	$-\frac{1}{4\sqrt{3}}\left(t + 3\sqrt{3}\right)$	$-\frac{1}{4\sqrt{3}}\left(t + \frac{5}{\sqrt{3}}\right)$	$\frac{1}{12}\left(t^2 + 4t\sqrt{3} + 13\right)$

Under the assumption $|\theta| \leq \frac{\pi}{6}$ we have that

$$t = \tan(\theta) \in \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \quad \text{and} \quad \sec(\theta) \in \left[1, \frac{2}{\sqrt{3}}\right].$$

So for each p, we need to calculate the bounds on a, b and the size of the bounding box

$$\left(\min(a) - \frac{1}{\sqrt{3}},\ \max(a) + \frac{1}{\sqrt{3}}\right) \times \left(\min(b) - \frac{1}{3},\ \max(b) + \frac{1}{3}\right).$$

We then need to show that

$$\begin{aligned} g_p^{2,3}(u,v) &= (u-a)^2 + 3(v-b)^2 - \frac{\sec^2(\theta)}{4} \\ &= u^2 - 2au + 3v^2 - 6bv + (a^2+3b^2) - \left(\frac{t^2+1}{4}\right) \geq 0 \end{aligned}$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$ inside the bounding box.

For $p = \mathrm{Id}$, we have $a = 0$ and $b = 0$. The bounding box

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \subset (-1, 1) \times (-1, 1)$$

contains the point $(0, 0)$ which corresponds to the excluded case $f = \mathrm{Id}$.

For $p = j_1(0)$, we have $a = \frac{1}{2\sqrt{3}}\left(t - \sqrt{3}\right) \in \left[-\frac{2}{3}, -\frac{1}{3}\right]$ and $b = -\frac{1}{2\sqrt{3}}\left(t + \frac{1}{\sqrt{3}}\right) \in \left[-\frac{1}{3}, 0\right]$. The bounding box

$$\left(-\frac{2}{3} - \frac{1}{\sqrt{3}}, -\frac{1}{3} + \frac{1}{\sqrt{3}}\right) \times \left(-\frac{1}{3} - \frac{1}{3}, \frac{1}{3}\right) \subset (-2, 1) \times (-1, 1)$$

contains the point $(0,0)$. The function

$$g_1^{2,3}(u,v) = u^2 + \left(1 - \frac{t}{\sqrt{3}}\right) u + 3v^2 + \left(1 + t\sqrt{3}\right) v + \frac{t^2+1}{12}$$

evaluated at this point is non-negative: $g_1^{2,3}(0,0) = \frac{t^2+1}{12} \geq \frac{1}{12} > 0$.

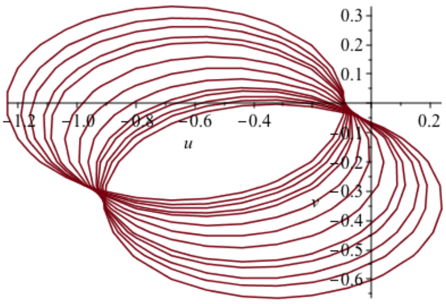

Figure 4.7: The level curves $g_1^{2,3}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$.

For $p = j_2(0)$, we have $a = \frac{1}{2}$ and $b = -\frac{t}{2\sqrt{3}} \in \left[-\frac{1}{6}, \frac{1}{6}\right]$. The bounding box

$$\left(\frac{1}{2} - \frac{1}{\sqrt{3}}, \frac{1}{2} + \frac{1}{\sqrt{3}}\right) \times \left(-\frac{1}{6} - \frac{1}{3}, \frac{1}{6} + \frac{1}{3}\right) \subset (-1,2) \times (-1,1)$$

contains the point $(0,0)$. The function

$$g_2^{2,3}(u,v) = u^2 - u + 3v^2 + tv\sqrt{3}$$

evaluated at this point is non-negative: $g_2^{2,3}(0,0) = 0$.

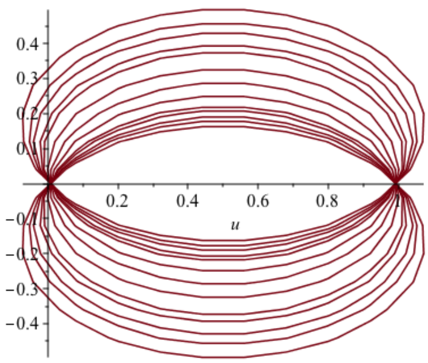

Figure 4.8: The level curves $g_2^{2,3}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$.

For $p = j_{12}(0)$, we have $a = \frac{1}{2\sqrt{3}}\left(t - 2\sqrt{3}\right) \in \left[-\frac{7}{6}, -\frac{5}{6}\right]$ and $b = -\frac{1}{6}$. The bounding box

$$\left(-\frac{7}{6} - \frac{1}{\sqrt{3}}, -\frac{5}{6} + \frac{1}{\sqrt{3}}\right) \times \left(-\frac{1}{6} - \frac{1}{3}, -\frac{1}{6} + \frac{1}{3}\right) \subset (-2, 0) \times (-1, 1)$$

contains no points $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$.

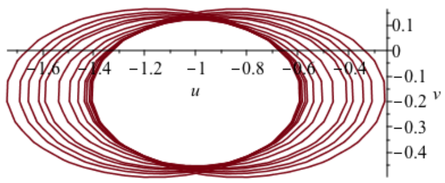

Figure 4.9: The level curves $g_{12}^{2,3}(u, v) = 0$ for several $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$.

For $p = j_{22}(0)$, we have $a = \frac{1}{4}\left(1 + t\sqrt{3}\right) \in \left[0, \frac{1}{2}\right]$ and $b = \frac{1}{4\sqrt{3}}\left(\sqrt{3} - t\right) \in \left[\frac{1}{6}, \frac{1}{3}\right]$. The bounding box

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{2} + \frac{1}{\sqrt{3}}\right) \times \left(\frac{1}{6} - \frac{1}{3}, \frac{1}{3} + \frac{1}{3}\right) \subset (-1, 2) \times (-1, 1)$$

contains the point $(0, 0)$. The function

$$g_{22}^{2,3}(u, v) = u^2 - \frac{u}{2}\left(1 + t\sqrt{3}\right) + 3v^2 - \frac{v}{2}\left(3 - t\sqrt{3}\right)$$

evaluated at this point is non-negative: $g_{22}^{2,3}(0, 0) = 0$.

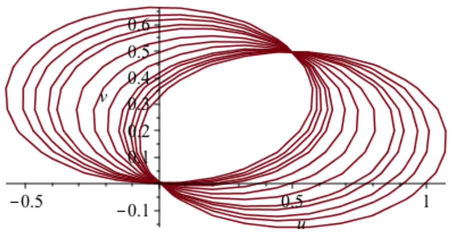

Figure 4.10: The level curves $g^{2,3}_{22}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$.

Finally for $p = j_{122}(0)$, we have $a = -\frac{1}{4\sqrt{3}}\left(t + 3\sqrt{3}\right) \in \left[-\frac{5}{6}, -\frac{2}{3}\right]$ and $b = -\frac{1}{4\sqrt{3}}\left(t + \frac{5}{\sqrt{3}}\right) \in \left[-\frac{1}{2}, -\frac{1}{3}\right]$. The bounding box

$$\left(-\frac{5}{6} - \frac{1}{\sqrt{3}}, -\frac{2}{3} + \frac{1}{\sqrt{3}}\right) \times \left(-\frac{1}{2} - \frac{1}{3}, -\frac{1}{3} + \frac{1}{3}\right) \subset (-2,0) \times (-1,0)$$

contains no points $(u,v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$.

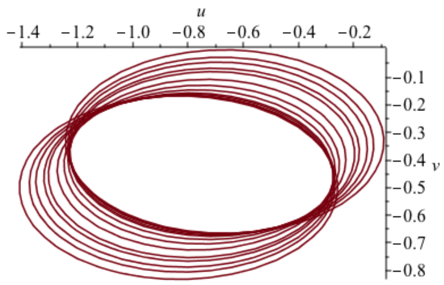

Figure 4.11: The level curves $g^{2,3}_{122}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$.

Therefore, as $g^{2,3}_{p}(u,v) \geq 0$ for all p, we have that

$$\left|f_p(x,y)\right|^2 = \left|p + xv_1 + yv_2\right|^2 \geq 3r^2 \geq 4$$

under the assumption that $r \geq \frac{2}{\sqrt{3}}$. That is, $\left|f_p(x,y)\right| \geq 2$ for all $f \in \Lambda \backslash \{\mathrm{Id}\}$. Hence the conditions of Lemma 3.2.4 are satisfied, and we can conclude that the complex hyperbolic ultra-parallel $[m, m, 0; 2, 3, 2]$-triangle group is discrete for

$$\cos(\alpha) \leq -\frac{1}{2} \quad \text{and} \quad m \geq \log(3).$$

$\square$

4.3 The case $[m, m, 0; 2, 4, 2]$

Proposition 4.3.1. *Every Heisenberg translation in $\Gamma' = \langle \iota_1, \iota_2 \rangle$ is of the form $T_1^x T_2^y H^z$, where T_1 and T_2 are Heisenberg translations, H is a vertical Heisenberg translation and $x, y, z \in \mathbb{Z}$. Every vertical Heisenberg translation in Γ' is of the form $H^z, z \in \mathbb{Z}$. In particular, the shortest non-trivial vertical translations in Γ' are*

$$H^{\pm 1} = [T_1, T_2]^{\pm 1} = (\iota_{12})^{\pm 4}.$$

Proof. The complex reflections ι_1 and ι_2 are of the form (4.1) with

$$r = \cosh(m/2), \quad \varphi_1 = re^{i\theta}, \quad \varphi_2 = -re^{-i\theta}, \quad \delta = 1 - \mu,$$

$$\phi = 1 - \lambda, \quad \mu = \exp(i \cdot \pi) \quad \text{and} \quad \lambda = \exp(i \cdot \pi/2).$$

As μ is a second root on unity and λ is a fourth root of unity, we will obtain Heisenberg translations with words containing x ι_1 and y ι_2 where $\frac{y}{2} + x \equiv 0 \mod 2$. Straightforward computation shows that the elements ι_{122}, ι_{212} and ι_{221} are Heisenberg translations. Let $\mathcal{T}$ be the group generated by these 3 Heisenberg translations. The group $\mathcal{T}$ is generated by $T_1 = \iota_{212}$ and $T_2 = \iota_{122}$ since $\iota_{221} = T_2^{-1}$.

The reduced length 3 words which are not Heisenberg translations can be expressed in terms of the generators T_1, T_2 and a remainder term of length at most 2:

$$\iota_{121} = T_2 T_1^{-1} \iota_2 \quad \text{and} \quad \iota_{222} = T_1^{-1} \iota_{21}.$$

Therefore, given any reduced word in ι_1 and ι_2, we are able to break it down into a sequence of Heisenberg translations T_1 and T_2 and their inverses, followed by a word of length at most 2. Hence $\mathcal{T}$ contains all Heisenberg translations in $\Gamma' = \langle \iota_1, \iota_2 \rangle$.

The group $\mathcal{T}$ has the presentation

$$\langle T_1, T_2, H \mid [T_1, T_2] = H, [T_1, H] = [T_2, H] = 1 \rangle.$$

Note that $\mathcal{T}$ is isomorphic to N_1 (see Definition 2.1.18). Any vertical translation in Γ' belongs to the subgroup $\mathcal{T}$. Computing the commutator $H = [T_1, T_2] = T_1^{-1} T_2^{-1} T_1 T_2$ we obtain the Heisenberg translation of the form (4.2) which is a vertical translation in $\mathcal{N}$. Using the identities

$$T_1 H = H T_1, \quad T_2 H = H T_2 \quad \text{and} \quad T_1 T_2 = T_2 T_1 H$$

every element of N_1 can be written in the form $T_1^x T_2^y H^z$ for some $x, y, z \in \mathbb{Z}$. If we project to $\mathbb{C} \times \{0\}$, the element $T_1^x T_2^y H^z$ acts as a translation by $xv_1 + yv_2$. That is, this element is a vertical translation if and only if it is a power of H. Direct computation should that $H^{\pm 1} = (\iota_{12})^{\pm 4}$. Note that this is inline with the results of Dekimpe [1] (Chapter 7, Case 10). $\square$

Proposition 4.3.2. *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 4, 2]$*-triangle group is discrete if the following conditions on the angular invariant* α *and on* m *are satisfied:*

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \textit{and} \quad m \geq \log\left(3 + 2\sqrt{2}\right).$$

Proof. We will use Lemma 3.2.4. Direct computation shows that the Heisenberg translations T_k for $k = 1, 2$ by (ξ_k, ν_k) are given as

$$\xi_1 = i \cdot 4r\cos(\theta) \text{ and } \nu_1 = 16r^2\cos^2(\theta),$$
$$\xi_2 = 4r\cos(\theta) \text{ and } \nu_2 = 16r^2\sin(\theta)\cos(\theta).$$

Substituting ξ_1 and ξ_2 into (4.2) we have that the vertical Heisenberg translation H is given as $\left(0, 64r^2\cos^2(\theta)\right)$. To satisfy the second part of Lemma 3.2.4, we need the displacement of every vertical translation $H^z, z \neq 0$, to be at least the height of the spinal sphere, i.e.

$$64r^2\cos^2(\theta) \geq 2 \Leftrightarrow r^2\cos^2(\theta) \geq \frac{1}{32}.$$

By our assumption, $\cos(\alpha) \leq -\frac{\sqrt{3}}{2}$, hence $\frac{5\pi}{6} \leq \alpha \leq \frac{7\pi}{6}$. Using $\alpha = \pi - 2\theta$, we have that $|\theta| \leq \frac{\pi}{12}$. We also have that $m = 2\cosh^{-1}(r) \geq \log(3 + 2\sqrt{2})$, hence $r \geq \sqrt{2}$. So we have

$$r^2\cos^2(\theta) \geq \left(\sqrt{2}\right)^2 \cdot \left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)^2 = 1 + \frac{\sqrt{3}}{2} > \frac{1}{32},$$

hence the condition $\left|h(0)\right| \geq 2$ is satisfied for all vertical translations $h \in \Gamma' \backslash \{\mathrm{Id}\}$.

To satisfy the first part of Lemma 3.2.4, we project ι_1 and ι_2 to $\mathbb{C}$ to obtain rotations j_1 and j_2 of $\mathbb{C} \times \{0\}$ through π and $\frac{\pi}{2}$ around φ_1 and φ_2 respectively. We can write every element $f \in \Lambda$ as a word in the generators j_1 and $j_2^{\pm 1, 2}$. Using the relations $j_2^{-2} = j_2^2$ and $j_2^{-1} = j_2^3$, we can rewrite every element f as a word in terms of j_1 and j_2.

Figure 4.12 shows the points $f(0)$ for all reduced words f of length up to 6 in the case $r = 1$ and $\theta = 0$.

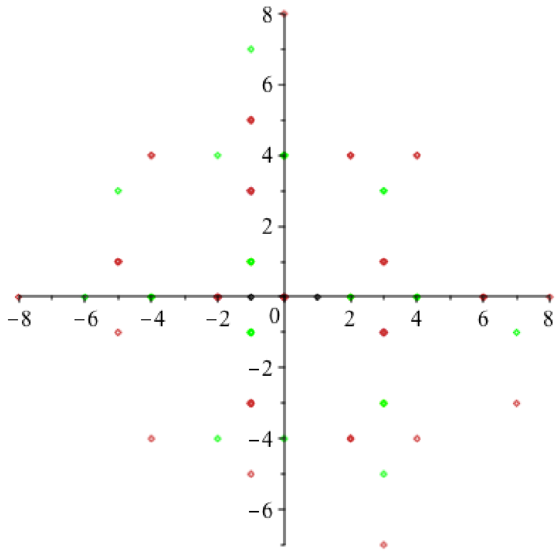

Figure 4.12: Points $f(0)$ for all words f up to length 6.

Projecting the Heisenberg translations to $\mathbb{C}$ we obtain Euclidean translations. The explicit formulas for these Euclidean translations are as follows

$$\begin{aligned} j_{122}(z) &= z + 2\left(\varphi_1 - \varphi_2\right), \\ j_{212}(z) &= z + 2i\left(\varphi_1 - \varphi_2\right), \\ j_{221}(z) &= z - 2\left(\varphi_1 - \varphi_2\right). \end{aligned}$$

These translations generate the subgroup of all translations in the group Λ. This subgroup is generated by two translations

$$j_{212}(z) = z + v_1 \quad \text{and} \quad j_{122}(z) = z + v_2,$$

where $v_1 = i \cdot 2\left(\varphi_1 - \varphi_2\right)$ and $v_2 = 2\left(\varphi_1 - \varphi_2\right)$.

Under the translations, we will be able to break down any element of Λ, written as a word in the generators j_1 and j_2, into a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word of length at most 2, so that every point in the orbit of 0 under Λ is of the form

$$f_p(x, y) := p + xv_1 + yv_2,$$

where $p = w(0)$ for some word w of length at most 2 and $x, y \in \mathbb{Z}$.

Remark 4.3.3. For words of length 3 which are not translation maps, notice that these maps are equal to maps of greater length which can be broken down into a sequence of translations followed by a word of length at most 2:

$$j_{121}(z) = j_{122}\left(j_{212}^{-1}\left(j_2(z)\right)\right) \quad \text{and} \quad j_{222}(z) = j_{212}^{-1}\left(j_{21}(z)\right).$$

Therefore the form of the orbit of 0 under Λ is still valid.

We can further reduce the choices of p. Notice that

$$j_{12}(z) = j_{122}\left(j_{212}^{-1}\left(j_{21}(z)\right)\right) \quad \text{and} \quad j_{22}(z) = j_{122}^{-1}\left(j_1(z)\right).$$

Therefore, we can write any element $f \in \Lambda$ as a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word $p = w(0)$ for some word $w \in \{\mathrm{Id}, j_1, j_2, j_{21}\}$.

To apply lemma 3.2.4, we want to show that

$$\left|f_p(x,y)\right|^2 = \left|p + xv_1 + yv_2\right|^2 \geq 2r^2 \geq 4$$

for all possible choices of p and for all $x, y \in \mathbb{Z}$. Using (4.3) we have

$$\left|f_p(x,y)\right|^2 = |p|^2 + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2}).$$

Calculating the terms that do not depend on p, we have

$$v_1 = i \cdot 2\left(\varphi_1 - \varphi_2\right) = i \cdot 4r\cos(\theta) \quad \text{and} \quad v_2 = 2\left(\varphi_1 - \varphi_2\right) = 4r\cos(\theta).$$

We then obtain

$$|v_1|^2 = |v_2|^2 = 16r^2\cos^2(\theta), \quad v_1\bar{v_2} = i \cdot 16r^2\cos^2(\theta) \quad \text{and} \quad \mathrm{Re}(v_1\bar{v_2}) = 0.$$

Hence, $x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2})$ is equal to $16r^2\cos^2(\theta)\left(x^2+y^2\right)$. So we have

$$\left|f_p(x,y)\right|^2 = 16r^2\cos^2(\theta)\left(x^2+y^2\right) + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + |p|^2.$$

We want to minimise this expression. In order to do so, we apply the coordinate change (4.4). Under this coordinates change, we have

$$\begin{aligned}
\left|f_p(u,v)\right|^2 &= 16r^2\cos^2(\theta)\left(\left(\frac{v-u}{2}\right)^2 + \left(\frac{u+v}{2}\right)^2\right) + (v-u)\,\mathrm{Re}(p\bar{v_1}) \\
&\quad + (u+v)\,\mathrm{Re}(p\bar{v_2}) + |p|^2 \\
&= 4r^2\cos^2(\theta)\left(v^2 - 2uv + u^2 + u^2 + 2uv + v^2\right) + v\left(\mathrm{Re}(p\bar{v_1}) + \mathrm{Re}\left(p\bar{v_2}\right)\right) \\
&\quad + u\left(\mathrm{Re}(p\bar{v_2}) - \mathrm{Re}(p\bar{v_1})\right) + |p|^2 \\
&= 8r^2\cos^2(\theta)\left(u^2+v^2\right) + v\left(\mathrm{Re}(p(\bar{v_1}+\bar{v_2}))\right) + u\left(\mathrm{Re}(p(\bar{v_2}-\bar{v_1}))\right) + |p|^2.
\end{aligned}$$

This can be rewritten as

$$\left|f_p(u,v)\right|^2 = 8r^2\cos^2(\theta)\left((u-a)^2 + (v-b)^2\right),$$

where

$$\begin{aligned}
a &= -\frac{\mathrm{Re}(p(\bar{v_2}-\bar{v_1}))}{16r^2\cos^2(\theta)} = -\frac{\mathrm{Re}\left(p(1+i)\right)}{4r\cos(\theta)}, \\
b &= -\frac{\mathrm{Re}(p(\bar{v_1}+\bar{v_2}))}{16r^2\cos^2(\theta)} = -\frac{\mathrm{Re}\left(p(1-i)\right)}{4r\cos(\theta)}, \\
a^2 + b^2 &= \frac{|p|^2}{8r^2\cos^2(\theta)}.
\end{aligned}$$

Our aim is to show that $|p + xv_1 + yv_2|^2 \geq 2r^2$ for all $(x, y) \in \mathbb{Z}^2$ excluding the case $p = x = y = 0$ which corresponds to the identity case. That is,

$$8r^2\cos^2(\theta)\left((u-a)^2 + (v-b)^2\right) \geq 2r^2 \Leftrightarrow (u-a)^2 + (v-b)^2 \geq \frac{\sec^2(\theta)}{4}$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$, excluding the case $a = b = u = v = 0$. Notice that this inequality is always satisfied if $|u - a| \geq \frac{\sec(\theta)}{2}$ or $|v - b| \geq \frac{\sec\theta}{2}$ and so we only need to check that

$$g_p^{2,4}(u, v) = (u-a)^2 + (v-b)^2 - \frac{\sec^2(\theta)}{4} \geq 0$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$ inside the bounding box

$$\left(a - \frac{\sec(\theta)}{2}, a + \frac{\sec(\theta)}{2}\right) \times \left(b - \frac{\sec(\theta)}{2}, b + \frac{\sec(\theta)}{2}\right).$$

For the choices of p, we look at the words $w \in \{\mathrm{Id}, j_1, j_2, j_{21}\}$. We have 4 possibilities: the identity Id and

$$\begin{aligned} j_1(z) &= 2\varphi_1 - z, \\ j_2(z) &= iz + (1-i)\varphi_2, \\ j_{21}(z) &= -iz + 2i\varphi_1 + (1-i)\varphi_2. \end{aligned}$$

Figure 4.13 shows the points $w(0)$ for all words $w \in \{\mathrm{Id}, j_1, j_2, j_{21}\}$ in the case $r = 1$ and $\theta = 0$.

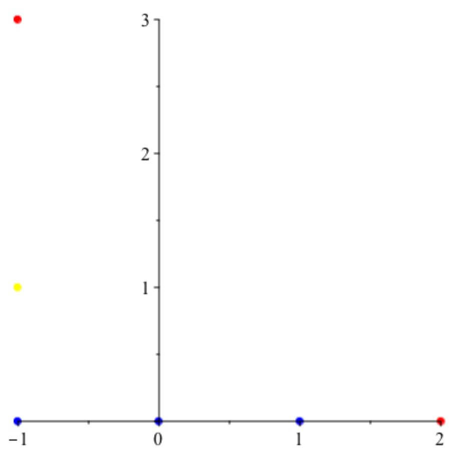

Figure 4.13: Points $w(0)$ for all words $w \in \{\mathrm{Id}, j_1, j_2, j_{21}\}$.

Evaluating these words at $z = 0$, we have four possible choices for p:

$$\begin{aligned} p &= 0, \\ p &= 2\varphi_1, \\ p &= (1-i)\varphi_2, \\ p &= 2i\varphi_1 + (1-i)\varphi_2. \end{aligned}$$

For each choice of p, the following table shows the values of a, b and $a^2 + b^2$ in terms of $t = \tan(\theta)$:

$p = w(0)$	a	b	$a^2 + b^2$
Id	0	0	0
$j_1(0)$	$\frac{1}{2}(t-1)$	$-\frac{1}{2}(t+1)$	$\frac{1}{2}\left(t^2+1\right)$
$j_2(0)$	$\frac{1}{2}$	$-\frac{t}{2}$	$\frac{1}{4}\left(t^2+1\right)$
$j_{21}(0)$	$\frac{1}{2}(t+2)$	$-\frac{1}{2}$	$\frac{1}{4}\left(t^2+4t+5\right)$

Under the assumption $|\theta| \leq \frac{\pi}{12}$ we have that

$$t = \tan(\theta) \in \left[\sqrt{3}-2, 2-\sqrt{3}\right] \quad \text{and} \quad \sec(\theta) \in \left[1, \sqrt{2}\left(\sqrt{3}-1\right)\right].$$

So for each p, we need to calculate the bounds on a, b and the size of the bounding box

$$\left(\min(a) - \frac{\sqrt{3}-1}{\sqrt{2}}, \max(a) + \frac{\sqrt{3}-1}{\sqrt{2}}\right) \times \left(\min(b) - \frac{\sqrt{3}-1}{\sqrt{2}}, \max(b) + \frac{\sqrt{3}-1}{\sqrt{2}}\right).$$

We then need to show that

$$\begin{aligned} g_p^{2,4}(u,v) &= (u-a)^2 + (v-b)^2 - \frac{\sec^2(\theta)}{4} \\ &= u^2 - 2au + v^2 - 2bv + (a^2+b^2) - \left(\frac{t^2+1}{4}\right) \geq 0 \end{aligned}$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$ inside the bounding box. For the purposes of the following calculations, we will denote $t_{\pi/12} = \tan\left(\frac{\pi}{12}\right)$ and $\gamma = \frac{\sqrt{3}-1}{\sqrt{2}}$.

For $p = \text{Id}$, we have $a = 0$ and $b = 0$. The bounding box

$$(-\gamma, \gamma) \times (-\gamma, \gamma) \subset (-1, 1) \times (-1, 1)$$

contains the point $(0, 0)$ which corresponds to the excluded case $f = \text{Id}$.

For $p = j_1(0)$, we have $a = \frac{1}{2}(t-1) \in \left[-\frac{1}{2} - \frac{t_{\pi/12}}{2}, -\frac{1}{2} + \frac{t_{\pi/12}}{2}\right]$ and $b = -\frac{1}{2}(t+1) \in \left[-\frac{1}{2} - \frac{t_{\pi/12}}{2}, -\frac{1}{2} + \frac{t_{\pi/12}}{2}\right]$. The bounding box

$$\left(-\frac{1}{2} - \frac{t_{\pi/12}}{2} - \gamma, -\frac{1}{2} + \frac{t_{\pi/12}}{2} + \gamma\right) \times \left(-\frac{1}{2} - \frac{t_{\pi/12}}{2} - \gamma, -\frac{1}{2} + \frac{t_{\pi/12}}{2} + \gamma\right)$$

$$\subset (-2, 1) \times (-2, 1)$$

contains the points $(-1, -1)$ and $(0, 0)$. The function

$$g_1^{2,4}(u, v) = u^2 + (1-t)\,u + v^2 + (1+t)\,v + \frac{t^2+1}{4}$$

evaluated at these points is non-negative: $g_1^{2,4}(-1, -1) = g_1^{2,4}(0, 0) = \frac{t^2+1}{4} \geq \frac{1}{4} > 0$.

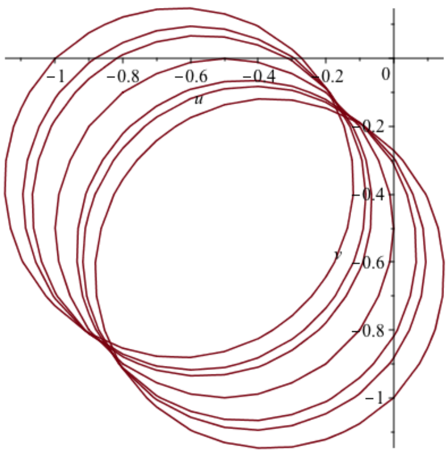

Figure 4.14: The level curves $g_1^{2,4}(u, v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

For $p = j_2(0)$, we have $a = \frac{1}{2}$ and $b = -\frac{t}{2} \in \left[-\frac{t_{\pi/12}}{2}, \frac{t_{\pi/12}}{2}\right]$. The bounding box

$$\left(\frac{1}{2} - \gamma, \frac{1}{2} + \gamma\right) \times \left(-\frac{t_{\pi/12}}{2} - \gamma, \frac{t_{\pi/12}}{2} + \gamma\right)$$

$$\subset (-1, 2) \times (-1, 1)$$

contains the point $(0, 0)$. The function

$$g_2^{2,4}(u, v) = u^2 - u + v^2 + tv$$

evaluated at this point is non-negative: $g_2^{2,4}(0, 0) = 0$.

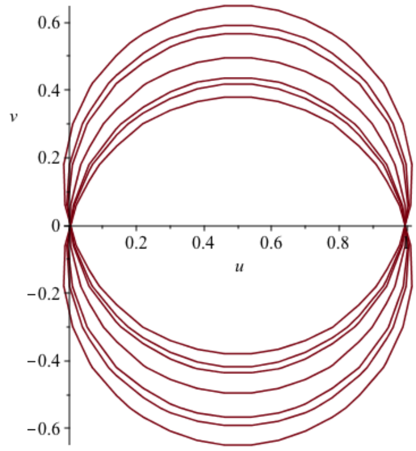

Figure 4.15: The level curves $g_2^{2,4}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

Finally for $p = j_{21}(0)$, we have $a = \frac{1}{2}(t+2) \in \left[1 - \frac{t_{\pi/12}}{2}, 1 + \frac{t_{\pi/12}}{2}\right]$ and $b = -\frac{1}{2}$. The bounding box

$$\left(1 - \frac{t_{\pi/12}}{2} - \gamma, 1 + \frac{t_{\pi/12}}{2} + \gamma\right) \times \left(-\frac{1}{2} - \gamma, -\frac{1}{2} + \gamma\right)$$
$$\subset (0,2) \times (-2,1)$$

contains the point $(1,-1)$. The function

$$g_{21}^{2,4}(u,v) = u^2 - (2+t)\,u + v^2 + v + 1 + t$$

evaluated at this point is non-negative: $g_{21}^{2,4}(1,-1) = 0$.

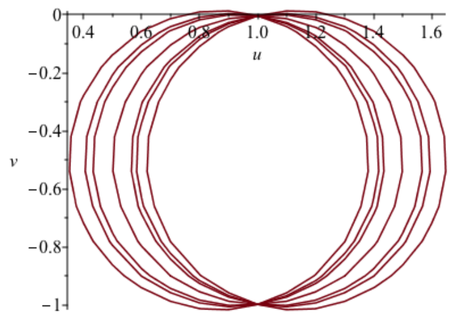

Figure 4.16: The level curves $g_{21}^{2,4}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

Therefore, as $g_p^{2,4}(u,v) \geq 0$ for all p, we have that

$$\left|f_p(x,y)\right|^2 = |p + xv_1 + yv_2|^2 \geq 2r^2 \geq 4$$

under the assumption that $r \geq \sqrt{2}$. That is, $\left|f_p(x, y)\right| \geq 2$ for all $f \in \Lambda \backslash \{\mathrm{Id}\}$. Hence the conditions of Lemma 3.2.4 are satisfied, and we can conclude that the complex hyperbolic ultra-parallel $[m, m, 0; 2, 4, 2]$-triangle group is discrete for

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \text{and} \quad m \geq \log\left(3 + 2\sqrt{2}\right).$$

$\square$

4.4 The case $[m, m, 0; 4, 4, 2]$

Proposition 4.4.1. *Every Heisenberg translation in* $\Gamma' = \langle \iota_1, \iota_2 \rangle$ *is of the form* $T_1^x T_2^y H^z$, *where* T_1 *and* T_2 *are Heisenberg translations,* H *is a vertical Heisenberg translation and* $x, y, z \in \mathbb{Z}$. *Every vertical Heisenberg translation in* Γ' *is of the form* $H^z, z \in \mathbb{Z}$. *In particular, the shortest non-trivial vertical translations in* Γ' *are*

$$H^{\pm 1} = (\iota_{12})^{\pm 2}.$$

Proof. The complex reflections ι_1 and ι_2 are of the form (4.1) with

$$r = \cosh(m/2), \quad \varphi_1 = re^{i\theta}, \quad \varphi_2 = -re^{-i\theta}, \quad \delta = \phi = 1 - \mu$$
$$\text{and} \quad \mu = \lambda = \exp(i \cdot \pi/2).$$

As μ is a fourth root of unity, we will obtain Heisenberg translations from words in ι_1 and ι_2 with length divisible by 4. Straightforward computation shows that the elements $\iota_{1112}, \iota_{1121}, \iota_{1211}, \iota_{2111}, \iota_{2211}, \iota_{1221}, \iota_{1122}, \iota_{2112}, \iota_{1212}, \iota_{2121}, \iota_{1222}, \iota_{2122}, \iota_{2212}$ and ι_{2221} are Heisenberg translations. Let $\mathcal{T}$ be the group generated by these 14 Heisenberg translations. The group $\mathcal{T}$ is generated by $T_1 = \iota_{1112}, T_2 = \iota_{2111}$ and $H = \iota_{1212}$ since all other generators can be expressed in terms of T_1, T_2 and H:

$$\iota_{1121} = T_1 H^{-1} T_2^{-1} T_1^{-1},\ \iota_{1211} = H T_1^{-1},\ \iota_{2211} = T_2 H T_1^{-1},\ \iota_{1221} = T_2^{-1} T_1^{-1},$$
$$\iota_{1122} = T_1 H^{-1} T_2^{-1},\ \iota_{2112} = T_2 T_1,\ \iota_{2121} = T_2 T_1 H^{-1} T_2^{-1} T_1^{-1},\ \iota_{1222} = T_2^{-1},$$
$$\iota_{2122} = T_2 T_1 H^{-1} T_2^{-1},\ \iota_{2212} = T_2 H \quad \text{and} \quad \iota_{2221} = T_1^{-1}.$$

Therefore, given any reduced word in ι_1 and ι_2, we are able to break it down into a sequence of Heisenberg translations T_1, T_2, H and their inverses, followed by a word of length at most 3. Hence $\mathcal{T}$ contains all Heisenberg translations in $\Gamma' = \langle \iota_1, \iota_2 \rangle$.

Direct computation shows that T_k for $k = 1, 2, 3$, where $T_3 = H$, is a Heisenberg translation by (ξ_k, ν_k), where

$$\begin{aligned}
&\xi_1 = 2r(1+i)\cos(\theta) \ \text{and} \ \nu_1 = 8r^2 \sin(\theta)\cos(\theta), \\
&\xi_2 = -2r(1-i)\cos(\theta) \ \text{and} \ \nu_2 = -8r^2 \sin(\theta)\cos(\theta), \\
&\xi_3 = 0 \ \text{and} \ \nu_3 = 16r^2\cos^2(\theta).
\end{aligned}$$

Computing the commutator $[T_2, T_1] = T_2^{-1}T_1^{-1}T_2T_1$ we obtain the Heisenberg translation of the form

$$(\zeta, \omega) \mapsto \big(\zeta, \omega + 4\,\mathrm{Im}(\xi_2\bar{\xi_1})\big) \tag{4.5}$$

which is a vertical translation in $\mathcal{N}$. Substituting ξ_1 and ξ_2 into (4.5) we have the vertical Heisenberg translation by $(0, 32r^2\cos^2(\theta))$. That is, $[T_2, T_1] = H^2$.

The group $\mathcal{T}$ has the presentation

$$\langle T_1, T_2, H \mid [T_2, T_1] = H^2, [H, T_1] = [H, T_2] = 1\rangle.$$

Note that $\mathcal{T}$ is isomorphic to N_2 (see Definition 2.1.18). Any vertical translation in Γ' belongs to the subgroup $\mathcal{T}$. Using the identities

$$T_1H = HT_1, \quad T_2H = HT_2 \quad \text{and} \quad T_2T_1 = T_1T_2H^2$$

every element of N_2 can be written in the form $T_1^xT_2^yH^z$ for some $x, y, z \in \mathbb{Z}$. If we project to $\mathbb{C} \times \{0\}$, the element $T_1^xT_2^yH^z$ acts as a translation by $xv_1 + yv_2$. That is, this element is a vertical translation if and only if it is a power of H. We can see that $H^{\pm 1} = (\iota_{12})^{\pm 2}$. Note that this is inline with the results of Dekimpe [1] (Chapter 7, Case 10). □

Proposition 4.4.2. *A complex hyperbolic ultra-parallel* $[m, m, 0; 4, 4, 2]$*-triangle group is discrete if the following conditions on the angular invariant* α *and on* m *are satisfied:*

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \textit{and} \quad m \geq \log\left(3 + 2\sqrt{2}\right).$$

Proof. We will use Lemma 3.2.4. To satisfy the second part of Lemma 3.2.4, we need the displacement of every vertical translation $H^z, z \neq 0$, to be at least the height of the spinal sphere, i.e.

$$16r^2\cos^2(\theta) \geq 2 \Leftrightarrow r^2\cos^2(\theta) \geq \frac{1}{8}.$$

Under our assumptions, $\cos(\alpha) \leq -\frac{\sqrt{3}}{2}$ and $m \geq \log\left(3 + 2\sqrt{2}\right)$, hence as in the previous section $|\theta| \leq \frac{\pi}{12}$ and $r \geq \sqrt{2}$. So we have

$$r^2\cos^2(\theta) \geq \left(\sqrt{2}\right)^2 \cdot \left(\frac{1 + \sqrt{3}}{2\sqrt{2}}\right)^2 = 1 + \frac{\sqrt{3}}{2} > \frac{1}{8},$$

hence the condition $\big|h(0)\big| \geq 2$ is satisfied for all vertical translation $h \in \Gamma' \backslash \{\mathrm{Id}\}$.

To satisfy the first part of Lemma 3.2.4, we project ι_1 and ι_2 to $\mathbb{C}$ to obtain rotations j_1 and j_2 of $\mathbb{C} \times \{0\}$ through $\frac{\pi}{2}$ around φ_1 and φ_2 respectively. We can write every element $f \in \Lambda$ as a word in the generators $j_1^{\pm 1,2}$ and $j_2^{\pm 1,2}$. Using the relations $j_k^{-1} = j_k^3$ and $j_k^{-2} = j_k^2$, for $k = 1, 2$, we can rewrite every element f as a word in terms of j_1 and j_2.

Figure 4.17 shows the points $f(0)$ for all reduced words f of length up to 6 in the case $r = 1$ and $\theta = 0$.

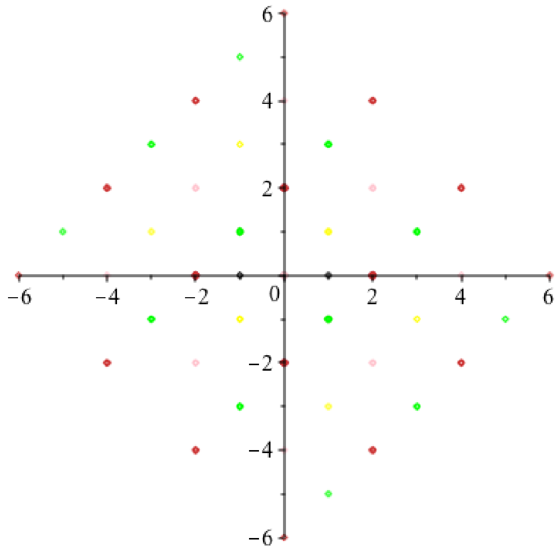

Figure 4.17: Points $f(0)$ for all words f up to length 6.

Projecting the Heisenberg translations to $\mathbb{C}$ we obtain Euclidean translations. Straightforward computation gives that

$$j_{a_1 a_2 a_3 a_4}(z) = z + (1 - \mu)(\mu^3 \varphi_{a_4} + \mu^2 \varphi_{a_3} + \mu \varphi_{a_2} + \varphi_{a_1}),$$

where

$$\mu = e^{\frac{\pi i}{2}}, \quad \varphi_1 = re^{i\theta} \quad \text{and} \quad \varphi_2 = -re^{-i\theta}.$$

The explicit formulas for the Euclidean translations are as follows

$$\begin{aligned}
j_{1112}(z) = j_{2122}(z) &= z + (1 + i)(\varphi_1 - \varphi_2), \\
j_{1211}(z) = j_{2221}(z) &= z - (1 + i)(\varphi_1 - \varphi_2), \\
j_{1222}(z) = j_{1121}(z) &= z + (1 - i)(\varphi_1 - \varphi_2), \\
j_{2111}(z) = j_{2212}(z) &= z - (1 - i)(\varphi_1 - \varphi_2), \\
j_{1122}(z) &= z + 2(\varphi_1 - \varphi_2), \\
j_{2211}(z) &= z - 2(\varphi_1 - \varphi_2), \\
j_{2112}(z) &= z + i \cdot 2(\varphi_1 - \varphi_2), \\
j_{1221}(z) &= z - i \cdot 2(\varphi_1 - \varphi_2),
\end{aligned}$$

Remark 4.4.3. The remaining maps $j_{1111}(z), j_{1212}(z), j_{2121}(z)$ and $j_{2222}(z)$ are equal to the identity map.

These translations generate the subgroup of all translations in the group Λ. This subgroup can be generated by two translations

$$j_{2111}(z) = z + v_1 \quad \text{and} \quad j_{1112}(z) = z + v_2.$$

where $v_1 = -(1-i)(\varphi_1 - \varphi_2)$ and $v_2 = (1+i)(\varphi_1 - \varphi_2)$.

Using the translations $j_{a_1a_2a_3a_4}$, we will be able to break down any element of Λ, written as a word in the generators j_1 and j_2, into a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word of length at most 3, so that every point in the orbit of 0 under Λ is of the form

$$f_p(x,y) := p + xv_1 + yv_2,$$

where $p = w(0)$ for some word w of length at most 3 and $x, y \in \mathbb{Z}$. We can further reduce the choices of p. Notice that for $T_1 = \iota_{1112}, T_2 = \iota_{2111}$ and $H = \iota_{1212}$, we have

$$\iota_2 = T_2\iota_1, \quad \iota_{12} = HT_1^{-1}\iota_{11}, \quad \iota_{21} = T_2\iota_{11}, \quad \iota_{22} = T_2HT_1^{-1}\iota_{11},$$
$$\iota_{112} = T_1H^{-1}T_2^{-1}T_1^{-1}\iota_{111}, \quad \iota_{121} = HT_1^{-1}\iota_{111}, \quad \iota_{211} = T_2\iota_{111},$$
$$\iota_{221} = T_2HT_1^{-1}\iota_{111} \quad \text{and} \quad \iota_{122} = T_2^{-1}T_1^{-1}\iota_{111}.$$

We are able to rewrite the translation element of each map in the form $T_1^xT_2^yH^z$, for $x, y, z \in \mathbb{Z}$, so that every map can be written as $T_1^xT_2^yH^z \cdot \iota_{a_1}$ for $x, y, z \in \mathbb{Z}$ and rotation element ι_{a_1}. Projecting to $\mathbb{C} \times \{0\}$ (i.e. setting H to the identity map, Id), we obtain the maps

$$\begin{aligned}
j_2(z) &= j_{2111}\left(j_1(z)\right), \\
j_{12}(z) &= j_{1112}^{-1}\left(j_{11}(z)\right), \\
j_{21}(z) &= j_{2111}\left(j_{11}(z)\right), \\
j_{22}(z) &= j_{2111}\left(j_{1112}^{-1}\left(j_{11}(z)\right)\right), \\
j_{112}(z) &= j_{2111}^{-1}\left(j_{111}(z)\right), \\
j_{121}(z) &= j_{1112}^{-1}\left(j_{111}(z)\right), \\
j_{211}(z) &= j_{2111}\left(j_{111}(z)\right), \\
j_{221}(z) &= j_{2111}\left(j_{1112}^{-1}\left(j_{111}(z)\right)\right), \\
j_{122}(z) &= j_{2111}^{-1}\left(j_{1112}^{-1}\left(j_{111}(z)\right)\right).
\end{aligned}$$

Therefore, we can write any element $f \in \Lambda$ as a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word $p = w(0)$ for some word $w \in \{\mathrm{Id}, j_1, j_{11}, j_{111}\}$.

To apply lemma 3.2.4, we want to show that

$$\left|f_p(x,y)\right|^2 = |p + xv_1 + yv_2|^2 \geq 2r^2 \geq 4$$

for all possible choices of p and for all $x, y \in \mathbb{Z}$. Using (4.3) we have

$$\left|f_p(x,y)\right|^2 = |p|^2 + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2}).$$

Calculating the terms that do not depend on p, we have

$$\begin{aligned} v_1 &= j_{2111}(0) = -(1-i)(\varphi_1 - \varphi_2) = -(1-i)\left(re^{i\theta} + re^{-i\theta}\right) = -2r(1-i)\cos(\theta), \\ v_2 &= j_{1112}(0) = (1+i)(\varphi_1 - \varphi_2) = (1+i)\left(re^{i\theta} + re^{-i\theta}\right) = 2r(1+i)\cos(\theta). \end{aligned}$$

We then obtain

$$\begin{gathered} |v_1|^2 = |v_2|^2 = 8r^2\cos^2(\theta), \quad v_1\bar{v_2} = -4r^2(1-i)^2\cos^2(\theta) = i \cdot 8r^2\cos^2(\theta) \\ \text{and} \quad \mathrm{Re}(v_1\bar{v_2}) = 0. \end{gathered}$$

Hence, $x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2})$ is equal to

$$x^2\left(8r^2\cos^2(\theta)\right) + y^2\left(8r^2\cos^2(\theta)\right) = 8r^2\cos^2(\theta)\left(x^2 + y^2\right).$$

So we have

$$\begin{aligned} \left|f_p(x,y)\right|^2 &= |p|^2 + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2}) \\ &= 8r^2\cos^2(\theta)\left(x^2 + y^2\right) + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + |p|^2. \end{aligned}$$

We want to minimise this expression. In order to do so, we apply the coordinate change (4.4). Under this coordinate change, we have

$$\begin{aligned} \left|f_p(u,v)\right|^2 = {} & 8r^2\cos^2(\theta)\left(\left(\frac{v-u}{2}\right)^2 + \left(\frac{u+v}{2}\right)^2\right) + 2\left(\frac{v-u}{2}\right)\mathrm{Re}(p\bar{v_1}) \\ & + 2\left(\frac{u+v}{2}\right)\mathrm{Re}(p\bar{v_2}) + |p|^2 \\ = {} & 2r^2\cos^2(\theta)\left(v^2 - 2uv + u^2 + u^2 + 2uv + v^2\right) + v\left(\mathrm{Re}(p\bar{v_1}) + \mathrm{Re}\left(p\bar{v_2}\right)\right) \\ & + u\left(\mathrm{Re}(p\bar{v_2}) - \mathrm{Re}(p\bar{v_1})\right) + |p|^2 \\ = {} & 4r^2\cos^2(\theta)\left(u^2 + v^2\right) + v\left(\mathrm{Re}(p(\bar{v_1} + \bar{v_2}))\right) + u\left(\mathrm{Re}(p(\bar{v_2} - \bar{v_1}))\right) + |p|^2. \end{aligned}$$

This can be rewritten as

$$\left|f_p(u,v)\right|^2 = 4r^2\cos^2(\theta)\left((u-a)^2 + (v-b)^2\right),$$

where

$$a = \frac{\mathrm{Re}(p(\bar{v_1} - \bar{v_2}))}{8r^2\cos^2(\theta)} = -\frac{\mathrm{Re}(p)}{2r\cos(\theta)}, \quad b = -\frac{\mathrm{Re}(p(\bar{v_1} + \bar{v_2}))}{8r^2\cos^2(\theta)} = -\frac{\mathrm{Im}(p)}{2r\cos(\theta)},$$

$$\text{and} \quad a^2 + b^2 = \frac{|p|^2}{4r^2\cos^2(\theta)}.$$

Our aim is to show that $|p + xv_1 + yv_2|^2 \geq 2r^2$ for all $(x, y) \in \mathbb{Z}^2$ excluding the case $p = x = y = 0$ which corresponds to the identity case. That is,

$$4r^2\cos^2(\theta)\left((u-a)^2 + (v-b)^2\right) \geq 2r^2 \Leftrightarrow (u-a)^2 + (v-b)^2 \geq \frac{\sec^2(\theta)}{2}$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$, excluding the case $a = b = u = v = 0$. Notice that this inequality is always satisfied if $|u - a| \geq \frac{\sec(\theta)}{\sqrt{2}}$ or $|v - b| \geq \frac{\sec\theta}{\sqrt{2}}$ and so we only need to check that

$$g_p^{4,4}(u, v) = (u-a)^2 + (v-b)^2 - \frac{\sec^2(\theta)}{2} \geq 0$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$ inside the bounding box

$$\left(a - \frac{\sec(\theta)}{\sqrt{2}}, a + \frac{\sec(\theta)}{\sqrt{2}}\right) \times \left(b - \frac{\sec(\theta)}{\sqrt{2}}, b + \frac{\sec(\theta)}{\sqrt{2}}\right).$$

For the choices of p, we look at the words $w \in \{\mathrm{Id}, j_1, j_{11}, j_{111}\}$. We have 4 possibilities: the identity Id and

$$\begin{aligned} j_1(z) &= i \cdot z + (1-i)\varphi_1, \\ j_{11}(z) &= -z + 2\varphi_1, \\ j_{111}(z) &= -i \cdot z + (1+i)\varphi_1. \end{aligned}$$

Figure 4.18 shows the points $w(0)$ for all words $w \in \{\mathrm{Id}, j_1, j_{11}, j_{111}\}$ in the case $r = 1$ and $\theta = 0$.

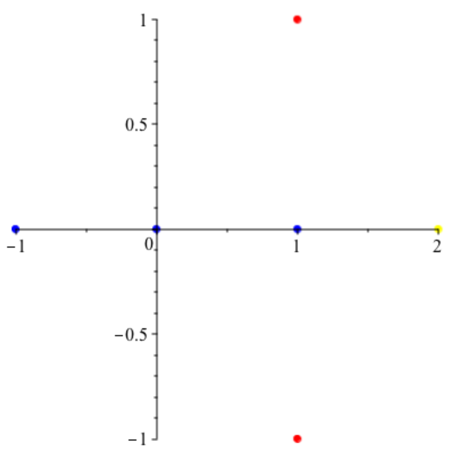

Figure 4.18: Points $w(0)$ for all words $w \in \{\mathrm{Id}, j_1, j_{11}, j_{111}\}$.

Evaluating these words at $z = 0$, we have four possible choices for p:

$$p = 0, \quad p = (1 \pm i)\varphi_1 \quad \text{and} \quad p = 2\varphi_1.$$

For each choice of p, the following table shows the values of a, b and $a^2 + b^2$ in terms of $t = \tan(\theta)$:

$p = w(0)$	a	b	$a^2 + b^2$
Id	0	0	0
$j_1(0)$	$-\frac{1}{2}(1+t)$	$\frac{1}{2}(1-t)$	$\frac{1}{2}\left(t^2+1\right)$
$j_{11}(0)$	-1	$-t$	t^2+1
$j_{111}(0)$	$\frac{1}{2}(t-1)$	$-\frac{1}{2}(t+1)$	$\frac{1}{2}\left(t^2+1\right)$

Under the assumption $|\theta| \leq \frac{\pi}{12}$ we have that

$$t = \tan(\theta) \in \left[\sqrt{3} - 2, 2 - \sqrt{3}\right] \quad \text{and} \quad \sec(\theta) \in \left[1, \sqrt{2}\left(\sqrt{3} - 1\right)\right].$$

So for each p, we need to calculate the bounds on a, b and the size of the bounding box

$$\left(\min(a) - \left(\sqrt{3}-1\right), \max(a) + \left(\sqrt{3}-1\right)\right) \times \left(\min(b) - \left(\sqrt{3}-1\right), \max(b) + \left(\sqrt{3}-1\right)\right).$$

We then need to show that

$$\begin{aligned} g_p^{4,4}(u,v) &= (u-a)^2 + (v-b)^2 - \frac{\sec^2(\theta)}{2} \\ &= u^2 - 2au + v^2 - 2bv + (a^2 + b^2) - \left(\frac{t^2+1}{2}\right) \geq 0 \end{aligned}$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$ inside the bounding box. For the purposes of the following calculations, we will denote $t_{\pi/12} = \tan\left(\frac{\pi}{12}\right)$ and $\gamma = \sqrt{3} - 1$.

For $p = \text{Id}$, we have $a = 0$ and $b = 0$. The bounding box

$$(-\gamma, \gamma) \times (-\gamma, \gamma) \subset (-1, 1) \times (-1, 1)$$

contains the point $(0, 0)$ which corresponds to the excluded case $f = \text{Id}$.

For $p = j_1(0)$, we have $a = -\frac{1}{2}(t+1) \in \left[-\frac{1}{2} - \frac{t_{\pi/12}}{2}, -\frac{1}{2} + \frac{t_{\pi/12}}{2}\right]$ and $b = \frac{1}{2}(1-t) \in \left[\frac{1}{2} - \frac{t_{\pi/12}}{2}, \frac{1}{2} + \frac{t_{\pi/12}}{2}\right]$. The bounding box

$$\left(-\frac{1}{2} - \frac{t_{\pi/12}}{2} - \gamma, -\frac{1}{2} + \frac{t_{\pi/12}}{2} + \gamma\right) \times \left(\frac{1}{2} - \frac{t_{\pi/12}}{2} - \gamma, \frac{1}{2} + \frac{t_{\pi/12}}{2} + \gamma\right)$$

$$\subset (-2,1) \times (-1,2)$$

contains the points $(-1,1)$ and $(0,0)$. The function

$$g_1^{4,4}(u,v) = u^2 + (1+t)\,u + v^2 - (1-t)\,v$$

evaluated at these points is non-negative: $g_1^{4,4}(-1,1) = g_1^{4,4}(0,0) = 0$.

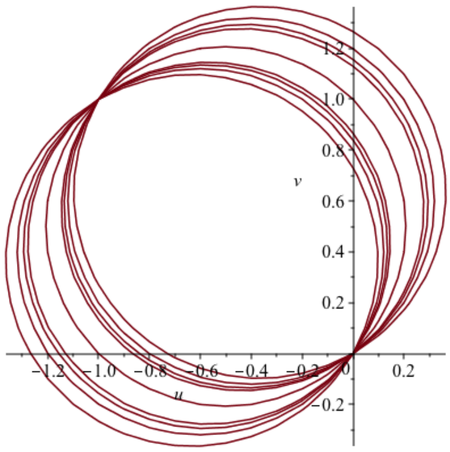

Figure 4.19: The level curves $g_1^{4,4}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

For $p = j_{11}(0)$, we have $a = -1$ and $b = -t \in \left[-t_{\pi/12}, t_{\pi/12}\right]$. The bounding box

$$(-1-\gamma, -1+\gamma) \times \left(-t_{\pi/12} - \gamma, t_{\pi/12} + \gamma\right) \subset (-2,0) \times (-1,1)$$

contains no points $(u,v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$.

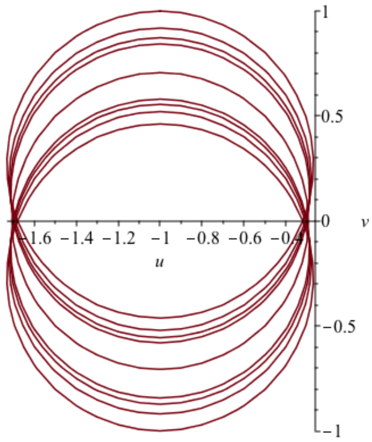

Figure 4.20: The level curves $g_{11}^{4,4}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

For $p = j_{111}(0)$, we have $a = \frac{1}{2}(t-1) \in \left[-\frac{1}{2} - \frac{t_{\pi/12}}{2}, -\frac{1}{2} + \frac{t_{\pi/12}}{2}\right]$ and $b = -\frac{1}{2}(t+1) \in \left[-\frac{1}{2} - \frac{t_{\pi/12}}{2}, -\frac{1}{2} + \frac{t_{\pi/12}}{2}\right]$. The bounding box

$$\left(-\frac{1}{2} - \frac{t_{\pi/12}}{2} - \gamma, -\frac{1}{2} + \frac{t_{\pi/12}}{2} + \gamma\right) \times \left(-\frac{1}{2} - \frac{t_{\pi/12}}{2} - \gamma, -\frac{1}{2} + \frac{t_{\pi/12}}{2} + \gamma\right)$$

$$\subset (-2, 1) \times (-2, 1)$$

contains the points $(-1, -1)$ and $(0, 0)$. The function

$$g_{111}^{4,4}(u, v) = u^2 + (1-t)\,u + v^2 + (1+t)\,v$$

evaluated at these points is non-negative: $g_{111}^{4,4}(-1, -1) = g_{111}^{4,4}(0, 0) = 0$.

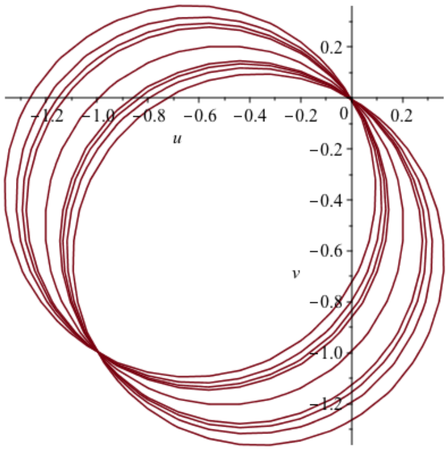

Figure 4.21: The level curves $g_{111}^{4,4}(u, v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

Therefore, as $g_p^{4,4}(u, v) \geq 0$ for all p, we have that

$$\left|f_p(x, y)\right|^2 = \left|p + xv_1 + yv_2\right|^2 \geq 2r^2 \geq 4$$

under the assumption that $r \geq \sqrt{2}$. That is, $\left|f_p(x, y)\right| \geq 2$ for all $f \in \Lambda \backslash \{\mathrm{Id}\}$. Hence the conditions of Lemma 3.2.4 are satisfied, and we can conclude that the complex hyperbolic ultra-parallel $[m, m, 0; 4, 4, 2]$-triangle group is discrete for

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \text{and} \quad m \geq \log\left(3 + 2\sqrt{2}\right).$$

$\square$

4.5 The case $[m, m, 0; 2, 6, 2]$

Proposition 4.5.1. *Every Heisenberg translation in $\Gamma' = \langle \iota_1, \iota_2 \rangle$ is of the form $T_1^x T_2^y H^z$, where T_1 and T_2 are Heisenberg translations, H is a vertical Heisenberg translation and $x, y, z \in \mathbb{Z}$. Every vertical Heisenberg translation in Γ' is of the form $H^z, z \in \mathbb{Z}$. In particular, the shortest non-trivial vertical translations in Γ' are*

$$H^{\pm 1} = (\iota_{12})^{\pm 3}.$$

Proof. The complex reflections ι_1 and ι_2 are of the form (4.1) with

$$r = \cosh(m/2), \quad \varphi_1 = re^{i\theta}, \quad \varphi_2 = -re^{-i\theta}, \quad \delta = 1 - \mu,$$

$$\phi = 1 - \lambda, \quad \mu = \exp(i \cdot \pi) \quad \text{and} \quad \lambda = \exp(i \cdot \pi/3).$$

As μ is a second root of unity and λ is a sixth root of unity, we will obtain Heisenberg translations with words containing x ι_1 and y ι_2 where $\frac{y}{3} + x \equiv 0 \mod 2$. Straightforward computation shows that the elements $\iota_{1222}, \iota_{2122}, \iota_{2212}, \iota_{2221}$ and ι_{121212} are Heisenberg translations. Let $\mathcal{T}$ be the group generated by these 5 Heisenberg translations. The group $\mathcal{T}$ is generated by $T_1 = \iota_{2122}, T_2 = \iota_{2212}$ and $H = \iota_{121212}$ since all other generators can be expressed in terms of T_1, T_2 and H:

$$\iota_{1222} = HT_2^{-1}T_1, \quad \text{and} \quad \iota_{2221} = T_1^{-1}T_2H^{-1}.$$

The reduced length 4 words which are not Heisenberg translations can be expressed in terms of the generators, T_1, T_2, H and a remainder term of length at most 3:

$$\iota_{1212} = HT_2^{-1}\iota_{22}, \;\; \iota_{1221} = HT_2^{-1}T_1T_2^{-1}\iota_{22}, \;\; \iota_{2121} = T_1T_2^{-1}\iota_{22} \quad \text{and} \quad \iota_{2222} = T_1^{-1}\iota_{21}.$$

Therefore, given any reduced word in ι_1 and ι_2, we are able to break it down into a sequence of Heisenberg translations T_1, T_2, H and their inverses, followed by a word of length at most 3. Hence $\mathcal{T}$ contains all Heisenberg translations in $\Gamma' = \langle \iota_1, \iota_2 \rangle$. Direct computation shows that T_k for $k = 1, 2, 3$, where $T_3 = H$, is a Heisenberg translation by (ξ_k, ν_k), where

$$\xi_1 = 2r(1 + i\sqrt{3})\cos(\theta) \text{ and } \nu_1 = 8\sqrt{3}r^2\cos^2(\theta) + 8r^2\sin(\theta)\cos(\theta),$$
$$\xi_2 = -2r(1 - i\sqrt{3})\cos(\theta) \text{ and } \nu_2 = 8\sqrt{3}r^2\cos^2(\theta) - 8r^2\sin(\theta)\cos(\theta),$$
$$\xi_3 = 0 \text{ and } \nu_2 = 16\sqrt{3}r^2\cos^2(\theta).$$

Computing the commutator $[T_2, T_1] = T_2^{-1}T_1^{-1}T_2T_1$ we obtain the Heisenberg translation of the form (4.5) which is a vertical translation in $\mathcal{N}$. Substituting ξ_1 and ξ_2 into (4.5) we have the vertical Heisenberg translation by $(0, 32\sqrt{3}r^2\cos^2(\theta))$. That is, $[T_2, T_1] = H^2$.

The group $\mathcal{T}$ has the presentation

$$\langle T_1, T_2, H \mid [T_2, T_1] = H^2, [H, T_1] = [H, T_2] = 1 \rangle.$$

Note that $\mathcal{T}$ is isomorphic to N_2 (see Definition 2.1.18). Any vertical translation in Γ' belongs to the subgroup $\mathcal{T}$. Using the identities

$$T_1H = HT_1, \quad T_2H = HT_2 \quad \text{and} \quad T_2T_1 = T_1T_2H^2$$

every element of N_2 can be written in the form $T_1^xT_2^yH^z$ for some $x, y, z \in \mathbb{Z}$. If we project to $\mathbb{C} \times \{0\}$, the element $T_1^xT_2^yH^z$ acts as a translation by $xv_1 + yv_2$. That is, this element is a vertical translation if and only if it is a power of H. We can see that $H^{\pm 1} = (\iota_{12})^{\pm 3}$. Note that this is inline with the results of Dekimpe [1] (Chapter 7, Case 16). $\square$

Proposition 4.5.2. *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 6, 2]$*-triangle group is discrete if the following conditions on the angular invariant* α *and on* m *are satisfied:*

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \textit{and} \quad m \geq \log\left(7 + 4\sqrt{3}\right).$$

Proof. We will use Lemma 3.2.4. To satisfy the second part of Lemma 3.2.4, we need the displacement of every vertical translation $H^z, z \neq 0$, to be at least the height of the spinal sphere, i.e.

$$16\sqrt{3}r^2\cos^2(\theta) \geq 2 \Leftrightarrow r^2\cos^2(\theta) \geq \frac{1}{8\sqrt{3}}.$$

By our assumption, $\cos(\alpha) \leq -\frac{\sqrt{3}}{2}$, hence $|\theta| \leq \frac{\pi}{12}$. We also have that $m = 2\cosh^{-1}(r) \geq \log\left(7 + 4\sqrt{3}\right)$ hence $r \geq 2$. So we have

$$r^2\cos^2(\theta) \geq 2^2 \cdot \left(\frac{1 + \sqrt{3}}{2\sqrt{2}}\right)^2 = 2 + \sqrt{3} > \frac{1}{8\sqrt{3}},$$

hence the condition $\left|h(0)\right| \geq 2$ is satisfied for all vertical translation $h \in \Gamma' \backslash \{\mathrm{Id}\}$.

To satisfy the first part of Lemma 3.2.4, we project ι_1 and ι_2 to $\mathbb{C}$ to obtain rotations j_1 and j_2 of $\mathbb{C} \times \{0\}$ through π and $\frac{\pi}{3}$ around φ_1 and φ_2 respectively. We can write every element $f \in \Lambda$ as a word in the generators j_1 and $j_2^{\pm 1,2,3}$. Using the relations $j_2^{-3} = j_2^3, j_2^{-2} = j_2^4$ and $j_2^{-1} = j_2^5$, we can rewrite every element f as a word in terms of j_1 and j_2.

Figure 4.22 shows the points $f(0)$ for all reduced words f of length up to 6 in the case $r = 1$ and $\theta = 0$.

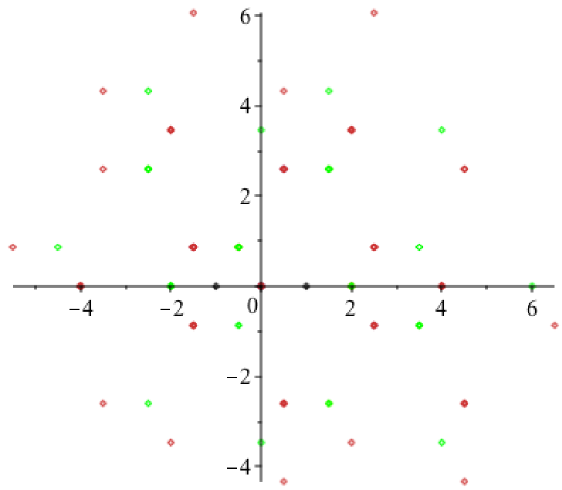

Figure 4.22: Points $f(0)$ for all words f up to length 6.

Projecting the Heisenberg translations to $\mathbb{C}$ we obtain Euclidean translations. The explicit formulas for the Euclidean translations are as follows

$$\begin{aligned}
j_{1222}(z) &= z + 2\left(\varphi_1 - \varphi_2\right), \\
j_{2122}(z) &= z + \left(1 + i\sqrt{3}\right)\left(\varphi_1 - \varphi_2\right), \\
j_{2212}(z) &= z - \left(1 - i\sqrt{3}\right)\left(\varphi_1 - \varphi_2\right), \\
j_{2221}(z) &= z - 2\left(\varphi_1 - \varphi_2\right).
\end{aligned}$$

Remark 4.5.3. The remaining map $j_{121212}(z)$ is equal to the identity map.
These translations generate the subgroup of all translations in the group Λ. This subgroup is generated by two translations

$$j_{2122}(z) = z + v_1 \quad \text{and} \quad j_{2212}(z) = z + v_2,$$

where $v_1 = \left(1 + i\sqrt{3}\right)(\varphi_1 - \varphi_2)$ and $v_2 = -\left(1 - i\sqrt{3}\right)(\varphi_1 - \varphi_2)$.

Under the translations, we will be able to break down any element of Λ, written as a word in the generators j_1 and j_2, into a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word of length at most 3, so that every point in the orbit of 0 under Λ is of the form

$$f_p(x, y) := p + xv_1 + yv_2,$$

where $p = w(0)$ for some word w of length at most 3 and $x, y \in \mathbb{Z}$.

Remark 4.5.4. For words of length 4 which are not translation maps, notice that these maps are equal to maps of greater length which can be broken down into a sequence of translations followed by a word of length at most 3:

$$
\begin{aligned}
j_{2222}(z) &= j_{2122}^{-1}\left(j_{21}(z)\right),\\
j_{1221}(z) &= j_{2212}^{-1}\left(j_{2122}\left(j_{2212}^{-1}\left(j_{22}(z)\right)\right)\right),\\
j_{1212}(z) &= j_{2212}^{-1}\left(j_{22}(z)\right),\\
j_{2121}(z) &= j_{2122}\left(j_{2212}^{-1}\left(j_{22}(z)\right)\right).
\end{aligned}
$$

Therefore the form of the orbit of 0 under Λ is still valid.

We can further reduce the choices of p. Notice that for $T_1 = \iota_{2122}, T_2 = \iota_{2212}$ and $H = \iota_{121212}$, we have

$$
\iota_{12} = HT_2^{-1}\iota_{21}, \quad \iota_{121} = HT_2^{-1}\iota_2, \quad \iota_{122} = HT_2^{-1}T_1T_2^{-1}\iota_{221},
$$
$$
\iota_{212} = T_1T_2^{-1}\iota_{221} \quad \text{and} \quad \iota_{222} = T_1^{-1}T_2H^{-1}\iota_1.
$$

We are able to rewrite the translation element of each map in the form $T_1^x T_2^y H^z$, for $x, y, z \in \mathbb{Z}$, so that every map above can be written as $T_1^x T_2^y H^z \cdot \iota_{a_1}$ for $x, y, z \in \mathbb{Z}$ and rotation element ι_{a_1}. Projecting to $\mathbb{C} \times \{0\}$ (i.e. setting H to the identity map, Id), we obtain the maps

$$
\begin{aligned}
j_{12}(z) &= j_{2212}^{-1}\left(j_{21}(z)\right),\\
j_{121}(z) &= j_{2212}^{-1}\left(j_{2}(z)\right),\\
j_{122}(z) &= j_{2212}^{-2}\left(j_{2122}\left(j_{221}(z)\right)\right),\\
j_{212}(z) &= j_{2122}\left(j_{2212}^{-1}\left(j_{221}(z)\right)\right),\\
j_{222}(z) &= j_{2122}^{-1}\left(j_{2212}\left(j_{1}(z)\right)\right).
\end{aligned}
$$

Therefore, we can write any element $f \in \Lambda$ as a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word $p = w(0)$ for some word $w \in \{\mathrm{Id}, j_1, j_2, j_{21}, j_{22}, j_{221}\}$.

To apply lemma 3.2.4, we want to show that

$$
\left|f_p(x, y)\right|^2 = |p + xv_1 + yv_2|^2 \geq r^2 \geq 4
$$

for all possible choices of p and for all $x, y \in \mathbb{Z}$. Using (4.3) we have

$$
\left|f_p(x, y)\right|^2 = |p|^2 + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + x^2|v_1|^2 + y^2|v_2|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2}).
$$

Calculating the terms that do not depend on p, we have

$$
\begin{aligned}
v_1 &= (1 + i\sqrt{3})\,(\varphi_1 - \varphi_2) = 2r(1 + i\sqrt{3})\cos(\theta)\\
v_2 &= -(1 - i\sqrt{3})\,(\varphi_1 - \varphi_2) = -2r(1 - i\sqrt{3})\cos(\theta).
\end{aligned}
$$

We then obtain

$$|v_1|^2 = |v_2|^2 = 16r^2\cos^2(\theta), \quad v_1\bar{v_2} = 8r^2\cos^2(\theta) - i \cdot 8\sqrt{3}r^2\cos^2(\theta)$$

$$\text{and} \quad \operatorname{Re}(v_1\bar{v_2}) = 8r^2\cos^2(\theta).$$

Hence,

$$x^2|v_1|^2 + y^2|v_2|^2 + 2xy\operatorname{Re}(v_1\bar{v_2}) = 16r^2\cos^2(\theta)\left(x^2 + xy + y^2\right).$$

So we have

$$\left|f_p(x,y)\right|^2 = 16r^2\cos^2(\theta)\left(x^2 + xy + y^2\right) + 2x\operatorname{Re}(p\bar{v_1}) + 2y\operatorname{Re}(p\bar{v_2}) + |p|^2.$$

We want to minimise this expression. In order to do so, we apply the coordinate change (4.4). Under this coordinate change, we have

$$\begin{aligned}
\left|f_p(u,v)\right|^2 &= 16r^2\cos^2(\theta)\left(\left(\frac{v-u}{2}\right)^2 + \left(\frac{v-u}{2}\right)\left(\frac{u+v}{2}\right) + \left(\frac{u+v}{2}\right)^2\right) \\
&\quad + (v-u)\operatorname{Re}(p\bar{v_1}) + (u+v)\operatorname{Re}(p\bar{v_2}) + |p|^2 \\
&= 4r^2\cos^2(\theta)\left(v^2 - 2uv + u^2 + v^2 - u^2 + u^2 + 2uv + v^2\right) \\
&\quad + v\left(\operatorname{Re}(p\bar{v_1}) + \operatorname{Re}(p\bar{v_2})\right) + u\left(\operatorname{Re}(p\bar{v_2}) - \operatorname{Re}(p\bar{v_1})\right) + |p|^2 \\
&= 4r^2\cos^2(\theta)\left(u^2 + 3v^2\right) + v\left(\operatorname{Re}(p(\bar{v_1} + \bar{v_2}))\right) + u\left(\operatorname{Re}(p(\bar{v_2} - \bar{v_1}))\right) + |p|^2.
\end{aligned}$$

This can be rewritten as

$$\left|f_p(u,v)\right|^2 = 4r^2\cos^2(\theta)\left((u-a)^2 + 3(v-b)^2\right),$$

where

$$\begin{aligned}
a &= -\frac{\operatorname{Re}(p(\bar{v_2} - \bar{v_1}))}{8r^2\cos^2(\theta)} = \frac{\operatorname{Re}(p)}{2r\cos(\theta)}, \\
b &= -\frac{\operatorname{Re}(p(\bar{v_1} + \bar{v_2}))}{24r^2\cos^2(\theta)} = -\frac{\operatorname{Im}(p)}{2\sqrt{3}r\cos(\theta)}, \\
a^2 + 3b^2 &= \frac{|p|^2}{4r^2\cos^2(\theta)}.
\end{aligned}$$

Our aim is to show that $|p + xv_1 + yv_2|^2 \geq r^2$ for all $(x,y) \in \mathbb{Z}^2$ excluding the case $p = x = y = 0$ which corresponds to the identity case. That is,

$$4r^2\cos^2(\theta)\left((u-a)^2 + 3(v-b)^2\right) \geq r^2 \Leftrightarrow (u-a)^2 + 3(v-b)^2 \geq \frac{\sec^2(\theta)}{4}$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$, excluding the case $a = b = u = v = 0$. Notice that this inequality is always satisfied if $|u - a| \geq \frac{\sec(\theta)}{2}$ or $|v - b| \geq \frac{\sec\theta}{2\sqrt{3}}$ and so we only need to check that

$$g_p^{2,6}(u, v) = (u - a)^2 + 3(v - b)^2 - \frac{\sec^2(\theta)}{4} \geq 0$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$ inside the bounding box

$$\left(a - \frac{\sec(\theta)}{2}, a + \frac{\sec(\theta)}{2}\right) \times \left(b - \frac{\sec(\theta)}{2\sqrt{3}}, b + \frac{\sec(\theta)}{2\sqrt{3}}\right).$$

For the choices of p, we look at the words $w \in \{\mathrm{Id}, j_1, j_2, j_{21}, j_{22}, j_{221}\}$. We have 6 possibilities: the identity Id and

$$\begin{aligned}
j_1(z) &= -z + 2\varphi_1, \\
j_2(z) &= \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) z + \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \varphi_2, \\
j_{21}(z) &= -\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) z + (1 + i\sqrt{3})\varphi_1 + \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \varphi_2, \\
j_{22}(z) &= -\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) z + \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right) \varphi_2, \\
j_{221}(z) &= \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) z - (1 - i\sqrt{3})\varphi_1 + \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right) \varphi_2.
\end{aligned}$$

Figure 4.23 shows the points $w(0)$ for all words $w \in \{\mathrm{Id}, j_1, j_2, j_{21}, j_{22}, j_{221}\}$ in the case $r = 1$ and $\theta = 0$.

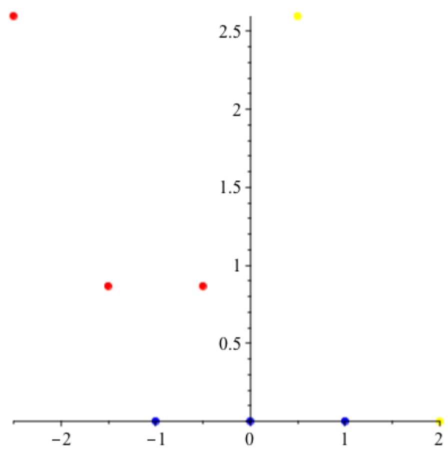

Figure 4.23: Points $w(0)$ for all words $w \in \{\mathrm{Id}, j_1, j_2, j_{21}, j_{22}, j_{221}\}$.

Evaluating these words at $z = 0$, we have 6 possible choices for p:

$$
\begin{aligned}
& p = 0, \quad p = 2\varphi_1, \quad p = \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\varphi_2, \\
& p = (1 + i\sqrt{3})\varphi_1 + \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\varphi_2, \\
& p = \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)\varphi_2, \\
& p = -(1 - i\sqrt{3})\varphi_1 + \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)\varphi_2.
\end{aligned}
$$

For each choice of p, the following table shows the values of a, b and $a^2 + 3b^2$ in terms of $t = \tan(\theta)$:

$p = w(0)$	a	b	$a^2 + 3b^2$
Id	0	0	0
$j_1(0)$	1	$-\frac{t}{\sqrt{3}}$	$t^2 + 1$
$j_2(0)$	$\frac{1}{4}\left(t\sqrt{3} - 1\right)$	$-\frac{1}{4\sqrt{3}}\left(t + \sqrt{3}\right)$	$\frac{1}{4}\left(t^2 + 1\right)$
$j_{21}(0)$	$\frac{1}{4}\left(1 - t\sqrt{3}\right)$	$-\frac{1}{4}\left(t\sqrt{3} + 3\right)$	$\frac{1}{4}\left(3t^2 + 4t\sqrt{3} + 7\right)$
$j_{22}(0)$	$\frac{1}{4}\left(t\sqrt{3} - 3\right)$	$-\frac{1}{4}\left(t\sqrt{3} + 1\right)$	$\frac{3}{4}\left(t^2 + 1\right)$
$j_{221}(0)$	$-\frac{1}{4}\left(t\sqrt{3} + 5\right)$	$-\frac{1}{4\sqrt{3}}\left(t + 3\sqrt{3}\right)$	$\frac{1}{4}\left(t^2 + 4t\sqrt{3} + 13\right)$

Under the assumption $|\theta| \leq \frac{\pi}{12}$ we have that

$$
t = \tan(\theta) \in \left[\sqrt{3} - 2, 2 - \sqrt{3}\right] \quad \text{and} \quad \sec(\theta) \in \left[1, \sqrt{2}\left(\sqrt{3} - 1\right)\right].
$$

So for each p, we need to calculate the bounds on a, b and the size of the bounding box

$$
\left(\min(a) - \frac{\sqrt{3} - 1}{\sqrt{2}}, \max(a) + \frac{\sqrt{3} - 1}{\sqrt{2}}\right) \times \left(\min(b) - \frac{\sqrt{3} - 1}{\sqrt{6}}, \max(b) + \frac{\sqrt{3} - 1}{\sqrt{6}}\right).
$$

We then need to show that

$$
\begin{aligned}
g_p^{2,6}(u, v) &= (u - a)^2 + 3(v - b)^2 - \frac{\sec^2(\theta)}{4} \\
&= u^2 - 2au + v^2 - 6bv + (a^2 + 3b^2) - \left(\frac{t^2 + 1}{4}\right) \geq 0
\end{aligned}
$$

for all $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$ inside the bounding box. For the purposes of the following calculations, we will denote $t_{\pi/12} = \tan\left(\frac{\pi}{12}\right), \gamma_1 = \frac{\sqrt{3}-1}{\sqrt{2}}$ and $\gamma_2 = \frac{\sqrt{3}-1}{\sqrt{6}}$.

For $p = \mathrm{Id}$, we have $a = 0$ and $b = 0$. The bounding box

$$(-\gamma_1, \gamma_1) \times (-\gamma_2, \gamma_2) \subset (-1, 1) \times (-1, 1)$$

contains the point $(0, 0)$ which corresponds to the excluded case $f = \mathrm{Id}$.

For $p = j_1(0)$, we have $a = 1$ and $b = -\frac{t}{\sqrt{3}} \in \left[-\frac{t_{\pi/12}}{\sqrt{3}}, \frac{t_{\pi/12}}{\sqrt{3}}\right]$. The bounding box

$$(1 - \gamma_1, 1 + \gamma_1) \times \left(-\frac{t_{\pi/12}}{\sqrt{3}} - \gamma_2, \frac{t_{\pi/12}}{\sqrt{3}} + \gamma_2\right) \subset (0, 2) \times (-1, 1)$$

contains no points $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$.

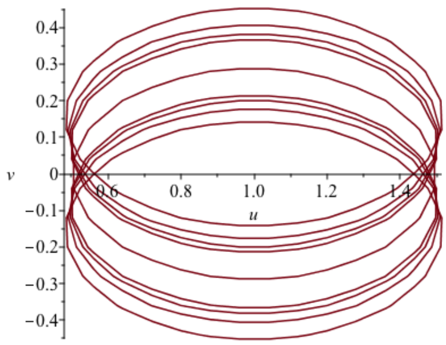

Figure 4.24: The level curves $g_1^{2,6}(u, v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

For $p = j_2(0)$, we have $a = \frac{1}{4}\left(t\sqrt{3} - 1\right) \in \left[-\frac{1}{4} - \frac{t_{\pi/12}\sqrt{3}}{4}, -\frac{1}{4} + \frac{t_{\pi/12}\sqrt{3}}{4}\right]$ and $b = -\frac{1}{4\sqrt{3}}\left(t + \sqrt{3}\right) \in \left[-\frac{1}{4} - \frac{t_{\pi/12}}{4\sqrt{3}}, -\frac{1}{4} + \frac{t_{\pi/12}}{4\sqrt{3}}\right]$. The bounding box

$$\left(-\frac{1}{4} - \frac{t_{\pi/12}\sqrt{3}}{4} - \gamma_1, -\frac{1}{4} + \frac{t_{\pi/12}\sqrt{3}}{4} + \gamma_1\right) \times \left(-\frac{1}{4} - \frac{t_{\pi/12}}{4\sqrt{3}} - \gamma_2, -\frac{1}{4} + \frac{t_{\pi/12}}{4\sqrt{3}} + \gamma_2\right)$$
$$\subset (-1, 1) \times (-1, 1)$$

contains the point $(0, 0)$. The function

$$g_2^{2,6}(u, v) = u^2 + \left(\frac{1 - t\sqrt{3}}{2}\right) u + 3v^2 + \left(\frac{3 - t\sqrt{3}}{2}\right) v$$

evaluated at this point is non-negative: $g_2^{2,6}(0,0) = 0$.

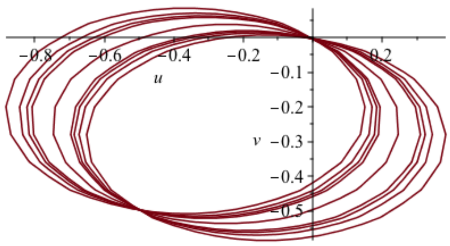

Figure 4.25: The level curves $g_2^{2,6}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

For $p = j_{21}(0)$, we have $a = \frac{1}{4}\left(1 - t\sqrt{3}\right) \in \left[\frac{1}{4} - \frac{t_{\pi/12}\sqrt{3}}{4}, \frac{1}{4} + \frac{t_{\pi/12}\sqrt{3}}{4}\right]$ and $b = -\frac{1}{4}\left(t\sqrt{3} + 3\right) \in \left[-\frac{3}{4} - \frac{t_{\pi/12}\sqrt{3}}{4}, -\frac{3}{4} + \frac{t_{\pi/12}\sqrt{3}}{4}\right]$. The bounding box

$$\left(\frac{1}{4} - \frac{t_{\pi/12}\sqrt{3}}{4} - \gamma_1, \frac{1}{4} + \frac{t_{\pi/12}\sqrt{3}}{4} + \gamma_1\right) \times \left(-\frac{3}{4} - \frac{t_{\pi/12}\sqrt{3}}{4} - \gamma_2, -\frac{3}{4} + \frac{t_{\pi/12}\sqrt{3}}{4} + \gamma_2\right)$$
$$\subset (-1,1) \times (-2,0)$$

contains no points $(u,v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$.

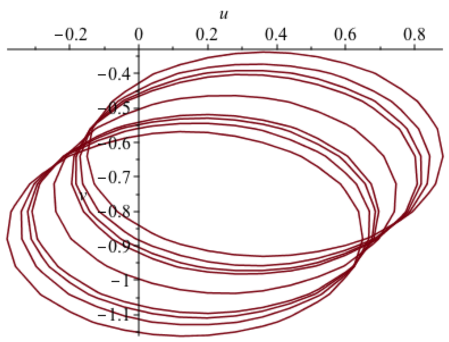

Figure 4.26: The level curves $g_{21}^{2,6}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

For $p = j_{22}(0)$, we have $a = \frac{1}{4}\left(t\sqrt{3} - 3\right) \in \left[-\frac{3}{4} - \frac{t_{\pi/12}\sqrt{3}}{4}, -\frac{3}{4} + \frac{t_{\pi/12}\sqrt{3}}{4}\right]$ and $b = -\frac{1}{4}\left(t\sqrt{3} + 1\right) \in \left[-\frac{1}{4} - \frac{t_{\pi/12}\sqrt{3}}{4}, -\frac{1}{4} + \frac{t_{\pi/12}\sqrt{3}}{4}\right]$. The bounding box

$$\left(-\frac{3}{4} - \frac{t_{\pi/12}\sqrt{3}}{4} - \gamma_1, -\frac{3}{4} + \frac{t_{\pi/12}\sqrt{3}}{4} + \gamma_1\right) \times \left(-\frac{1}{4} - \frac{t_{\pi/12}\sqrt{3}}{4} - \gamma_2, -\frac{1}{4} + \frac{t_{\pi/12}\sqrt{3}}{4} + \gamma_2\right)$$
$$\subset (-2, 0) \times (-1, 1)$$

contains no points $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$.

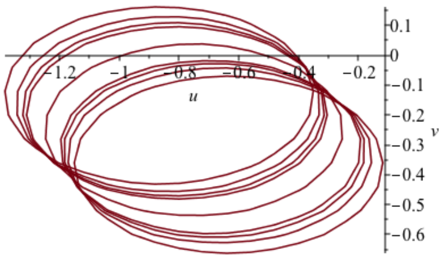

Figure 4.27: The level curves $g^{2,6}_{22}(u, v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

Finally for $p = j_{221}(0)$, we have $a = -\frac{1}{4}\left(t\sqrt{3} + 5\right) \in \left[-\frac{5}{4} - \frac{t_{\pi/12}\sqrt{3}}{4}, -\frac{5}{4} + \frac{t_{\pi/12}\sqrt{3}}{4}\right]$ and $b = -\frac{1}{4\sqrt{3}}\left(t + 3\sqrt{3}\right) \in \left[-\frac{3}{4} - \frac{t_{\pi/12}}{4\sqrt{3}}, -\frac{3}{4} + \frac{t_{\pi/12}}{4\sqrt{3}}\right]$. The bounding box

$$\left(-\frac{5}{4} - \frac{t_{\pi/12}\sqrt{3}}{4} - \gamma_1, -\frac{5}{4} + \frac{t_{\pi/12}\sqrt{3}}{4} + \gamma_1\right) \times \left(-\frac{3}{4} - \frac{t_{\pi/12}}{4\sqrt{3}} - \gamma_2, -\frac{3}{4} + \frac{t_{\pi/12}}{4\sqrt{3}} + \gamma_2\right)$$
$$\subset (-2, 0) \times (-2, 0)$$

contains the point $(-1, -1)$. The function

$$g^{2,6}_{221}(u, v) = u^2 - \left(\frac{5 + t\sqrt{3}}{2}\right) u + 3v^2 + \left(\frac{9 + t\sqrt{3}}{2}\right) v + 3 + t\sqrt{3}$$

evaluated at this point is non-negative: $g^{2,6}_{221}(-1, -1) = 0$.

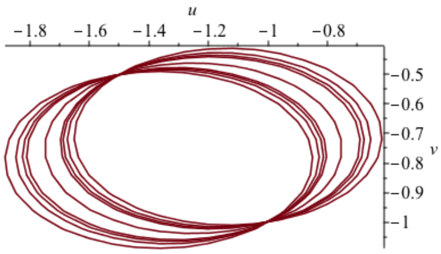

Figure 4.28: The level curves $g_{221}^{2,6}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

Therefore, as $g_p^{2,6}(u,v) \geq 0$ for all p, we have that

$$\left|f_p(x,y)\right|^2 = \left|p + xv_1 + yv_2\right|^2 \geq r^2 \geq 4$$

under the assumption that $r \geq 2$. That is, $\left|f_p(x,y)\right| \geq 2$ for all $f \in \Lambda \backslash \{\mathrm{Id}\}$. Hence the conditions of Lemma 3.2.4 are satisfied, and we can conclude that the complex hyperbolic ultra-parallel $[m, m, 0; 2, 6, 2]$-triangle group is discrete for

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \text{and} \quad m \geq \log\left(7 + 4\sqrt{3}\right).$$

□

4.6 The case $[m, m, 0; 3, 6, 2]$

Proposition 4.6.1. *Every Heisenberg translation in* $\Gamma' = \langle \iota_1, \iota_2 \rangle$ *is of the form* $T_1^x T_2^y H^z$*, where* T_1 *and* T_2 *are Heisenberg translations,* H *is a vertical Heisenberg translation and* $x, y, z \in \mathbb{Z}$*. Every vertical Heisenberg translation in* Γ' *is of the form* $H^z, z \in \mathbb{Z}$*. In particular, the shortest non-trivial vertical translations in* Γ' *are*

$$H^{\pm 1} = (\iota_{12})^{\pm 2}.$$

Proof. The complex reflections ι_1 and ι_2 are of the form (4.1) with

$$r = \cosh(m/2), \quad \varphi_1 = re^{i\theta}, \quad \varphi_2 = -re^{-i\theta}, \quad \delta = 1 - \mu,$$

$$\phi = 1 - \lambda, \quad \mu = \exp(i \cdot 2\pi/3) \quad \text{and} \quad \lambda = \exp(i \cdot \pi/3).$$

As μ is a third root of unity and λ is a sixth root of unity, we will obtain Heisenberg translations with words containing x ι_1 and y ι_2 where $\frac{y}{2} + x \equiv 0 \mod 3$. Straightforward computation shows that the elements $\iota_{1122}, \iota_{1212}, \iota_{1221}, \iota_{2112}, \iota_{2121}$ and ι_{2211} are

Heisenberg translations. Consider the elements $T_1 = \iota_{1122}, T_2 = \iota_{2211}$ and $H = \iota_{1212}$. Direct computation shows that T_k for $k = 1, 2, 3$, where $T_3 = H$, is a Heisenberg translation by (ξ_k, ν_k), where

$$\begin{aligned}
&\xi_1 = r(3 + i\sqrt{3})\cos(\theta) \text{ and } \nu_1 = 12r^2 \sin(\theta)\cos(\theta),\\
&\xi_2 = -r(3 - i\sqrt{3})\cos(\theta) \text{ and } \nu_2 = -12r^2 \sin(\theta)\cos(\theta),\\
&\xi_3 = 0 \text{ and } \nu_3 = 8\sqrt{3}r^2 \cos^2(\theta).
\end{aligned}$$

Computing the commutator $[T_2, T_1] = T_2^{-1}T_1^{-1}T_2T_1$ we obtain the Heisenberg translation of the form (4.5) which is a vertical translation in $\mathfrak{N}$. Substituting ξ_1 and ξ_2 into (4.5) we have the vertical Heisenberg translation by $(0, 24\sqrt{3}r^2\cos^2(\theta))$. That is, $[T_2, T_1] = H^3$.

Let $\mathcal{T}$ be the group generated by the 6 Heisenberg translations above. The group $\mathcal{T}$ is generated by $T_1 = \iota_{1122}, T_2 = \iota_{2211}$ and $H = \iota_{1212}$ since all other generators can be expressed in terms of T_1, T_2 and H:

$$\iota_{1221} = T_2^{-1}T_1^{-1}, \;\; \iota_{2112} = T_1T_2H^2 \quad \text{and} \quad \iota_{2121} = H.$$

Remark 4.6.2. It is difficult to show that the words ι_{1212} and ι_{2121} are equal by expressing ι_{2121} in terms of T_1, T_2 and H. However, direct computation shows that the word ι_{2121} is a Heisenberg translation by (ξ, ν), where

$$\xi = 0 \text{ and } \nu = 8\sqrt{3}r^2\cos^2(\theta).$$

This is the exact Heisenberg translation for the word ι_{1212}. Given that $\iota_{1212} = \iota_{2121}$ we are able to show the equality of the element ι_{2112}:

$$\begin{aligned}
T_1 \cdot T_2 \cdot H^2 &= \iota_{1122} \cdot \iota_{2211} \cdot \iota_{1212} \cdot \iota_{1212}\\
&= \iota_{112222\underline{111}2121212} = \iota_{1122222121212}\\
&= \iota_{1122222} \cdot \underline{\iota_{1212}} \cdot \iota_{12}\\
&= \iota_{1122222} \cdot \underline{\iota_{2121}} \cdot \iota_{12}\\
&= \iota_{11\underline{222222}12112} = \iota_{\underline{111}2112} = \iota_{2112}.
\end{aligned}$$

The reduced length 4 words which are not Heisenberg translations can be expressed in terms of the generators T_1, T_2, H and a remainder term of length at most 3:

$$\iota_{1121} = T_2^{-1}H\iota_2, \;\; \iota_{1211} = H^{-1}T_2^{-1}T_1^{-1}\iota_2, \;\; \iota_{1222} = T_2^{-1}H^{-1}\iota_{121}, \;\; \iota_{2122} = H\iota_{112}$$

$$\iota_{2212} = T_2\iota_{112}, \;\; \iota_{2221} = T_2HT_1^{-1}\iota_{112} \quad \text{and} \quad \iota_{2222} = T_1^{-1}\iota_{11}.$$

Remark 4.6.3. To show that $\iota_{1121} = T_2^{-1}H\iota_2$, $\iota_{1211} = H^{-1}T_2^{-1}T_1^{-1}\iota_2$ and $\iota_{2122} = H\iota_{112}$, one must use the fact that $\iota_{1212} = \iota_{2121}$.

Therefore, given any reduced word in ι_1 and ι_2, we are able to break it down into a sequence of Heisenberg translations T_1, T_2, H and their inverses, followed by a word of length at most 3. Hence $\mathcal{T}$ contains all Heisenberg translations in $\Gamma' = \langle \iota_1, \iota_2 \}$.

The group $\mathcal{T}$ has the presentation

$$\langle T_1, T_2, H \mid [T_2, T_1] = H^3, [H, T_1] = [H, T_2] = 1 \rangle.$$

Note that $\mathcal{T}$ is isomorphic to N_3 (see Definition 2.1.18). Any vertical translation in Γ' belongs to the subgroup $\mathcal{T}$. Using the identities

$$T_1 H = H T_1, \quad T_2 H = H T_2 \quad \text{and} \quad T_2 T_1 = T_1 T_2 H^3$$

every element of N_3 can be written in the form $T_1^x T_2^y H^z$ for some $x, y, z \in \mathbb{Z}$. If we project to $\mathbb{C} \times \{0\}$, the element $T_1^x T_2^y H^z$ acts as a translation by $xv_1 + yv_2$. That is, this element is a vertical translation if and only if it is a power of H. We can see that $H^{\pm 1} = (\iota_{12})^{\pm 2}$. Note that this is inline with the results of Dekimpe [1] (Chapter 7, Case 16). □

Proposition 4.6.4. *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 6, 2]$*-triangle group is discrete if the following conditions on the angular invariant* α *and on* m *are satisfied:*

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \textit{and} \quad m \geq \log\left(7 + 4\sqrt{3}\right).$$

Proof. We will use Lemma 3.2.4. To satisfy the second part of Lemma 3.2.4, we need the displacement of every vertical translation $H^z, z \neq 0$, to be at least the height of the spinal sphere, i.e.

$$8\sqrt{3} r^2 \cos^2(\theta) \geq 2 \Leftrightarrow r^2 \cos^2(\theta) \geq \frac{1}{4\sqrt{3}}.$$

By our assumption, $\cos(\alpha) \leq -\frac{\sqrt{3}}{2}$ and $m = 2\cosh^{-1}(r) \geq \log\left(7 + 4\sqrt{3}\right)$, hence, as in the previous section, we have that $|\theta| \leq \frac{\pi}{12}$ and $r \geq 2$. So we have

$$r^2 \cos^2(\theta) \geq 2^2 \cdot \left(\frac{1 + \sqrt{3}}{2\sqrt{2}}\right)^2 = 2 + \sqrt{3} > \frac{1}{4\sqrt{3}},$$

hence the condition $\left|h(0)\right| \geq 2$ is satisfied for all vertical translation $h \in \Gamma' \backslash \{\mathrm{Id}\}$.

To satisfy the first part of Lemma 3.2.4, we project ι_1 and ι_2 to $\mathbb{C}$ to obtain rotations j_1 and j_2 of $\mathbb{C} \times \{0\}$ through $\frac{2\pi}{3}$ and $\frac{\pi}{3}$ around φ_1 and φ_2 respectively. We can write every element $f \in \Lambda$ as a word in the generators $j_1^{\pm 1, 2}$ and $j_2^{\pm 1, 2, 3}$. Using the relations $j_1^{-2} = j_1, j_1^{-1} = j_1^2, j_2^{-3} = j_2^3, j_2^{-2} = j_2^4$ and $j_2^{-1} = j_2^5$, we can rewrite every element f as a word in terms of j_1 and j_2.

Figure 4.29 shows the points $f(0)$ for all reduced words f of length up to 6 in the case $r = 1$ and $\theta = 0$.

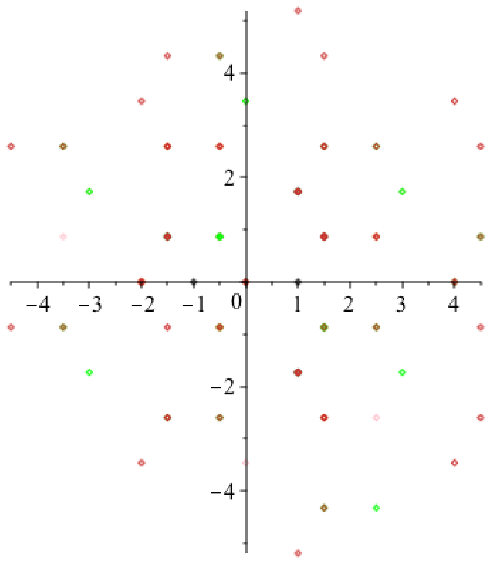

Figure 4.29: Points $f(0)$ for all words f up to length 6.

Projecting the Heisenberg translations to $\mathbb{C}$ we obtain Euclidean translations. The explicit formulas for the Euclidean translations are as follows

$$\begin{aligned} j_{1122}(z) &= z + \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2), \\ j_{2211}(z) &= z - \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2), \\ j_{1221}(z) &= z - i\sqrt{3}(\varphi_1 - \varphi_2), \\ j_{2112}(z) &= z + i\sqrt{3}(\varphi_1 - \varphi_2). \end{aligned}$$

Remark 4.6.5. The remaining maps $j_{1212}(z)$ and $j_{2121}(z)$ are equal to the identity map.

These translations generate the subgroup of all translations in the group Λ. This subgroup can be generated by two translations

$$j_{1122}(z) = z + v_1 \quad \text{and} \quad j_{2211}(z) = z + v_2,$$

where $v_1 = \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2)$ and $v_2 = -\left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2)$.

Under the translations, we will be able to break down any element of Λ, written as a word in the generators j_1 and j_2, into a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word of length at most 3, so that every point in the orbit of 0 under Λ is of the form

$$f_p(x, y) := p + xv_1 + yv_2,$$

where $p = w(0)$ for some word w of length at most 3 and $x, y \in \mathbb{Z}$.

Remark 4.6.6. For words of length 4 which are not translation maps, notice that these maps are equal to maps of greater length which can be broken down into a sequence of translations followed by a word of length at most 3:

$$\begin{aligned}
j_{1121}(z) &= j_{2211}^{-1}\left(j_2(z)\right),\\
j_{1211}(z) &= j_{2211}^{-1}\left(j_{1122}^{-1}\left(j_2(z)\right)\right),\\
j_{1222}(z) &= j_{2211}^{-1}\left(j_{1122}^{-1}\left(j_{112}(z)\right)\right),\\
j_{2122}(z) &= j_{112}(z),\\
j_{2212}(z) &= j_{2211}\left(j_{112}(z)\right),\\
j_{2221}(z) &= j_{2211}\left(j_{1122}^{-1}\left(j_{112}(z)\right)\right),\\
j_{2222}(z) &= j_{1122}^{-1}\left(j_{2211}^{-1}\left(j_{221}(z)\right)\right).
\end{aligned}$$

Therefore the form of the orbit of 0 under Λ is still valid.

We can further reduce the choices of p. Notice that for $T_1 = \iota_{1122}, T_2 = \iota_{2211}$ and $H = \iota_{1212}$, we have that

$$\iota_1 = T_2^{-1}\iota_{22}, \quad \iota_{11} = T_2^{-1}\iota_{221}, \quad \iota_{12} = T_2^{-1}\iota_{222}, \quad \iota_{21} = HT_1\iota_{222},$$
$$\iota_{121} = HT_1^{-1}\iota_{112}, \quad \iota_{122} = T_2^{-1}T_1^{-1}T_2^{-1}\iota_{221} \quad \text{and} \quad \iota_{211} = HT_1T_2HT_1^{-1}\iota_{112}.$$

We are able to rewrite the translation element of each map in the form $T_1^xT_2^yH^z$, for $x, y, z \in \mathbb{Z}$, so that every map above can be written as $T_1^xT_2^yH^z \cdot \iota_{a_1}$ for $x, y, z \in \mathbb{Z}$ and rotation element ι_{a_1}. Projecting to $\mathbb{C} \times \{0\}$ (i.e. setting H to the identity map, Id), we obtain the maps

$$\begin{aligned}
j_1(z) &= j_{2211}^{-1}\left(j_{22}(z)\right),\\
j_{11}(z) &= j_{2211}^{-1}\left(j_{221}(z)\right),\\
j_{12}(z) &= j_{2211}^{-1}\left(j_{222}(z)\right),\\
j_{21}(z) &= j_{1122}\left(j_{222}(z)\right),\\
j_{121}(z) &= j_{1122}^{-1}\left(j_{112}(z)\right),\\
j_{122}(z) &= j_{2211}^{-2}\left(j_{1122}^{-1}\left(j_{221}(z)\right)\right),\\
j_{211}(z) &= j_{2211}\left(j_{112}(z)\right).
\end{aligned}$$

Therefore, we can write any element $f \in \Lambda$ as a sequence of translations by $\pm v_1$ and $\pm v_2$, followed by a word $p = w(0)$ for some word $w \in \{\mathrm{Id}, j_2, j_{22}, j_{112}, j_{221}, j_{222}\}$.

To apply lemma 3.2.4, we want to show that

$$\left|f_p(x,y)\right|^2 = \left|p + xv_1 + yv_2\right|^2 \geq r^2 \geq 4$$

for all possible choices of p and for all $x, y \in \mathbb{Z}$. Using (4.3) we have

$$\left|f_p(x,y)\right|^2 = \left|p\right|^2 + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + x^2\left|v_1\right|^2 + y^2\left|v_2\right|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2}).$$

Calculating the terms that do not depend on p, we have

$$\begin{aligned} v_1 &= \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2) = r(3 + i\sqrt{3})\cos(\theta) \\ v_2 &= -\left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)(\varphi_1 - \varphi_2) = r(-3 + i\sqrt{3})\cos(\theta). \end{aligned}$$

We then obtain

$$\left|v_1\right|^2 = \left|v_2\right|^2 = 12r^2\cos^2(\theta), \quad v_1\bar{v_2} = -r^2(3+i\sqrt{3})^2\cos^2(\theta) = -6r^2\cos^2(\theta)\left(1 + i\sqrt{3}\right)$$

$$\text{and} \quad \mathrm{Re}(v_1\bar{v_2}) = -6r^2\cos^2(\theta).$$

Hence,

$$x^2\left|v_1\right|^2 + y^2\left|v_2\right|^2 + 2xy\,\mathrm{Re}(v_1\bar{v_2}) = 12r^2\cos^2(\theta)\left(x^2 - xy + y^2\right).$$

So we have

$$\left|f_p(x,y)\right|^2 = 12r^2\cos^2(\theta)\left(x^2 - xy + y^2\right) + 2x\,\mathrm{Re}(p\bar{v_1}) + 2y\,\mathrm{Re}(p\bar{v_2}) + \left|p\right|^2.$$

We want to minimise this expression. In order to do so, we apply the coordinate change (4.4). Under this coordinate change, we have

$$\begin{aligned} \left|f_p(u,v)\right|^2 &= 12r^2\cos^2(\theta)\left(\left(\frac{v-u}{2}\right)^2 - \left(\frac{v-u}{2}\right)\left(\frac{u+v}{2}\right) + \left(\frac{u+v}{2}\right)^2\right) \\ &\quad + (v-u)\,\mathrm{Re}(p\bar{v_1}) + (u+v)\,\mathrm{Re}(p\bar{v_2}) + \left|p\right|^2 \\ &= 3r^2\cos^2(\theta)\left(v^2 - 2uv + u^2 - v^2 + u^2 + u^2 + 2uv + v^2\right) \\ &\quad + v\left(\mathrm{Re}(p\bar{v_1}) + \mathrm{Re}\left(p\bar{v_2}\right)\right) + u\left(\mathrm{Re}(p\bar{v_2}) - \mathrm{Re}(p\bar{v_1})\right) + \left|p\right|^2 \\ &= 3r^2\cos^2(\theta)\left(3u^2 + v^2\right) + v\left(\mathrm{Re}(p(\bar{v_1} + \bar{v_2}))\right) + u\left(\mathrm{Re}(p(\bar{v_2} - \bar{v_1}))\right) + \left|p\right|^2. \end{aligned}$$

This can be rewritten as

$$\left|f_p(u,v)\right|^2 = 3r^2\cos^2(\theta)\left(3(u-a)^2+(v-b)^2\right),$$

where

$$\begin{aligned} a &= -\frac{\operatorname{Re}(p(\bar{v}_2-\bar{v}_1))}{18r^2\cos^2(\theta)} = \frac{\operatorname{Re}(p)}{3r\cos(\theta)}, \\ b &= -\frac{\operatorname{Re}(p(\bar{v}_1+\bar{v}_2))}{6r^2\cos^2(\theta)} = -\frac{\operatorname{Im}(p)}{r\sqrt{3}\cos(\theta)}, \\ 3a^2+b^2 &= \frac{|p|^2}{3r^2\cos^2(\theta)}. \end{aligned}$$

Our aim is to show that $|p+xv_1+yv_2|^2 \geq r^2$ for all $(x,y)\in\mathbb{Z}^2$ excluding the case $p=x=y=0$ which corresponds to the identity case. That is,

$$3r^2\cos^2(\theta)\left(3(u-a)^2+(v-b)^2\right) \geq r^2 \Leftrightarrow 3(u-a)^2+(v-b)^2 \geq \frac{\sec^2(\theta)}{3}$$

for all $(u,v)\in\mathbb{Z}^2$ with $u\equiv v \mod 2$, excluding the case $a=b=u=v=0$. Notice that this inequality is always satisfied if $|u-a|\geq\frac{\sec(\theta)}{3}$ or $|v-b|\geq\frac{\sec\theta}{\sqrt{3}}$ and so we only need to check that

$$g_p^{3,6}(u,v) = 3(u-a)^2+(v-b)^2-\frac{\sec^2(\theta)}{3} \geq 0$$

for all $(u,v)\in\mathbb{Z}^2$ with $u\equiv v \mod 2$ inside the bounding box

$$\left(a-\frac{\sec(\theta)}{3}, a+\frac{\sec(\theta)}{3}\right)\times\left(b-\frac{\sec(\theta)}{\sqrt{3}}, b+\frac{\sec(\theta)}{\sqrt{3}}\right).$$

For the choices of p, we look at the words $w\in\{\mathrm{Id}, j_2, j_{22}, j_{112}, j_{221}, j_{222}\}$. We have 6 possibilities: the identity Id and

$$\begin{aligned} j_2(z) &= \left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)z+\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\varphi_2, \\ j_{22}(z) &= -\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)z+\left(\frac{3}{2}-\frac{i\sqrt{3}}{2}\right)\varphi_2, \\ j_{112}(z) &= \left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)z+\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)\varphi_1-\varphi_2, \\ j_{221}(z) &= -\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)z+i\sqrt{3}\varphi_1+\left(\frac{3}{2}-\frac{i\sqrt{3}}{2}\right)\varphi_2, \\ j_{222}(z) &= -z+2\varphi_2. \end{aligned}$$

Figure 4.30 shows the points $w(0)$ for all words $w \in \{\mathrm{Id}, j_2, j_{22}, j_{112}, j_{221}, j_{222}\}$ in the case $r = 1$ and $\theta = 0$.

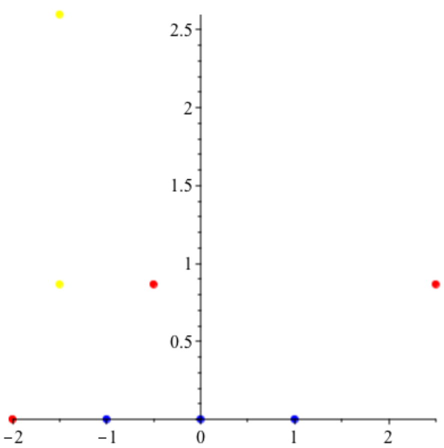

Figure 4.30: Points $w(0)$ for all words $w \in \{\mathrm{Id}, j_2, j_{22}, j_{112}, j_{221}, j_{222}\}$.

Evaluating these words at $z = 0$, we have six possible choices for p:

$$p = 0, \quad p = \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\varphi_2, \quad p = \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)\varphi_2,$$

$$p = \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)\varphi_1 - \varphi_2, \quad p = i\sqrt{3}\varphi_1 + \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)\varphi_2 \quad \text{and} \quad p = 2\varphi_2.$$

For each choice of p, the following table shows the values of a, b and $3a^2+b^2$ in terms of $t = \tan(\theta)$:

$p = w(0)$	a	b	$3a^2 + b^2$
Id	0	0	0
$j_2(0)$	$\frac{1}{2\sqrt{3}}\left(t - \frac{1}{\sqrt{3}}\right)$	$-\frac{1}{2\sqrt{3}}\left(t + \sqrt{3}\right)$	$\frac{1}{3}\left(t^2 + 1\right)$
$j_{22}(0)$	$\frac{1}{2\sqrt{3}}\left(t - \sqrt{3}\right)$	$-\frac{1}{2}\left(t\sqrt{3} + 1\right)$	$t^2 + 1$
$j_{112}(0)$	$\frac{1}{2\sqrt{3}}\left(\frac{5}{\sqrt{3}} - t\right)$	$-\frac{1}{2\sqrt{3}}\left(t + \sqrt{3}\right)$	$\frac{1}{3}\left(t^2 - 2t\sqrt{3} + 7\right)$
$j_{221}(0)$	$-\frac{1}{2\sqrt{3}}\left(t + \sqrt{3}\right)$	$-\frac{1}{2}\left(t\sqrt{3} + 3\right)$	$t^2 + 2t\sqrt{3} + 3$
$j_{222}(0)$	$-\frac{2}{3}$	$-\frac{2t}{\sqrt{3}}$	$\frac{4}{3}\left(t^2 + 1\right)$

Under the assumption $|\theta| \leq \frac{\pi}{12}$ we have that

$$t = \tan(\theta) \in \left[\sqrt{3} - 2, 2 - \sqrt{3}\right] \quad \text{and} \quad \sec(\theta) \in \left[1, \sqrt{2}\left(\sqrt{3} - 1\right)\right].$$

So for each p, we need to calculate the bounds on a, b and the size of the bounding box

$$\left(\min(a) - \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{3}, \max(a) + \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{3}\right)$$
$$\times \left(\min(b) - \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{\sqrt{3}}, \max(b) + \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{\sqrt{3}}\right).$$

We then need to show that

$$\begin{aligned} g_p^{3,6}(u,v) &= 3(u-a)^2 + (v-b)^2 - \frac{\sec^2(\theta)}{3} \\ &= 3u^2 - 6au + v^2 - 2bv + (3a^2+b^2) - \left(\frac{t^2+1}{3}\right) \geq 0 \end{aligned}$$

for all $(u,v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$ inside the bounding box. For the purposes of the following calculations, we will denote $t_{\pi/12} = \tan\left(\frac{\pi}{12}\right), \gamma_1 = \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{3}$ and $\gamma_2 = \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{\sqrt{3}}$.

For $p = \mathrm{Id}$, we have $a = 0$ and $b = 0$. The bounding box

$$(-\gamma_1, \gamma_1) \times (-\gamma_2, \gamma_2) \subset (-1,1) \times (-1,1)$$

contains the point $(0,0)$ which corresponds to the excluded case $f = \mathrm{Id}$.

For $p = j_2(0)$, we have $a = \frac{1}{2\sqrt{3}}\left(t - \frac{1}{\sqrt{3}}\right) \in \left[-\frac{1}{6} - \frac{t_{\pi/12}}{2\sqrt{3}}, -\frac{1}{6} + \frac{t_{\pi/12}}{2\sqrt{3}}\right]$ and $b = -\frac{1}{2\sqrt{3}}\left(t + \sqrt{3}\right) \in \left[-\frac{1}{2} - \frac{t_{\pi/12}}{2\sqrt{3}}, -\frac{1}{2} + \frac{t_{\pi/12}}{2\sqrt{3}}\right]$. The bounding box

$$\left(-\frac{1}{6} - \frac{t_{\pi/12}}{2\sqrt{3}} - \gamma_1, -\frac{1}{6} + \frac{t_{\pi/12}}{2\sqrt{3}} + \gamma_1\right) \times \left(-\frac{1}{2} - \frac{t_{\pi/12}}{2\sqrt{3}} - \gamma_2, -\frac{1}{2} + \frac{t_{\pi/12}}{2\sqrt{3}} + \gamma_2\right)$$
$$\subset (-1,1) \times (-2,1)$$

contains the point $(0,0)$. The function

$$g_2^{3,6}(u,v) = 3u^2 + (1 - t\sqrt{3})u + v^2 + \left(1 + \frac{t}{\sqrt{3}}\right)v$$

evaluated at this point is non-negative: $g_2^{3,6}(0,0) = 0$.

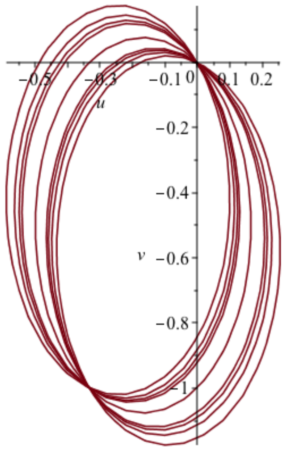

Figure 4.31: The level curves $g_2^{3,6}(u, v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

For $p = j_{22}(0)$, we have $a = \frac{1}{2\sqrt{3}}\left(t - \sqrt{3}\right) \in \left[-\frac{1}{2} - \frac{t_{\pi/12}}{2\sqrt{3}}, -\frac{1}{2} + \frac{t_{\pi/12}}{2\sqrt{3}}\right]$ and $b = -\frac{1}{2}\left(t\sqrt{3} + 1\right) \in \left[-\frac{1}{2} - \frac{t_{\pi/12}\sqrt{3}}{2}, -\frac{1}{2} + \frac{t_{\pi/12}\sqrt{3}}{2}\right]$. The bounding box

$$\left(-\frac{1}{2} - \frac{t_{\pi/12}}{2\sqrt{3}} - \gamma_1, -\frac{1}{2} + \frac{t_{\pi/12}}{2\sqrt{3}} + \gamma_1\right) \times \left(-\frac{1}{2} - \frac{t_{\pi/12}\sqrt{3}}{2} - \gamma_2, -\frac{1}{2} + \frac{t_{\pi/12}\sqrt{3}}{2} + \gamma_2\right)$$
$$\subset (-1, 0) \times (-2, 1)$$

contains no points $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$.

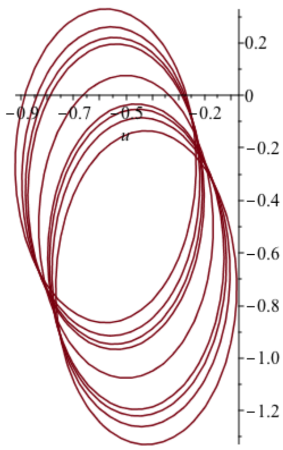

Figure 4.32: The level curves $g_{22}^{3,6}(u, v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

For $p = j_{112}(0)$, we have $a = \frac{1}{2\sqrt{3}}\left(\frac{5}{\sqrt{3}} - t\right) \in \left[\frac{5}{6} - \frac{t_{\pi/12}}{2\sqrt{3}}, \frac{5}{6} + \frac{t_{\pi/12}}{2\sqrt{3}}\right]$ and $b = -\frac{1}{2\sqrt{3}}\left(t + \sqrt{3}\right)$

$\in \left[-\frac{1}{2} - \frac{t_{\pi/12}}{2\sqrt{3}}, -\frac{1}{2} + \frac{t_{\pi/12}}{2\sqrt{3}}\right]$. The bounding box

$$\left(\frac{5}{6} - \frac{t_{\pi/12}}{2\sqrt{3}} - \gamma_1, \frac{5}{6} + \frac{t_{\pi/12}}{2\sqrt{3}} + \gamma_1\right) \times \left(-\frac{1}{2} - \frac{t_{\pi/12}}{2\sqrt{3}} - \gamma_2, -\frac{1}{2} + \frac{t_{\pi/12}}{2\sqrt{3}} + \gamma_2\right)$$

$$\subset (0,2) \times (-2,1)$$

contains the point $(1,-1)$. The function

$$g_{112}^{3,6}(u,v) = 3u^2 - (5 - t\sqrt{3})u + v^2 + \left(1 + \frac{t}{\sqrt{3}}\right)v + 2 - \frac{2t}{\sqrt{3}}$$

evaluated at this point is non-negative: $g_{112}^{3,6}(1,-1) = 0$.

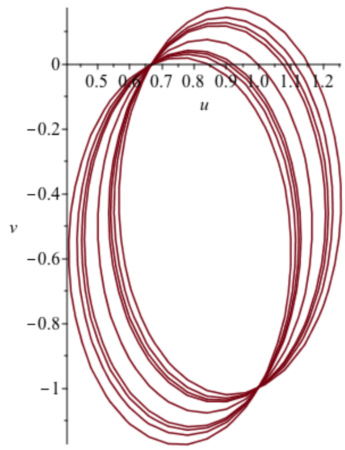

Figure 4.33: The level curves $g_{112}^{3,6}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

For $p = j_{221}(0)$, we have $a = -\frac{1}{2\sqrt{3}}\left(t + \sqrt{3}\right) \in \left[-\frac{1}{2} - \frac{t_{\pi/12}}{2\sqrt{3}}, -\frac{1}{2} + \frac{t_{\pi/12}}{2\sqrt{3}}\right]$ and $b = -\frac{1}{2}\left(t\sqrt{3} + 3\right) \in \left[-\frac{3}{2} - \frac{t_{\pi/12}\sqrt{3}}{2}, -\frac{3}{2} + \frac{t_{\pi/12}\sqrt{3}}{2}\right]$. The bounding box

$$\left(-\frac{1}{2} - \frac{t_{\pi/12}}{2\sqrt{3}} - \gamma_1, -\frac{1}{2} + \frac{t_{\pi/12}}{2\sqrt{3}} + \gamma_1\right) \times \left(-\frac{3}{2} - \frac{t_{\pi/12}\sqrt{3}}{2} - \gamma_2, -\frac{3}{2} + \frac{t_{\pi/12}\sqrt{3}}{2} + \gamma_2\right)$$

$$\subset (-1,0) \times (-3,0)$$

contains no points $(u,v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$.

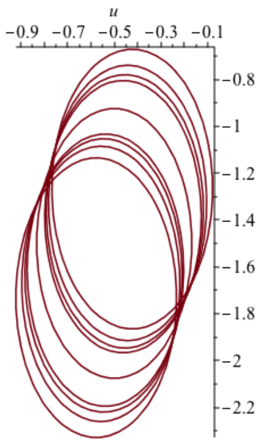

Figure 4.34: The level curves $g_{221}^{3,6}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

Finally for $p = j_{222}(0)$, we have $a = -\frac{2}{3}$ and $b = -\frac{2t}{\sqrt{3}} \in \left[-\frac{2t_{\pi/12}}{\sqrt{3}}, \frac{2t_{\pi/12}}{\sqrt{3}}\right]$. The bounding box

$$\left(-\frac{2}{3} - \gamma_1, -\frac{2}{3} + \gamma_1\right) \times \left(-\frac{2t_{\pi/12}}{\sqrt{3}} - \gamma_2, \frac{2t_{\pi/12}}{\sqrt{3}} + \gamma_2\right)$$

$$\subset (-2, 0) \times (-1, 1)$$

contains no points $(u, v) \in \mathbb{Z}^2$ with $u \equiv v \mod 2$.

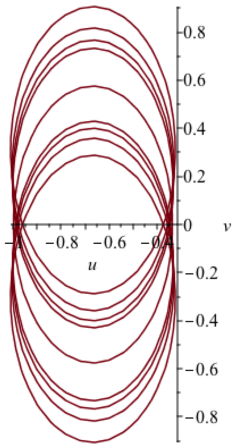

Figure 4.35: The level curves $g_{222}^{3,6}(u,v) = 0$ for several $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

Therefore, as $g_p^{3,6}(u,v) \geq 0$ for all p, we have that

$$\left|f_p(x,y)\right|^2 = |p + xv_1 + yv_2|^2 \geq r^2 \geq 4$$

under the assumption that $r \geq 2$. That is, $\left|f_p(x,y)\right| \geq 2$ for all $f \in \Lambda \backslash \{\mathrm{Id}\}$. Hence the conditions of Lemma 3.2.4 are satisfied, and we can conclude that the complex hyperbolic ultra-parallel $[m, m, 0; 3, 6, 2]$-triangle group is discrete for

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \text{and} \quad m \geq \log\left(7 + 4\sqrt{3}\right).$$

$\square$

Chapter 5

Non-Discreteness Results

We have found conditions on m and α for the complex hyperbolic ultra-parallel $[m, m, 0; n_1, n_2, 2]$-triangle group to be discrete. On the other hand, we can use Shimizu's Lemma to find sufficient conditions for m and α for the group to be non-discrete. We first need the following definition.

Definition 5.0.1. For an element $h = (h_{ij})_{1 \leq i,j \leq 3} \in \mathrm{SU}(2,1)$ with $h(\infty) \neq \infty$ we can define the *isometric sphere* of h as the sphere with respect to the Cygan metric with centre $h^{-1}(\infty)$ and radius

$$r_h = \sqrt{\frac{2}{|h_{22} - h_{23} + h_{32} - h_{33}|}}.$$

We will use the complex hyperbolic version of Shimizu's Lemma introduced in [16].

Lemma 5.0.2. *Let G be a discrete subgroup of* $\mathrm{PU}(2,1)$*. Let $g \in G$ be a Heisenberg translation by (ξ, ν) and $h \in G$ be an element that satisfies $h(\infty) \neq \infty$, then*

$$r_h^2 \leq \rho_0(g(h^{-1}(\infty)), h^{-1}(\infty))\rho_0(g(h(\infty)), h(\infty)) + 4|\xi|^2,$$

where ρ_0 is the Cygan metric on $\mathfrak{N}$ and r_h is the radius of the isometric sphere of h.

5.1 Construction

For an ultra-parallel triangle group $\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle$ we will apply Lemma 5.0.2 to translation elements g and the element $h = \iota_3$. The matrix of a Heisenberg translation g by (ξ, ν) is of the form

$$g = \begin{pmatrix} 1 & \xi & \xi \\ -\overline{\xi} & 1 - \frac{|\xi|^2 - i\nu}{2} & -\frac{|\xi|^2 - i\nu}{2} \\ \overline{\xi} & \frac{|\xi|^2 - i\nu}{2} & 1 + \frac{|\xi|^2 - i\nu}{2} \end{pmatrix}. \tag{5.1}$$

The element $h = \iota_3$ is given as

$$h = h^{-1} = \iota_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The radius of the isometric sphere of h is $r_h = 1$.

To calculate $h(\infty)$ we first map ∞ from the Heisenberg space to the boundary of the complex hyperbolic 2-space. That is,

$$\infty \mapsto [0 : 1 : -1] \in \partial \mathbb{H}^2_{\mathbb{C}}.$$

We apply h to this point

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = [0 : 1 : 1] \in \partial \mathbb{H}^2_{\mathbb{C}}.$$

Note that $h(\infty) \neq \infty$. Mapping this point back to the Heisenberg space

$$[0 : 1 : 1] \mapsto (0, 0) \in \mathfrak{N}.$$

Next, we want to find $g(h(\infty))$. As g is the matrix of the Heisenberg translation by $(\xi, \nu) \in \mathfrak{N}$, $g(h(\infty))$ is simply $(\xi, \nu) + (0, 0) = (\xi, \nu)$. Finally notice that $\rho_0(g(h(\infty)), h(\infty)) = \rho_0(g(h^{-1}(\infty)), h^{-1}(\infty))$, since $h = h^{-1}$.

The distance $\rho_0((\xi_1, \nu_1), (\xi_2, \nu_2))$ is given as

$$\rho_0((\xi_1, \nu_1), (\xi_2, \nu_2)) = \left| |\xi_1 - \xi_2|^2 - i(\nu_1 - \nu_2) - 2i \operatorname{Im}(\xi_1 \bar{\xi_2}) \right|^{\frac{1}{2}}.$$

So the distance $\rho_0(g(h(\infty)), h(\infty))$ is equal to

$$\rho_0((\xi, \nu), (0, 0)) = \left| |\xi|^2 - i\nu \right|^{\frac{1}{2}} = \sqrt{\left| |\xi|^2 - i\nu \right|}.$$

Substituting these values into the inequality given in Lemma 5.0.2, we obtain that if the group is discrete then

$$1 \leq \sqrt{|\xi|^4 + \nu^2} + 4|\xi|^2.$$

Hence the group is not discrete if there exists a Heisenberg translation element $T_{(\zeta, \nu)} \in \Gamma$ with

$$\sqrt{|\xi|^4 + \nu^2} + 4|\xi|^2 < 1 \Leftrightarrow \sqrt{|\xi|^4 + \nu^2} < 1 - 4|\xi|^2. \tag{5.2}$$

We can further simplify this inequality. For each case, we will consider the vertical Heisenberg translation. That is, the generator H of the form $(\xi, \nu) = (0, \nu)$. Substituting this into (5.2), we have that the group is not discrete if

$$|\nu| < 1. \tag{5.3}$$

5.2 The case $[m, m, 0; 3, 3, 2]$

Proposition 5.2.1. *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 3, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{12\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Proof. We consider the translation element $g = H = [T_1, T_2]$. In this case the matrix g is of the form (5.1) with $\xi = 0$ and $\nu = 24\sqrt{3}r^2\cos^2(\theta)$. Substituting this into the inequality (5.3), we have that the group is not discrete if

$$24\sqrt{3}r^2\cos^2(\theta) < 1 \Leftrightarrow \cos^2(\theta) < \frac{1}{24\sqrt{3}r^2}.$$

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta) + 1\big) = \frac{1}{2}\big(1 - \cos(\alpha)\big)$ we obtain

$$\frac{1}{2}\big(1 - \cos(\alpha)\big) < \frac{1}{24\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{12\sqrt{3}r^2}.$$

Therefore, we conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{12\sqrt{3}r^2} = 1 - \frac{1}{12\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

□

5.3 The case $[m, m, 0; 2, 3, 2]$

Proposition 5.3.1. *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 3, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{48\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Proof. We consider the translation element $g = H = [T_1, T_2]$. In this case the matrix g is of the form (5.1) with $\xi = 0$ and $\nu = 96\sqrt{3}r^2\cos^2(\theta)$. Substituting this into the inequality (5.3), we have that the group is not discrete if

$$96\sqrt{3}r^2\cos^2(\theta) < 1 \Leftrightarrow \cos^2(\theta) < \frac{1}{96\sqrt{3}r^2}.$$

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta) + 1\big) = \frac{1}{2}\big(1 - \cos(\alpha)\big)$ we obtain

$$\frac{1}{2}\big(1 - \cos(\alpha)\big) < \frac{1}{96\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{48\sqrt{3}r^2}.$$

Therefore, we conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{48\sqrt{3}r^2} = 1 - \frac{1}{48\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

□

5.4 The case $[m, m, 0; 2, 4, 2]$

Proposition 5.4.1. *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 4, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{32 \cdot \cosh^2\left(\frac{m}{2}\right)}$$

Proof. We consider the translation element $g = H = [T_1, T_2]$. In this case the matrix g is of the form (5.1) with $\xi = 0$ and $\nu = 64r^2 \cos^2(\theta)$. Substituting this into the inequality (5.3), we have that the group is not discrete if

$$64r^2 \cos^2(\theta) < 1 \Leftrightarrow \cos^2(\theta) < \frac{1}{64r^2}.$$

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta) + 1\big) = \frac{1}{2}\big(1 - \cos(\alpha)\big)$ we obtain

$$\frac{1}{2}\big(1 - \cos(\alpha)\big) < \frac{1}{64r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{32r^2}.$$

Therefore, we conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{32r^2} = 1 - \frac{1}{32 \cdot \cosh^2\left(\frac{m}{2}\right)}.$$

□

5.5 The case $[m, m, 0; 4, 4, 2]$

Proposition 5.5.1. *A complex hyperbolic ultra-parallel* $[m, m, 0; 4, 4, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{8 \cdot \cosh^2\left(\frac{m}{2}\right)}.$$

Proof. We consider the translation element $g = H = \iota_{1212}$. In this case the matrix g is of the form (5.1) with $\xi = 0$ and $\nu = 16r^2 \cos^2(\theta)$. Substituting this into the inequality (5.3), we have that the group is not discrete if

$$16r^2 \cos^2(\theta) < 1 \Leftrightarrow \cos^2(\theta) < \frac{1}{16r^2}.$$

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta) + 1\big) = \frac{1}{2}\big(1 - \cos(\alpha)\big)$ we obtain

$$\frac{1}{2}\big(1 - \cos(\alpha)\big) < \frac{1}{16r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{8r^2}.$$

Therefore, we conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{8r^2} = 1 - \frac{1}{8 \cdot \cosh^2\left(\frac{m}{2}\right)}.$$

□

5.6 The case $[m, m, 0; 2, 6, 2]$

Proposition 5.6.1. *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 6, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{8\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Proof. We consider the translation element $g = H = \iota_{121212}$. In this case the matrix g is of the form (5.1) with $\xi = 0$ and $\nu = 16\sqrt{3}r^2\cos^2(\theta)$. Substituting this into the inequality (5.3), we have that the group is not discrete if

$$16\sqrt{3}r^2\cos^2(\theta) < 1 \Leftrightarrow \cos^2(\theta) < \frac{1}{16\sqrt{3}r^2}.$$

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta) + 1\big) = \frac{1}{2}\big(1 - \cos(\alpha)\big)$ we obtain

$$\frac{1}{2}\big(1 - \cos(\alpha)\big) < \frac{1}{16\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{8\sqrt{3}r^2}.$$

Therefore, we conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{8\sqrt{3}r^2} = 1 - \frac{1}{8\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

□

5.7 The case $[m, m, 0; 3, 6, 2]$

Proposition 5.7.1. *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 6, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Proof. We consider the translation element $g = H = \iota_{1212}$. In this case the matrix g is of the form (5.1) with $\xi = 0$ and $\nu = 8\sqrt{3}r^2\cos^2(\theta)$. Substituting this into the inequality (5.3), we have that the group is not discrete if

$$8\sqrt{3}r^2\cos^2(\theta) < 1 \Leftrightarrow \cos^2(\theta) < \frac{1}{8\sqrt{3}r^2}.$$

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta) + 1\big) = \frac{1}{2}\big(1 - \cos(\alpha)\big)$ we obtain

$$\frac{1}{2}\big(1 - \cos(\alpha)\big) < \frac{1}{8\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{4\sqrt{3}r^2}.$$

Therefore, we conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{4\sqrt{3}r^2} = 1 - \frac{1}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

□

5.8 Further Non-Discreteness Results

We are able to further these non-discreteness results by considering the following results of Parker [15] (section 3.1):

Theorem 5.8.1. *Let Γ be a discrete subgroup of $\mathrm{PU}(n,1)$ containing a vertical translation g by $t > 0$. Let h be any element of Γ not fixing q_∞, a distinguished point at infinity, and let r_h be the radius of its isometric sphere. Then either*

$$t/r_h^2 \geq 2 \quad \textit{or} \quad t/r_h^2 = 2\cos(\pi/q) \quad \textit{for some integer} \quad q \geq 3.$$

For each ultra-parallel $[m, m, 0; n_1, n_2, 2]$-triangle group we will apply Theorem 5.8.1 to the vertical Heisenberg translation element H. Recall from the previous construction that for each case, if we again let $h = \iota_3$, then this does not fix q_∞ and the radius of the isometric sphere of h is $r_h = 1$. Therefore, if the group is discrete then

$$t \geq 2 \quad \text{or} \quad t = 2\cos(\pi/q) \quad \text{for some integer} \quad q \geq 3.$$

Hence the group is not discrete if

$$t < 2 \quad \text{and} \quad t \neq 2\cos(\pi/q) \quad \text{for some integer} \quad q \geq 3. \tag{5.4}$$

5.8.1 The case $[m, m, 0; 3, 3, 2]$

Proposition 5.8.2. *A complex hyperbolic ultra-parallel $[m, m, 0; 3, 3, 2]$-triangle group with angular invariant α is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{6\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{6\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proof. The vertical Heisenberg translation element H is given as $(0, 24\sqrt{3}r^2\cos^2(\theta))$. Substituting this into (5.4) with $t = 24\sqrt{3}r^2\cos^2(\theta)$ we have that the group is not discrete if

$$\begin{aligned} &24\sqrt{3}r^2\cos^2(\theta) < 2 \quad \text{and} \quad 24\sqrt{3}r^2\cos^2(\theta) \neq 2\cos(\pi/q) \\ \Leftrightarrow\ &\cos^2(\theta) < \frac{1}{12\sqrt{3}r^2} \quad \text{and} \quad \cos^2(\theta) \neq \frac{\cos(\pi/q)}{12\sqrt{3}r^2} \end{aligned}$$

for some integer $q \geq 3$.

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta) + 1\big) = \frac{1}{2}\big(1 - \cos(\alpha)\big)$ we are able to rewrite the first inequality as

$$\frac{1}{2}\big(1 - \cos(\alpha)\big) < \frac{1}{12\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{6\sqrt{3}r^2}.$$

We can rewrite the second inequality as

$$\frac{1}{2}\big(1-\cos(\alpha)\big) \neq \frac{\cos(\pi/q)}{12\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{6\sqrt{3}r^2}.$$

Therefore, we can conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{6\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \text{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{6\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$. □

5.8.2 The case $[m, m, 0; 2, 3, 2]$

Proposition 5.8.3. *A complex hyperbolic ultra-parallel $[m, m, 0; 2, 3, 2]$-triangle group with angular invariant α is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{24\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{24\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proof. The vertical Heisenberg translation element H is given as $(0, 96\sqrt{3}r^2\cos^2(\theta))$. Substituting this into (5.4) with $t = 96\sqrt{3}r^2\cos^2(\theta)$ we have that the group is not discrete if

$$\begin{aligned} & 96\sqrt{3}r^2\cos^2(\theta) < 2 \quad \text{and} \quad 96\sqrt{3}r^2\cos^2(\theta) \neq 2\cos(\pi/q) \\ \Leftrightarrow \; & \cos^2(\theta) < \frac{1}{48\sqrt{3}r^2} \quad \text{and} \quad \cos^2(\theta) \neq \frac{\cos(\pi/q)}{48\sqrt{3}r^2} \end{aligned}$$

for some integer $q \geq 3$.

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta)+1\big) = \frac{1}{2}\big(1-\cos(\alpha)\big)$ we are able to rewrite the first inequality as

$$\frac{1}{2}\big(1-\cos(\alpha)\big) < \frac{1}{48\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{24\sqrt{3}r^2}.$$

We can rewrite the second inequality as

$$\frac{1}{2}\big(1-\cos(\alpha)\big) \neq \frac{\cos(\pi/q)}{48\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{24\sqrt{3}r^2}.$$

Therefore, we can conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{24\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \text{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{24\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$. □

5.8.3 The case $[m, m, 0; 2, 4, 2]$

Proposition 5.8.4. *A complex hyperbolic ultra-parallel $[m, m, 0; 2, 4, 2]$-triangle group with angular invariant α is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{16\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{16\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proof. The vertical Heisenberg translation element H is given as $(0, 64r^2\cos^2(\theta))$. Substituting this into (5.4) with $t = 64r^2\cos^2(\theta)$ we have that the group is not discrete if

$$\begin{aligned} 64r^2\cos^2(\theta) < 2 \quad &\text{and} \quad 64r^2\cos^2(\theta) \neq 2\cos(\pi/q) \\ \Leftrightarrow \ \cos^2(\theta) < \frac{1}{32r^2} \quad &\text{and} \quad \cos^2(\theta) \neq \frac{\cos(\pi/q)}{32r^2} \end{aligned}$$

for some integer $q \geq 3$.

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta) + 1\big) = \frac{1}{2}\big(1 - \cos(\alpha)\big)$ we are able to rewrite the first inequality as

$$\frac{1}{2}\big(1 - \cos(\alpha)\big) < \frac{1}{32r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{16r^2}.$$

We can rewrite the second inequality as

$$\frac{1}{2}\big(1 - \cos(\alpha)\big) \neq \frac{\cos(\pi/q)}{32r^2} \Leftrightarrow \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{16r^2}.$$

Therefore, we can conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{16\cosh^2\left(\frac{m}{2}\right)} \quad \text{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{16\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$. $\square$

5.8.4 The case $[m, m, 0; 4, 4, 2]$

Proposition 5.8.5. *A complex hyperbolic ultra-parallel $[m, m, 0; 4, 4, 2]$-triangle group with angular invariant α is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{4\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{4\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proof. The vertical Heisenberg translation element H is given as $(0, 16r^2\cos^2(\theta))$. Substituting this into (5.4) with $t = 16r^2\cos^2(\theta)$ we have that the group is not discrete if

$$\begin{aligned} &16r^2\cos^2(\theta) < 2 \quad \text{and} \quad 16r^2\cos^2(\theta) \neq 2\cos(\pi/q) \\ \Leftrightarrow\ &\cos^2(\theta) < \frac{1}{8r^2} \quad \text{and} \quad \cos^2(\theta) \neq \frac{\cos(\pi/q)}{8r^2} \end{aligned}$$

for some integer $q \geq 3$.

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta)+1\big) = \frac{1}{2}\big(1-\cos(\alpha)\big)$ we are able to rewrite the first inequality as

$$\frac{1}{2}\big(1-\cos(\alpha)\big) < \frac{1}{8r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{4r^2}.$$

We can rewrite the second inequality as

$$\frac{1}{2}\big(1-\cos(\alpha)\big) \neq \frac{\cos(\pi/q)}{8r^2} \Leftrightarrow \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{4r^2}.$$

Therefore, we can conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{4\cosh^2\left(\frac{m}{2}\right)} \quad \text{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{4\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$. □

5.8.5 The case $[m, m, 0; 2, 6, 2]$

Proposition 5.8.6. *A complex hyperbolic ultra-parallel $[m, m, 0; 2, 6, 2]$-triangle group with angular invariant α is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proof. The vertical Heisenberg translation element H is given as $(0, 16\sqrt{3}r^2\cos^2(\theta))$. Substituting this into (5.4) with $t = 16\sqrt{3}r^2\cos^2(\theta)$ we have that the group is not discrete if

$$\begin{aligned} &16\sqrt{3}r^2\cos^2(\theta) < 2 \quad \text{and} \quad 16\sqrt{3}r^2\cos^2(\theta) \neq 2\cos(\pi/q) \\ \Leftrightarrow\ &\cos^2(\theta) < \frac{1}{8\sqrt{3}r^2} \quad \text{and} \quad \cos^2(\theta) \neq \frac{\cos(\pi/q)}{8\sqrt{3}r^2} \end{aligned}$$

for some integer $q \geq 3$.

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta)+1\big) = \frac{1}{2}\big(1-\cos(\alpha)\big)$ we are able to rewrite the first inequality as

$$\frac{1}{2}\big(1-\cos(\alpha)\big) < \frac{1}{8\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{4\sqrt{3}r^2}.$$

We can rewrite the second inequality as

$$\frac{1}{2}\big(1-\cos(\alpha)\big) \neq \frac{\cos(\pi/q)}{8\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{4\sqrt{3}r^2}.$$

Therefore, we can conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \text{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$. □

5.8.6 The case $[m, m, 0; 3, 6, 2]$

Proposition 5.8.7. *A complex hyperbolic ultra-parallel $[m, m, 0; 3, 6, 2]$-triangle group with angular invariant α is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{2\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{2\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proof. The vertical Heisenberg translation element H is given as $(0, 8\sqrt{3}r^2\cos^2(\theta))$. Substituting this into (5.4) with $t = 8\sqrt{3}r^2\cos^2(\theta)$ we have that the group is not discrete if

$$\begin{aligned} & 8\sqrt{3}r^2\cos^2(\theta) < 2 \quad \text{and} \quad 8\sqrt{3}r^2\cos^2(\theta) \neq 2\cos(\pi/q) \\ \Leftrightarrow \; & \cos^2(\theta) < \frac{1}{4\sqrt{3}r^2} \quad \text{and} \quad \cos^2(\theta) \neq \frac{\cos(\pi/q)}{4\sqrt{3}r^2} \end{aligned}$$

for some integer $q \geq 3$.

Using $\cos^2(\theta) = \frac{1}{2}\big(\cos(2\theta)+1\big) = \frac{1}{2}\big(1-\cos(\alpha)\big)$ we are able to rewrite the first inequality as

$$\frac{1}{2}\big(1-\cos(\alpha)\big) < \frac{1}{4\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) > 1 - \frac{1}{2\sqrt{3}r^2}.$$

We can rewrite the second inequality as

$$\frac{1}{2}\left(1-\cos(\alpha)\right) \neq \frac{\cos(\pi/q)}{4\sqrt{3}r^2} \Leftrightarrow \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{2\sqrt{3}r^2}.$$

Therefore, we can conclude that the group Γ is not discrete provided that

$$\cos(\alpha) > 1 - \frac{1}{2\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \text{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{2\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$. $\square$

Chapter 6

Brief Summary

In this thesis we studied complex hyperbolic ultra-parallel triangle groups of type $[m, m, 0; n_1, n_2, 2]$. That is, groups of isometries of the complex hyperbolic plane generated by complex reflections ι_1, ι_2 and ι_3 of orders n_1, n_2 and 2 respectively in the complex geodesics with pairwise distances m, m and 0.

We first used the work of Hersonsky and Paulin [9] and Parker [17] to classify what possible orders we can have for the complex reflections ι_1 and ι_2 for the complex hyperbolic ultra-parallel triangle group to be discrete:

Theorem. (3.3.2). *A complex hyperbolic ultra-parallel $[m_1, m_2, 0; n_1, n_2, 2]$-triangle group can only be discrete if the unordered pair of orders of the complex reflections ι_1 and ι_2 is one of*

$$\{2, 2\},\ \{2, 3\},\ \{2, 4\},\ \{2, 6\},\ \{3, 3\},\ \{3, 6\}\ \textit{and}\ \{4, 4\}.$$

Next we introduced a compression property and set out sufficient discreteness conditions for the angular invariant α and the distance m for all possible orders of the complex reflections ι_1 and ι_2:

Proposition. (4.1.2) & (4.2.2). *A complex hyperbolic ultra-parallel $[m, m, 0; n, 3, 2]$-triangle group with $n \in \{2, 3\}$ is discrete if the following conditions on the angular invariant α and on m are satisfied:*

$$\cos(\alpha) \leq -\frac{1}{2} \quad \textit{and} \quad m \geq \log(3).$$

Proposition. (4.3.2) & (4.4.2). *A complex hyperbolic ultra-parallel $[m, m, 0; n, 4, 2]$-triangle group with $n \in \{2, 4\}$ is discrete if the following conditions on the angular invariant α and on m are satisfied:*

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \textit{and} \quad m \geq \log\left(3 + 2\sqrt{2}\right).$$

Proposition. (4.5.2) & (4.6.4). *A complex hyperbolic ultra-parallel* $[m, m, 0; n, 6, 2]$*-triangle group with* $n \in \{2, 3\}$ *is discrete if the following conditions on the angular invariant* α *and on* m *are satisfied:*

$$\cos(\alpha) \leq -\frac{\sqrt{3}}{2} \quad \textit{and} \quad m \geq \log\left(7 + 4\sqrt{3}\right).$$

These conditions were chosen to ensure that the projections of the images of the unit spinal sphere were disjoint. One way to improve these results would be to work with the images of the unit spinal sphere themselves rather than with their projections.

In contrast to these discreteness results we then found non-discreteness conditions for the angular invariant α and the distance m. We used the complex hyperbolic version of Shimizu's Lemma introduced in [16] to find such conditions:

Proposition. (5.2.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 3, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{12\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Proposition. (5.3.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 3, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{48\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Proposition. (5.4.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 4, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{32 \cdot \cosh^2\left(\frac{m}{2}\right)}.$$

Proposition. (5.5.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 4, 4, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{8 \cdot \cosh^2\left(\frac{m}{2}\right)}.$$

Proposition. (5.6.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 6, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{8\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

Proposition. (5.7.1). *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 6, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}.$$

We then furthered these non-discreteness results by considering the results of Parker [15] to obtain further conditions on the angular invariant α and the distance m:

Proposition. (5.8.2). *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 3, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{6\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{6\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proposition. (5.8.3). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 3, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{24\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{24\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proposition. (5.8.4). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 4, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{16\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{16\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proposition. (5.8.5). *A complex hyperbolic ultra-parallel* $[m, m, 0; 4, 4, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{4\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{4\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proposition. (5.8.6). *A complex hyperbolic ultra-parallel* $[m, m, 0; 2, 6, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \textit{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{4\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Proposition. (5.8.7). *A complex hyperbolic ultra-parallel* $[m, m, 0; 3, 6, 2]$*-triangle group with angular invariant* α *is non-discrete if*

$$\cos(\alpha) > 1 - \frac{1}{2\sqrt{3}\cosh^2\left(\frac{m}{2}\right)} \quad \text{and} \quad \cos(\alpha) \neq 1 - \frac{\cos(\pi/q)}{2\sqrt{3}\cosh^2\left(\frac{m}{2}\right)}$$

for some integer $q \geq 3$.

Notice that for the angular invariant α and m there is a gap between the discreteness and non-discreteness results for each case. One of the main ways to improve these results would be to attempt to close this gap. A first approach would be to consider these results in the context of the conjecture put forward by Schwartz in his ICM talk [23]:

Conjecture 6.0.1. *A complex hyperbolic triangle group is discrete if the elements* $\omega_A = \iota_1\iota_3\iota_2\iota_3$ *and* $\omega_B = \iota_1\iota_2\iota_3$ *are not elliptic.*

These conjectural results are illustrated in the figure below for the $[m, m, 0; 3, 3, 2]$-case.

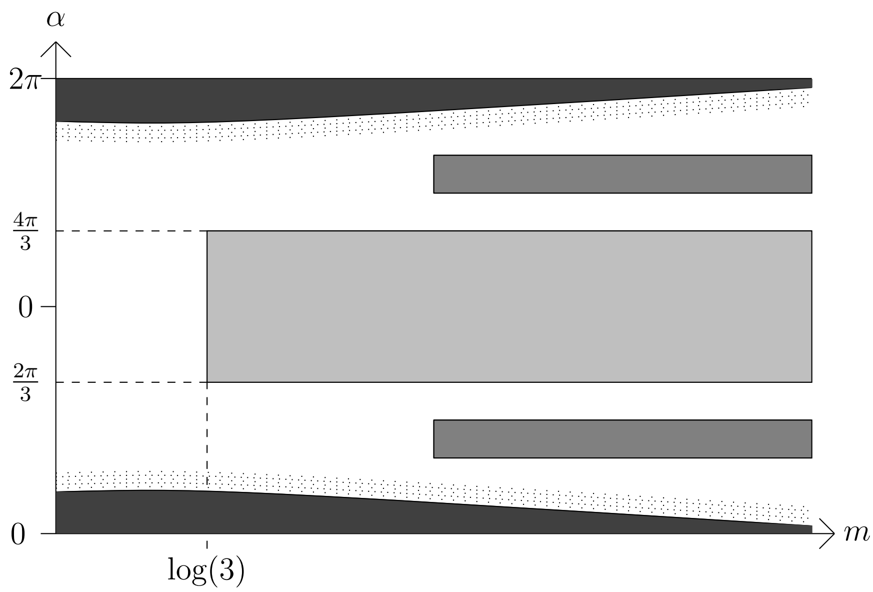

Figure 6.1: Gap between discreteness (light grey), non-discreteness (dark grey) and conjectural discreteness (grey) results.

To further this work, one particularly interesting concept would be to consider increasing the order of the complex reflection ι_3. As we have established all possible orders of the complex reflections ι_1 and ι_2 for the case $[m, m, 0; n_1, n_2, 2]$, one could increase the order of the complex reflection ι_3 and consider the more general case $[m, m, 0; n_1, n_2, n_3]$. The possible orders for $\{\iota_1, \iota_2\}$ would remain the same as in

Theorem 3.3.2, however, initial considerations suggest that increasing the order of ι_3 would have an impact on the conditions for the angular invariant α. Another direction to take would be to consider the case $[m_1, m_2, 0; n_1, n_2, 2]$ with $m_1 \neq m_2$.

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