International Journal of Fracture manuscript No.
(will be inserted by the editor)

Delamination growth in buckled composite struts

Anton Köllner · Rolf Jungnickel · Christina Völlmecke

Received: date / Accepted: date

Abstract Existing analytical models dealing with buckling and postbuckling phenomena of delaminated composites comprise one limitation: the restriction to stationary delaminations. In the current work, an analytical framework is presented which allows to model the postbuckling response of composites without such limitation. Therefore, the well-known problem of a composite strut with a through-the-width delamination is studied. The system is fully described by a set of I generalized coordinates. The postbuckling response for a stationary delamination is modelled using the conventional total potential energy approach. The postbuckling response for a non-stationary delamination, *i.e.* once delamination growth occurs, is modelled using an extended total potential energy functional in which the delamination length is expressed by the generalized coordinates and the load parameters. By solving the underlying variational principle the postbuckling response is obtained. Implementing the Rayleigh-Ritz method yields a set of non-linear algebraic equations which is solved numerically. Postbuckling responses for a cross-ply laminate are provided until the strut fails. Depending on delamination depth and length additional load bearing capacities of such composite struts are documented before failure due to unstable delamination growth occurs.

A. Köllner, C. Völlmecke
Technische Universität Berlin, Institute of Mechanics,
Stability and Failure of Functionally Optimized Structures Group, Einsteinufer 5, 10587 Berlin, Germany.
Tel.: +49-30-31421716
E-mail: anton.koellner@tu-berlin.de

R. Jungnickel
Technische Universität Berlin, Institute of Mechanics,
Material Theory and Continuum Mechanics Group,
Einsteinufer 5, 10587 Berlin, Germany.

Keywords Buckling · Postbuckling · Delamination · Growth · Composites · Struts · Analytical model

1 Introduction

Since the early 1980s, with the work of Chai *et al.* [2], the problem of delamination buckling in composites has received much attention. The issue of possible delamination growth is also already mentioned in [2]. Studying the behaviour of the energy release rate provided information about whether delamination growth would occur during the postbuckling response and whether such growth would be stable or unstable. As done in the literature (*e.g.*, see [1,2]), delamination growth is termed *stable* if under constant loading the growth stops, otherwise it is termed *unstable*. A quasi-brittle fracture process, *i.e.* a Griffith-type crack problem, is commonly assumed (*e.g.* see [1,15]) dealing with delaminations in composites. Hence, stable delamination growth does not lead to failure of the structure, whereas unstable growth does.

Within the current work, the terminology *stationary* and *non-stationary* delamination describes whether the size of a delamination increases (non-stationary) during consecutive loading steps (quasi-static process) or remains constant (stationary). Non-stationary delaminations may be characterized by stable or unstable growth. In contrast to unstable growth, stable growth may be described by a quasi-static formulation as the delamination growth would stop under a constantly kept state of loading.

Taking a closer look at analytical models documented within the literature, studies endeavoured to derive detailed information about buckling loads [5,17], postbuckling responses for stationary delaminations [8,16] and information about the energy release rate [2,15]. Extensions to different delamination types, such as embedded rectangular [20] and elliptical [1,9] delaminations, are also documented. However, to the authors' knowledge, the deformation behaviour once delamination growth occurs is solely investigated by finite element models (*e.g.*, see [7,11]).

In the current work, the postbuckling response of a composite strut with a through-the-width delamination is determined with the aid of an analytical model. This is done up to the state of loading where failure occurs in form of unstable delamination growth. First, the problem is solved for a stationary delamination with the conventional total potential energy formulation. Subsequently, the state where growth would occur is determined. Thereafter, an extended total potential energy formulation implementing the condition of stable growth is used to determine the ensuing postbuckling response and delamination growth by just a set of generalized coordinates and the load parameter(s). Results obtained are compared to findings from a finite element (FE) simulation using the commercial software code Abaqus [4].

2 Analytical framework

2.1 The geometrical model

For the model of the composite strut shown in Fig. 1, a description for an isotropic strut documented in [6] is taken as a benchmark model. Thus, the (post-)buckling behaviour of the composite strut is treated as a one-dimensional problem. A central pre-existing delamination is assigned to the strut which allows to subdivide the structure into four parts.

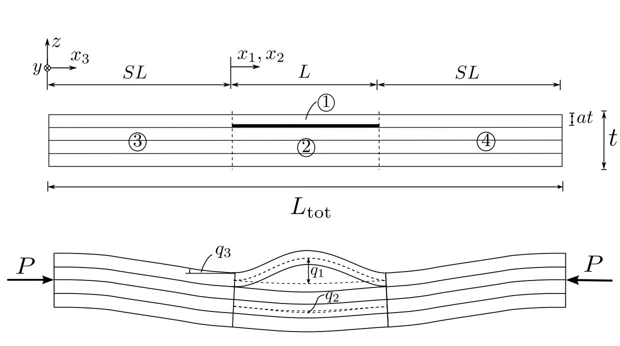

Fig. 1 One-dimensional model of the composite strut.

Parts 3 and 4 are the undelaminated regions and parts 1 and 2 describe the upper und lower sublaminate respectively. Subsequently, owing to the symmetry of the problem, the contributions from part 4 will be incorporated in part 3. The strut has a total length of L_{tot}, an initial delamination length of L and a thickness of t. The delamination is assigned in between two layers of the multi-layered strut and its depth $(a\,t)$ is described by the parameter a.

The system is fully described by four generalized coordinates (GCs) $q_1 ... q_4$ (*cf.* Fig. 1). The amplitudes of the upper and lower sublaminate are given by q_1 and q_2 respectively. The rotation at the interface between undelaminated and delaminated parts is described by q_3. A fourth generalized coordinate (q_4) which is not shown in Fig. 1 is assigned to the model representing the total end-shortening of the delaminated region [6].

2.2 The energy formulation

2.2.1 Stationary delamination

The buckling and postbuckling behaviour for a stationary delamination is modelled with the aid of the total potential energy (Π). Assuming a quasi-static deformation process and using a set of I generalized coordinates to describe the displacement field, the total potential energy [18] reads

$$\Pi(q_i) = W(q_i) - P\mathcal{E}(q_i) \quad i = 1, 2, ..., I, \tag{1}$$

where W is the strain energy and P is a conservative load with its conjugate displacement $\mathcal{E}$ (end-shortening of the strut). If the system is in an equilibrium state, the first variation of the total potential energy vanishes [18], *i.e.* $\delta\Pi = 0$, yielding the deformation path $P(q_i)$.

The strain energy is derived using the Classical Laminate Theory [10] in which the in-plane stiffness A_{ij} relates to the in-plane strains $\{\varepsilon_0\}$,

$$\left\{\varepsilon_0\right\}^{ⓚ} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{ⓚ} \tag{2}$$

and the bending stiffness D_{ij} to the curvatures $\{\kappa_0\}$,

$$\left\{\kappa_0\right\}^{ⓚ} = \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}^{ⓚ} = \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y}. \end{Bmatrix}^{ⓚ} . \tag{3}$$

The respective parts of the strut (*cf.* Fig. 1) are indicated in Eqs. (2) and (3) by the circled index k. Coupling effects due to asymmetric layups caused by the delamination are also considered in the model by incorporating the coupling stiffness matrix, B_{ij}.

In the current work, just $\varepsilon_{xx}^{ⓚ}$ and $\kappa_{xx}^{ⓚ}$ are considered in the analytical model (one-dimensional formulation). However, entries in the stiffness matrices (D_{ij},

A_{ij}) responsible for bending-twisting, bending-bending, as well as stretching-shearing and stretching-stretching are considered and incorporated by calculating effective parameters for in-plane (A_{eff}), coupling (B_{eff}) and bending (D_{eff}) stiffness. This is done assuming that the forces n_{yy} and n_{xy}, as well as the moments m_{yy} and m_{xy} are zero which allows to determine the resulting strains ($\varepsilon_{yy}, \varepsilon_{xy}$) and curvatures ($\kappa_{yy}, \kappa_{xy}$). Such strains and curvatures are subsequently replaced in the relevant expressions for n_{xx} and m_{xx}. Moreover, it is assumed that the undelaminated strut has a symmetric layup and that the in-plane displacement of part 3 is negligible [6].

In order to derive the in-plane strains of the delaminated parts $\varepsilon_{xx}^{ⓚ}$, a non-linear kinematic approach [6], which is not dissimilar to the well-known VON KÁRMÁN approach [10], is implemented, such that further in-plane stretching is considered in the postbuckling regime [19]. Thus, the respective in-plane displacements $u^{ⓚ}$ consist of the total end-shortening of the delaminated region q_4, the purely geometric displacement due to buckling [19] and the displacement due to the rotation at the interface, *i.e.*

$$
\begin{aligned}
u^{①} &= q_4 - \frac{1}{2}\int_0^L \left(\frac{\partial w_1}{\partial x_1}\right)^2 \mathrm{d}x_1 - (1-a)tq_3, \\
u^{②} &= q_4 - \frac{1}{2}\int_0^L \left(\frac{\partial w_2}{\partial x_2}\right)^2 \mathrm{d}x_2 + atq_3.
\end{aligned} \tag{4}
$$

Dividing Eq. (4) by the delamination length L yields the in-plane strains of the delaminated parts $\varepsilon_{xx}^{ⓚ}$.

The strain energy W is equal to the sum of stretching (W_{u}) and bending energy (W_κ) of the respective parts and reads

$$
\begin{aligned}
W &= W_{\text{u}}^{ⓚ} + W_{\kappa}^{ⓚ} = \\
&\frac{1}{2}b\Bigg(2D_{\text{eff}}^{③}\int_0^{SL}\left(\kappa_{xx}^{③}\right)^2 \mathrm{d}x_3 + \\
&\int_0^L \left(D_{\text{eff}}^{①}\left(\kappa_{xx}^{①}\right)^2 + A_{\text{eff}}^{①}\left(\varepsilon_{xx}^{①}\right)^2 + \right. \\
&\left. 2B_{\text{eff}}^{①}\kappa_{xx}^{①}\varepsilon_{xx}^{①}\right)\mathrm{d}x_1 + \int_0^L \left(D_{\text{eff}}^{②}\left(\kappa_{xx}^{②}\right)^2 + \right. \\
&\left. A_{\text{eff}}^{②}\left(\varepsilon_{xx}^{②}\right)^2 + 2B_{\text{eff}}^{②}\kappa_{xx}^{②}\varepsilon_{xx}^{②}\right)\mathrm{d}x_2\Bigg).
\end{aligned} \tag{5}
$$

As can be seen in Eq. (5), owing to the assumptions made no coupling effects are considered for part 3 (all entries of $B_{ij}^{③}$ are zero) and its stretching energy contribution is omitted.

As the in-plane displacement of the undelaminated region is omitted the end-shortening of the strut $\mathcal{E}$ can be written as

$$
\mathcal{E} = q_4 + \int_0^{SL}\left(\frac{\partial w_3}{\partial x_3}\right)^2 \mathrm{d}x_3, \tag{6}
$$

in which the second term describes the end-shortening associated to the buckling (geometric) displacement of parts 3 and 4 (*cf.* Fig. 1).

Hence, all terms of the total potential energy (Eq. (1)) are determined. In order to derive the total potential energy as a function of I generalized coordinates (GCs), a RAYLEIGH-RITZ formulation is employed approximating the buckling displacements (for all parts) while enforcing the geometric boundary and the continuity conditions. Using trigonometric functions for the respective buckling displacement of each part and adding a polynomial function which satisfies the boundary conditions (equal displacement and slope at the interface) for the delaminated parts yields

$$
\begin{aligned}
w_i &= q_i \sin^2\left(\frac{\pi x_i}{L}\right) + C_0 x_i^3 + C_1 x_i^2 + C_2 x_i + C_3 \\
&\quad \text{with} \quad i = 1,2 \quad \text{and} \\
C_0 &= \frac{q_3}{2L^3}\left(\frac{L_{\text{tot}}}{\pi}\tan\left(\frac{\pi SL}{L_{\text{tot}}}\right) - \right. \\
&\quad \left. \frac{L_{\text{tot}}}{\pi\cos\left(\frac{\pi SL}{L_{\text{tot}}}\right)}\sin\left(\frac{\pi(SL+L)}{L_{\text{tot}}}\right)\right), \\
C_1 &= \frac{1}{2}\Big(-\frac{2q_3}{L} - 3C_0 L\Big), \\
C_2 &= q_3, \quad C_3 = \frac{q_3 L_{\text{tot}}}{2\pi}\tan\left(\frac{\pi SL}{L_{\text{tot}}}\right), \\
w_3 &= q_3 \frac{L_{\text{tot}}}{2\pi\cos\left(\frac{\pi SL}{L_{\text{tot}}}\right)\sin\left(\frac{\pi SL}{L_{\text{tot}}}\right)}\sin^2\left(\frac{\pi x_3}{L_{\text{tot}}}\right),
\end{aligned} \tag{7}
$$

in which w_1 and w_2 describe the buckling displacement for the upper and lower sublaminate respectively. The buckling displacement of the undelaminated part is approximated with w_3.

Inserting Eqs. (4) and (7) in Eq. (1) and applying the variational principle, $\delta\Pi = 0$, yields a set of four non-linear algebraic equations which comprises the buckling and postbuckling response of the composite strut, *i.e.* $P(q_i)$. The set of equations is solved numerically using the software package AUTO-07P [3] which applies a continuous NEWTON method.

From the solution obtained, the deformation of the system in terms of load *vs.* buckling displacement (P *vs.* w) and load *vs.* end-shortening (P *vs.* $\mathcal{E}$) may be readily calculated using Eqs. (6) and (7).

2.2.2 Non-stationary delamination

Delamination growth occurs if the condition

$$G \geq G_{\mathrm{c}} \tag{8}$$

is met, *i.e.* the energy release rate G equals or exceeds the critical energy release rate G_{c} [2].

The energy release rate G is the thermodynamic force [12,14] available for delamination growth. The force required for a change in structure is the critical energy release rate G_{c} which is a material parameter depending on the mode mixity [7]. However, for reasons of simplicity G_{c} is assumed constant in the current work. The energy release rate is calculated as

$$G = -\frac{\partial \Pi(q_i, P, L)}{b \partial L}, \tag{9}$$

using the solution obtained for a stationary delamination, $P(q_i)$. In Eq. (9), b describes the width of the strut and L is the delamination length. The delamination length L may also be understood as a damage parameter describing the current state of damage within the structure (no further damage is assumed within the struture).

Once the critical energy release rate is reached delamination growth occurs. Thus, for the current state of loading (quasi-static process) associated to the critical energy release rate G_{c} the delamination length increases from L to $L+\Delta L$. For the new configuration, *i.e.* the constantly kept state of loading and a delamination length of $L+\Delta L$, the energy release rate changes from G_{c} to another value, which we will refer to as G_{new}. If the energy release rate decreases ($G_{\mathrm{new}} < G_{\mathrm{c}}$) growth stops and the process is termed *stable*. If $G_{\mathrm{new}} > G_{\mathrm{c}}$, unstable growth occurs yielding failure of the strut. If stable delamination growth occurs further loading may be applied until G_{c} is reached again. Thus, the postbuckling response under stable delamination growth follows the condition $G = G_{\mathrm{c}}$ which also dictates the load which may be applied to the system. The condition of stable delamination growth can be written as

$$G = -\frac{\partial \Pi(q_i, P, L)}{b \partial L} = G_{\mathrm{c}}, \tag{10}$$

yielding a function D,

$$D(q_i, P, L) = G - G_{\mathrm{c}} \equiv 0, \tag{11}$$

from which the delamination length $L = L(q_i, P)$ is implicitly given assuming that an unique solution of $L = L(q_i, P)$ exists. Thus, Eq. (11) may be rewritten as

$$D\big(q_i, P, L(q_i, P)\big) = G - G_{\mathrm{c}} \equiv 0. \tag{12}$$

However, to derive an explicit form of $L(q_i, P)$ a Taylor series approximation around the state where growth occurs first is employed. This state will be referred to as "damage state" which is characterized by the set of generalized coordinates q_i^0 and the load P^0. Thus, the Taylor series approximation of $L(q_i, P)$ yields

$$\begin{aligned} L(q_i, P) = {} & L^0 + \\ & \left.\frac{\partial L}{\partial q_i}\right|_{\substack{q_i^0 \\ P^0}} (q_i - q_i^0) + \left.\frac{\partial L}{\partial P}\right|_{\substack{q_i^0 \\ P^0}} (P - P^0) + \\ & \frac{1}{2} \left.\frac{\partial^2 L}{\partial q_i \partial q_j}\right|_{\substack{q_i^0 \\ P^0}} (q_i - q_i^0)(q_j - q_j^0) + \\ & \frac{1}{2} \left.\frac{\partial^2 L}{\partial P^2}\right|_{\substack{q_i^0 \\ P^0}} (P - P^0)^2 + \\ & \left.\frac{\partial^2 L}{\partial q_i \partial P}\right|_{\substack{q_i^0 \\ P^0}} (q_i - q_i^0)(P - P^0) + \mathcal{O}(3), \end{aligned} \tag{13}$$

which may be extended to higher orders, if required. In the current study, an approximation up to the third order is implemented as it was found optimal with regards to accuracy and computational cost. The derivatives of L with respect to q_i and P are provided by Eq. (12) by implicit differentiation with respect to q_i and P. Thus, by applying the chain rule and rearranging, the derivatives of L used in Eq. (13) may be readily obtained.

The postbuckling response while delamination growth occurs is determined by, first, identifying the "damage state" (q_i^0, P^0) where the conventional total potential energy formulation (Eq. (1)) loses its validity. This is done with the aid of Eq. (9) and the solution for the stationary delamination $P(q_i)$. Subsequently, an extended total potential energy (Π^*) formulation is derived by adding the dissipative energy term W_{d} associated with delamination growth [13,14],

$$W_{\mathrm{d}} = W_{\mathrm{d}}(L) = G_{\mathrm{c}}(L - L^0) b, \tag{14}$$

to Eq. (1) in which L^0 is the initial delamination length and b is the width of the delamination area (assumed to be constant). In a next step, the damage parameter L is replaced in Eqs. (1) and (14) by the expression derived in Eq. (13). For the condition of stable delamination growth, *i.e.* $G = G_{\mathrm{c}}$, the extended total potential energy,

$$\begin{aligned} \Pi^*(q_i) = {} & W\big(q_i, L(q_i, P)\big) + W_{\mathrm{d}}\big(L(q_i, P)\big) - \\ & P \mathcal{E}\big(q_i, L(q_i, P)\big), \end{aligned} \tag{15}$$

is a true potential [13], such that $\partial \Pi^* / \partial P = -\mathcal{E}$. Hence, the variational principle using the extended total potential energy may be written as

$$\delta\Pi^*(q_i) = \delta\Big(W\big(q_i, L(q_i, P)\big) + W_\mathrm{d}\big(L(q_i, P)\big) - P\mathcal{E}\big(q_i, L(q_i, P)\big)\Big) = 0, \tag{16}$$

from which the solution in terms of $P(q_i)$ starting at the "damage state" (q_i^0, P^0) is obtained by solving the resulting set of algebraic equations numerically by applying the NEWTON method.

During the solution process, an iterative scheme is implemented in which a new expression for the delamination length is derived and inserted in Eq. (16) once the condition of $G = G_\mathrm{c}$ is violated due to the approximation by the TAYLOR series.

3 Results

A composite strut with a cross-ply layup ($[0°/(90°/0°)_7]$) is examined in this study. The dimensions and material parameters are taken from Ref. [15] and are listed in Tab. 1. First, solutions obtained for a stationary delamination are compared to findings documented within the literature. Subsequently, postbuckling responses without the limitation to stationary delamination are presented and compared to results obtained by a FE-simulation using the commercial software code Abaqus [4].

Table 1 Dimensions and Material parameters of the composite strut.

Dimensions		Material Parameters	
L_tot	96.52 mm	E_{11}	137.90 GPa
b	12.7 mm	E_{22}	8.98 GPa
t	1.337 mm	G_{12}	7.20 GPa
a	4/15	ν_{12}	0.30
h	0.0889 mm	G_c^I	190 Nm/m^2

3.1 Stationary delamination

The postbuckling response of a strut containing a pre-existing delamination length of $L = 50.8\,\mathrm{mm}$ is illustrated in Fig. 2 in terms of load *vs.* midpoint deflection (P *vs.* $w_{1,2}(x_{1,2} = L/2)$). Findings documented by Ref. [15] are indicated by "□" symbols.

In Fig. 2, normalization of the load is carried out, as documented in Ref. [15], with respect to the EULER buckling load of an undelaminated strut with 0° layers only. All results to follow will be normalized to the EULER buckling load of the cross-ply laminate used. The midpoint deflection is normalized with respect to the total thickness of the strut t.

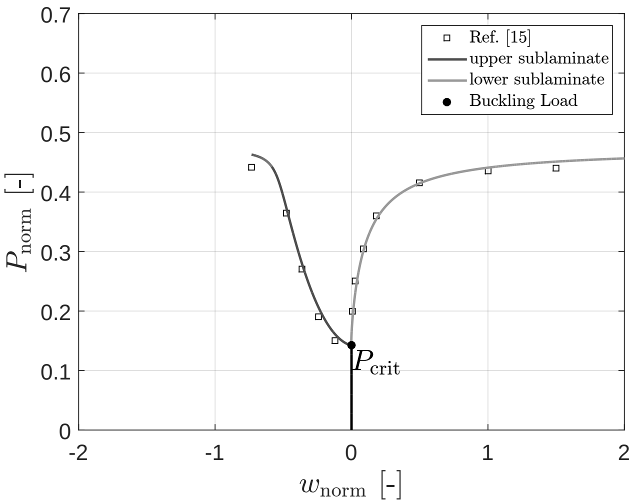

Fig. 2 Postbuckling response for a stationary delamination length: normalized load P_norm *vs.* normalized midpoint deflection w_norm.

As can be seen in Fig. 2, the model proposed in section 2.2.2 yields results which are in excellent agreement with Ref. [15].

3.2 Non-stationary delamination

First, the energy release rate G is determined and evaluated with respect to the delamination length for different states of end-shortening ($\mathcal{E}$) applied to the strut (Fig. 3). Normalization of the energy release rate is carried out with respect to the critical energy release rate given in Tab. 1. As described in section 2.2.2, mode mixity is not considered in this study, thus the critical energy release rate of Mode I is taken as a conservative measure for G_c, *i.e.* $G_\mathrm{c} = G_\mathrm{c}^\mathrm{I}$. Hence, if $G_\mathrm{norm} \geq 1$ delamination growth is assumed to occur.

Depending on the length of the pre-existing delamination, Fig. 3 indicates regions of stable and unstable delamination growth. For a better understanding, two vertical dashed lines are added to Fig. 3 at $L_\mathrm{norm} = 0.36$ and $L_\mathrm{norm} = 0.45$. Taking a closer look at the case of $L_\mathrm{norm} = 0.36$, loading (here understood as increasing end-shortening) would initially follow the vertical dashed line as no delamination growth occurs, *i.e.* $G_\mathrm{norm} < 1$. At an end-shortening of $\mathcal{E}_\mathrm{norm} = 6.58$, $G_\mathrm{norm} = 1$ which yields delamination growth. As described in section 2.2.2, stable or unstable growth is characterized whether the energy release rate increases (unstable) or decreases (stable) for a constant state of

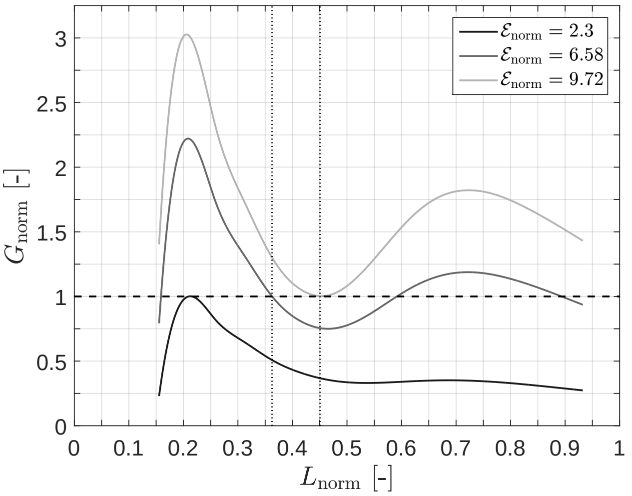

Fig. 3 Normalized energy release rate G_{norm} *vs.* normalized delamination length L_{norm} for different states of end-shortening.

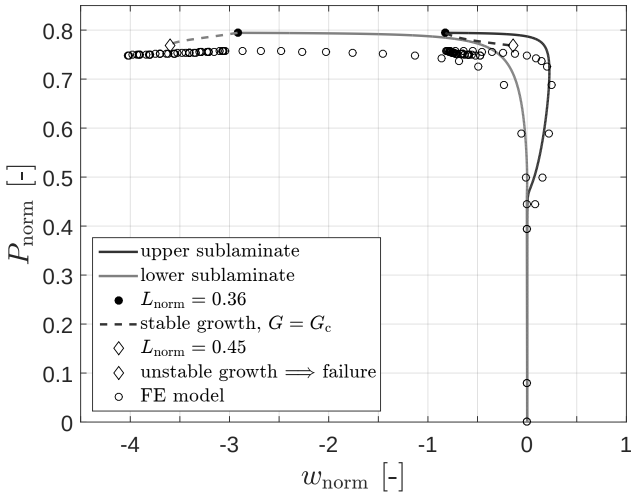

Fig. 4 Normalized load P_{norm} *vs.* normalized midpoint deflection w_{norm} for an initial delamination length $L_{\mathrm{norm}} = 0.36$ and depth $a = 4/15$.

loading and increasing delamination length. Thus, stable delamination growth is present from $L_{\mathrm{norm}} = 0.36$ to 0.45 which dictates that loading will follow the horizontal dashed line in this region (from $\mathcal{E}_{\mathrm{norm}} = 6.58$ to $\mathcal{E}_{\mathrm{norm}} = 9.72$). At $L_{\mathrm{norm}} = 0.45$, the energy release rate increases for a constant state of loading causing unstable growth and failure.

Subsequently, the postbuckling response of the composite strut without the limitation of a stationary delamination length is presented. Therefore, a pre-existing delamination of $L_{\mathrm{norm}} = 0.36$ is assigned to the strut without changing the depth of the delamination ($a = 4/15$).

Fig. 4 describes the postbuckling behaviour in terms of $P_{\mathrm{norm}}(w_{\mathrm{norm}})$. Initially, Fig. 4 shows a local buckling response (local buckling load of approx. $P_{\mathrm{norm}} = 0.45$), thus the upper (less stiff) sublaminate shows a buckling deflection in positive direction whereas the lower sublaminate remains unaffected. Once global buckling is triggered, the system shifts towards the negative direction showing the characteristic asymptotic behaviour of a strut. Shortly after the global buckling load is reached, delamination growth occurs which is indicated by a "•" symbol in Fig. 4. The response of the system changes abruptly, causing deflections of the upper and lower sublaminate in opposite directions. Stable delamination growth is present and indicated by the dashed line in Fig. 4. The postbuckling response during delamination growth illustrates a loss in load bearable by the strut off approximately 3.3% (from "•" to "⋄"). The delamination grows from $L_{\mathrm{norm}} = 0.36$ to 0.45. At the "⋄" symbol ($L_{\mathrm{norm}} = 0.45$), unstable growth causes failure of strut as indicated by Fig. 3. Material failure is indicated by "⋄" which is reached if a displacement-controlled setup is given. For a load-controlled setup, the strut fails due to a loss of stability as the "•" symbol describes a limit point [18].

In order to verify the results obtained, a FE-simulation is carried out using the software Abaqus. Therefore, the strut is built-up by shell elements (type S4R) with an element size of 0.2 mm by 0.2 mm and a total of 62790 nodes. Delamination propagation is modelled using the virtual crack closure technique in which the fracture criterion is adjusted, such that $G_{\mathrm{c}} = G_{\mathrm{c}}^{\mathrm{I}}$.

As can be seen in Fig. 4, the FE-simulation illustrated by "∘" symbols is in good agreement with the analytical model. However, the FE-simulation yields smaller values of load compared to the analytical model. The global buckling load is approximately 4-5 % smaller. Such deviations might be expected considering the enormous difference in degrees of freedom between the FE-model and the analytical model. The analytical model also omits shear deformations which might contribute to the deviations obtained.

Fig. 5 provides a detailed look at the postbuckling response once global buckling occurred and delamination growth is triggered. Both models show similar qualitative results for the buckling displacements of both sublaminates. As described before, both sublaminates deflect in opposite direction during delamination growth. The prediction of the buckling displacement at which growth occurs (•) almost coincides for the upper sublaminate and shows small deviations between the analytical and the FE model for the lower sublaminate.

Further information can be gained by evaluating the load-displacement (end-shortening) behaviour of the

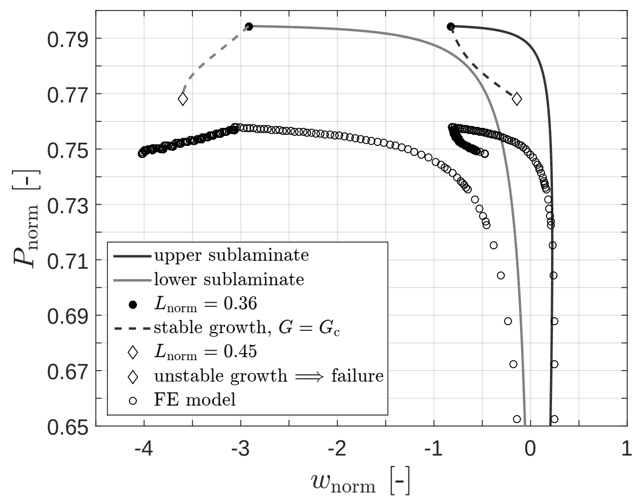

Fig. 5 Detailed look at the postbuckling response $P_{\text{norm}}(w_{\text{norm}})$ around the region where delamination growth occurs.

strut which is illustrated in Fig. 6. First, comparing the values for the end-shortening at the initiation of delamination growth (•) and at failure (◇) to the predictions made in Fig. 3 verifies the formulation used. Moreover, Fig. 6 describes a certain additional capacity of end-shortening which could be applied to the strut (from $\mathcal{E}_{\text{norm}} = 6.58$ to 9.72) before failure occurs even though the delamination grows from $L_{\text{norm}} = 0.36$ to 0.45. During this deformation process the load would just decrease by approximately 3.3%. As discussed before, such additional loading in form of end-shortening can be applied only if a displacement-controlled setup is given.

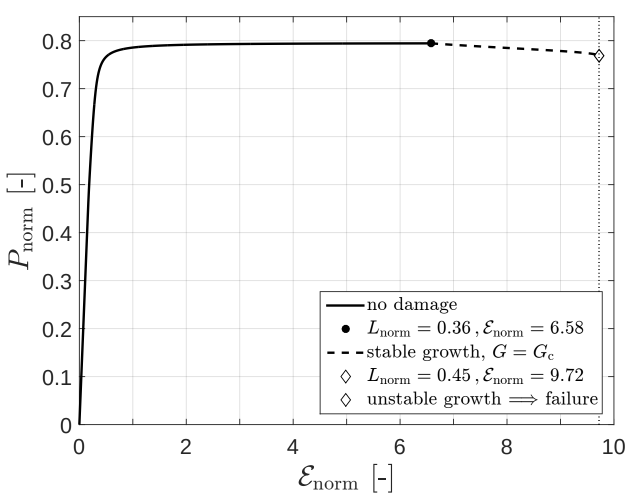

Fig. 6 Normalized load P_{norm} *vs.* normalized end-shortening $\mathcal{E}_{\text{norm}}$.

4 Conclusions

The postbuckling behaviour of composite struts incorporating possible delamination growth (*i.e.* no limitation to stationary delamination lengths) has been modelled analytically. This represents an extension to pre-existing analytical models within the literature which are limited to stationary delaminations or at most describe whether delamination growth would occur and whether such growth would be stable or unstable.

The model proposed intends to describe the postbuckling behaviour by means of a few generalized coordinates. As a consequence, contributions to the displacement field such as shear deformations are omitted as the influence is arguably small and computational cost would increase significantly, specifically in deriving the delamination length in terms of the generalized coordinates and the load parameter. However, if intended, such extensions may be readily added to the model. The comparison to the FE-simulation has shown that such simplifications most likely explain the deviations obtained in maximum load bearable by the system of up to 5 %. Such a deviation appears satisfactory considering the simplicity and efficiency of the model. On the other hand, the approach presented does not satisfy the local asymptotic behaviour of the displacement field around the delamination tip and the crack interface which is, however, beyond the scope of the work and assumed to be nonessential with regards to the physical quantities calculated and hence the results obtained.

The prediction of the onset of delamination growth as well as the postbuckling behaviour during delamination growth are in good agreement with the results obtained by the FE-simulation. Differences obtained appear inevitable owing to deviations in the maximum load as the formulation of the energy release rate depends on load and generalized coordinates. Characteristic changes obtained in the postbuckling response once delamination grows, such as the deflection of both sublaminates in opposite directions, have also been verified by the FE-simulation.

In summary, an analytical framework for studying the postbuckling response of delaminated composite struts is presented. As possible delamination growth is considered within the formulation, the postbuckling response is not restricted to stationary delaminations. Thus, postbuckling responses so far solely described by FE-simulations are determined analytically.

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