Top Trading Cycles, Consistency, and Acyclic Priorities for House Allocation with Existing Tenants*

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Abstract

We study the house allocation with existing tenants model (Abdulkadiroğlu and Sönmez, 1999) and consider rules that allocate houses based on priorities. We introduce a new acyclicity requirement and show that for house allocation with existing tenants a top trading cycles (TTC) rule is *consistent* if and only if its underlying priority structure satisfies our acyclicity condition. Next we give an alternative description of TTC rules based on ownership-adapted acyclic priorities in terms of two specific rules, *YRMH-IGYT (you request my house - I get your turn)* and *efficient priority rules*, that are applied in two steps. Moreover, even if no priority structure is *a priori* given, we show that a rule is a top trading cycles rule based on ownership-adapted acyclic priorities if and only if it satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *consistency*, and either *reallocation-proofness* or *non-bossiness*.

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1 Introduction

We consider the allocation of indivisible objects when agents have preferences over the objects and objects possibly have priorities for the agents. This problem occurs in many applications,

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e.g., for school choice, the allocation of social or university housing to tenants, or kidney exchange. In many applications, some of the agents additionally have existing claims for an object that have to be respected. When allocating social housing, there might be existing tenants that already occupy a housing unit, but may want to switch if they can obtain a unit that they prefer, while new applicants have to be newly assigned to a vacant unit. When assigning teachers to jobs (Combe et al., 2018), employed teachers may have the option to be reassigned to a new job, while new teachers have to be assigned to their first job. In many school choice districts students have a right to attend a neighborhood school that has to be respected in the assignment.

These situations are captured by the model of *house allocation problems with existing tenants* which is introduced by Abdulkadiroğlu and Sönmez (1999): A finite set of houses has to be allocated to a finite set of agents without using monetary transfers. Each agent is either a tenant who occupies a house or an applicant, and each house is either occupied or vacant. Furthermore, each agent has strict preferences over all houses and the so-called null house (or outside option). An outcome for a house allocation problem with existing tenants is a matching that assigns to each agent either a real house or his outside option, such that no real house is assigned to more than one agent. A rule selects a matching for each house allocation problem with existing tenants.

The model of house allocation problems with existing tenants is a hybrid of two models, *housing markets* and *house allocation problems*. A house allocation problem with existing tenants reduces to a *housing market* (Shapley and Scarf, 1974) if there are no applicants and no vacant houses, i.e., all agents are tenants and all houses are occupied. A house allocation problem with existing tenants reduces to a *house allocation problem* (Hylland and Zeckhauser, 1979) if there are no tenants and no occupied houses, i.e., all agents are applicants and all houses are vacant.

Ideally, any rule used in practice to allocate indivisible objects, with or without existing claims, would be *efficient*, *strategically robust* and *fair*. In the case of housing markets, *efficiency* in the form of *Pareto optimality*,[1] *strategical robustness* in the form of *strategy-proofness*[2], and *fairness* in the limited sense that existing rights are respected, can be achieved by the top-trading cycles rule (Ma, 1994; Roth, 1982). However, for more general settings and for fairness in the stronger sense of *no justified envy* (or *stability*),[3] no rule satisfying all three desiderata exists (Ergin, 2002). Since no ideal allocation rule exists, it is important to study the trade-offs of the various properties that represent *efficiency*, *fairness*, and *strategic robustness*. This research agenda uses the *axiomatic method* (see Thomson, 2001) to understand the various normative trade-offs when implementing certain

[1]A rule is *Pareto-optimal* if the matching chosen by the rule is such that there is no other matching that makes some agents better off without hurting the others.

[2]A rule is *strategy-proof* if no agent can ever benefit by misrepresenting his preferences unilaterally.

[3]A matching *eliminates justified envy* if whenever an agent prefers another agent's match to his own, he has a lower priority than the other agent for that house. Existing rights of tenants are incorporated by giving each tenant top priority at the house that he occupies. A matching is *stable* if it *eliminates justified envy*, is *individually-rational*, and *non-wasteful* (see Balinski and Sönmez, 1999).

types of rules instead of others. To place our paper within this research agenda, we would like to refer to the characterizations of deferred acceptance mechanisms (Ehlers and Klaus, 2014, 2016; Kojima and Manea, 2010) and immediate acceptance mechanisms (Doğan and Klaus, 2018; Kojima and Ünver, 2014): in all those characterizations the priorities (or more generally, choice functions) are obtained together with the rule using a set of normative criteria that reflect our desiderata. Similar characterizations for top trading cycles rules, without exogenously fixed priorities, have not been established yet. We are interested in completing the research agenda of a full normative understanding of standard house allocation and school choice rules. To this end, in this paper, we study rules that allocate houses to agents for house allocation problems with existing tenants with a particular focus on top trading cycles (TTC) rules.

Next, we survey the state of the art of the axiomatic approach for housing market, house allocation, and house allocation with existing tenants problems. Then, we formulate the exact research question we tackle in this paper and explain our results against the background of the literature.

Related Literature

Our paper is related to studies that consider not only house allocation problems with existing tenants but also housing markets and house allocation problems. We first look at some key results for housing markets and house allocation problems.

An important question that emerged in both the literature on housing markets as well as house allocation problems is the characterization of rules that allocate houses in a *Pareto-optimal* and *strategy-proof* way. For housing markets, Roth and Postlewaite (1977, Theorem 2') showed that the *core* of a housing market[4] is unique and it is the outcome of the *top trading cycles (TTC) algorithm*.[5] Roth (1982) showed that the core/TTC rule is *strategy-proof*, and Bird (1984) showed that it is also *group strategy-proof*.[6] Ma (1994, Theorem 1) characterized the core/TTC rule of a housing market by *Pareto-optimality*, *individual-rationality (for tenants)*,[7] and *strategy-proofness*; see also Sönmez (1999, Corollary 3) and Svensson (1999, Theorem 2).

For house allocation problems, Svensson (1999, Theorem 1) showed that a rule satisfies *strategy-proofness*, *non-bossiness*[8], and *neutrality*[9] if and only if it is a *simple serial dictator-*

[4] A matching for a housing market is in the *core* (or *core stable*) if no subset of agents exists such that some of them strictly benefit by reallocating their occupied houses among themselves, without hurting other agents in the group.

[5] The TTC algorithm was defined in Shapley and Scarf (1974) and attributed to David Gale.

[6] A rule is *group strategy-proof* if no group of agents can ever benefit by misrepresenting their preferences.

[7] A rule for housing markets satisfies *individual-rationality (for tenants)* if no agent is assigned a house that is worse for him than his occupied house.

[8] A rule satisfies *non-bossiness* if an agent by changing his preferences gets the same allocation under the rule, then the change in his preferences does not affect the allocation of other agents, and therefore the allocation of each agent by the rule remains the same.

[9] A rule for house allocation problems is *neutral* if the matching selected by the rule is independent of

ship rule.[10] The following two research contributions show what can happen when *neutrality* is dropped in house allocation problems.

Pápai (2000) introduced *hierarchical exchange rules*: hierarchical exchange rules extend the way TTC rules work by specifying ownership rights for the houses in an iterative hierarchical manner and by allowing for associated iterative top trades. Pápai (2000) showed that a rule for house allocation problems satisfies *Pareto-optimality*, *strategy-proofness*, *non-bossiness*, and *reallocation-proofness*[11] if and only if it is a hierarchical exchange rule.[12] Pycia and Ünver (2017, Theorem 1) extended this result by providing a full characterization of the class of *Pareto-optimal*, *strategy-proof*, and *non-bossy* rules, called *trading cycles (TC) rules*. The set of TC rules extends the set of hierarchical exchange rules by allowing agents to not only own houses throughout the iterative trading cycles allocation procedure but to also have a different "control right" called "brokerage" (a broker cannot necessarily consume a brokered house directly himself but he can trade it for another house he desires).

Another important precursor of our study is the work of Sönmez and Ünver (2010) on *YRMH-IGYT (you request my house - I get your turn) rules* introduced by Abdulkadiroğlu and Sönmez (1999). Sönmez and Ünver (2010, Theorem 1) showed that a rule for house allocation problems with existing tenants satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *weak neutrality*,[13] and *consistency* if and only if it is a YRMH-IGYT rule. YRMH-IGYT rules are essentially simple serial dictatorship rules that adapt to the ownership rights of tenants. Therefore, one question that motivated our work was what rules emerge for house allocation problems with existing tenants when dropping *weak neutrality*.

For house allocation problems, if *consistency*[14] is considered in addition to *Pareto-optimality* and *(group) strategy-proofness*, then rules based on acyclic priorities become focal.For house allocation with quotas problems, which reduce to house allocation problems when the quota of each house is one, Ergin (2002) and Kesten (2006) studied allocation rules and their properties in relation to (*acyclic*) priorities. Ergin (2002, Theorem 1) showed that for the agents-proposing deferred acceptance rule (Gale and Shapley, 1962) based on a priority structure π, denoted by DA^π, the following are equivalent: DA^π is *Pareto-optimal*,

the names of the houses.

[10] A *simple serial dictatorship* determines a matching based on an exogenously given ordering of the agents such that the first agent chooses his favorite house, the second agents chooses his favorite house among those that remain, etc.

[11] A rule for house allocation problems is *reallocation-proof* if there do not exist two agents who gain by first misreporting their preferences and then swapping their assigned houses, such that neither of the two agents can change his assignment by misreporting alone.

[12] A rule satisfies *strategy-proofness* and *non-bossiness* if and only if it satisfies *group strategy-proofness* (Pápai, 2000).

[13] A rule for house allocation problems with existing tenants is *weakly neutral* if it is independent of the names of the vacant houses.

[14] A rule for house allocation problems is *consistent* if the following holds: suppose that after houses are allocated according to the rule, some agents leave the house allocation problem with their assigned houses. Then, if the remaining agents were to allocate the remaining houses according to the rule, each of them would receive the same house.

DA^π is *group strategy-proof*, DA^π is *consistent*, and π is Ergin acyclic. Kesten (2006, Theorems 1 and 2) strengthened Ergin's acyclicity condition in two ways and showed that the TTC rule based on the priority structure is the agents-proposing deferred acceptance rule based on the priority structure if and only if the priority structure is Kesten acyclic, and the TTC rule based on the priority structure is *consistent* if and only if the priority structure is strongly Kesten acyclic. Ergin (2002) and Kesten (2006) use different notions of acyclicity that coincide for house allocation problems. Furthermore, in their studies the priority structure is exogenously given. However, the class of TTC rules based on priority structures is a subclass of the hierarchical exchange rules studied by Pápai (2000). Hence, under *group strategy-proofness* and *reallocation-proofness*, priorities can endogenously arise. This is also the case under *consistency*; Ehlers and Klaus (2006, Proposition 2 and Theorem 1) characterized *efficient priority rules*[15] for house allocation problems by *Pareto-optimality*, *strategy-proofness*, and *reallocation-consistency*.[16] Ehlers and Klaus (2007) and Velez (2014) characterize a slightly larger class of rules by weakening the characterizing properties either to *consistency* (Ehlers and Klaus, 2006) or to versions of *Pareto-optimality* and *consistency* that pertain to two agent (reduced) problems only. Ergin's (2000, Theorem 1 and Corollary 1) results imply that a rule for house allocation problems satisfies *Pareto-optimality*, *neutrality*, and *consistency* if and only if it is a simple serial dictatorship rule (he uses somewhat weaker properties to show his result).

Table 1 summarizes the properties of various house allocation rules that we have just reviewed (some of them characterize the respective rules).

Our Paper

Motivated by these previous works, we extend the analysis of *Pareto-optimal*, *strategy-proof*, and *consistent* rules to the more general model of house allocation with existing tenants. We observe that in house allocation problems with and without exiting tenants, requiring *(weak) neutrality* together with these properties results in simple serial dictatorship rules (here we interpret the YRMH-IGYT rules as ownership adapted simple serial dictatorship rules). We explore what happens when *(weak) neutrality* is dropped.

First, we extend the notion of (Ergin/Kesten) acyclicity to house allocation problems with existing tenants and show that a TTC rule based on ownership-adapted priorities[17] is *consistent*[18] if and only if the priority structure is acyclic (Theorem 1). Moreover, a TTC rule

[15]A rule is an *efficient priority rule* if it is an agents-proposing deferred acceptance rule based on an Ergin acyclic priority structure.

[16]A rule for house allocation problems is *reallocation-consistent* if the following holds: suppose that after houses are allocated according to the rule some agents are removed from the problem with their assigned houses, then, if these removed agents were to allocate their assigned houses (the removed houses) among themselves according to the rule, each of them would receive the same house.

[17]Priorities are *ownership-adapted* if each tenant has top priority at his occupied house.

[18]A rule for house allocation problems with existing tenants is *consistent* if the following holds: suppose that after houses are allocated according to the rule we remove some agents with their assignments and some unassigned houses from the problem in a way that the reduced problem contains the occupied houses of all

House allocation rules	PO	IR	SP	NB	RP	RCON	NEU
simple serial dictatorship	✓	✓	✓	✓	✓	✓	✓
efficient priority	✓	✓	✓	✓	✓	✓	
hierarchical exchange	✓	✓	✓	✓	✓		
trading cycles	✓	✓	✓	✓			

Notation:
PO stands for *Pareto-optimality*,
IR stands for *individual-rationality*,
SP stands for *strategy-proofness*,
NB stands for *non-bossiness*,
RP stands for *reallocation-proofness*,
RCON stands for *reallocation-consistency*, and
NEU stands for *neutrality*.

Table 1: House allocation rules and their properties.

based on ownership-adapted acyclic priorities can be interpreted as a *two-step rule* where the first rule is an *almost YRMH-IGYT rule* and the second rule is an *efficient priority rule* of Ehlers and Klaus (2006) (Proposition 2).

For house allocation problems *consistency* implies *non-bossiness* and hence the set of rules satisfying *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, and *consistency* is a (strict) subset of the trading cycles (TC) rules defined and characterized by Pycia and Ünver (2017). However, for house allocation problems with existing tenants *consistency* does *not* imply *non-bossiness* and we show that the class of rules that satisfy *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, and *consistency* is neither a sub- nor a superset of Pycia and Ünver's trading cycles rules (Example 6). However, by adding either *reallocation-proofness* (Theorem 2) or *non-bossiness* (Theorem 3), we reduce the class of rules to the set of TTC rules based on ownership-adapted acyclic priorities.

One small but important difference between the model considered by Svensson (1999), Pápai (2000), Velez (2014), and Pycia and Ünver (2017) and our model, apart from the presence of tenants, lies in how the "not receiving a house" or outside option is treated. In our model the outside option can be freely ranked by agents and hence they can divide the set of houses into acceptable and unacceptable houses while in the previously mentioned papers all houses are acceptable, i.e., the outside option is ranked last by default. Due to this difference one has to be careful when comparing results for these models.

Table 2 summarizes the properties of various house allocation with existing tenants rules that we have just reviewed (some of them characterize the respective rules).

remaining tenants, i.e., we never remove an occupied house without its tenant. Then, if the remaining agents

House allocation with existing tenants rules	**PO**	**IR**	**SP**	**NB**	**RP**	**CON**	**WNEU**
YRMH-IGYT	✓	✓	✓	✓	✓	✓	✓
TTC based on acyclic priorities*	✓	✓	✓	✓	✓	✓	
Example 6	✓	✓	✓			✓	
hierarchical exchange*	✓	✓	✓	✓	✓		
trading cycles*	✓	✓	✓	✓			

Notation:
PO stands for *Pareto-optimality*,
IR stands for *individual-rationality*,
SP stands for *strategy-proofness*,
NB stands for *non-bossiness*,
RP stands for *reallocation-proofness*,
CON stands for *consistency*, and
WNEU stands for *weak neutrality*.
$*$ indicates that rules are adapted to give all tenants their corresponding ownership rights.

Table 2: House allocation with existing tenants rules and their properties.

The paper is organized as follows. In Section 2 we introduce the house allocation with existing tenants model and basic properties of rules. In Section 3 we introduce priority structures and TTC rules, and show that a TTC rule based on ownership-adapted priorities satisfies *Pareto-optimality*, *individual-rationality*, *group strategy-proofness*, and *reallocation-proofness* (Proposition 1). Furthermore, in Subsection 3.1 we show that a TTC rule based on ownership-adapted priorities is *consistent* if and only if the priority structure is *acyclic* (Theorem 1). We also show (Subsection 3.2) that any TTC rule based on ownership-adapted acyclic priorities can be described as a two-step rule where the first rule is an almost YRMH-IGYT rule and the second rule is an efficient priority rule (Proposition 2). In Section 4 we first demonstrate that *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, and *consistency* neither imply *reallocation-proofness* nor *non-bossiness* (Example 6). Then, we state and prove our characterizations of TTC rules that are based on ownership-adapted acyclic priorities by *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *consistency*, and either *reallocation-proofness* (Theorem 2) or *non-bossiness* (Theorem 3).

were to allocate the remaining houses according to the rule, each of them would receive the same house.

2 House Allocation with Existing Tenants and Basic Properties

We mostly follow Sönmez and Ünver (2010) in this section. Let $\mathcal{I}$ be a finite **set of potential agents** and $\mathcal{H}$ be a finite **set of potential houses**. We assume that $|\mathcal{I}| \geq 3$ and $|\mathcal{H}| \geq 2$. Let h_0 denote the **null house**. We interpret the null house as the outside option of an agent if he does not receive any house. We fix a global **ownership structure** $h : \mathcal{I} \to \mathcal{H} \cup \{h_0\}$. An agent $i \in \mathcal{I}$ is either a **tenant**, i.e., he already occupies a house $h(i) \in \mathcal{H}$, or an **applicant**, i.e., $h(i) = h_0$. No two agents can occupy the same house in $\mathcal{H}$, i.e., for each $i, j \in \mathcal{I}$ with $h(i) = h(j) \neq h_0$ we have $i = j$. Let $\mathcal{I}_T$ denote the set of **potential tenants** and $\mathcal{I}_A$ the set of **potential applicants**; the set of potential agents $\mathcal{I}$ is partitioned into the sets $\mathcal{I}_T$ and $\mathcal{I}_A$.[19]

For each agent $i \in \mathcal{I}$ and set of houses $H \subseteq \mathcal{H}$, let $\mathcal{R}(i, H)$ denote the set of all linear orders over $H \cup \{h_0\}$.[20] For each agent $i \in \mathcal{I}$, we interpret $R_i \in \mathcal{R}(i, H)$ as agent i's (strict) **preferences** over houses in H and the null house h_0; e.g., for $h, h', h'' \in H$, $[R_i : h\ P_i\ h'\ P_i\ h_0\ P_i\ h''\ P_i \ldots]$ means that agent i would first like to have house h, then to have h', and then i would prefer to have the null house h_0 rather than house h'', etc. An agent $i \in \mathcal{I}$ finds a house $h \in \mathcal{H}$ **acceptable** if $h\ P_i\ h_0$. We assume that every tenant $i \in \mathcal{I}_T$ finds the house that he already occupies acceptable, i.e., $h(i)\ P_i\ h_0$. Hence, for tenants the set of preferences $\mathcal{R}(i, H)$ is smaller than for applicants. Let $\mathcal{R}(I, H)$ denote the **set of all preference profiles** over $H \cup \{h_0\}$ for agents in I, i.e., $\mathcal{R}(I, H) = \prod_{i \in I} \mathcal{R}(i, H)$.

Given $R \in \mathcal{R}(I, H)$ and $\widetilde{I} \subseteq I$, let $R_{\widetilde{I}}$ denote the preference profile $(R_i)_{i \in \widetilde{I}}$; it is the **restriction of $\boldsymbol{R}$ to the set of agents $\widetilde{\boldsymbol{I}}$**. We also use the notation $R_{-\widetilde{I}} = R_{I \setminus \widetilde{I}}$ and $R_{-i} = R_{I \setminus \{i\}}$.

A **house allocation problem with existing tenants** is a list (I, H, R), where

(i) $I \subseteq \mathcal{I}$ is a finite set of agents,

(ii) $H \subseteq \mathcal{H}$ is a finite set of houses such that for each tenant $i \in I \cap \mathcal{I}_T$, $h(i) \in H$, and

(iii) $R = (R_i)_{i \in I} \in \mathcal{R}(I, H)$ is a preference profile.

Note that by (ii), if a tenant is present, then so is the house he occupies.

Throughout the paper we will consider the **domain of all house allocation problems with existing tenants** (as introduced by Abdulkadiroğlu and Sönmez, 1999). We will abbreviate the term **house allocation problem with existing tenants** simply as **problem**.

[19] In contrast to Sönmez and Ünver (2010) we do not require that there exists at least one house that is not occupied by a potential tenant, i.e., we do not require $|\mathcal{H}| > |\mathcal{I}_T|$.

[20] A linear order over $H \cup \{h_0\}$ is a binary relation $\widehat{R}$ that is *antisymmetric* (for each $h, h' \in H \cup \{h_0\}$, if $h\,\widehat{R}\,h'$ and $h'\,\widehat{R}\,h$, then $h = h'$), *transitive* (for each $h, h', h'' \in H \cup \{h_0\}$, if $h\,\widehat{R}\,h'$ and $h'\,\widehat{R}\,h''$, then $h\,\widehat{R}\,h''$), and *complete* (for each $h, h' \in H \cup \{h_0\}$, $h\,\widehat{R}\,h'$ or $h'\,\widehat{R}\,h$). By $\widehat{P}$ we denote the asymmetric part of $\widehat{R}$. Hence, given $h, h' \in H \cup \{h_0\}$, $h\,\widehat{P}\,h'$ means that h is *strictly preferred* to h'; $h\,\widehat{R}\,h'$ means that $h\,\widehat{P}\,h'$ or $h = h'$ and that h is *weakly preferred* to h'.

Given a problem (I,H,R), $I_T = \mathcal{I}_T \cap I$ denotes the **set of tenants**, $I_A = I \setminus I_T = \mathcal{I}_A \cap I$ denotes the **set of applicants**, $H_O = \{h(i)\}_{i \in I_T}$ denotes the **set of occupied houses**, and $H_V = H \setminus H_O$ denotes the **set of vacant houses**.

A problem is called a **problem with only applicants** when there are no tenants and hence no occupied houses, i.e., $I_T = \emptyset$ and $H_O = \emptyset$. A problem with only applicants is traditionally called a **house allocation problem** (house allocation problems were first analyzed in Hylland and Zeckhauser, 1979).

A problem is called a **problem with only tenants** when there are no applicants and no vacant houses, i.e., $I_A = \emptyset$ and $H_V = \emptyset$. A problem with only tenants is traditionally called a **housing market** (housing markets were first introduced by Shapley and Scarf, 1974).

A **matching** for a problem (I,H,R) is a function $\mu : I \to H \cup \{h_0\}$ such that no two agents are assigned to the same house in H, i.e., for each $h \in H$, $|\mu^{-1}(h)| \leq 1$ (the null house can be assigned to more than one agent). Given a matching μ for problem (I,H,R) and an agent $i \in I$, $\mu(i) \in (H \cup \{h_0\})$ denotes the house agent i is matched to under μ and is referred to as the **allotment of agent** $\boldsymbol{i}$. For each agent $i \in I$ and matchings μ, μ', we let $\mu \; R_i \; \mu'$ if and only if $\mu(i) \; R_i \; \mu'(i)$, i.e., agents only care about their own allotments but not how the remaining houses are allocated.

A **rule** ϕ is a function that associates with each problem (I,H,R) a matching $\phi(I,H,R)$. Given a problem (I,H,R), an agent $i \in I$, and a rule ϕ, $\phi_i(I,H,R)$ denotes the allotment of agent i at matching $\phi(I,H,R)$. For a group of agents $I' \subseteq I$ we define $\phi_{I'}(I,H,R) := \bigcup_{i \in I'} \phi_i(I,H,R)$.

The first property of a rule we introduce is the well-known condition of *Pareto-optimality*.

Definition 1 (**Pareto-Optimality**). A matching μ is *Pareto-optimal* for problem (I,H,R) if there is no other matching μ' for problem (I,H,R) such that for each agent $i \in I$, $\mu' \; R_i \; \mu$ and for some $j \in I$, $\mu' \; P_j \; \mu$. A rule ϕ is *Pareto-optimal* if it only assigns *Pareto-optimal* matchings.

Next, we introduce voluntary participation conditions based on the idea that no tenant can be forced to be assigned a house that is worse than the house he already occupies and no agent can be forced to be matched to a house that is unacceptable to him.

Definition 2 (**Individual-Rationality**). A matching μ is *individually-rational for tenants* for problem (I,H,R) if for each tenant $i \in I_T$, $\mu(i) \; R_i \; h(i)$. A matching μ is *individually-rational* for problem (I,H,R) if for each agent $i \in I$, $\mu(i) \; R_i \; h(i)$. A rule ϕ is *individually-rational (for tenants)* if it only assigns *individually-rational (for tenants)* matchings.

Note that *Pareto-optimality* and *individual-rationality for tenants* together imply *individual-rationality* (since every tenant finds his occupied house acceptable and if an applicant receives an unacceptable allotment, then we can make him better off by assigning the null house to him without making any other agent worse off). For simplicity we use the stronger notion of *individual-rationality* but we could use the weaker version of *individual-rationality for tenants* throughout.

The well-known non-manipulability property *strategy-proofness* requires that no agent can ever benefit from misrepresenting his preferences.

Definition 3 (**Strategy-Proofness**). A rule ϕ is *strategy-proof* if for each problem (I, H, R), each agent $i \in I$, and each preference relation $\widetilde{R}_i \in \mathcal{R}(i, H)$,

$$\phi_i(I, H, R)\, R_i\, \phi_i(I, H, (\widetilde{R}_i, R_{-i})).$$

The next property was introduced by Pápai (2000) to exclude joint preference manipulation by two individuals who plan to swap objects ex post under the condition that the collusion changed both their allotments and is self enforcing in the sense that neither agent changes his allotment in case he misreports while the other agents reports the truth.

Definition 4 (**Reallocation-Proofness**). A rule ϕ is *reallocation-proof* if for each problem (I, H, R) and each pair of agents $i, j \in I$, there exist no preference relations $\widetilde{R}_i \in \mathcal{R}(i, H)$ and $\widetilde{R}_j \in \mathcal{R}(j, H)$ such that

$$\phi_j(I, H, (\widetilde{R}_i, \widetilde{R}_j, R_{-\{i,j\}}))\, R_i\, \phi_i(I, H, R),$$

$$\phi_i(I, H, (\widetilde{R}_i, \widetilde{R}_j, R_{-\{i,j\}}))\, P_j\, \phi_j(I, H, R),$$

and

$$\phi_k(I, H, R) = \phi_k(I, H, (\widetilde{R}_k, R_{-k})) \neq \phi_k(I, H, (\widetilde{R}_i, \widetilde{R}_j, R_{-\{i,j\}})) \text{ for } k = i, j.$$

Next, we formulate a *consistency* notion for house allocation with existing tenants (as introduced by Sönmez and Ünver, 2010): if some agents leave a house allocation problem with existing tenants with their allotments and possibly some unassigned houses are removed, as long as no tenant is left behind while his occupied house is removed, the rule should allocate the remaining houses among the agents who did not leave in the same way as in the original house allocation problem with existing tenants. For problems with only applicants, *consistency* for house allocation problem with existing tenants implies *reallocation-consistency*[16] as introduced by Ehlers and Klaus (2006) as well as the standard *consistency*[14] property (e.g., Ehlers and Klaus, 2007; Ergin, 2000, 2002). We introduce some notation before defining *consistency*.

For each agent $i \in I$, preference relation $R_i \in \mathcal{R}(i, H)$, and set of houses $\widehat{H} \subseteq H$, let $R_i^{\widehat{H}} \in \mathcal{R}(i, \widehat{H})$ denote the **restriction of $\boldsymbol{R_i}$ to houses in $\boldsymbol{\widehat{H} \cup \{h_0\}}$**, i.e., for each $h, \widehat{h} \in \widehat{H} \cup \{h_0\}$, $h\, R_i^{\widehat{H}}\, \widehat{h}$ if and only if $h\, R_i\, \widehat{h}$. Given fixed $H \subseteq \mathcal{H}$, $\widehat{H} \subsetneq H$, and $R_i \in \mathcal{R}(i, H)$, we denote the restriction of R_i to houses in $(H \setminus \widehat{H}) \cup \{h_0\}$ by $R_i^{-\widehat{H}}$, i.e., $R_i^{H \setminus \widehat{H}} = R_i^{-\widehat{H}}$. For each $\widehat{I} \subseteq I$ and $\widehat{H} \subseteq H$, let $R_{\widehat{I}}^{\widehat{H}} = (R_i^{\widehat{H}})_{i \in \widehat{I}}$ denote the **restriction of preference profile $\boldsymbol{R}$ to agents in $\boldsymbol{\widehat{I}}$ and houses in $\boldsymbol{\widehat{H} \cup \{h_0\}}$**.

Given a problem (I, H, R), $\widehat{I} \subseteq I$, and $\widehat{H} \subseteq H$, $(\widehat{I}, \widehat{H}, R_{\widehat{I}}^{\widehat{H}})$ is the **restriction of $\boldsymbol{(I, H, R)}$ to agents in $\boldsymbol{\widehat{I}}$ and houses in $\boldsymbol{\widehat{H} \cup \{h_0\}}$**. The restricted problem $(\widehat{I}, \widehat{H}, R_{\widehat{I}}^{\widehat{H}})$ is a **reduced problem** if the occupied house of each tenant in $\widehat{I}$ belongs to $\widehat{H}$, that is, for each $i \in \widehat{I}_T$, $h(i) \in \widehat{H}$.

Definition 5 (**Consistency**)**.** A rule ϕ is *consistent* if for each problem (I, H, R) and each removal of a set of agents $\widehat{I} \subsetneq I$ together with their allotments under ϕ, $\widehat{H} = \phi_{\widehat{I}}(I, H, R)$, and some unassigned houses $\widetilde{H} \subseteq H$ that results in a reduced problem $(I \setminus \widehat{I}, H \setminus (\widehat{H} \cup \widetilde{H}), R_{-\widehat{I}}^{-(\widehat{H} \cup \widetilde{H})})$, it follows that for each agent $i \in (I \setminus \widehat{I})$,

$$\phi_i(I, H, R) = \phi_i(I \setminus \widehat{I}, H \setminus (\widehat{H} \cup \widetilde{H}), R_{-\widehat{I}}^{-(\widehat{H} \cup \widetilde{H})}).$$

For problems with only applicants, a well-known property that is implied by *consistency*, called *non-bossiness*, requires that whenever a change in an agent's preference relation does not bring about a change in his allotment, it does not bring about a change in anybody's allotment (Satterthwaite and Sonnenschein, 1981).

Definition 6 (**Non-Bossiness**)**.** A rule ϕ is *non-bossy* if for each problem (I, H, R), each agent $i \in I$, and each preference relation $\widetilde{R}_i \in \mathcal{R}(i, H)$, if

$$\phi_i(I, H, R) = \phi_i(I, H, (\widetilde{R}_i, R_{-i})),$$

then

$$\phi(I, H, R) = \phi(I, H, (\widetilde{R}_i, R_{-i})).$$

Remark 1 (**Consistency does not imply Non-Bossiness**)**.** For problems with only applicants we can easily see why *consistency* implies *non-bossiness*: If an applicant unilaterally changes his preferences such that he receives the same allotment, then, since this allotment is a vacant house, in each of the two problems he can leave with his allotment and two reduced problems in the domain of house allocation problems result. These two problems are identical and hence have to have the same matching. Thus, by *consistency*, also the matchings in the two original problems have to have been identical.

For house allocation problems with existing tenants, *consistency* does *not* imply *non-bossiness* anymore. The reason is that if an agent i now unilaterally changes his preferences such that he receives the same allotment, and this allotment is an occupied house, when leaving with only the occupied house, its tenant would be left behind and hence the resulting reduced problems are not in the domain of house allocation with existing tenants problems. Thus, *consistency* has no bite. One then could try to remove the smallest set of tenants together with agent i such that a well-defined house allocation with existing tenants problems results. However, the set of removed tenants or their allotments need not be the same and different reduced house allocation with existing tenants problems might result. Again, we cannot conclude that the matchings in the two original problems have to have been identical.

We will show in Section 4, Example 6 that even if we also assume *Pareto-optimality*, *individual-rationality*, and *strategy-proofness*, *consistency* does *not* imply *non-bossiness*. □

Strategy-proofness and *non-bossiness* are equivalent to *group strategy-proofness* (see, e.g., Pápai, 2000).

Definition 7 (**Group Strategy-Proofness**)**.** A rule ϕ is *group strategy-proof* if for each problem (I, H, R), there is no group of agents $\widetilde{I} \subseteq I$ and no preference profile $\widetilde{R} \in \mathcal{R}(\widetilde{I}, H)$, such that for all $i \in \widetilde{I}$,

$$\phi_i(I, H, (\widetilde{R}, R_{-\widetilde{I}}) \; R_i \; \phi_i(I, H, R)$$

and for some $j \in \widetilde{I}$,

$$\phi_j(I, H, (\widetilde{R}, R_{-\widetilde{I}}) \; P_j \; \phi_j(I, H, R).$$

A recent survey (Thomson, 2016) discusses many other logical relationships of *non-bossiness* with well-known normative or strategic properties.

3 Priority Structures and Top Trading Cycles Rules

For $I \subseteq \mathcal{I}$ let Π^I denote all one-to-one functions from $\{1, \ldots, |I|\}$ to I. For the set of all agents $\mathcal{I}$ and each house $h \in \mathcal{H}$, $\pi^h \in \Pi^{\mathcal{I}}$ denotes the **priority ordering for house $\boldsymbol{h}$**. Here agent $\pi^h(1)$ has the top priority at house h, agent $\pi^h(2)$ has the second priority at h, and so on. By a slight abuse of notation we will also denote the inverse function $(\pi^h)^{-1}$ by π^h such that for an agent $i \in \mathcal{I}$, $\pi^h(i) \in \{1, \ldots, |\mathcal{I}|\}$ denotes his rank in the priority. For a subset of agents $I \subseteq \mathcal{I}$ we define the **restriction of $\boldsymbol{\pi^h}$ to $\boldsymbol{I}$** to be a one-to-one function $\pi^h_I \in \Pi^I$ such that $\pi^h_I(i) < \pi^h_I(j)$ if and only if $\pi^h(i) < \pi^h(j)$; $\pi^h_I(i)$ indicates the rank of agent i in set I. A **priority structure** is a list $\pi \equiv \{\pi^h \mid \pi^h \in \Pi^{\mathcal{I}}\}_{h \in \mathcal{H}}$ of priority orderings, one for each house in $\mathcal{H}$. For a set of agents $I \subseteq \mathcal{I}$, $\pi_I \equiv \{\pi^h_I\}_{h \in \mathcal{H}}$ denotes a **restricted priority structure**. Note that $\pi_{\mathcal{I}} = \pi$.

For each problem (I, H, R) and each priority structure π we define the **top trading cycles (TTC) rule based on priority structure $\boldsymbol{\pi}$** recursively using Gale's **top trading cycles (TTC) algorithm** (Shapley and Scarf, 1974, attributed the TTC algorithm that finds a core allocation in housing markets to David Gale):

Input. A problem (I, H, R) and a priority structure π.

Step 1. Let $I_1 := I$ and $H_1 := H$. We construct a (directed) graph with the set of nodes $I_1 \cup H_1 \cup \{h_0\}$. For each agent $i \in I_1$ we add a directed edge to his most preferred house in $H_1 \cup \{h_0\}$. For each directed edge (i, h) ($i \in I_1$ and $h \in H_1$) we say that agent i points to house h. For each house $h \in H_1$ we add a directed edge to the highest ranked agent in I_1 in its priority ordering, i.e., to $\pi^h_{I_1}(1)$. For the null house we add a directed edge to each agent in I_1.

A **trading cycle** is a directed cycle in the graph. Given the finite number of nodes, at least one trading cycle exists for the graph. We assign to each agent in a trading cycle the house he points to and remove all trading cycle agents and houses. We define I_2 to be the set of remaining agents and H_2 to be the set of remaining houses and, if $I_2 \neq \emptyset$, we continue with Step 2. Otherwise we stop.

In general at Step t we have the following:

Step t. We construct a (directed) graph with the set of nodes $I_t \cup H_t \cup \{h_0\}$ where $I_t \subseteq I$ is the set of agents that remain after Step $t-1$ and $H_t \subseteq H$ is the set of houses that remain after Step $t-1$.

For each agent $i \in I_t$ we add a directed edge to his most preferred house in $H_t \cup \{h_0\}$. For each house $h \in H_t$ we add a directed edge to the highest ranked agent in I_t in its priority ordering, i.e., to $\pi^h_{I_t}(1)$. For the null house we add a directed edge to each agent in I_t.

At least one trading cycle exists for the graph and we assign to each agent in a trading cycle the house he points to and remove all trading cycle agents and houses. We define I_{t+1} to be the set of remaining agents and H_{t+1} to be the set of remaining houses and, if $I_{t+1} \neq \emptyset$, we continue with Step $t+1$. Otherwise we stop.

Output. The TTC algorithm terminates when all agents in I are assigned a house in $H \cup \{h_0\}$ (it takes at most $|I|$ steps). We denote the house in $H \cup \{h_0\}$ that agent $i \in I$ obtains in the TTC algorithm by $\varphi^\pi_i(I, H, R)$.

The **TTC rule based on priority structure $\boldsymbol{\pi}$, $\boldsymbol{\varphi^\pi}$**, associates with each problem (I, H, R) the matching determined by the TTC algorithm.

We say that a priority structure π is **adapted to the ownership structure** if each tenant has top priority at his own house, i.e., for each $i \in \mathcal{I}_T$, $\pi^{h(i)}(1) = i$.

Any priority structure π can be adapted to the ownership structure by moving every tenant to the top of the priority ordering of his own house without changing the ordering of other agents. Formally, for each priority structure $\pi \equiv \{\pi^h \mid \pi^h \in \Pi^{\mathcal{I}}\}_{h\in\mathcal{H}}$ the **ownership-adapted priority structure $\boldsymbol{\widehat{\pi}}$** $\equiv \{\widehat{\pi}^h \mid \widehat{\pi}^h \in \Pi^{\mathcal{I}}\}_{h\in\mathcal{H}}$ is such that

(a) for each vacant house $h \in \mathcal{H}$, $\widehat{\pi}^h := \pi^h$ and

(b) for each occupied house $h(i) \in \mathcal{H}$,

$\widehat{\pi}^{h(i)}(1) = i$ and

for each $j, k \in \mathcal{I} \setminus \{i\}$, $\widehat{\pi}^{h(i)}(j) < \widehat{\pi}^{h(i)}(k)$ if and only if $\pi^{h(i)}(j) < \pi^{h(i)}(k)$.

Given a problem (I, H, R), a priority structure π, and a matching μ, we say that μ **violates the priority of agent $\boldsymbol{i \in I}$ for house $\boldsymbol{h \in H}$** if there exists an agent $j \in I$ such that $\mu(j) = h$, $\pi^h(i) < \pi^h(j)$, and $h\ P_i\ \mu(i)$, i.e., agent i has higher priority for house h than agent j but j receives h and i justifiably envies j. A rule ϕ **adapts to a priority structure $\boldsymbol{\pi}$** if for each problem (I, H, R), $\phi(I, H, R)$ does not violate the priority of any agent for any house.

All TTC rules based on ownership-adapted priority structures are *Pareto-optimal*, *individually-rational*, *group strategy-proof*, and *reallocation-proof*.

Proposition 1 ($\boldsymbol{\varphi^\pi}$: Pareto-Optimality, Individual-Rationality, Group Strategy-Proofness, Reallocation-Proofness). *For each priority structure π that is adapted to the ownership structure, the TTC rule based on π, φ^π, satisfies* Pareto-optimality, individual-rationality, group strategy-proofness, *and* reallocation-proofness.

Proof. *Individual-rationality* of φ^π follows from the facts that π is adapted to the ownership structure and no agent points to a house that is unacceptable for him at any step of the TTC algorithm. *Pareto-optimality* and *group strategy-proofness* follow from the well-known fact that TTC rules satisfy these properties. In particular, each TTC rule based on a priority structure is a "hierarchical exchange rule" and therefore satisfies *Pareto-optimality*, *group strategy-proofness*, and *reallocation-proofness* (see Pápai, 2000). □

3.1 Acyclic Priorities, Top Trading Cycles Rules, and Consistency

For more general house allocation problems where each house can have multiple identical copies, the house allocation with quotas model (also known as school choice model), Ergin (2002) and Kesten (2006) introduced acyclicity conditions for priority structures that coincide for problems with only applicants. We extend their acyclicity notion to house allocation problems with existing tenants.

Definition 8 (**Acyclicity**)**.** For a set of agents $I \subseteq \mathcal{I}$ and a restricted priority structure π_I that is adapted to the ownership structure, π_I is *acyclic* if for agents $i, j, k \in I$ and houses $h, h' \in \mathcal{H}$ such that h' is not owned by any of the three agents, i.e., $h' \notin \{h(i), h(j), h(k)\}$,

$$\pi^h(i) < \pi^h(j) < \pi^h(k) \text{ implies } [\pi^{h'}(i) < \pi^{h'}(k) \text{ or } \pi^{h'}(j) < \pi^{h'}(k)].$$

As already mentioned above, for problems with only applicants our definition of acyclicity coincides with Ergin and Kesten acyclicity.[21]

The following is an example of an (ownership-adapted) acyclic priority structure.

Example 1 (**An Acyclic Priority Structure**)**.** Table 3 gives an example of an (ownership-adapted) acyclic priority structure π for a problem with three tenants a, b, c, five applicants d, e, f, g, i and eight houses $h(a), h(b), h(c), h_1, h_2, h_3, h_4, h_5$.

n	$\pi^{h(a)}(n)$	$\pi^{h(b)}(n)$	$\pi^{h(c)}(n)$	$\pi^{h_1}(n)$	$\pi^{h_2}(n)$	$\pi^{h_3}(n)$	$\pi^{h_4}(n)$	$\pi^{h_5}(n)$
1	a	b	c	d	d	d	d	d
2	d	d	d	b	b	b	b	b
3	b	c	b	c	c	c	c	c
4	c	a	e	e	a	a	e	e
5	e	e	a	a	e	e	a	a
6	f	g	f	f	f	f	g	f
7	g	f	g	g	g	g	f	g
8	i	i	i	i	i	i	i	i

Table 3: An acyclic priority structure.

□

[21]In general, if a priority structure is Kesten acyclic, then it is also Ergin acyclic (note that in the more general house allocation with quotas model additional "scarcity conditions" are used to define Ergin and Kesten cycles).

The next example illustrates the TTC rule with the acyclic priority structure of Example 1 (Table 3).

Example 2 (TTC Rule based on Acyclic Priorities). Let $I = \{a, b, c, d, e, f, g, i\}$ and $H = \{h(a), h(b), h(c), h_1, h_2, h_3, h_4, h_5\}$. Agents $I_T = \{a, b, c\}$ are tenants. Consider the preference profile $R \in \mathcal{R}(I, H)$ defined as in Table 4.

R_a	R_b	R_c	R_d	R_e	R_f	R_g	R_i
h_1	h_1	$h(b)$	$h(c)$	h_1	h_1	h_1	h_1
h_2	$h(b)$	$h(c)$	h_0	$h(c)$	h_2	$h(a)$	h_2
$h(a)$				h_5	h_4	h_0	h_3
				h_0	h_0		h_0

Table 4: A preference profile.

We consider the TTC assignment with priority structure π given in Table 3 for problem (I, H, R). Table 5 gives the set of agents and houses present in the steps of the TTC and the top trading cycles that form in each step.

t	I_t	$H_t \cup \{h_0\}$	trading cycles in Step t
1	$\{a, b, c, d, e, f, g, i\}$	$\{h(a), h(b), h(c), h_1, h_2, h_3, h_4, h_5, h_0\}$	$[d, h(c), c, h(b), b, h_1]$
2	$\{a, e, f, g, i\}$	$\{h(a), h_2, h_3, h_4, h_5, h_0\}$	$[a, h_2]$, $[e, h_5]$
3	$\{f, g, i\}$	$\{h(a), h_3, h_4, h_0\}$	$[f, h_4, g, h(a)]$
4	$\{i\}$	$\{h_3, h_0\}$	$[i, h_3]$

Table 5: Steps of the TTC algorithm.

In the final assignment $\mu = \varphi^{\pi}(I, H, R)$, we have $\mu(a) = h_2$, $\mu(b) = h_1$, $\mu(c) = h(b)$, $\mu(d) = h(c)$, $\mu(e) = h_5$, $\mu(f) = h_4$, $\mu(g) = h(a)$, and $\mu(i) = h_3$. □

Next, one could think that to obtain acyclic priorities for problems with existing tenants one can take any acyclic priority structure for problems with only applicants and adapt it to the ownership structure. However, the following example shows that this might not result in acyclic priorities for problems with existing tenants and that the associated TTC rule violates *consistency*.

Example 3 (Cyclic Priorities when Adapted to Ownership and Inconsistency). Let (I, H, R) be a problem where $I = \{i, j, k\}$, $H = \{h, h'\}$, and $\pi^h(i) < \pi^h(j) < \pi^h(k)$ and $\pi^{h'}(j) < \pi^{h'}(i) < \pi^{h'}(k)$. Note that if none of the agents in I is a tenant of any house in H, then π_I is acyclic for agents i, j, k and houses h, h'.

Now assume that agent k is the tenant of house h, i.e., $h = h(k)$, while house h' is vacant. Then, the ownership-adapted priority structure $\widehat{\pi}$ is such that $\widehat{\pi}^{h(k)}(k) < \widehat{\pi}^{h(k)}(i) < \widehat{\pi}^{h(k)}(j)$ and $\widehat{\pi}^{h'}(j) < \widehat{\pi}^{h'}(i) < \widehat{\pi}^{h'}(k)$, which is not acyclic anymore. We now show that with the TTC based on the ownership adapted but cyclic priorities $\widehat{\pi}$ is not *consistent*.

Assume that preferences $R = (R_i, R_j, R_k)$ are such that

- $R_i : h(k)\ P_i\ h_0\ P_i\ h'$,
- $R_j : h(k)\ P_j\ h_0\ P_j\ h'$, and
- $R_k : h'\ P_k\ h(k)\ P_k\ h_0$.

The TTC rule based on the priority structure $\widehat{\pi}$ for problem (I, H, R) results in allotments $\varphi_i^{\widehat{\pi}}(I, H, R) = h_0$, $\varphi_j^{\widehat{\pi}}(I, H, R) = h(k)$, and $\varphi_k^{\widehat{\pi}}(I, H, R) = h'$.

We now consider the reduced problem (I', H', R') obtained from (I, H, R) by removing agent k with his allotment $\varphi_k^{\widehat{\pi}}(I, H, R) = h'$ (h' is a vacant house). Therefore, $I' = \{i, j\}$, $H' = \{h(k)\}$ (now $h(k)$ is vacant), and preferences are $R' = (R_i^{H'}, R_j^{H'})$. The TTC rule based on the priority structure $\widehat{\pi}$ for problem (I', H', R') results in allotments $\varphi_i^{\widehat{\pi}}(I', H', R') = h(k)$ and $\varphi_j^{\widehat{\pi}}(I', H', R') = h_0$. For agent j we have $\varphi_j^{\widehat{\pi}}(I, H, R) = h(k) \neq h_0 = \varphi_j^{\widehat{\pi}}(I', H', R')$. Hence, the TTC rule based on (cyclic) priority structure $\widehat{\pi}$ violates *consistency*. □

The fact that the TTC rule based on a cyclic priority structure in Example 3 is not *consistent* is not a coincidence: any TTC rule based on ownership-adapted priorities is *consistent* if and only if the priority structure is acyclic.

Theorem 1 (φ^π: Consistency $\Leftrightarrow$ π is Acyclic). *Let π be a priority structure that is adapted to the ownership structure. Then, the TTC rule based on π, φ^π, is consistent if and only if π is acyclic.*

***Proof*.** Let π be a priority structure that is adapted to the ownership structure.

Only If Part: Assume that φ^π is *consistent*. We show that then π is acyclic.

Assume for the sake of contradiction that π is cyclic. Hence, there exist agents $i, j, k \in \mathcal{I}$ and houses $h, h' \in \mathcal{H}$ with $h' \notin \{h(i), h(j), h(k)\}$ such that

$$\pi^h(i) < \pi^h(j) < \pi^h(k)$$

and

$$\pi^{h'}(k) < \pi^{h'}(i) \text{ and } \pi^{h'}(k) < \pi^{h'}(j).$$

Since π is adapted to the ownership structure, if $h \in \{h(i), h(j), h(k)\}$, then $h = h(i)$ (since among the three agents agent i has the top priority for house h).

Consider the problem (I, H, R) where $I = \{i, j, k\}$, $H \cup \{h_0\} = \{h, h', h(i), h(j), h(k), h_0\}$, and preferences are such that

- $R_i : h'\ P_i\ h\ R_i\ h(i)\ P_i \ldots$,
- $R_j : h\ P_j\ h'\ P_j\ h(j)\ P_j \ldots$, and
- $R_k : h\ P_k\ h(k)\ P_k \ldots$.

Then, the TTC rule φ^π assigns h' to i, h to k (because agent i has the top priority for house h, agent k has the top priority for house h', and then they trade), and $h(j)$ to j. Next, consider the reduced problem $(I', H', R_{I'}^{H'})$ where agent i leaves with his allotment h', i.e., $I' = \{j, k\}$ and $H' \cup \{h_0\} = \{h, h(i), h(j), h(k), h_0\}$. Now, the TTC rule φ^π assigns h to j (because agent j has the top priority for house h in I' and house h is the best house among H' for agent j) and $h(k)$ to k; contradicting *consistency*.

If Part: Assume that π is acyclic. We show that then φ^π is *consistent*.

Consider a problem (I, H, R) and remove a set of agents $\widehat{I} \subsetneq I$ together with their allotments $\widehat{H}$ as well as some unassigned houses $\widetilde{H}$ to obtain a reduced problem $(I', H', R^{H'}_{I'}) = \Big(I \setminus \widehat{I}, H \setminus (\widehat{H} \cup \widetilde{H}), R^{-(\widehat{H} \cup \widetilde{H})}_{-\widehat{I}}\Big)$; that is, the occupied houses of tenants in I' are in H': for each $i \in I'_T$, $h(i) \in H'$. We will show that for each $j \in I'$, $\varphi^\pi_j(I, H, R) = \varphi^\pi_j(I', H', R^{H'}_{I'})$. It suffices to consider the following four cases (all other cases can be obtained by iteratively applying these four cases):

Case 1. Only one unassigned house is removed: for $h \in H_V$ and $h \notin \varphi^\pi_I(I, H, R)$, $\widehat{I} = \emptyset$, $\widehat{H} = \emptyset$, and $\widetilde{H} = \{h\}$.

Case 2. An applicant who is assigned the null house is removed: for $i \in I_A$ with $\varphi^\pi_i(I, H, R) = h_0$, $\widehat{I} = \{i\}$, $\widehat{H} = \{h_0\}$, and $\widetilde{H} = \emptyset$.

Case 3. A "trading cycle" $[i_0, h(i_1), i_1, \ldots, i_K, h(i_0)]$ is removed: for $0 \leq k \leq K$ with $\varphi^\pi_{i_k}(I, H, R) = h(i_{k+1})$ (modulo $K+1$), $\widehat{I} = \{i_0, \ldots, i_K\} \subseteq I_T$, $\widehat{H} = \{h(i_0), \ldots h(i_K)\} \subseteq H_O$, and $\widetilde{H} = \emptyset$.

Case 4. A recipient of a vacant house is removed together with his allotment: for $h \in H_V$ with $\varphi^\pi_i(I, H, R) = h$, $\widehat{I} = \{i\}$, $\widehat{H} = \{h\}$, and $\widetilde{H} = \emptyset$.

In *Case 1*, note that during any step of the TTC algorithm with priorities π applied to the problem (I, H, R), no agent points to house h; otherwise h would be assigned. Thus, removing house h from the problem does not change the outcome of the algorithm and $\varphi^\pi(I, H, R) = \varphi^\pi(I', H', R^{H'}_{I'})$.

In *Case 2*, note that during any step of the TTC algorithm with priorities π applied to the problem (I, H, R), agents who point to h_0 can obtain it independently of whether i points to it or not. Thus, removing agent i from the problem does not change the outcome of the algorithm for agents in $I \setminus \{i\}$ and $\varphi^\pi_{-i}(I, H, R) = \varphi^\pi_{-i}(I', H', R^{H'}_{I'})$.

In *Case 3*, consider a preference profile $\widetilde{R} \in \mathcal{R}(I, H)$ such that $\widetilde{R}_{-\widehat{I}} = R_{-\widehat{I}}$ and for each $i \in \widehat{I}$ and each $h \in H' \setminus \{\varphi^\pi_i(I, H, R), h(i)\}$ we have

- $\widetilde{R}_i : \varphi^\pi_i(I, H, R)\ \widetilde{P}_i\ h(i)\ \widetilde{P}_i\ h.$

Starting from problem (I, H, R), if any of the agents $i \in \widehat{I}$ changes his preferences from R_i to $\widetilde{R}_i$, by *strategy-proofness* of the TTC rule, he will receive the same allotment before and after. Then, by *non-bossiness* of the TTC rule, the allotments of all other agents will also not change. This argument can be applied step by step for all agents in $\widehat{I}$ to move from problem (I, H, R) to problem $(I, H, \widetilde{R})$. Hence, by *group strategy-proofness* of the TTC rule, we have $\varphi^\pi(I, H, \widetilde{R}) = \varphi^\pi(I, H, R)$. By the definition of preferences $\widetilde{R}$, in the first step of the TTC algorithm with priorities π applied to the problem $(I, H, \widetilde{R})$, the trading cycle $[i_0, h(i_1), i_1, \ldots, i_K, h(i_0)]$ forms. After allocating houses according to this trading cycle and after removing it, the problem becomes the reduced problem $(I', H', R^{H'}_{I'})$. Note that other trading cycles that formed in Step 1 for problems $(I, H, \widetilde{R})$ will form again in Step 1 for the reduced problem $(I', H', R^{H'}_{I'})$. Thus, for each $i \in I'$ we have $\varphi^\pi_i(I, H, \widetilde{R}) = \varphi^\pi_i(I', H', R^{H'}_{I'})$. Since $\varphi^\pi(I, H, R) = \varphi^\pi(I, H, \widetilde{R})$ this concludes the proof for *Case 3*.

In *Case 4*, consider a preference profile $\widetilde{R} \in \mathcal{R}(I,H)$ such that $\widetilde{R}_i = R_i$ and for each $j \in I' = I \setminus \{i\}$ preferences $\widetilde{R}_j$ are obtained from R_j by making house h unacceptable, i.e., $h(j)\,\widetilde{P}_j\,h$ while leaving preferences over $H' = H \setminus \{h\}$ unchanged and thus $\widetilde{R}^{H'}_{I'} = R^{H'}_{I'}$.

Starting from problem (I,H,R), if any of the agents $j \in I'$ changes his preferences from R_j to $\widetilde{R}_j$, by *strategy-proofness* of the TTC rule, he will receive the same allotment before and after. Then, by *non-bossiness* of the TTC rule, the allotments of all other agents will also not change. This argument can be applied step by step for all agents in I' to move from problem (I,H,R) to problem $(I,H,\widetilde{R})$. Hence, by *group strategy-proofness* of the TTC rule, we have $\varphi^{\pi}(I,H,\widetilde{R}) = \varphi^{\pi}(I,H,R)$. During the TTC algorithm with priorities π applied to the problem $(I,H,\widetilde{R})$, a trading cycle including i and h forms. Let $[i_0,h_0,i_1,h_1,\ldots,i_K,h_K]$ with $i_0 = i, h_0 = h$ be this trading cycle and t the step in which it forms.

By the definition of preferences $\widetilde{R}$, the same trading cycles form in the first $t-1$ steps of the TTC algorithm with priorities π applied to the two problems $(I,H,\widetilde{R})$ and $(I',H',\widetilde{R}^{H'}_{I'})$. Next, we consider Step t of the TTC algorithm with priorities π in the two problems.

If $K = 1$, i.e., if agent i points to h and house h points to i in Step t in problem $(I,H,\widetilde{R})$, then the only difference between the two problems is that in problem $(I,H,\widetilde{R})$ we have an additional trading cycle consisting of i and h. Otherwise, the same trading cycles form in the two problems. Moreover, in the consecutive Steps $t+1, t+2, \ldots$ the same trading cycles form in the two problems. Thus, for each $j \in I' = I \setminus \{i\}$, we have $\varphi^{\pi}_j(I',H',R^{H'}_{I'}) = \varphi^{\pi}_j(I,H,\widetilde{R}) = \varphi^{\pi}_j(I,H,R)$.

If $K > 1$, then we show that in Step t of the TTC algorithm applied to $(I',H',R^{H'}_{I'})$ the trading cycle $i_1,h_1,\ldots,i_K,h_K$ forms. If this is true, then it follows immediately that all other trading cycles in the two problems are the same and moreover, in the consecutive Steps $t+1, t+2, \ldots$ the same trading cycles form in the two problems. To show that trading cycle $i_1,h_1,\ldots,i_K,h_K$ forms, it suffices to show that h_K points to i_1 in Step t of the TTC algorithm applied to problem $(I',H',R^{H'}_{I'})$. Suppose not and house h_K points to an agent $j \neq i_1$. Then, $\pi^{h_K}(i_0) < \pi^{h_K}(j) < \pi^{h_K}(i_1)$. Note however that $\pi^{h_0}(i_1) < \pi^{h_0}(j)$ and $\pi^{h_0}(i_1) < \pi^{h_0}(i_0)$, since otherwise $h_0 = h$ would not point to i_1 in Step t of the TTC algorithm with priorities π applied to problem $(I,H,\widetilde{R})$. Thus, there is a cycle in π, contradicting its acyclicity. □

3.2 A Representation of TTC Rules based on Ownership-Adapted Acyclic Priorities as Two-Step Rules

We show that a TTC rule based on ownership-adapted acyclic priorities can be decomposed into a so-called two-step rule where the first rule is an almost YRMH-IGYT rule and the second rule is an efficient priority rule (Proposition 2). We first introduce the class of *efficient priority rules.*

A rule **adapts to the priority structure** if and only if it chooses stable matchings, or equivalently, no justified envy occurs (see Balinski and Sönmez, 1999, Lemma 2). A rule is an **efficient priority rule** if the assignment of houses to agents are determined by the agents-proposing deferred acceptance rule and it adapts to an (Ergin) acyclic priority structure. Since Ergin and Kesten acyclicity coincide for problems with only applicants, by Kesten

(2006, Theorem 1) the agents-proposing deferred acceptance rule based on acyclic priorities is the TTC rule based on acyclic priorities. Hence, for problems with only applicants, the class of **efficient priority rules** is the subclass of TTC rules where for each problem with only applicants there is either one agent with the highest priority for all houses or there are two agents who share the first and second priorities of each house, i.e., the acyclic priority structure π is such that

- for each problem with only applicants (I, H, R) there is an agent $i \in I$ such that for each $h \in H$, $\pi_I^h(1) = i$, or there are two agents $i, j \in I$ such that for each $h \in H$, $\{\pi_I^h(1), \pi_I^h(2)\} = \{i, j\}$.

Hence, at each step of the TTC algorithm, each trading cycle contains at most two agents and two houses. That is, at each step of the TTC algorithm, either the top priorities of remaining houses are assigned to exactly one agent (a dictator) or the top priorities of remaining houses are divided between two agents. If there is an agent (a dictator) who has top priority for all remaining houses then he gets his best house among the remaining houses. If the top priorities of remaining houses are divided between two agents, then, at that step of the TTC algorithm, either (i) both agents get a house for which they have the top priority, or (ii) they swap two houses for which they have top priorities, or (iii) only one of them gets one of his top priority houses and the other gets nothing. In the last case, the agent who gets nothing becomes a dictator at the next step of the TTC algorithm because of acyclicity of the priority structure.

For problems with only applicants, Ehlers and Klaus (2006, Proposition 2 and Theorem 1) characterized the class of efficient priority rules by *Pareto-optimality*, *strategy-proofness*, and *reallocation-consistency*.

For more general house allocation problems with existing tenants, TTC rules based on ownership-adapted acyclic priorities have the property that at each step of the TTC algorithm at most two *applicants* and two *vacant houses* are involved in a trading cycle. A very natural subclass of TTC rules based on ownership-adapted acyclic priorities is the class of rules where at each step of the TTC algorithm, at most one trading cycle involving a vacant house appears and this trading cycle contains at most one applicant and at most one vacant house. This class was introduced under the name of **YRMH-IGYT (you request my house - I get your turn) rules** by Abdulkadiroğlu and Sönmez (1999). Sönmez and Ünver (2010) showed that a rule satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *weak neutrality*, and *consistency* if and only if it is a YRMH-IGYT rule.

The class of *YRMH-IGYT rules* is the subclass of TTC rules based on ownership-adapted acyclic priorities where for each problem there is a single agent who has top priority at each vacant house, i.e., the priority structure π is such that

(i) for each problem (I, H, R) and for each $i \in I_T$, $\pi_I^{h(i)}(1) = i$ and

(ii) for each problem (I, H, R) there exists an agent $i \in I$ such that for each $h \in H_V$, $\pi_I^h(1) = i$.

Another way to describe the set of YRMH-IGYT rule is as follows. Let π be a serial dictatorship priority structure. Then, for any $h, h' \in \mathcal{H}$, $\pi^h = \pi^{h'}$, i.e., every house has the same priority ordering. Let $\widehat{\pi}$ denote the priority structure obtained from π by adapting it to the ownership structure. It is easy to see that, since π is a serial dictatorship priority structure, $\widehat{\pi}$ is acyclic. Then, a rule ϕ is a YRMH-IGYT rule if and only if there exists a serial dictatorship priority structure π such that $\phi = \varphi^{\widehat{\pi}}$.

Next we give an alternative description of TTC rules based on ownership-adapted acyclic priorities in terms of two specific rules, YRMH-IGYT and efficient priority rules, that are applied in two steps. That is, the class of TTC rules based on ownership-adapted acyclic priorities is equivalent to a class of **two-step rules**. Essentially, these rules can be described as follows: The agents are split into two groups, where the first group contains all tenants and some applicants and the second group consists of only applicants. In the first step, houses are allocated to the first group of agents according to - essentially - the YRMH-IGYT rule. In the second step, houses that have not been allocated in the first step are allocated to the second group of agents according to an efficient priority rule.

We used the term "essentially" in the previous paragraph, because the allocation in the first step is generated according to a rule that might slightly differ from a YRMH-IGYT rule because we allow for the possibility that two agents have top priority at different vacant houses in steps of the TTC where only these two agents are left in the problem.

Formally, we define a TTC rule φ^{π} based on ownership-adapted acyclic priorities π to be an **almost YRMH-IGYT rule** if

(i) for each problem (I, H, R) and for each $i \in I_T$, $\pi_I^{h(i)}(1) = i$ and

(ii) for each problem (I, H, R) **with** $|\boldsymbol{I}| > \mathbf{2}$ there exists an agent $i \in I$ such that for each $h \in H_V$, $\pi_I^h(1) = i$.

That is, for each problem each tenant has the top priority for his own house and for each problem with more than two agents there exists an agent who has top priority for all vacant houses. This means that at each step of the TTC algorithm a tenant has the top priority for his own house and at each step of the TTC algorithm with more than two remaining agents there exists an agent who has top priority for all remaining vacant houses. Note that then the difference between a YRMH-IGYT rule and an almost YRMH-IGYT rule is that for the latter rule, the underlying priorities are only almost serial dictatorship priorities because the two lowest ranked agents might share ownership of remaining vacant houses. For problems with a large number of agents, such a rule behaves essentially like a YRMH-IGYT rule.

Example 4 (**An almost YRMH-IGYT rule**)**.** Consider agents $\mathcal{I} := \{a, b, c, d, e\}$ with $\mathcal{I}_T = \{a, b, c\}$, houses $\{h(a), h(b), h(c), h_1, h_2, h_3, h_4, h_5\}$ and the following priority structure:

Note that the TTC algorithm based on Table 6 priorities assigns allotments equal to the YRMH-IGYT rule for any Step t with $|I_t| > 2$. However, for a last Step t with $I_t = \{a, e\}$, the TTC algorithm might assign different allotments because agents a and e have different priorities at different vacant houses (the YRMH-IGYT rule would assign priorities dictatorially at that step as well). □

n	$\pi^{h(a)}(n)$	$\pi^{h(b)}(n)$	$\pi^{h(c)}(n)$	$\pi^{h_1}(n)$	$\pi^{h_2}(n)$	$\pi^{h_3}(n)$	$\pi^{h_4}(n)$	$\pi^{h_5}(n)$
1	a	b	c	d	d	d	d	d
2	d	d	d	b	b	b	b	b
3	b	c	b	c	c	c	c	c
4	c	a	e	e	a	a	e	e
5	e	e	a	a	e	e	a	a

Table 6: An acyclic priority structure: the top part of the priorities of Table 3.

Definition 9 (**Two-Step Rules**)**.** A rule ϕ is a **two-step rule** if there are

- a partition of the set of agents $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ such that the first group contains all tenants $\mathcal{I}_T \subseteq \mathcal{I}_1$ and
- rules ϕ^1 and ϕ^2 such that

  rule ϕ^1 is an almost YRMH-IGYT rule that is defined for all problems (I, H, R) with $I \subseteq \mathcal{I}_1$ and

  rule ϕ^2 is an efficient priority rule that is defined for all problems (I, H, R) with $I \subseteq \mathcal{I}_2$,

and these rules are applied in two steps as follows:

Step 1. for each problem (I, H, R) and each $i \in I_1 := \mathcal{I}_1 \cap I$ we have $\phi_i(I, H, R) = \phi_i^1(I_1, H, R_{I_1})$ and

Step 2. for each $i \in I_2 := \mathcal{I}_2 \cap I$ we have $\phi_i(I, H, R) = \phi_i^2(I_2, H \setminus \widehat{H}, R_{I_2})$ where $\widehat{H} := \phi_{I_1}^1(I_1, H, R_{I_1})$.

Example 5 (**A TTC Rule based on Ownership-Adapted Acyclic Priorities as a Two-Step Rule**)**.** We reconsider the TTC rule based on ownership-adapted acyclic priorities π defined by Table 3 that we discussed in Example 2. It can be reinterpreted as a two-step rule as follows: To determine the first group of agents $\mathcal{I}_1$, we consider the lowest priority that any tenant has for any house. For the priorities π given in Table 3, this lowest priority of 5 is given to tenant a (e.g., for house h_1). Then, we consider all agents that have as least as high a priority at all houses: $\mathcal{I}_1 := \{j \in \mathcal{I} \mid \pi^h(j) \leq 5 \text{ for all } h \in \mathcal{H}\} = \{a, b, c, d, e\}$. The remaining agents form the second group $\mathcal{I}_2 := \{f, g, i\}$.

The rule ϕ^1 is the almost YRMH-IGYT rule based on Table 6 priorities previously described in Example 4. Rule ϕ^2 is now defined through the (acyclic) priorities consisting of the last three rows of Table 3:

n	$\pi^{h(a)}(n)$	$\pi^{h(b)}(n)$	$\pi^{h(c)}(n)$	$\pi^{h_1}(n)$	$\pi^{h_2}(n)$	$\pi^{h_3}(n)$	$\pi^{h_4}(n)$	$\pi^{h_5}(n)$
1	f	g	f	f	f	f	g	f
2	g	f	g	g	g	g	f	g
3	i	i	i	i	i	i	i	i

Table 7: An acyclic priority structure: the bottom part of the priorities of Table 3.

Note that for any problem (I, H, R) with $I \subseteq \mathcal{I}_2$ the TTC rule based on Table 7 priorities is an efficient priority rule. $\square$

We now show that the correspondence between TTC rules based on ownership-adapted acyclic priorities and two-step rules holds in general.

Proposition 2 (Characterizing TTC Rules based on Ownership-Adapted Acyclic Priorities as Two-Step Rules). *The class of TTC rules based on ownership-adapted acyclic priorities is the class of two-step rules.*

Proof. Let φ^π be a TTC rule based on ownership-adapted acyclic priorities π. Consider the lowest priority assigned to a tenant at π, i.e., $m := \max_{h \in \mathcal{H}, i \in \mathcal{I}_T} \pi^h(i)$. Let $i^* \in \mathcal{I}_T$ be a tenant and $h^* \in \mathcal{H}$ be a house such that $\pi^{h^*}(i^*) = m$. We define the set $\mathcal{I}_1$ to be the set of agents who have higher or equal priority for house h^* than agent i^*, i.e., $\mathcal{I}_1 := \{j \in \mathcal{I} \mid \pi^{h^*}(j) \leq m\}$. The set $\mathcal{I}_2$ is the set of agents who have lower priority for house h^* than agent i^*, i.e., $\mathcal{I}_2 := \{j \in \mathcal{I} \mid \pi^{h^*}(j) > m\} = \mathcal{I} \setminus \mathcal{I}_1$. By definition, the two sets are disjoint, we have $\mathcal{I}_T \subseteq \mathcal{I}_1$ and $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$.

First we show that agents in $\mathcal{I}_1$ have higher priority than agents in $\mathcal{I}_2$ at all houses, i.e., for each $h \in \mathcal{H}$ we have $\mathcal{I}_1 = \{j \in \mathcal{I} \mid \pi^h(j) \leq m\}$ and $\mathcal{I}_2 = \{j \in \mathcal{I} \mid \pi^h(j) > m\}$. Consider $j \in \mathcal{I}_1 \setminus \{i^*\}$ and $k \in \mathcal{I}_2$ and assume by contradiction that there is a house $h \in \mathcal{H}$ with $\pi^h(k) < \pi^h(j)$. Since $\pi^{h^*}(j) < \pi^{h^*}(i^*) < \pi^{h^*}(k)$ and $k \notin \mathcal{I}_T$, acyclicity and $\pi^h(k) < \pi^h(j)$ imply that $\pi^h(i^*) < \pi^h(k) < \pi^h(j)$. Moreover, since $\pi^h(i^*) < \pi^h(k) < \pi^h(j)$ and $\pi^{h^*}(j) < \pi^{h^*}(i^*)$, acyclicity implies that h^* is occupied with j as a tenant. Since i^*'s priority at h^* is the lowest priority of any tenant at any house, tenant j cannot have lower priority at h than i^* has at h^*, i.e., $\pi^h(j) \leq \pi^{h^*}(i^*)$. Thus, there exists an agent ℓ with $\pi^{h^*}(j) < \pi^{h^*}(\ell) < \pi^{h^*}(i^*) < \pi^{h^*}(k)$ and $\pi^h(i^*) < \pi^h(k) < \pi^h(j) < \pi^h(\ell)$. But then agents k, j, ℓ with houses h, h^* form a cycle; a contradiction.

Next let $k \in \mathcal{I}_2$ and suppose there is a house $h \in \mathcal{H}$ with $\pi^h(k) < \pi^h(i^*)$. We may assume that there is another agent $j \in \mathcal{I}_1$ with $\pi^{h^*}(j) < \pi^{h^*}(i^*)$ and $\pi^h(i^*) < \pi^h(j)$. Otherwise i^* would have lower priority at h than at h^*. But since $k \notin \mathcal{I}_T$, we immediately get a contradiction with acyclicity.

Second, we show that the rule φ^π restricted to problems with agents in $\mathcal{I}_1$ is an almost YRMH-IGYT rule and restricted to problems with agents in $\mathcal{I}_2$ is an efficient priority rule. This will imply that φ^π is a two-step rule. Thus, we have to show that

- for each problem (I, H, R) with $I \subseteq \mathcal{I}_1$ and $|I| > 2$ there is an agent $i \in I$ such that for each $h \in H_V$, $\pi^h_I(i) = 1$,
- for each problem (I, H, R) with $I \subseteq \mathcal{I}_2$ there is an agent $i \in I$ such that for $h \in H$, $\pi^h_I(1) = i$, or there are two agents $i, j \in I$ such that for each $h \in H$, $\{\pi^h_I(1), \pi^h_I(2)\} = \{i, j\}$.

Since $\mathcal{I}_2$ contains only applicants and the priority structure π is acyclic, the second item follows immediately from the fact that for problems with only applicants the class of TTC rules based on acyclic priorities is the class of efficient priority rules.

Recall that i^* is a tenant who is assigned the lowest priority at π for some houses and such a house is h^*. To show the first item, consider the problem (I', H', R') with $I' = I \cup \{i^*\}$, $H' = H \cup \{h(i^*), h^*\}$ and R' arbitrary. Let $j := \pi_{I'}^{h(i^*)}(2)$. We claim that at any $h \in H_V$ we have $\pi_I^h(j) = 1$. The claim trivially holds if $h = h(i^*)$. If $h \neq h(i^*)$, then by acyclicity we have $\pi_{I'}^h(1) \in \{i^*, j\}$. If $\pi_{I'}^h(1) = j$, then also $\pi_I^h(1) = j$ and we are finished. Thus, it suffices to show that $\pi_{I'}^h(1) = i^*$ yields a contradiction. Since $|I| > 2$, there is an agent $k \neq i^*, j$ in the problem (I', H', R'). Since $I' \subseteq \mathcal{I}_1$ and i^* has lowest priority among agents in $\mathcal{I}_1$ at house h^*, we have $\pi^{h^*}(k) < \pi^{h^*}(i^*)$ and $\pi^{h^*}(j) < \pi^{h^*}(i^*)$. Thus, if $\pi_{I'}^h(i^*) = 1$, then we have a cycle involving houses h, h^* and agents i^*, j, k and hence, a contradiction.

Finally, let ϕ be a two-step rule induced by rules ϕ^1 and ϕ^2. Let π^1 be the priority structure associated with rule ϕ^1 and π^2 be the priority structure associated with rule ϕ^2. Define a priority structure π as the concatenation of the two priority structures, i.e., for each $h \in \mathcal{H}$ we let

$$\pi^h(i) = \begin{cases} \pi^{1,h}(i), & \text{if } i \in \mathcal{I}_1 \\ \pi^{2,h}(i) + |\mathcal{I}_1|, & \text{if } i \in \mathcal{I}_2. \end{cases}$$

Since π^1 and π^2 are acyclic priority structures, priority structure π is acyclic as well. Moreover, $\phi = \varphi^\pi$. □

4 Pareto-Optimality, Group Strategy-Proofness, Consistency, and Reallocation-Proofness

In the introduction we reviewed several classes of rules for house allocation with and without tenants and their properties, see Tables 1 and 2. We observed that *(weak) neutrality* is the key property that distinguishes simple serial dictatorship and YRMH-IGYT rules from other rules. For house allocation rules various sets of rules have been characterized by subsets of the following properties: *Pareto-optimality*, *strategy-proofness*, *non-bossiness* (Pycia and Ünver, 2017), and *reallocation-proofness* (Pápai, 2000) or *reallocation-consistency* (Ehlers and Klaus, 2006). For house allocation with existing tenants only one characterization of the YRMH-IGYT rules using *(weak) neutrality* is known (Sönmez and Ünver, 2010). We aim to understand what rules emerge without *(weak) neutrality*.

First, recall that for house allocation problems, Ehlers and Klaus (2006) characterized efficient priority rules for house allocation problems by *Pareto-optimality*, *strategy-proofness*, and *reallocation-consistency*. Since efficient priority rules are a subset of hierarchical exchange rules (Pápai, 2000) and of trading cycles rules (Pycia and Ünver, 2017), *Pareto-optimality*, *strategy-proofness*, and *reallocation-consistency*[22] imply *reallocation-proofness* and *non-bossiness*. We first show that this is not the case for house allocation problems with existing tenants. In the next example we define a rule, $\widetilde{\phi}$, for problems with one

[22]Note that *consistency* for house allocation problems with existing tenants implies *reallocation-consistency* for problems when no tenants are present.

tenant and at least two applicants and show that it satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, and *consistency* but that it violates *reallocation-proofness* and *non-bossiness*.

Example 6 (A Rule that is Consistent but neither Non-Bossy nor Reallocation-Proof). Let $\mathcal{I}$ be the set of potential agents with $\mathcal{I}_T = \{i\}$ and $\{j,k\} \subseteq \mathcal{I}_A$. Let $\mathcal{H}$ be the set of potential houses such that $h(i)$ is the only occupied house. We consider two priority structures π and $\widetilde{\pi}$ defined below.

We let $\pi_{\mathcal{I}\setminus\{i,j,k\}} = \widetilde{\pi}_{\mathcal{I}\setminus\{i,j,k\}}$ be an acyclic priority structure.

Let $\pi_{\{i,j,k\}}$ be a priority structure (for agents i, j, and k) defined by

$$\pi^{h(i)}(i) < \pi^{h(i)}(k) < \pi^{h(i)}(j), \text{ and for each vacant house } h \in \mathcal{H}, \pi^h(j) < \pi^h(i) < \pi^h(k).$$

Let $\widetilde{\pi}_{\{i,j,k\}}$ be a priority structure (for agents i, j, and k) defined by

$$\widetilde{\pi}^{h(i)}(i) < \widetilde{\pi}^{h(i)}(k) < \widetilde{\pi}^{h(i)}(j), \text{ and for each vacant house } h \in \mathcal{H}, \widetilde{\pi}^h(i) < \widetilde{\pi}^h(j) < \widetilde{\pi}^h(k).$$

Hence, $\widetilde{\pi}^{h(i)} = \pi^{h(i)}$ but priorities of vacant houses differ. We let π be the concatenation of $\pi_{\{i,j,k\}}$ and $\pi_{\mathcal{I}\setminus\{i,j,k\}}$ and $\widetilde{\pi}$ be the concatenation of $\widetilde{\pi}_{\{i,j,k\}}$ and $\widetilde{\pi}_{\mathcal{I}\setminus\{i,j,k\}}$. Note that agents i, j, and k are the highest ranked agents under priorities π and $\widetilde{\pi}$.

For each problem (I,H,R), $I \subseteq \mathcal{I}$ and $H \subseteq \mathcal{H}$, we partition $\mathcal{R}(I,H) = \mathcal{R}_1(I,H) \cup \mathcal{R}_2(I,H)$ into two sets as follows:

- If $i,k \in I$ we let $\mathcal{R}_2(I,H)$ be the set of all profiles $R \in \mathcal{R}(I,H)$ such that one of the two following statements is true:
  1. agent k ranks $h(i)$ best and i does not rank $h(i)$ best;
  2. agent k ranks $h(i)$ second best and both agents i and k rank the same house best.

  We let $\mathcal{R}_1(I,H) = \mathcal{R}(I,H) \setminus \mathcal{R}_2(I,H)$.
- If $i \notin I$ or $k \notin I$, then we let $\mathcal{R}_1(I,H) = \mathcal{R}(I,H)$ and $\mathcal{R}_2(I,H) = \emptyset$.

We use the partition of preference profiles to define our rule $\widetilde{\phi}$ as follows: For each problem (I,H,R),

$$\widetilde{\phi}(I,H,R) = \begin{cases} \varphi^{\widetilde{\pi}}(I,H,R), & \text{if } i,k \in I, \text{ and } R \in \mathcal{R}_2(I,H) \\ \varphi^{\pi}(I,H,R), & \text{otherwise,} \end{cases}$$

that is, if agents i and k are present and preferences R belong to $\mathcal{R}_2(I,H)$, we use the TTC rule based on $\widetilde{\pi}$ and otherwise we use the TTC rule based on π. We prove in Appendix A that rule $\widetilde{\phi}$ satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, and *consistency* but *neither reallocation-proofness nor non-bossiness*. □

4.1 A Characterization of TTC Rules based on Ownership-Adapted Acyclic Priorities with Reallocation-Proofness

Our next main result is that TTC rules based on ownership-adapted acyclic priorities are the only rules that satisfy *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *consistency*, and *reallocation-proofness*.

Theorem 2 (**A Characterization of φ^π with Reallocation-Proofness**). *A rule ϕ satisfies Pareto-optimality, individual-rationality, strategy-proofness, reallocation-proofness, and consistency if and only if there exists an ownership-adapted acyclic priority structure π such that $\phi = \varphi^\pi$.*

Note that for the only-if-part of Theorem 2 it suffices to require *reallocation-proofness* only for pairs of agents that contain at least one tenant (see proof of Lemma 4). Furthermore, the proof of Theorem 2 reveals that in the characterization, *reallocation-proofness* is not needed if $\mathcal{I} = \mathcal{I}_A$.

We prove Theorem 2 through a sequence of lemmata (which we prove in Appendix B). Throughout the remainder of this section we assume that rule ϕ satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *reallocation-proofness*,[23] and *consistency*.

Using *Pareto-optimality* and *individual-rationality*, we derive a **priority structure** $\pi = (\pi^h)_{h\in\mathcal{H}}$ from ϕ.

For each house $h \in \mathcal{H}$ we call a preference profile at which each agent likes house h best and only finds houses in $\{h, h(i)\}$ *individually-rational* a **version of a maximal conflict preference profile** for h. Formally, $R^h \in \mathcal{R}(\mathcal{I}, \mathcal{H})$ is a version of a maximal conflict preference profile for h if for each $i \in \mathcal{I}$ and $h' \in \mathcal{H} \setminus \{h, h(i)\}$ we have

- $R^h_i : h \; R^h_i \; h(i) \; P^h_i \; h'$.

Note that there can be multiple versions of a maximal conflict preference profile for h that differ in the ranking of houses that are not *individually-rational*. After we have defined π^h we will show that the definition is independent of which versions of maximal conflict preference profiles we choose.

We consider the problem $(\mathcal{I}, \mathcal{H}, R^h)$ where R^h is some version of a maximal conflict preference profile. By *Pareto-optimality*, for some $i \in \mathcal{I}$ we have $\phi_i(\mathcal{I}, \mathcal{H}, R^h) = h$. We assign the top priority of house h to agent i, i.e., $\pi^h(1) = i$.

Note that, by *individual-rationality*, if house h is occupied by a tenant i, then agent i will have the top priority at house h, i.e., if for some $i \in \mathcal{I}_T$, $h = h(i)$, then $\pi^{h(i)}(1) = i$. Hence, the priority structure π we are constructing will be adapted to the ownership structure, i.e., for each $i \in \mathcal{I}_T$, $\pi^{h(i)}(1) = i$.

We next remove agent $\pi^h(1) = i$ and consider the remaining maximal conflict problem $(\mathcal{I} \setminus \{i\}, \mathcal{H}, R^h_{-i})$. Again, by *Pareto-optimality*, for some $j \in \mathcal{I} \setminus \{i\}$ we have $\phi_j(\mathcal{I} \setminus \{i\}, \mathcal{H}, R^h_{-i}) =$

[23] However, note that *reallocation-proofness* is used only in the proof of Lemma 4 and, by the use of Lemma 4, in Lemma 5.

h. We assign the second priority of house h to agent j, i.e., $\pi^h(2) = j$. We next remove agent $\pi^h(2) = j$ and consider the remaining maximal conflict problem $(\mathcal{I} \setminus \{i, j\}, \mathcal{H}, R^h_{-\{i,j\}})$, etc. We iterate in this way until we have considered all agents in $\mathcal{I}$. In this way we obtain a priority ordering π^h for each $h \in \mathcal{H}$.

Next, we establish a sequence of lemmata about properties of π.

Lemma 1 (**Maximal Conflict Preference Profile Independence**). *Each version of a maximal conflict preference profile for a house together with rule ϕ induces the same priority ranking for the house.*

From now on, we simply refer to maximal conflict preference profiles with the understanding that it does not matter which version is used.

Lemma 2 (**Consistent Reduction of Maximal Conflict Preference Profiles**). *Let $i, j \in \mathcal{I}$ be two different agents and house $h \in \mathcal{H}$. Consider a problem (I, H, R) such that $I = \{i, j\}$, $\{h, h(i), h(j)\} \subseteq H \cup \{h_0\}$, and $R \in \mathcal{R}(I, H)$ is a maximal conflict preference profile for h restricted to I and H. Then, $\pi^h(i) < \pi^h(j)$ implies $\phi_i(I, H, R) = h$ and $\phi_j(I, H, R) = h(j)$.*

Lemma 3 (**Acyclicity for Vacant Houses**). *Let $i, j, k \in \mathcal{I}$ be three different agents and assume that houses $h, h' \in \mathcal{H}$ are not owned by any of them, i.e., $h, h' \notin \{h(i), h(j), h(k)\}$. Then,*

$$\pi^h(i) < \pi^h(j) < \pi^h(k) \textit{ implies } [\pi^{h'}(i) < \pi^{h'}(k) \textit{ or } \pi^{h'}(j) < \pi^{h'}(k)].$$

Lemma 4 (**Acyclicity for Occupied Houses**). *Let $i, j, k \in \mathcal{I}$ be three different agents and assume that house $h(i) \in \mathcal{H}$ is occupied by agent i and house $h' \in \mathcal{H}$ is not owned by any of the three agents, i.e., $h' \notin \{h(i), h(j), h(k)\}$. Then,*

$$\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k) \textit{ implies } [\pi^{h'}(i) < \pi^{h'}(k) \textit{ or } \pi^{h'}(j) < \pi^{h'}(k)].$$

Lemmata 3 and 4 together imply that our constructed **priority structure $\boldsymbol{\pi}$ is acyclic**. We are now ready to start proving that if a rule ϕ satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *reallocation-proofness*, and *consistency*, then it is a TTC rule based on ownership-adapted acyclic priorities, i.e., $\phi = \varphi^\pi$. To this end, we first show that ϕ adapts to the priority structure π for top priority agents.

Lemma 5 (**Top Priority Adaptation**). *For each problem (I, H, R), if agent $i \in I$ has the top priority in I for a vacant house $h \in H_V$, i.e., for each $j \in I \setminus \{i\}$, $\pi^h(i) < \pi^h(j)$, then $\phi_i(I, H, R)\ R_i\ h$.*

We now prove that rule ϕ is the TTC rule φ^π.

Proposition 3 ($\boldsymbol{\phi = \varphi^\pi}$). *If a rule ϕ satisfies Pareto-optimality, individual-rationality, strategy-proofness, reallocation-proofness, and consistency, then it is the TTC rule that is based on the ownership-adapted acyclic priority structure π, i.e., $\phi = \varphi^\pi$.*

Proof. Let (I, H, R) be a problem. We show that $\varphi^\pi(I, H, R) = \phi(I, H, R)$. First note that by *consistency*, it suffices to consider the first step of TTC rule $\varphi^\pi(I, H, R)$: Once we have shown that for each trading cycle that forms in the first step, each agent in the cycle obtains the same allotment under φ^π and ϕ, we can consider the reduced problem (I', H', R') where these agents are removed together with their allotments under φ^π and ϕ. Then, when we subsequently consider the first step of $\varphi^\pi(I', H', R')$, by consistency of φ^π (which follows from Lemmata 3 and 4 and Theorem 1), each trading cycle that forms in the first step of $\varphi^\pi(I', H', R')$ also forms in the second step of $\varphi^\pi(I, H, R)$ and vice versa. Thus, if in each trading cycle that forms in the first step of $\varphi^\pi(I', H', R')$ the agents in the cycle obtain the same allotment under φ^π and ϕ, then in each trading cycle that forms in the second step of $\varphi^\pi(I, H, R)$ the agents in the cycle obtain the same allotment under φ^π and ϕ, and so on.

Thus, consider a trading cycle that forms in the first step of $\varphi^\pi(I, H, R)$, consisting of agents $i_0, \ldots, i_K$ and houses $\varphi^\pi_{i_0}(I, H, R), \ldots, \varphi^\pi_{i_K}(I, H, R)$. Note that each agent $i_k \in \{i_0, \ldots, i_K\}$ prefers $\varphi^\pi_{i_k}(I, H, R)$ most among houses in H. For every $i_k \in \{i_0, \ldots, i_K\}$ we define preferences $\widetilde{R}_{i_k} \in \mathcal{R}(i_k, H)$ such that

- $\widetilde{R}_{i_k} : \varphi^\pi_{i_k}(I, H, R)\, \widetilde{P}_{i_k}\, \varphi^\pi_{i_{k-1}}(I, H, R)\, \widetilde{R}_{i_k}\, h(i_k)\, \widetilde{R}_{i_k} \ldots$ (modulo $K + 1$),

e.g., by moving $\varphi^\pi_{i_{k-1}}(I, H, R)$ just after $\varphi^\pi_{i_k}(I, H, R)$ and, if $\varphi^\pi_{i_{k-1}}(I, H, R) \neq h(i_k)$ (i.e., when $\varphi^\pi_{i_{k-1}}(I, H, R)$ is vacant), by moving $h(i_k)$ just after $\varphi^\pi_{i_{k-1}}(I, H, R)$ without changing the ordering of other houses.

First, we consider the preference profile $R^0 = (\widetilde{R}_{\{i_0,\ldots,i_K\}}, R_{-\{i_0,\ldots,i_K\}})$.

Let $0 \leq k \leq K$. If $\varphi^\pi_{i_{k-1}}(I, H, R) = h(i_k)$ (modulo $K+1$), then by *individual-rationality*, we have $\phi_{i_k}(I, H, R^0)\, R_{i_k}\, \varphi^\pi_{i_{k-1}}(I, H, R)$ and $\phi_{i_k}(I, H, R^0) \in \{\varphi^\pi_{i_{k-1}}(I, H, R), \varphi^\pi_{i_k}(I, H, R)\}$. If $\varphi^\pi_{i_{k-1}}(I, H, R) \neq h(i_k)$, then $\varphi^\pi_{i_{k-1}}(I, H, R)$ is a vacant house and agent i_k has the top priority for it. Hence, by Lemma 5, we have $\phi_{i_k}(I, H, R^0)\, R_{i_k}\, \varphi^\pi_{i_{k-1}}(I, H, R)$ and $\phi_{i_k}(I, H, R^0) \in \{\varphi^\pi_{i_{k-1}}(I, H, R), \varphi^\pi_{i_k}(I, H, R)\}$. To summarize, for each $0 \leq k \leq K$, $\phi_{i_k}(I, H, R^0) \in \{\varphi^\pi_{i_{k-1}}(I, H, R), \varphi^\pi_{i_k}(I, H, R)\}$ (modulo $K + 1$). So, $\phi_{\{i_0,\ldots,i_K\}}(I, H, R^0) = \varphi^\pi_{\{i_0,\ldots,i_K\}}(I, H, R)$. By *Pareto-optimality*, for each $i_k \in \{i_0, \ldots, i_K\}$, $\phi_{i_k}(I, H, R^0) = \varphi^\pi_{i_k}(I, H, R)$.

Next, let $l_1 \in \{i_0, \ldots, i_K\}$, and consider the preference profile $R^1 = (\widetilde{R}_{\{i_0,\ldots,i_K\}\setminus\{l_1\}}, R_{-(\{i_0,\ldots,i_K\}\setminus\{l_1\})}) = (R_{l_1}, R^0_{-l_1})$. We start by showing that $\phi_{l_1}(I, H, R^1) = \varphi^\pi_{l_1}(I, H, R)$. That is, when agent l_1 changes his preferences from $\widetilde{R}_{l_1}$ to R_{l_1} at the preference profile R^0, house $\varphi^\pi_{l_1}(I, H, R)$ is still assigned to him under rule ϕ at the changed preference profile R^1. By *strategy-proofness* we have $\phi_{l_1}(I, H, R^1)\, R_{l_1}\, \phi_{l_1}(I, H, R^0)$. Since $\phi_{l_1}(I, H, R^0) = \varphi^\pi_{l_1}(I, H, R)$ is agent l_1's best house at preference profiles R^0 and R^1, we have $\phi_{l_1}(I, H, R^1) = \phi_{l_1}(I, H, R^0) = \varphi^\pi_{l_1}(I, H, R)$. By *individual-rationality*, for each $i_k \in \{i_0, \ldots, i_K\} \setminus \{l_1\}$, $\phi_{i_k}(I, H, R^1) \in \{\varphi^\pi_{i_{k-1}}(I, H, R), \varphi^\pi_{i_k}(I, H, R)\}$ (modulo $K + 1$). So, $\phi_{\{i_0,\ldots,i_K\}}(I, H, R^1) = \varphi^\pi_{\{i_0,\ldots,i_K\}}(I, H, R)$. By *Pareto-optimality*, for each $i_k \in \{i_0, \ldots, i_K\}$, $\phi_{i_k}(I, H, R^1) = \varphi^\pi_{i_k}(I, H, R)$.

Now, let $l_2 \in \{i_0, \ldots, i_K\} \setminus \{l_1\}$, and we consider the preference profile $R^2 = (\widetilde{R}_{\{i_0,\ldots,i_K\}\setminus\{l_1,l_2\}}, R_{-(\{i_0,\ldots,i_K\}\setminus\{l_1,l_2\})}) = (R_{l_2}, R^1_{-l_2})$. We first show that $\phi_{l_2}(I, H, R^2) = \varphi^\pi_{l_2}(I, H, R)$. That is, when agent l_2 changes his preferences from $\widetilde{R}_{l_2}$ to R_{l_2} at the preference profile R^1, house $\varphi^\pi_{l_2}(I,H,R)$ is still assigned to him under rule ϕ at the changed preference profile R^2. By *strategy-proofness* we have $\phi_{l_2}(I,H,R^2)\ R_{l_2}\ \phi_{l_2}(I,H,R^1)$. Since $\phi_{l_2}(I,H,R^1)=\varphi^\pi_{l_2}(I,H,R)$ is agent l_2's best house at preference profiles R^1 and R^2, we have $\phi_{l_2}(I,H,R^2)=\phi_{l_2}(I,H,R^1)=\varphi^\pi_{l_2}(I,H,R)$.

Since the choice of agents $\{l_1,l_2\}\subseteq\{i_0,\dots,i_K\}$ was arbitrary, we obtain by the same argument changing the roles of l_1 and l_2 that $\phi_{l_1}(I,H,R^2)=\varphi^\pi_{l_1}(I,H,R)$.

By *individual-rationality*, for each $i_k \in \{i_0,\dots,i_K\}\setminus\{l_1,l_2\}$, $\phi_{i_k}(I,H,R^2) \in \{\varphi^\pi_{i_{k-1}}(I,H,R),\varphi^\pi_{i_k}(I,H,R)\}$ (modulo $K+1$). So, $\phi_{\{i_0,\dots,i_K\}}(I,H,R^2)=\varphi^\pi_{\{i_0,\dots,i_K\}}(I,H,R)$. By *Pareto-optimality*, for each $i_k\in\{i_0,\dots,i_K\}$, $\phi_{i_k}(I,H,R^2)=\varphi^\pi_{i_k}(I,H,R)$.

We continue to replace the preferences of agents in $\{i_0,\dots,i_K\}\setminus\{l_1,l_2\}$ one at a time as above and reach the preference profile R such that for each $i_k\in\{i_0,\dots,i_K\}$, $\phi_{i_k}(I,H,R)=\varphi^\pi_{i_k}(I,H,R)$. □

Finally, recall that by Proposition 1, the TTC rule φ^π satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, and *reallocation-proofness* and by Theorem 1 and the acyclicity of π (Lemmata 3 and 4), φ^π is *consistent*. The proof of Theorem 2 is now complete. We prove the independence of properties used in the characterization (Theorem 2) in Appendix D.

4.2 A Characterization of TTC Rules based on Ownership-Adapted Acyclic Priorities with Non-Bossiness

Our last main result is that TTC rules based on ownership-adapted acyclic priorities are essentially the only rules that satisfy *Pareto-optimality*, *individual-rationality*, *group strategy-proofness*, and *consistency*.

Theorem 3 (A Characterization of $\boldsymbol{\varphi^\pi}$ with Non-Bossiness). *Let $[|\mathcal{H}|=2$ and $\mathcal{I}_T=\emptyset]$, or $[|\mathcal{H}|=3$ and $|\mathcal{I}_T|\leq 1]$, or $|\mathcal{H}|\geq 4$. Then, a rule ϕ satisfies Pareto-optimality, individual-rationality, strategy-proofness, non-bossiness, and consistency if and only if there exists an ownership-adapted acyclic priority structure π such that $\phi=\varphi^\pi$.*

If $[|\mathcal{H}|=2$ and $\mathcal{I}_T=\emptyset]$, then $\mathcal{I}=\mathcal{I}_A$ and *non-bossiness* is not needed in the characterization (because in the proof of Theorem 2, *reallocation-proofness* is used only to show acyclicity of occupied houses, Lemma 4, which is not needed if $\mathcal{I}=\mathcal{I}_A$). In Appendix E we provide examples that show that the other restrictions on the number of potential houses in Theorem 3 are necessary.

Throughout the remainder of this section we assume that either $[|\mathcal{H}|=2$ and $\mathcal{I}_T=\emptyset]$, or $[|\mathcal{H}|=3$ and $|\mathcal{I}_T|\leq 1]$, or $|\mathcal{H}|\geq 4$,[24] and that rule ϕ satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *non-bossiness*, and *consistency*. Recall that in the previous section *reallocation-proofness* was used only in the proofs of Lemmata 4 and 5, hence we can use the same construction of priority structure π and Lemmata 1, 2, and 3 remain correct without *reallocation-proofness*. We prove Theorem 3 through a sequence of additional

[24]However, note that this assumption is used only in the proof of Lemma 8.

lemmata (which we prove in Appendix C). First we introduce some terminology due to Pycia and Ünver (2017).

In the following, we will assume that a **pair** $(I, H) \in \mathcal{I} \times \mathcal{H}$ is such that for each tenant $i \in I_T$, $h(i) \in H$. Then, under rule ϕ, agent $i \in I$ **brokers*** house $h \in H$ at pair (I, H) if for each preference profile $R \in \mathcal{R}(I, H)$ such that all agents rank house h top, the house assigned to agent i, $\phi_i(I, H, R)$, is his second best house according to R_i (it is possible that $\phi_i(I, H, R) = h_0$). The following lemma connects brokerage* to cycles in priority structure π.

Lemma 6 (**Broker* Lemma**). *Let $i, j, k \in \mathcal{I}$ be three different agents and assume that house $h \in \mathcal{H}$ is not owned by any of them, i.e., $h \notin \{h(i), h(j), h(k)\}$. If agent i is a tenant, i.e., $i \in \mathcal{I}_T$,*

$$\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k), \text{ and } \pi^h(k) < \pi^h(i), \pi^h(j),$$

then under rule ϕ, agent j brokers house h at each pair (I, H) with $I = \{i, j, k\}$ and $h \in H$.*

A first consequence of Lemma 6 is that cycles have a particular structure if they occur at all.

Lemma 7. *Let $i, j, k \in \mathcal{I}$ be three different agents and assume that house $h \in \mathcal{H}$ is not owned by any of them, i.e., $h \notin \{h(i), h(j), h(k)\}$. If agent i is a tenant, i.e., $i \in \mathcal{I}_T$,*

$$\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k), \text{ and } \pi^h(k) < \pi^h(i), \pi^h(j),$$

then

$$\pi^h(k) < \pi^h(i) < \pi^h(j)$$

and agent j is an applicant, i.e., $j \in \mathcal{I}_A$.

The two previous lemmata imply a version of Lemma 4 where *reallocation-proofness* is replaced by *non-bossiness*.

Lemma 8 (**Acyclicity for Occupied Houses, Non-Bossiness Version**). *Let $i, j, k \in \mathcal{I}$ be three different agents and assume that house $h(i) \in \mathcal{H}$ is occupied by agent i and house $h' \in \mathcal{H}$ is not owned by any of the three agents, i.e., $h' \notin \{h(i), h(j), h(k)\}$. Then,*

$$\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k) \text{ implies } [\pi^{h'}(i) < \pi^{h'}(k) \text{ or } \pi^{h'}(j) < \pi^{h'}(k)].$$

With Lemma 8, we can now complete the proof of Theorem 3. By Proposition 1, the TTC rule φ^π satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, and *non-bossiness*, and by Theorem 1 and the acyclicity of π (Lemmata 3 and 8), φ^π is *consistent*. For the other direction, note that the proof of Proposition 3 only uses *reallocation-proofness* by relying on Lemma 4, which uses *reallocation-proofness* in its proof. Thus replacing Lemma 4 by Lemma 8, we can use the same proof as for Proposition 3. We prove the independence of properties used in the characterization (Theorem 3) in Appendix D.

Appendix

A A Rule that is Strategy-Proof and Consistent but neither Reallocation-Proof nor Non-Bossy

Note that the two rules φ^π and $\varphi^{\widetilde{\pi}}$ defined in Example 6 are *Pareto-optimal*, *individually-rational*, and *strategy-proof*. This immediately implies that $\widetilde{\phi}$ is *Pareto-optimal*, *individually-rational* and can only be manipulated by agent i or agent k.

Rule $\widetilde{\phi}$ is Strategy-Proof. First we consider agent i. If $R \in \mathcal{R}_2(I,H)$, then, since $\widetilde{\phi}(I,H,R) = \varphi^{\widetilde{\pi}}(I,H,R)$ and agent i is the highest ranked agent for all houses in H under $\widetilde{\pi}$, agent i obtains his best house, so he cannot gain by misreporting his preferences. So, consider $R \in \mathcal{R}_1(I,H)$. Suppose agent i reports R'_i and $R' := (R'_i, R_{-i})$. We consider two cases, (i) $R' \in \mathcal{R}_1(I,H)$ and (ii) $R' \in \mathcal{R}_2(I,H)$.

(i): If $R' \in \mathcal{R}_1(I,H)$, then, since $R, R' \in \mathcal{R}_1(I,H)$, $\widetilde{\phi}(I,H,R) = \varphi^\pi(I,H,R)$, $\widetilde{\phi}(I,H,R') = \varphi^\pi(I,H,R')$, and φ^π is *strategy-proof*, agent i cannot gain by misreporting R'_i.

(ii): If $R' \in \mathcal{R}_2(I,H)$, then either i ranks $h(i)$ first under R_i but not under R'_i, or i and k rank different vacant houses first under R and the same vacant house first under R'. In the first case, $h(i) = \widetilde{\phi}_i(I,H,R)\ R_i\ \widetilde{\phi}_i(I,H,R')$ and i cannot improve by reporting R'_i instead of R_i. In the second case, since $R' \in \mathcal{R}_2(I,H)$, $\widetilde{\phi}(I,H,R') = \varphi^{\widetilde{\pi}}(I,H,R')$, and i is the highest ranked agent for all houses at $\widetilde{\pi}$, i obtains his best house under R'_i, and this best house (at R'_i) is not his best house under his original preferences R_i. However, in this case, house $h(i)$ is not the best house at R_i and agent i is the second ranked agent for all vacant houses at π; so i obtains at least his second best house under R_i at $\widetilde{\phi}(I,H,R) = \varphi^\pi(I,H,R)$. This, together with the fact that allotment $\widetilde{\phi}_i(I,H,R')$ is not the best house for agent i under R_i, implies that agent i cannot benefit by reporting R'_i instead of R_i.

Second, we consider agent k. If $R \in \mathcal{R}_2(I,H)$, then agent k obtains house $h(i)$, which is either his best house or his second best house under R_k. If house $h(i)$ is agent k's second best house, then his best house is also agent i's best house. However, note that agent i has higher priority than agent k for all houses under both π and $\widetilde{\pi}$. Thus, agent k cannot gain from misreporting his preferences in this case. So, consider $R \in \mathcal{R}_1(I,H)$. Suppose agent k reports R'_k and $R' := (R'_k, R_{-k})$. We consider two cases, (i) $R' \in \mathcal{R}_1(I,H)$ and (ii) $R' \in \mathcal{R}_2(I,H)$.

(i): If $R' \in \mathcal{R}_1(I,H)$, then, since $\widetilde{\phi}(I,H,R) = \varphi^\pi(I,H,R)$, $\widetilde{\phi}(I,H,R') = \varphi^\pi(I,H,R')$, and φ^π is *strategy-proof*, agent k cannot gain by misreporting R'_k.

(ii): If $R' \in \mathcal{R}_2(I,H)$, then agent k obtains house $h(i)$ under $\widetilde{\phi}(I,H,R') = \varphi^{\widetilde{\pi}}(I,H,R')$. Since $R \notin \mathcal{R}_2(I,H)$, house $h(i)$ is either not among k's top two houses under R_k or $h(i)$ is k's second best house under R_k and i and k have different best houses under R. In the first case, by the definition of π, k receives at least his third best house under R_k and cannot improve by reporting R'_k instead of R_k. In the second case, if $h(i)$ is agent k's second best house under R_k and agents i and k have different best houses under R, then by the definition of π, agent k obtains his best house or second best house under R_k at $\widetilde{\phi}(I, H, R) = \varphi^{\pi}(I, H, R)$. Thus, also in this case agent k cannot benefit from reporting R'_k instead of R_k. □

Rule $\widetilde{\phi}$ is Consistent. Consider a problem (I, H, R) and its reduced problem (I', H', R').

If $R \in \mathcal{R}_2(I, H)$, then either $R' \in \mathcal{R}_2(I', H')$ or we have $\widetilde{\pi}_{I'} = \pi_{I'}$ (recall that at R agent k obtains house $h(i)$ and he cannot leave with it while agent i remains). Thus, $\widetilde{\phi}(I, H, R) = \varphi^{\widetilde{\pi}}(I, H, R)$ and $\widetilde{\phi}(I', H', R') = \varphi^{\widetilde{\pi}}(I', H', R')$. Since $\widetilde{\pi}$ is acyclic, *consistency* of $\varphi^{\widetilde{\pi}}$ follows from Theorem 1.

If $R \in \mathcal{R}_1(I, H)$, then either $R' \in \mathcal{R}_1(I', H')$ or $\widetilde{\pi}_{I'} = \pi_{I'}$ (this latter case only occurs when agent j leaves). However, now π has cycles, involving $h(i)$, a vacant house h, and agents i, j, and k. The only case where such a cycle might cause a problem is if in a problem (I, H, R) agent i trades a vacant house h for his endowment $h(i)$ with agent j, so i obtains vacant house h and j obtains house $h(i)$. In this case, if we remove agent i with his allotment h, then for the reduced problem $h(i)$ becomes a vacant house and agent k has a higher priority than agent j for house $h(i)$. However, since $R \in \mathcal{R}_1(I, H)$, agent k ranks neither $h(i)$ as best at R nor h as best and $h(i)$ as second best. Hence, $h(i)$ is not a best house for agent k at R'. So, in the reduced problem agents j and k both obtain their best houses, which are different (and agent j obtains $h(i)$). The other cases can be handled as in the acyclic case because the cycle in priorities π does not cause any "*consistency* problem" (since it is not active when trading). □

Rule $\widetilde{\phi}$ is not Reallocation-Proof. We consider (I, H, R) with $I = \{i, j, k\}$, $\{h, h(i)\} \subseteq H$, $H_O = \{h(i)\}$, and the following preferences:

- $R_i : h(i)\ P_i \ldots,$
- $R_j : h\ P_j\ h_0\ P_j \ldots,$
- $R_k : h\ P_k\ h_0\ P_k \ldots,$
- $\widetilde{R}_i : h\ P_i\ h(i)\ P_i \ldots,$
- $\widetilde{R}_k : h(i)\ P_k\ h_0\ P_k \ldots.$

Let $R = (R_i, R_j, R_k)$ and $\widetilde{R} = (\widetilde{R}_i, R_j, \widetilde{R}_k)$. Then, since $R \in \mathcal{R}_1(I, H)$ and $\widetilde{R} \in \mathcal{R}_2(I, H)$,

$$\widetilde{\phi}_i(I, H, R) = h(i), \quad \widetilde{\phi}_j(I, H, R) = h, \quad \widetilde{\phi}_k(I, H, R) = h_0,$$
$$\widetilde{\phi}_i(I, H, \widetilde{R}) = h, \quad \widetilde{\phi}_j(I, H, \widetilde{R}) = h_0, \quad \widetilde{\phi}_k(I, H, \widetilde{R}) = h(i).$$

We now show that agents i and k by changing their preferences from (R_i, R_k) at R to $(\widetilde{R}_i, \widetilde{R}_k)$ at $\widetilde{R}$ cause a violation of *reallocation-proofness*.

Consider agent i changing his preferences at R from R_i to $\widetilde{R}_i$. For the resulting preference profile $R^1 = (\widetilde{R}_i, R_j, R_k) \in \mathcal{R}_1(I, H)$ we have $\widetilde{\phi}_i(I, H, R^1) = \widetilde{\phi}_i(I, H, R) = h(i)$. Hence, agent i does not change his allotment by unilaterally moving from R to R^1.

Consider agent k changing his preferences at R from R_k to $\widetilde{R}_k$. For the resulting preference profile $R^2 = (R_i, R_j, \widetilde{R}_k) \in \mathcal{R}_1(I, H)$ we have $\widetilde{\phi}_k(I, H, R^2) = \widetilde{\phi}_k(I, H, R) = h_0$. Hence, agent k does not change his allotment by unilaterally moving from R to R^2.

Finally, consider both agents i and k changing their preferences at the same time, moving from R to $\widetilde{R}$, and then swapping their allotments. Then, agent i receives the same allotment $\phi_k(I, H, \widetilde{R}) = h(i) = \phi_i(I, H, R)$ while agent k is better off receiving $\phi_i(I, H, \widetilde{R}) = h \; P_k \; h_0 = \phi_k(I, H, R)$; a contradiction to *reallocation-proofness*. □

Rule $\widetilde{\phi}$ is Bossy. We consider (I, H, R) with $I = \{i, j, k\}$, $\{h, h', h(i)\} \subseteq H$, $H_O = \{h(i)\}$, and the following preferences:

- $R_i : h \; P_i \; h' \; P_i \; h(i) \; P_i \; \ldots,$
- $R_j : h \; P_j \; h' \; P_j \; h(i) \; P_j \; h_0 \; P_j \; \ldots,$
- $R_k : h \; P_k \; h' \; P_k \; h(i) \; P_k \; h_0 \; P_k \; \ldots,$
- $\widehat{R}_k : h \; \widehat{P}_k \; h(i) \; \widehat{P}_k \; h' \; \widehat{P}_k \; h_0 \; \widehat{P}_k \; \ldots.$

Let $R = (R_i, R_j, R_k)$ and $R' = (R_i, R_j, \widehat{R}_k)$. Then, since $R \in \mathcal{R}_1(I, H)$ and $R' \in \mathcal{R}_2(I, H)$,

$$\widetilde{\phi}_i(I, H, R) = h', \quad \widetilde{\phi}_j(I, H, R) = h, \quad \widetilde{\phi}_k(I, H, R) = h(i),$$
$$\widetilde{\phi}_i(I, H, R') = h, \quad \widetilde{\phi}_j(I, H, R') = h', \quad \widetilde{\phi}_k(I, H, R') = h(i).$$

Consider agent k changing his preferences at R from R_k to $\widehat{R}_k$. For the resulting preference profile $R' = (R_i, R_j, \widehat{R}_k)$ we have $\widetilde{\phi}_k(I, H, R') = h(i) = \widetilde{\phi}_k(I, H, R)$, i.e., agent k does not change his allotment by unilaterally moving from R to R'. However, the allotments of agents i and j change when agent k unilaterally moves from R to R', i.e., $\widetilde{\phi}_i(I, H, R) = h' \neq h = \widetilde{\phi}_i(I, H, R')$ and $\widetilde{\phi}_j(I, H, R) = h \neq h' = \widetilde{\phi}_j(I, H, R')$. Hence, the rule $\widetilde{\phi}$ is bossy. □

B Proofs of Section 4.1 Lemmata

Proof of Lemma 1 **(Maximal Conflict Preference Profile Independence)**. Let R^h and $\bar{R}^h$ be different versions of a maximal conflict preference profile for h and π^h and $\bar{\pi}^h$ be the corresponding priority rankings obtained. We show that for each $i \in \{1, 2, \ldots, |\mathcal{I}|\}$ we have $\pi^h(i) = \bar{\pi}^h(i)$. We proceed by induction on i.

Induction Basis. Let $i = 1$ and suppose $\pi^h(1) \neq \bar{\pi}^h(1)$, $\phi_{\pi^h(1)}(\mathcal{I}, \mathcal{H}, R^h) = h$ but $\phi_{\bar{\pi}^h(1)}(\mathcal{I}, \mathcal{H}, \bar{R}^h) = h$. By *individual-rationality*, $\phi_{\bar{\pi}^h(1)}(\mathcal{I}, \mathcal{H}, R^h) = h(\bar{\pi}^h(1))$ and $\phi_{\pi^h(1)}(\mathcal{I}, \mathcal{H}, \bar{R}^h) = h(\pi^h(1))$.

First, consider the reduced problem (I, H, R) of $(\mathcal{I}, \mathcal{H}, R^h)$ where $I = \{\pi^h(1), \bar{\pi}^h(1)\}$, $H \cup \{h_0\} = \{h, h(\pi^h(1)), h(\bar{\pi}^h(1)), h_0\}$ and $R = (R^h)_I^H$. By *consistency*, $\phi_{\pi^h(1)}(I, H, R) = h$.

Second, consider the reduced problem $(I, H, \bar{R})$ of $(\mathcal{I}, \mathcal{H}, \bar{R}^h)$ where I and H are defined as before and $\bar{R} = (\bar{R}^h)_I^H$. By *consistency*, $\phi_{\bar{\pi}^h(1)}(I, H, \bar{R}) = h$.

Third, starting from (I,H,R), change agent $\pi^h(1)$'s preferences to $\bar{R}_{\pi^h(1)}$. By *strategy-proofness*, $\phi_{\pi^h(1)}(I,H,(\bar{R}_{\pi^h(1)},R_{\bar{\pi}^h(1)})) = h$. By *individual-rationality*, $\phi_{\bar{\pi}^h(1)}(I,H,(\bar{R}_{\pi^h(1)},R_{\bar{\pi}^h(1)})) = h(\bar{\pi}^h(1))$.

Fourth, starting from $(I,H,(\bar{R}_{\pi^h(1)},R_{\bar{\pi}^h(1)}))$ change agent $\bar{\pi}^h(1)$'s preferences to $\bar{R}_{\bar{\pi}^h(1)}$. This change results in preference profile $\bar{R}$. By *strategy-proofness*, $\phi_{\bar{\pi}^h(1)}(I,H,\bar{R}) = h(\bar{\pi}^h(1))$. By *Pareto-optimality*, $\phi_{\pi^h(1)}(I,H,\bar{R}) = h$; contradicting $\phi_{\bar{\pi}^h(1)}(I,H,\bar{R}) = h$.

Induction Hypothesis. We assume that for each $i' \leq i < |\mathcal{I}|$ we have $\pi^h(i') = \bar{\pi}^h(i')$.

Induction Step. We show that $\pi^h(i+1) = \bar{\pi}^h(i+1)$. Suppose $\pi^h(i+1) \neq \bar{\pi}^h(i+1)$.

Consider the problem $(I,\mathcal{H},R^h_I)$ where $I = \mathcal{I} \setminus \{\pi^h(1),\ldots,\pi^h(i)\}$ and the problem $(\bar{I},\mathcal{H},\bar{R}^h_{\bar{I}})$ where $\bar{I} = \mathcal{I} \setminus \{\bar{\pi}^h(1),\ldots,\bar{\pi}^h(i)\}$. By the induction assumption $I = \bar{I}$. Hence, $\phi_{\pi^h(i+1)}(I,\mathcal{H},R^h_I) = h$ but $\phi_{\bar{\pi}^h(i+1)}(I,\mathcal{H},\bar{R}^h_I) = h$. By *individual-rationality*, we have $\phi_{\bar{\pi}^h(i+1)}(I,\mathcal{H},R^h_I) = h(\bar{\pi}^h(i+1))$ and $\phi_{\pi^h(i+1)}(I,\mathcal{H},\bar{R}^h_{\bar{I}}) = h(\pi^h(i+1))$.

First, consider the reduced problem (I',H',R) of $(I,\mathcal{H},R^h_I)$ where $I' = \{\pi^h(i+1),\bar{\pi}^h(i+1)\}$, $H' \cup \{h_0\} = \{h, h(\pi^h(i+1)), h(\bar{\pi}^h(i+1)), h_0\}$ and $R = (R^h)^{H'}_{I'}$. By *consistency*, $\phi_{\pi^h(i+1)}(I',H',R') = h$.

Second, consider the reduced problem $(I',H',\bar{R})$ of $(\bar{I},\mathcal{H},\bar{R}^h_{\bar{I}})$ where I' and H' are defined as before and $\bar{R} = (\bar{R}^h)^{H'}_{I'}$. By *consistency*, $\phi_{\bar{\pi}^h(i+1)}(I',H',\bar{R}) = h$.

An analog argument as for the induction basis shows that when changing preferences step by step from R to $\bar{R}$, *Pareto-optimality*, *individual-rationality*, and *strategy-proofness* imply that $\phi_{\pi^h(i+1)}(I,H,\bar{R}) = h$; contradicting $\phi_{\bar{\pi}^h(i+1)}(I,H,\bar{R}) = h$. □

Proof of Lemma 2 **(Consistent Reduction of Maximal Conflict Preference Profiles)**. Let $i,j \in \mathcal{I}$ be two different agents and house $h \in \mathcal{H}$ and problem (I,H,R) be such that $I = \{i,j\}$, $\{h,h(i),h(j)\} \subseteq H \cup \{h_0\}$, and $R \in \mathcal{R}(I,H)$ is a maximal conflict preference profile for h restricted to I and H. We show that $\pi^h(i) < \pi^h(j)$ implies $\phi_i(I,H,R) = h$ and $\phi_j(I,H,R) = h(j)$.

Recall that π^h is generated by a maximal conflict preference profile for h. By Lemma 1, it is no loss of generality to assume that this maximal conflict preference profile is a preference profile $R^h \in \mathcal{R}(\mathcal{I},\mathcal{H})$ such that R is its restriction to I and H, i.e., $(R^h)^H_I = R$.

By our construction to calibrate π^h, there exists a set of agents $\widetilde{I} := \big(\mathcal{I} \setminus \{\pi^h(1),\ldots,\pi^h(l)\}\big)$ such that $i,j \in \widetilde{I}$, agent i has the highest priority for house h in $\widetilde{I}$, i.e., $\pi^h(l+1) = i$, and $\phi_i(\widetilde{I},\mathcal{H},R^h_{\widetilde{I}}) = h$. By *individual-rationality*, for all $k \in \widetilde{I} \setminus \{i\}$ we have $\phi_k(\widetilde{I},\mathcal{H},R^h_{\widetilde{I}}) = h(k)$; in particular, $\phi_j(\widetilde{I},\mathcal{H},R^h_{\widetilde{I}}) = h(j)$.

Note that (I,H,R) is a reduced problem of $(\widetilde{I},\mathcal{H},R^h_{\widetilde{I}})$ obtained by removing all agents $\widetilde{I} \setminus \{i,j\}$ with their allotments $\bigcup_{k \in \left(\widetilde{I} \setminus \{i,j\}\right)} \{h(k)\} = \widetilde{H}$ and also by removing all unassigned houses $\widetilde{h} \in (\mathcal{H} \setminus \widetilde{H}) \setminus H$ that are not occupied by remaining agents i and j, i.e., $\widetilde{h} \notin \{h(i),h(j)\}$. By *consistency*, $\phi_i(I,H,R) = \phi_i(\widetilde{I},\mathcal{H},R^h_{\widetilde{I}})$ and $\phi_j(I,H,R) = \phi_j(\widetilde{I},\mathcal{H},R^h_{\widetilde{I}})$. Hence, $\phi_i(I,H,R) = h$ and $\phi_j(I,H,R) = h(j)$. □

Proof of Lemma 3 **(Acyclicity for Vacant Houses)**. Let $i,j,k \in \mathcal{I}$ be three different agents and assume that houses $h,h' \in \mathcal{H}$ are not owned by any of them, i.e., $h,h' \notin$

$\{h(i), h(j), h(k)\}$. We show that $\pi^h(i) < \pi^h(j) < \pi^h(k)$ implies $[\pi^{h'}(i) < \pi^{h'}(k)$ or $\pi^{h'}(j) < \pi^{h'}(k)]$.

Assume for the sake of contradiction that $\pi^h(i) < \pi^h(j) < \pi^h(k)$, $\pi^{h'}(k) < \pi^{h'}(i)$ and $\pi^{h'}(k) < \pi^{h'}(j)$. Consider the problem (I, H, R) where $I = \{i, j, k\}$, $H \cup \{h_0\} = \{h, h', h(i), h(j), h(k), h_0\}$, and preferences are such that

- $R_i : h' \; P_i \; h \; P_i \; h(i) \; P_i \ldots$,
- $R_j : h \; P_j \; h(j) \; P_j \ldots$, and
- $R_k : h \; P_k \; h' \; P_k \; h(k) \; P_k \ldots$.

By *Pareto-optimality* and *individual-rationality*, either $\phi_i(I, H, R) = h'$ or $\phi_k(I, H, R) = h'$.
Case 1. $\phi_i(I, H, R) = h'$. By *Pareto-optimality*, either $\phi_j(I, H, R) = h$ or $\phi_k(I, H, R) = h$.
Case 1.1. If $\phi_j(I, H, R) = h$, then consider the reduced problem $(I', H', R_{I'}^{H'})$ where agent j leaves with his allotment h and furthermore (by *individual-rationality*) the unassigned house $h(j)$ is deleted from the problem, i.e., $I' = \{i, k\}$ and $H' \cup \{h_0\} = \{h', h(i), h(k), h_0\}$. By *consistency*, $\phi_i(I', H', R_{I'}^{H'}) = h'$. However, note that $R_{I'}^{H'}$ is the restriction of a maximal conflict preference profile for h' to I' and H'. Hence, by Lemma 2, $\pi^{h'}(k) < \pi^{h'}(i)$ implies $\phi_k(I', H', R_{I'}^{H'}) = h'$; a contradiction.
Case 1.2. If $\phi_k(I, H, R) = h$, then consider the reduced problem $(I', H', R_{I'}^{H'})$ where agent i leaves with his allotment h' and furthermore (by *individual-rationality*) the unassigned house $h(i)$ is deleted from the problem, i.e., $I' = \{j, k\}$ and $H' \cup \{h_0\} = \{h, h(j), h(k), h_0\}$. By *consistency*, $\phi_k(I', H', R_{I'}^{H'}) = h$. However, note that $R_{I'}^{H'}$ is the restriction of a maximal conflict preference profile for h to I' and H'. Hence, by Lemma 2, $\pi^h(j) < \pi^h(k)$ implies $\phi_j(I', H', R_{I'}^{H'}) = h$; a contradiction.
Case 2. $\phi_k(I, H, R) = h'$. By *Pareto-optimality*, $\phi_j(I, H, R) = h$. Consider the reduced problem $(I', H', R_{I'}^{H'})$ where agent k leaves with his allotment h' and furthermore (by *individual-rationality*) the unassigned house $h(k)$ is deleted from the problem, i.e., $I' = \{i, j\}$ and $H' \cup \{h_0\} = \{h, h(i), h(j), h_0\}$. By *consistency*, $\phi_j(I', H', R_{I'}^{H'}) = h$. However, note that $R_{I'}^{H'}$ is the restriction of a maximal conflict preference profile for h to I' and H'. Hence, by Lemma 2, $\pi^h(i) < \pi^h(j)$ implies $\phi_i(I', H', R_{I'}^{H'}) = h$; a contradiction. □

***Proof of Lemma 4* (Acyclicity for Occupied Houses).** Let $i, j, k \in \mathcal{I}$ be three different agents and assume that house $h(i) \in \mathcal{H}$ is occupied by agent i and house $h' \in \mathcal{H}$ is not owned by any of the three agents, i.e., $h' \notin \{h(i), h(j), h(k)\}$. We show that $\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k)$ implies $[\pi^{h'}(i) < \pi^{h'}(k)$ or $\pi^{h'}(j) < \pi^{h'}(k)]$.

Assume for the sake of contradiction that $\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k)$, $\pi^{h'}(k) < \pi^{h'}(i)$, and $\pi^{h'}(k) < \pi^{h'}(j)$. Consider the problem (I, H, R) where $I = \{i, j, k\}$, $H \cup \{h_0\} = \{h', h(i), h(j), h(k), h_0\}$, and preferences are such that

- $R_i : h' \; P_i \; h(i) \; P_i \ldots$,
- $R_j : h(i) \; P_j \; h(j) \; P_j \ldots$, and
- $R_k : h(i) \; P_k \; h' \; P_k \; h(k) \; P_k \ldots$.

By *Pareto-optimality* and *individual-rationality*, either $\phi_i(I,H,R) = h'$ or $\phi_k(I,H,R) = h'$.
Case 1. $\phi_i(I,H,R) = h'$. By *Pareto-optimality*, either $\phi_j(I,H,R) = h(i)$ or $\phi_k(I,H,R) = h(i)$.
Case 1.1. If $\phi_j(I,H,R) = h(i)$, then $\phi_k(I,H,R) = h(k)$. Now, consider $\widetilde{R}_i \in \mathcal{R}(i,H)$, $\widetilde{R}_j \in \mathcal{R}(j,H)$, and $\widetilde{R}_k \in \mathcal{R}(k,H)$ such that

- $\widetilde{R}_i : h(i)\ \widetilde{P}_i \ldots$,
- $\widetilde{R}_j : h'\ \widetilde{P}_j\ h(j)\ P_k \ldots$, and
- $\widetilde{R}_k : h'\ \widetilde{P}_k\ h(k)\ P_k \ldots$.

First, consider $R^1 = (R_i, R_j, \widetilde{R}_k)$. By *strategy-proofness*, $\phi_k(I,H,R^1) = h(k)$ and by *consistency*, $\phi_i(I,H,R^1) = h'$ and $\phi_j(I,H,R^1) = h(i)$. Second, consider $R^2 = (\widetilde{R}_i, \widetilde{R}_j, \widetilde{R}_k)$. By *individual-rationality*, $\phi_i(I,H,R^2) = h(i)$. Then, *consistency*, Lemma 2, and $\pi^{h'}(k) < \pi^{h'}(j)$ imply $\phi_k(I,H,R^2) = h'$ and $\phi_j(I,H,R^2) = h(j)$. We now show that agents i and j by changing their preferences from $(\widetilde{R}_i, \widetilde{R}_j)$ at R^2 to (R_i, R_j) at R^1 cause a violation of *reallocation-proofness*.[25]

Consider agent i changing his preferences at R^2 from $\widetilde{R}_i$ to R_i. The resulting preference profile $R^3 = (R_i, \widetilde{R}_j, \widetilde{R}_k)$ is the restriction of a maximal conflict preference profile for h' to I and H and Lemmata 1 and 2 together with $\pi^{h'}(k) < \pi^{h'}(i)$ and $\pi^{h'}(k) < \pi^{h'}(j)$ imply $\phi_k(I,H,R^3) = h'$ and $\phi_i(I,H,R^3) = \phi_i(I,H,R^2) = h(i)$. Hence, agent i does not change his allotment by unilaterally moving from R^2 to R^3.

Consider agent j changing his preferences at R^2 from $\widetilde{R}_j$ to R_j. The resulting preference profile is $R^4 = (\widetilde{R}_i, R_j, \widetilde{R}_k)$. By *individual-rationality*, $\phi_i(I,H,R^4) = h(i)$. Hence, by *individual-rationality*, $\phi_j(I,H,R^4) = h(j)$. Hence, agent j does not change his allotment by unilaterally moving from R^2 to R^4.

Finally, consider both agents i and j changing their preferences at the same time, moving from R^2 to R^1, and then swapping their allotments. Then, agent i receives the same allotment $\phi_j(I,H,R^1) = h(i) = \phi_i(I,H,R^2)$ while agent j is better off receiving $\phi_i(I,H,R^1) = h'\ \widetilde{P}_j\ h(j) = \phi_j(I,H,R^2)$; a contradiction to *reallocation-proofness*.
Case 1.2. If $\phi_k(I,H,R) = h(i)$, then consider the reduced problem $(I', H', R^{H'}_{I'})$ where agent i leaves with his allotment h', i.e., $I' = \{j,k\}$ and $H' \cup \{h_0\} = \{h(i), h(j), h(k), h_0\}$. By *consistency*, $\phi_k(I', H', R^{H'}_{I'}) = h(i)$. However, note that $R^{H'}_{I'}$ is the restriction of a maximal conflict preference profile for $h(i)$ to I' and H'. Hence, by Lemma 2, $\pi^{h(i)}(j) < \pi^{h(i)}(k)$ implies $\phi_j(I', H', R^{H'}_{I'}) = h(i)$; a contradiction.
Case 2. $\phi_k(I,H,R) = h'$. By *individual-rationality*, $\phi_i(I,H,R) = h(i)$, contradicting *Pareto-optimality* (agents i and k would like to swap allotments). □

***Proof of Lemma 5* (Top Priority Adaptation).** Suppose for the sake of contradiction that there is a top priority violation at a vacant house for some problem. Let (I,H,R) be

[25] Note that this is the only step in our proofs that uses *reallocation-proofness* and we can see that for this step to work, it would suffice to require *reallocation-proofness* only for pairs of agents that contain at least one tenant.

a smallest problem in terms of the number of agents such that the priority of a top ranked agent at a vacant house is violated, i.e., such that there is an agent $i \in I$ and a vacant house $h \in H_V$ with $\pi_I^h(1) = i$ but $h \, P_i \, \phi_i(I, H, R)$.

Let $R_i^h \in \mathcal{R}(i, H)$ be maximal conflict preferences for h restricted to H and define $R^0 := (R_i^h, R_{-i})$. Then, by *strategy-proofness*, agent i still does not receive house h when reporting R_i^h. By *Pareto-optimality*, there is an agent j_1 such that $\phi_{j_1}(I, H, R^0) = h$. Let $R_{j_1}^h \in \mathcal{R}(j_1, H)$ be maximal conflict preferences for h restricted to H and define $R^1 := (R_{j_1}^h, R_{-j_1}^0)$. Then, by *strategy-proofness*, agent j_1 still receives house h when reporting $R_{j_1}^h$. Thus, $\phi_{j_1}(I, H, R^1) = h$. By *individual-rationality*, $\phi_i(I, H, R^1) = h(i)$.

Since (I, H, R) is a smallest problem with a top priority violation for a vacant house, *consistency* implies that there are different agents $j_1, \ldots, j_K$ (possibly $K = 1$) such that $I = \{i, j_1, \ldots, j_K\}$, and $\phi_{j_1}(I, H, R^1) = h$, $\phi_{j_2}(I, H, R^1) = h(j_1) \in H_O$, $\phi_{j_3}(I, H, R^1) = h(j_2) \in H_O$, $\ldots$, $\phi_{j_K}(I, H, R^1) = h(j_{K-1}) \in H_O$ and either $h(j_K) = h_0$ or $h(j_K) \in H_O$ (if not, by *consistency*, we could reduce the problem and obtain a problem with fewer agents and a top priority violation at a vacant house).

Case 1. $K = 1$ and $I = \{i, j_1\}$. Recall that agents i and j_1 have maximal conflict preferences for h restricted to H, i.e., R^1 is a maximal conflict preference profile for h restricted to I and H. Hence, by Lemma 2, $\phi_i(I, H, R^1) = h$; a contradiction.

Case 2. $K > 1$ and $I = \{i, j_1, \ldots, j_K\}$. Let $R_{j_2}^{h(j_1)} \in \mathcal{R}(j_2, H)$ be maximal conflict preferences for $h(j_1)$ restricted to H and define $R^2 := (R_{j_2}^{h(j_1)}, R_{-j_2}^1)$. By *strategy-proofness*, $\phi_{j_2}(I, H, R^2) = h(j_1)$. This implies that $\phi_{j_1}(I, H, R^2) = h$ (if not, then by *individual-rationality* agent j_1's allotment would be his second best house $h(j_1)$, which is already allocated to agent j_2). By *individual-rationality*, $\phi_i(I, H, R^2) = h(i)$ (if agent i cannot get his best house h, he must receive $h(i)$). By our assumption that (I, H, R) was a minimal problem with a top priority violation at a vacant house and by *consistency*, it now follows that $\{j_3, \ldots, j_K\} = \{j'_3, \ldots, j'_K\}$ such that $\phi_{j'_3}(I, H, R^2) = h(j_2) \in H_O$, $\phi_{j'_4}(I, H, R^2) = h(j'_3) \in H_O$, $\ldots$, $\phi_{j'_K}(I, H, R^2) = h(j'_{K-1}) \in H_O$ and either $h(j'_K) = h_0$ or $h(j'_K) \in H_O$. To simplify notation, we will assume that for all $k \in \{3, \ldots, K\}$, $j_k = j'_k$, otherwise we can just rename agents.

Let $R_{j_3}^{h(j_2)} \in \mathcal{R}(j_3, H)$ be maximal conflict preferences for $h(j_2)$ restricted to H and define $R^3 := (R_{j_3}^{h(j_2)}, R_{-j_3}^2)$. By *strategy-proofness*, $\phi_{j_3}(I, H, R^3) = h(j_2)$. This implies that $\phi_{j_2}(I, H, R^3) = h(j_1)$ (if not, then by *individual-rationality* agent j_2's allotment would be his second best house $h(j_2)$, which is already allocated to agent j_3) and $\phi_{j_1}(I, H, R^3) = h$ (if not, then by *individual-rationality* agent j_1's allotment would be his second best house $h(j_1)$, which is already allocated to agent j_2). By *individual-rationality*, $\phi_i(I, H, R^3) = h(i)$ (if agent i cannot get his best house h, he must receive $h(i)$). By our assumption that (I, H, R) was a minimal problem with a top priority violation at a vacant house and by *consistency*, it now follows that $\{j_4, \ldots, j_K\} = \{j'_4, \ldots, j'_K\}$ such that $\phi_{j'_4}(I, H, R^3) = h(j_3) \in H_O$, $\phi_{j'_5}(I, H, R^3) = h(j'_4) \in H_O$, $\ldots$, $\phi_{j'_K}(I, H, R^3) = h(j'_{K-1}) \in H_O$ and either $h(j'_K) = h_0$ or $h(j'_K) \in H_O$. To simplify notation, we will assume that for all $k \in \{4, \ldots, K\}$, $j_k = j'_k$, otherwise we can just rename agents.

We continue to replace the preferences of agents $j_4, \ldots, j_K$ with maximal conflict preferences (and renaming agents if necessary) until we reach a preference profile R^K with $R_i^K = R_i^h$, $R_{j_1}^K = R_{j_1}^h$, $R_{j_2}^K = R_{j_2}^{h(j_1)}$, $\ldots$, $R_{j_K}^K = R_{j_K}^{h(j_{K-1})}$, and $\phi_i(I,H,R^K) = h(i)$, $\phi_{j_1}(I,H,R^K) = h$, $\phi_{j_2}(I,H,R^K) = h(j_1)$, $\phi_{j_3}(I,H,R^K) = h(j_2)$, $\ldots$, $\phi_{j_K}(I,H,R^K) = h(j_{K-1})$ and either $h(j_K) = h_0$ or $h(j_K) \in H_O$.

Next, consider $\widehat{R}_i$ that is obtained from $R_i^K = R_i^h$ by moving $h(j_1)$ between the best house h and the second best house $h(i)$, i.e.,

$$\widehat{R}_i : h \; \widehat{P}_i \; h(j_1) \; \widehat{P}_i \; h(i) \; \widehat{R}_i \; h_0 \; \widehat{P}_i \ldots.$$

By *strategy-proofness*, $\phi_i(I,H,(\widehat{R}_i,R_{-i}^K)) \neq h$ and by *individual-rationality*, $\phi_i(I,H,(\widehat{R}_i,R_{-i}^K)) \in \{h(i),h(j_1)\}$.

If $\phi_i(I,H,(\widehat{R}_i,R_{-i}^K)) = h(j_1)$, then by *individual-rationality* and *Pareto-optimality*, $\phi_{j_1}(I,H,(\widehat{R}_i,R_{-i}^K)) = h$, $\phi_{j_2}(I,H,(\widehat{R}_i,R_{-i}^K)) = h(j_2)$, $\phi_{j_3}(I,H,(\widehat{R}_i,R_{-i}^K)) = h(j_3)$, $\ldots$, $\phi_{j_K}(I,H,(\widehat{R}_i,R_{-i}^K)) = h(j_K)$. Now, consider the reduced problem where agents $j_2,\ldots,j_K$ leave with their allotments that are the houses they occupy, i.e., $\widehat{I} = \{i,j_1\}$ and $\widehat{H} = H \setminus \{h(j_2),\ldots,h(j_K)\}$. By *consistency*, $h \; \widehat{P}_i \; \phi_i(\widehat{I},\widehat{H},(\widehat{R}_i,R_{-i}^K)_{\widehat{I}}^{\widehat{H}}) = h(j_1)$. But then that reduced problem is a problem with fewer agents than problem (I,H,R) with a top priority violation at a vacant house; a contradiction.

Hence, $\phi_i(I,H,(\widehat{R}_i,R_{-i}^K)) = h(i)$ and by *individual-rationality* and *Pareto-optimality*, $\phi_{j_1}(I,H,(\widehat{R}_i,R_{-i}^K)) = h$. Since agent j_1 owns house $h(j_1)$, $\pi^{h(j_1)}(j_1) < \pi^{h(j_1)}(i)$ and for each $2 \leq k \leq K$, $\pi^{h(j_1)}(j_1) < \pi^{h(j_1)}(j_k)$. If $\pi^{h(j_1)}(j_1) < \pi^{h(j_1)}(j_k) < \pi^{h(j_1)}(i)$, then by acyclicity (Lemmata 3 and 4) $\pi^h(j_1) < \pi^h(i)$ or $\pi^h(j_k) < \pi^h(i)$; contradicting that agent i has the top priority for house h, i.e., $\pi^h(i) < \pi^h(j_1)$ and $\pi^h(i) < \pi^h(j_k)$. Hence, $\pi^{h(j_1)}(j_1) < \pi^{h(j_1)}(i) < \pi^{h(j_1)}(j_k)$ and agent i has the second highest priority for $h(j_1)$ (after agent j_1 who has the highest priority).

Now, consider the reduced problem where agents j_1 leaves with his allotment house h, i.e., $\widetilde{I} = I \setminus \{j_1\}$ and $\widetilde{H} = H \setminus \{h\}$ (note that now $h(j_1) \in \widetilde{H}_V$). By *consistency*, $\phi_{j_2}(\widetilde{I},\widetilde{H},\left(\widehat{R}_i,R_{-i}^K\right)_{\widetilde{I}}^{\widetilde{H}}) = h(j_1)$. Furthermore, agent i now has the top priority for $h(j_1)$. However, this is now a problem with fewer agents than problem (I,H,R) with a top priority violation at a vacant house. □

C Proofs of Section 4.2 Lemmata

Throughout this section we use that *strategy-proofness* and *non-bossiness* is equivalent to *group strategy-proofness.*

Proof of Lemma 6 (**Broker* Lemma**). Let $i,j,k \in \mathcal{I}$ be three different agents and assume that house $h \in \mathcal{H}$ is not owned by any of them, i.e., $h \notin \{h(i),h(j),h(k)\}$. We show that if $i \in \mathcal{I}_T$, $\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k)$, and $\pi^h(k) < \pi^h(i),\pi^h(j)$, then under rule ϕ, agent j brokers* house h at each pair (I,H) with $I = \{i,j,k\}$ and $h \in H$.

Let $I = \{i, j, k\}$, $H \subseteq \mathcal{H}$ with $\{h, h(i), h(j), h(k)\} \subseteq H \cup \{h_0\}$, and $R \in \mathcal{R}(I, H)$ such that each agent ranks h top. We show that $\phi_j(I, H, R)$ is j's second ranked house.

Step 1. Let preferences $R' \in \mathcal{R}(I, H)$ be such that

- $R'_i : h \; P'_i \; h(i) \; P'_i \ldots$,
- $R'_j : h \; P'_j \; h(i) \; P'_j \; h(j) \; P'_j \ldots$,
- $R'_k : h \; P'_k \ldots$.

We show that agent j gets his second best house, $\phi_j(I, H, R') = h(i)$.

Suppose not. Then, by *individual rationality*, $[\phi_i(I, H, R') = h$ or $\phi_i(I, H, R') = h(i)]$ and $\phi_j(I, H, R') \in \{h, h(j)\}$. More precisely, by *Pareto-optimality*, either $[\phi_i(I, H, R') = h$, $\phi_j(I, H, R') = h(j)$, and $\phi_k(I, H, R') = h(i)]$ or $\phi_i(I, H, R') = h(i)$.

In the first case, consider the reduced problem (I^0, H^0, R^0) of (I, H, R') where agent i leaves with his allotment h, i.e., $I^0 = \{j, k\}$, $H^0 := H \setminus \{h\}$ (with $h(i) \in H^0_V$), and $R^0 = R'^{H_0}_{I_0}$. By *consistency*, we have $\phi_j(I^0, H^0, R^0) = h(j)$ and $\phi_k(I^0, H^0, R^0) = h(i)$. Note that for maximal conflict preferences $R^{h(i)}_k \in \mathcal{R}(k, H^0)$ restricted to H^0, preference profile $(R^0_j, R^{h(i)}_k)$ is a maximal conflict preference profile for $h(i)$ restricted to I^0 and H^0. By *strategy-proofness*, $\phi_k(I^0, H^0, (R^0_j, R^{h(i)}_k)) = h(i)$. However, $\pi^{h(i)}(j) < \pi^{h(i)}(k)$. Thus, by Lemma 2, we have a contradiction.

In the second case, $\phi_i(I, H, R') = h(i)$, we have either $\phi_j(I, H, R') = h$ or $\phi_k(I, H, R') = h$. If $\phi_j(I, H, R') = h$, consider the reduced problem (I^1, H^1, R^1) of (I, H, R') where agent i leaves with his allotment $h(i)$, i.e., $I^1 = \{j, k\}$, $H^1 := H \setminus \{h(i)\}$, and $R^1 = R'^{H_1}_{I_1}$. By *consistency*, we have $\phi_j(I^1, H^1, R^1) = h$ and $h \; P^1_k \; \phi_k(I^1, H^1, R^1)$. Note that for maximal conflict preferences $R^h_k \in \mathcal{R}(k, H^1)$ restricted to H^1, preference profile (R^1_j, R^h_k) is a maximal conflict preference profile for h restricted to I^1 and H^1. By *strategy-proofness*, $h \; P^h_k \; \phi_k(I^1, H^1, (R^1_j, R^h_k))$. By *Pareto-optimality*, $\phi_j(I^1, H^1, R^h) = h$. However, $\pi^h(k) < \pi^h(j)$. Thus, by Lemma 2, we have a contradiction.

Thus, we may assume that $\phi_k(I, H, R') = h$. Now, suppose that agent k changes his preferences from R'_k to $R^2_k \in \mathcal{R}(k, H)$ such that

- $R^2_k : h(i) \; P^2_k \; h \; P^2_k \; h(k) \; P^2_k \ldots$.

Consider the problem (I, H, R^2) with $R^2 = (R'_i, R'_j, R^2_k)$. As $h = \phi_k(I, H, R') \; P'_k \; h(k)$, *strategy-proofness* implies that $\phi_k(I, H, R^2) \in \{h, h(i)\}$. By *individual-rationality*, $\phi_i(I, H, R^2) \in \{h, h(i)\}$. Hence, by *Pareto-optimality*, $\phi_i(I, H, R^2) = h$ and $\phi_k(I, H, R^2) = h(i)$. Then, by *individual-rationality*, $\phi_j(I, H, R^2) = h(j)$. Consider the reduced problem (I^3, H^3, R^3) of (I, H, R^2) where agent i leaves with his allotment h, i.e., $I^3 = \{j, k\}$, $H^3 = H \setminus \{h\}$ (with $h(i) \in H^3_V$), and $R^3 = (R^2)^{H^3}_{I^3}$. By *consistency*, $\phi_k(I^3, H^3, R^3) = h(i)$ and $\phi_j(I^3, H^3, R^3) = h(j)$. Note however that R^3 is a maximal conflict preference profile for $h(i)$ (restricted to I^3 and H^3) and that $\pi^{h(i)}(j) < \pi^{h(i)}(k)$. Thus, by Lemma 2, we have a contradiction.

We have now shown that $\phi_j(I, H, R') = h(i)$. By *individual-rationality*, $\phi_i(I, H, R') = h$.

Step 2. Let preferences $R'' \in \mathcal{R}(I, H)$ be such that all three agents rank h top and agent j ranks $h(i)$ second. By *group strategy-proofness* (for i and j), the result that $\phi_i(I, H, R') = h$

and $\phi_j(I,H,R') = h(i)$ for a profile R' as in Step 1 implies that $\phi_i(I,H,R'') = h$ and $\phi_j(I,H,R'') = h(i)$.

Step 3. Next we consider the general case of a profile $R \in \mathcal{R}(I,H)$ at which all three agents rank h top, but agent j does not necessarily rank $h(i)$ second.

If $\phi_j(I,H,R) = h$, then by *strategy-proofness* agent j would also obtain h if R_j is replaced by a profile R''_j where j ranks h top and $h(i)$ second; this would contradict Step 2 which shows that j then obtains $h(i)$. Hence, $\phi_j(I,H,R) \neq h$ and *Pareto-optimality* implies that either $\phi_j(I,H,R)$ is j's second ranked or third ranked house. Suppose, for the sake of contradiction, $\phi_j(I,H,R)$ is j's third ranked house and denote by $h' \in H \setminus \{h\}$ his second ranked house (note that $h' \neq h(i)$ by Step 2).

First, consider preferences $\overline{R}_j \in \mathcal{R}(j,H)$ such that

$$\overline{R}_j : h \overline{P}_j \, h(i) \overline{P}_j \ldots .$$

Then, by Step 2, $\phi_i(I,H,(R_i,\overline{R}_j,R_k)) = h$ and $\phi_j(I,H,(R_i,\overline{R}_j,R_k)) = h(i)$.

Second, consider preferences $\widetilde{R}_j \in \mathcal{R}(j,H)$ such that

$$\widetilde{R}_j : h \, \widetilde{P}_j \, h' \, \widetilde{P}_j \, h(i) \, \widetilde{P}_j \, h(j) \, \widetilde{P}_j \ldots .$$

By *strategy-proofness*, $\phi_j(I,H,(R_i,\widetilde{R}_j,R_k)) \notin \{h,h'\}$. Moreover, by *strategy-proofness*, $\phi_j(I,H,(R_i,\widetilde{R}_j,R_k)) = h(i)$, since otherwise j could change preferences from $\widetilde{R}_j$ to $\overline{R}_j$ and obtain $h(i)$. As $\phi_j(I,H,(R_i,\widetilde{R}_j,R_k)) = h(i) = \phi_j(I,H,(R_i,\overline{R}_j,R_k))$, *non-bossiness* implies $\phi_i(I,H,(R_i,\widetilde{R}_j,R_k)) = h = \phi_i(I,H,(R_i,\overline{R}_j,R_k))$.

Next, for maximal conflict preferences $\widetilde{R}_i = R^h_i \in \mathcal{R}(i,H)$ restricted to H, by *group strategy-proofness* (for i and j), we have $\phi(I,H,(\widetilde{R}_i,\widetilde{R}_j,R_k)) = \phi(I,H,(R_i,\widetilde{R}_j,R_k))$.

Now, consider preferences $\widehat{R}_j \in \mathcal{R}(j,H)$ such that

$$\widehat{R}_j : h \, \widehat{P}_j \, h' \, \widehat{P}_j \, h(j) \, \widehat{P}_j \ldots .$$

By *individual-rationality* and *strategy-proofness*, $\phi_j(I,H,(\widetilde{R}_i,\widehat{R}_j,R_k)) = h(j)$. By *individual-rationality* $\phi_i(I,H,(\widetilde{R}_i,\widehat{R}_j,R_k)) \neq h'$ and therefore by *Pareto-optimality*, $\phi_i(I,H,(\widetilde{R}_i,\widehat{R}_j,R_k)) = h$ and $\phi_k(I,H,(\widetilde{R}_i,\widehat{R}_j,R_k)) = h'$. Consider the reduced problem (I^4,H^4,R^4) of $(I,H,(\widetilde{R}_i,\widehat{R}_j,R_k))$ where agent j leaves with his allotment $h(j)$. By *consistency*, $\phi_i(I^4,H^4,R^4) = h$ and $\phi_k(I^4,H^4,R^4) = h'$. Recall that R^4_i are maximal conflict preferences for h restricted to H^4. Hence, for each maximal conflict preference profile $R^h \in \mathcal{R}(I^4,H^4)$ for h restricted to I^4 and H^4, by *Pareto-optimality* and *strategy-proofness*, we then have $\phi_i(I^4,H^4,R^h) = h$ and $h \, P^4_k \, \phi_k(I^4,H^4,R^h)$. However, $\pi^h(k) < \pi^h(i)$. Thus, by Lemma 2, we have a contradiction. □

Proof of Lemma 7. Let $i,j,k \in \mathcal{I}$ be three different agents and assume that house $h \in \mathcal{H}$ is not owned by any of them, i.e., $h \notin \{h(i),h(j),h(k)\}$. We show that if $i \in \mathcal{I}_T$, $\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k)$, and $\pi^h(k) < \pi^h(i), \pi^h(j)$, then $\pi^h(k) < \pi^h(i) < \pi^h(j)$ and $j \in \mathcal{I}_A$.

First, by means of contradiction, assume that $\pi^h(j) < \pi^h(i)$. Consider the problem (I,H,R) with $I = \{i,j,k\}$, $H_V := \{h\}$, and $R \in \mathcal{R}(I,H)$ such that

- $R_i : h\; P_i\; h(i)\; P_i \ldots,$
- $R_j : h\; P_j\; h(i)\; P_j\; h(j)\; P_j \ldots,$
- $R_k : h\; P_k\; h(k)\; P_k \ldots.$

By Lemma 6, under rule ϕ, agent j brokers* house h at pair (I, H) and hence $\phi_j(I, H, R) = h(i)$. Then, by *individual rationality*, we have $\phi_i(I, H, R) = h$ and $\phi_k(I, H, R) = h(k)$.

Consider the reduced problem (I^1, H^1, R^1) of (I, H, R) where agent k leaves with his allotment $h(k)$. By *consistency*, we have $\phi_i(I^1, H^1, R^1) = h$ and $\phi_j(I^1, H^1, R^1) = h(i)$.

Note that for maximal conflict preferences $R^h_j \in \mathcal{R}(j, H^1)$ for h restricted to H^1, preference profile (R^1_i, R^h_j) is a maximal conflict preference profile for h restricted to I^1 and H^1. By *strategy-proofness*, $h P^h_j \phi_j(I^1, H^1, (R^1_i, R^h_j))$. By *Pareto-optimality*, $\phi_i(I^1, H^1, (R^1_i, R^h_j)) = h$. However, $\pi^h(j) < \pi^h(i)$. Thus, by Lemma 2, we have a contradiction. Hence, we have $\pi^h(i) < \pi^h(j)$.

Next we show that agent j is an applicant. Suppose not. Then, j is a tenant and $h(j) \neq h_0$. Consider the problem $(I, H, \widetilde{R})$ with $I = \{i, j, k\}$, $H_V := \{h\}$, and $\widetilde{R} \in \mathcal{R}(I, H)$ such that

- $\widetilde{R}_i : h\; \widetilde{P}_i\; h(i)\; \widetilde{P}_i \ldots,$
- $\widetilde{R}_j : h\; \widetilde{P}_j\; h(j)\; \widetilde{P}_j \ldots,$
- $\widetilde{R}_k : h\; \widetilde{P}_k\; h(j)\; \widetilde{P}_k\; h(k)\; \widetilde{P}_k \ldots.$

By Lemma 6, under rule ϕ, agent j brokers* house h at pair (I, H) and hence $\phi_j(I, H, \widetilde{R}) = h(j)$. Consider the reduced problem (I^2, H^2, R^2) of $(I, H, \widetilde{R})$ where agent j leaves with his allotment $h(j)$. Note that R^2 is a maximal conflict preference profile for h restricted to I^2 and H^2. As $\pi^h(k) < \pi^h(i)$, Lemma 2 implies that $\phi_k(I^2, H^2, R^2) = h$. Thus, by *consistency*, $\phi_k(I, H, \widetilde{R}) = h$ and by *individual-rationality*, $\phi_i(I, H, \widetilde{R}) = h(i)$.

Now consider $R' \in \mathcal{R}(I, H)$ with

- $R'_i = \widetilde{R}_i : h\; \widetilde{P}_i\; h(i)\; \widetilde{P}_i \ldots,$
- $R'_j = \widetilde{R}_j : h\; \widetilde{P}_j\; h(j)\; \widetilde{P}_j \ldots,$
- $R'_k : h(j)\; P'_k\; h\; P'_k\; h(k)\; P'_k \ldots.$

By *strategy-proofness*, $\phi_k(I, H, R') \in \{h, h(j)\}$. If $\phi_k(I, H, R') = h$, then by *individual-rationality*, $\phi(I, H, \widetilde{R}) = \phi(I, H, R')$, contradicting *Pareto-optimality*. Hence, $\phi_k(I, H, R') = h(j)$ and by *individual-rationality*, $\phi_i(I, H, R') = h(i)$ and $\phi_j(I, H, R') = h$.

Now consider $\widehat{R} \in \mathcal{R}(I, H)$ with

- $\widehat{R}_i = R'_i = \widetilde{R}_i : h\; \widetilde{P}_i\; h(i)\; \widetilde{P}_i \ldots,$
- $\widehat{R}_j : h\; \widehat{P}_j\; h(i)\; \widehat{P}_j\; h(j)\; \widehat{P}_j \ldots,$
- $\widehat{R}_k : h\; \widehat{P}_k\; h(j)\; \widehat{P}_k\; h(k)\; \widehat{P}_k \ldots.$

By Lemma 6, under rule ϕ, agent j brokers* house h at pair (I, H) and hence $\phi_j(I, H, \widehat{R}) = h(i)$. By *individual-rationality* and *Pareto-optimality* this implies $\phi_i(I, H, \widehat{R}) = h$ and

$\phi_k(I,H,\widehat{R}) = h(j)$. However, since moving from preference profile $\widehat{R}$ to R' we have $\phi_j(I,H,R') = h\ \widehat{P}_j\ h(i) = \phi_j(I,H,\widehat{R})$ and $\phi_k(I,H,R') = h(j) = \phi_k(I,H,\widehat{R})$, we have a violation of *group strategy-proofness*. □

Proof of Lemma 8 (Acyclicity for Occupied Houses, Non-Bossiness Version). Let $i,j,k \in \mathcal{I}$ be three different agents and assume that house $h(i) \in \mathcal{H}$ is occupied by agent i and house $h' \in \mathcal{H}$ is not owned by any of the three agents, i.e., $h' \notin \{h(i), h(j), h(k)\}$. Thus, $[|\mathcal{H}| = 3$ and $|\mathcal{I}_T| \leq 1]$ or $|\mathcal{H}| \geq 4$ (since $i \in \mathcal{I}_T$, we cannot have $[|\mathcal{H}| = 2$ and $\mathcal{I}_T = \emptyset]$). We show that $\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k)$ implies $[\pi^{h'}(i) < \pi^{h'}(k)$ or $\pi^{h'}(j) < \pi^{h'}(k)]$.

Assume for the sake of contradiction that $\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k)$, $\pi^{h'}(k) < \pi^{h'}(i)$, and $\pi^{h'}(k) < \pi^{h'}(j)$. As $[|\mathcal{H}| = 3$ and $|\mathcal{I}_T| \leq 1]$ or $|\mathcal{H}| \geq 4$, there is a house $h \notin \{h(i), h(k), h'\}$. By Lemma 6, j is a broker* of h' at $(\mathcal{I}, \mathcal{H})$. By Lemma 7 applied to houses $h(i)$ and h', we have $\pi^{h'}(k) < \pi^{h'}(i) < \pi^{h'}(j)$ and j is an applicant, in particular, j does not own h.

By Lemma 3 applied to houses h and h', we have $\pi^h(k) < \pi^h(j)$ or $\pi^h(i) < \pi^h(j)$. In the first case, if we have $\pi^h(k) < \pi^h(i)$, then Lemma 7 applied to houses $h(i)$ and h would imply $\pi^h(k) < \pi^h(i) < \pi^h(j)$ and if we have $\pi^h(i) < \pi^h(k)$, then $\pi^h(i) < \pi^h(j)$. Hence, we may assume that $\pi^h(i) < \pi^h(j)$.

Consider the problem (I,H,R) with $I = \{i,j,k\}$, $H \cup \{h_0\} = \{h, h', h(i), h(j), h(k), h_0\}$, and preferences are such that

- $R_i : h'\ P_i\ h(i)\ P_i \ldots$,
- $R_j : h\ P_j\ h(i)\ P_j\ h(j)\ P_j \ldots$,
- $R_k : h'\ P_k\ h(k)\ P_k \ldots$.

By *Pareto-optimality* we have $\phi_j(I,H,R) = h$ and either $\phi_i(I,H,R) = h'$ or $\phi_k(I,H,R) = h'$. Consider the reduced problem (I',H',R') of (I,H,R) where agent j leaves with his allotment h, i.e., $I' = \{i,k\}$, $H' = H \setminus \{h\}$, and $R' = R^{H'}_{I'}$. Note that R' is a maximal conflict preference profile for h' (restricted to I' and H') and that $\pi^{h'}(k) < \pi^{h'}(i)$. Thus, Lemma 2 implies that $\phi_k(I',H',R') = h'$. Hence, *consistency* implies $\phi_k(I,H,R) = h'$ and $\phi_i(I,H,R) = h(i)$.

Consider the profile $R^1 = (R^1_i, R_j, R_k)$ such that

- $R^1_i : h'\ P^1_i\ h\ P^1_i\ h(i)\ P^1_i \ldots$.

By *strategy-proofness* and *individual-rationality* we have $\phi_i(I,H,R^1) \in \{h(i), h\}$.

First, consider the case that $\phi_i(I,H,R^1) = h(i)$. By *Pareto-optimality*, we have $\phi_j(I,H,R^1) = h$ and $\phi_k(I,H,R^1) = h'$. Consider the reduced problem (I^2,H^2,R^2) of (I,H,R^1) where agent k leaves with his allotment h', i.e., $I' = \{i,j\}$, $H^2 = H \setminus \{h'\}$. By *consistency*, we have $\phi_j(I^2,H^2,R^2) = h$ and $\phi_i(I^2,H^2,R^2) = h(i)$. Note that at preference profile R^2, agent i has maximal conflict preferences for h restricted to H^2. So, when we consider a maximal conflict preference profile $R^h \in \mathcal{R}(I^2,H^2)$ for h restricted to I^2 and H^2, only agent j changes his preferences from R^2_j to R^h_j. Then, by *strategy-proofness*, we also have $\phi_j(I^2,H^2,R^h) = h$ and $\phi_i(I^2,H^2,R^h) = h(i)$. However, as $\pi^h(i) < \pi^h(j)$, this contradicts Lemma 2.

Next, consider the case that $\phi_i(I, H, R^1) = h$. Then, by *Pareto-optimality*, we have $\phi_j(I, H, R^1) = h(i)$. Now consider preference profile $R^3 \in \mathcal{R}(I, H)$ such that

- $R_i^3 : h' \; P_i \; h(i) \; P_i \ldots,$
- $R_j^3 : h' \; P_j \; h(i) \; P_j \; h(j) \; P_j \ldots,$
- $R_k^3 = R_k : h' \; P_k \; h(k) \; P_k \ldots.$

By Lemma 6, under rule ϕ, agent j brokers* house h' at pair (I, H). Hence, $\phi_j(I, H, R^3) = h(i)$. Then, by *individual-rationality*, $\phi_i(I, H, R^3) = h'$. However, $\phi_i(I, H, R^3) = h' \; P_i^1 \; h = \phi_i(I, H, R^1)$ and $\phi_j(I, H, R^3) = h(i) = \phi_j(I, H, R^1)$, a violation of *group strategy-proofness*. □

D Independence of Properties in Theorems 2 and 3

For each of the examples introduced to establish independence below we indicate the property of Theorem 2 or 3 it fails (while it satisfies all remaining properties).

Pareto-Optimality. The **null rule** ϕ^0 assigns to each tenant his occupied house and to each applicant the null house. Hence, for each problem (I, H, R) and each agent $i \in I$, $\phi_i^0(I, H, R) = h(i)$. The null rule ϕ^0 satisfies *individual-rationality*, *strategy-proofness*, *non-bossiness*, *reallocation-proofness*, and *consistency*, but it violates *Pareto-optimality*. □

Individual Rationality (for Tenants). Let π be a priority structure such that for any $h, h' \in \mathcal{H}$, $\pi^h = \pi^{h'}$, i.e., every house has the same priority ordering or serial dictatorship ordering. The **serial dictatorship rule** φ^π now works as follows. For each problem (I, H, R), the highest serial dictatorship priority agent in I, let's say agent i, is assigned his best house in $H \cup \{h_0\}$, the highest serial dictatorship priority agent in $I \setminus \{i\}$, let's say agent j, is assigned his best house in $(H \setminus \{\varphi_i^\pi(I, H, R)\}) \cup \{h_0\}$, and so on. The serial dictatorship rule φ^π satisfies *Pareto-optimality*, *strategy-proofness*, *non-bossiness*, *reallocation-proofness*, and *consistency*, but it violates *individual-rationality for tenants* since the serial dictatorship priority structure is not adapted to the ownership structure. □

Strategy-Proofness. Let π be a priority structure such that for any $h, h' \in \mathcal{H}$, $\pi^h = \pi^{h'}$, i.e., every house has the same priority ordering or serial dictatorship ordering. Furthermore assume that all tenants have higher priority than all applicants, i.e., for each house $h \in \mathcal{H}$, each tenant $i \in \mathcal{I}_T$, and each applicant $j \in \mathcal{I}_A$, $\pi^h(i) < \pi^h(j)$. Let $\widehat{\pi}$ denote the priority structure obtained from π by adapting it to the ownership structure. Note that at $\widehat{\pi}$, again, all tenants have higher priority for all houses than all applicants and $\widehat{\pi}$ is acyclic.

We define rule $\widehat{\phi}$ as follows. For each problem (I, H, R), we first consider the problem consisting of tenants and all houses, i.e., we consider the problem (I_T, H, R_{I_T}), and we apply the TTC rule $\varphi^{\widehat{\pi}}$, i.e.,

$$\text{for each } i \in I_T, \; \widehat{\phi}_i(I, H, R) = \varphi_i^{\widehat{\pi}}(I_T, H, R_{I_T}).$$

Since the TTC rule $\varphi^{\widehat{\pi}}$ is used, we have *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *non-bossiness*, *reallocation-proofness*, and *consistency* among tenants.

Next, we apply the immediate acceptance algorithm to determine the matching for the remaining reduced problem $(I \setminus I_T, H \setminus \widehat{H}, R^{H\setminus\widehat{H}}_{I\setminus I_T})$ where $\widehat{H} = \widehat{\phi}_{I_T}(I, H, R)$.

Immediate Acceptance Algorithm

Step 1: Each applicant applies to his favorite house in $\left(H \setminus \widehat{H}\right) \cup \{h_0\}$. Each house in $H \setminus \widehat{H}$ accepts the highest priority applicant and rejects all others. The null house h_0 accepts all applicants.

Step $r \geq 2$: Each applicant who was rejected at Step $r-1$ applies to his favorite house in $\left(H \setminus \widehat{H}\right) \cup \{h_0\}$ that did not reject him yet. Each house in $H \setminus \widehat{H}$ not assigned in a previous step accepts the highest priority applicant and rejects all others. Each house in $H \setminus \widehat{H}$ that was assigned in a previous step rejects all applicants and the null house h_0 accepts all applicants.

The algorithm terminates when each applicant in $I \setminus I_T$ is accepted by a house in $\left(H \setminus \widehat{H}\right) \cup \{h_0\}$. The matching where each agent is assigned the house that he was accepted by at the end of the algorithm is called the **immediate acceptance matching** and denoted by $IA^{\widehat{\pi}}(I \setminus I_T, H \setminus \widehat{H}, R^{H\setminus\widehat{H}}_{I\setminus I_T})$. Hence,

$$\text{for each } i \in I_A,\ \widehat{\phi}_i(I, H, R) = IA^{\widehat{\pi}}(I \setminus I_T, H \setminus \widehat{H}, R^{H\setminus\widehat{H}}_{I\setminus I_T}).$$

Any immediate acceptance rule is *Pareto-optimal*, *individually-rational*, *non-bossy*, and *consistent* (Doğan and Klaus, 2018; Kojima and Ünver, 2014). Hence, we have *Pareto-optimality*, *individual-rationality*, *non-bossiness*, and *consistency* among applicants. Since the underlying priority structure for the immediate acceptance algorithm used here is a serial dictatorship ordering, it is easy to see that we also have *reallocation-proofness* among applicants.

Given the sequentiality of rule $\widehat{\phi}$, first using rule $\varphi^{\widehat{\pi}}$ for tenants and then rule $IA^{\widehat{\pi}}$ for applicants, it follows that $\widehat{\phi}$ satisfies *Pareto-optimality*, *individual-rationality*, *non-bossiness*, *reallocation-proofness*, and *consistency*. However, it is well-known that immediate acceptance rules are *not strategy-proof*. Hence, rule $\widehat{\phi}$ is not *strategy-proof*. □

Consistency. By Proposition 1, a TTC rule based on a cyclic priority structure that is adapted to the ownership structure satisfies *individual rationality*, *Pareto-optimality*, *strategy-proofness*, *non-bossiness*, and *reallocation-proofness*. By Theorem 1 it violates *consistency*. □

Reallocation-Proofness/Non-Bossiness. Example 6. □

E Trading Cycles with One Broker and One Vacant House

In Theorem 3 we require that $|\mathcal{H}| \geq 4$, or $[|\mathcal{H}| = 3$ and $|\mathcal{I}_T| \leq 1]$, or $[|\mathcal{H}| = 2$ and $\mathcal{I}_T = \emptyset]$. A violation of the requirements implies that (i) $[|\mathcal{H}| = 3$ and $|\mathcal{I}_T| = 3]$, or (ii) $[|\mathcal{H}| = 3$ and $|\mathcal{I}_T| = 2]$, or (iii) $[|\mathcal{H}| = 2$ and $|\mathcal{I}_T| = 2]$, or (iv) $[|\mathcal{H}| = 2$ and $|\mathcal{I}_T| = 1]$.

We now show that in all four cases, we may have additional rules that are *Pareto-optimal*, *individually-rational*, *group-strategy-proof*, and *consistent*.[26] They can be formulated as Trading Cycles with One Broker (TC-OB) rules that are similar to the TC rules introduced by Pycia and Ünver (2017, Appendix F) but adjusted to our model with the assumptions of one broker and that agents can rank the null house arbitrarily (in Pycia and Ünver, 2017, Appendix F, all houses are acceptable, but for some situations multiple brokers are allowed).

The TC-OB algorithm uses as input a **structure of control rights**, i.e., a family of mappings $c_{(I,H)} : H \to I \times \{\text{brokerage}, \text{ownership}\}$ for each pair (I, H). For agent $i \in I$ with $c_{(I,H)}(h) = (i, \text{brokerage})$ we call i the **broker** of house h at pair (I, H). For agent $i \in I$ with $c_{(I,H)}(h) = (i, \text{ownership})$ we call i the **owner** of house h at pair (I, H). If agent i either brokers or owns h at (I, H), then i **controls** house h at pair (I, H). We assume that c is such that for each (I, H) there is at most one broker.

Similarly, as in the TTC algorithm the TC-OB algorithm matches agents in a "cycle" according to a prescribed set of rules. For a problem (I, H, R) and a structure of control rights c with one broker, we construct a directed graph where each agent points to his most preferred house, each house points to the agent who controls it, and the null house points to each agent. A cycle in the directed graph is **simple** if one of the agents in the cycle owns the house pointing to him or if an agent points to the null house.

For each problem (I, H, R) and each structure of control rights c with one broker, the **trading cycles with one broker (TC-OB) algorithm based on *c*** is defined as follows:

Input. A problem (I, H, R) and a structure of control rights c with one broker.

Step 1. Let $I_1 := I$ and $H_1 := H$. Each agent $i \in I_1$ points to his most preferred house in $H_1 \cup \{h_0\}$. Each house $h \in H_1$ points to the agent in I_1 who controls it according to $c_{(I_1,H_1)}$. The null house h_0 points to each agent in I_1.

- **Step 1(A). Matching Simple Trading Cycles.** We assign to each agent in a simple trading cycle the house he points to and remove all simple trading cycle agents and houses (except the null house).

- **Step 1(B). Forcing the Broker to Downgrade His Pointing.** If there are no simple trading cycles in Step 1(A), and only then, the algorithm works as follows (otherwise we go to Step 2).

[26] If $|\mathcal{I}| = 3$, then in cases (i) and (iii) *non-bossiness* is not needed in the characterization (because acyclicity for occupied houses is vacuously satisfied). Hence, for Cases (i) and (iii) we additionally have to have at least four possible agents.

- If the broker is the only agent pointing at the brokered house, then he receives it and we remove both.
- Otherwise, we force the broker to point to his next best house and return to Step 1(A).

We define I_2 to be the set of remaining agents and H_2 to be the set of remaining houses and, if $I_2 \neq \emptyset$, continue with Step 2. Otherwise we stop.

In general at Step t we have the following:

Step t. Each agent $i \in I_t$ points to his most preferred house in $H_t \cup \{h_0\}$. Each house $h \in H_t$ points to the agent in I_t who controls it according to $c_{(I_t,H_t)}$. The null house h_0 points to each agent in I_t.

- **Step t(A). Matching Simple Trading Cycles.** We assign to each agent in a simple trading cycle the house he points to and remove all simple trading cycle agents and houses (except the null house).
- **Step t(B). Forcing the Broker to Downgrade His Pointing.** If there are no simple trading cycles in Step t(A), and only then, the algorithm works as follows (otherwise we go to Step $t+1$).
  - If the broker is the only agent pointing at the brokered house, then he receives it and we remove both.
  - Otherwise, we force the broker to point to his next best house and return to Step t(A).

We define I_{t+1} to be the set of remaining agents and H_{t+1} to be the set of remaining houses and, if $I_{t+1} \neq \emptyset$, continue with Step $t+1$. Otherwise we stop.

Output. The TC-OB algorithm terminates when all agents in I are assigned a house in $H \cup \{h_0\}$. We denote the house in $H \cup \{h_0\}$ that agent $i \in I$ obtains in the TC-OB algorithm by $\phi_i^c(I, H, R)$.

The **TC-OB rule based on the structure of control rights c, ϕ^c**, associates with each problem (I, H, R) the matching determined by the TC-OB algorithm. Note that our definition of the TC-OB algorithm slightly differs from that of Pycia and Ünver (2017, Appendix F) because we accommodated arbitrary rankings of the null house and in Step t(B) force the broker to downgrade his pointing when another agent points to the brokered house and not just an owner. We now give examples of TC-OB rules that are *Pareto-optimal*, *individually-rational*, *strategy-proof*, *non-bossy*, and *consistent* for each of our cases.

Example 7 (Case (i)). Consider $\mathcal{I} = \{i, j, k, \ell\}$, $\mathcal{I}_T = \{i, k, \ell\}$, and $\mathcal{H} = \{h(i), h(k), h(\ell)\}$. Define the structure of control rights c with one broker as follows. For each pair (I, H) with

$|I| > 1$ we let

$$c_{(I,H)}(h(i)) = \begin{cases} (i, \text{ownership}) & \text{if } i \in I, \\ (\ell, \text{ownership}) & \text{if } i \notin I, \ell \in I, \\ (j, \text{ownership}) & \text{if } i, \ell \notin I, j \in I, \\ (k, \text{ownership}) & \text{else,} \end{cases}$$

$$c_{(I,H)}(h(k)) = \begin{cases} (k, \text{ownership}) & \text{if } k \in I, \\ (\ell, \text{ownership}) & \text{if } k \notin I, \ell \in I, \\ (j, \text{ownership}) & \text{if } k, \ell \notin I, j \in I, \\ (i, \text{ownership}) & \text{else,} \end{cases}$$

and

$$c_{(I,H)}(h(\ell)) = \begin{cases} (\ell, \text{ownership}) & \text{if } \ell \in I, \\ (j, \text{brokerage}) & \text{if } I = \{i,j,k\}, \\ (k, \text{ownership}) & \text{if } I = \{i,k\} \text{ or } I = \{j,k\}, \\ (i, \text{ownership}) & \text{else.} \end{cases}$$

Note that according to c, agents i, k, and ℓ have ownership rights over their endowments in each problems where they are present. Hence, ϕ^c is *individually-rational*. Note that c satisfies requirements (R1)-(R6) given in Pycia and Ünver (2017). Thus, similarly as in Pycia and Ünver (2017, Theorem 8 in Appendix F) it follows that ϕ^c satisfies *Pareto-optimality*, *strategy-proofness*, and *non-bossiness*.

Next we show that ϕ^c is *consistent*. It suffices to consider the case of a problem (I, H, R) with $I = \{i, j, k\}$ and $H = \mathcal{H}$ and a reduced problem (I', H', R') obtained from (I, H, R) by removing one agent with his allotment (all other cases are easy to check, since in these cases the rule behaves like a TTC rule with acyclic priorities). Let $\mu = \phi^c(I, H, R)$ and $\mu' = \phi^c(I', H', R')$.

We distinguish between four cases:

1. according to R_i, agent i ranks $h(i)$ top,
2. according to R_k, agent k ranks $h(k)$ top,
3. according to R, agent i ranks $h(k)$ top and agent k ranks $h(i)$ top,
4. according to R, agent i or agent k or both rank $h(\ell)$ top.

Case 1. In this case, we have $\mu(i) = h(i)$ and in case that $i \in I'$ we have $\mu'(i) = h(i)$. Thus, for $I' = \{i, j\}$ and $I' = \{i, k\}$, *Pareto-optimality* implies that agent j, or respectively agent k, obtains the same allotment as before. It remains to consider the case that $I' = \{j, k\}$ and $H' = \{h(k), h(\ell)\}$. Moreover, it suffices to consider the case that j ranks $h(\ell)$ above $h(k)$ and h_0 and k ranks $h(\ell)$ above $h(k)$, because otherwise *consistency* follows by *Pareto-optimality* and *individual-rationality*.

Then, in the TC-OB algorithm for (I, H, R), first a simple trading cycle with i and $h(i)$ forms ($\mu(i) = h(i)$). Next, k points to $h(\ell)$ and since $I_2 = \{j, k\}$, $h(\ell)$ points to k, and a simple trading cycle $[k, h(\ell)]$ forms. Hence, k is assigned $h(\ell)$ and j is either assigned $h(k)$ or h_0, depending on which of the two he prefers ($\mu(k) = h(\ell)$ and $\mu(j) \in \{h(k), h_0\}$). In the TC-OB algorithm for the reduced problem (I', H', R'), both j and k point to $h(\ell)$ and similarly as before both agents obtain the same allotments ($\mu'(k) = h(\ell)$ and $\mu'(j) \in \{h(k), h_0\}$).

Case 2. This case can be handled with a completely analogous argument to Case 1 by switching the roles of agents i and k.

Case 3. In this case, in the TC-OB algorithm for (I, H, R), first the simple trading cycle $[i, h(k), k, h(i)]$ forms and hence we have $\mu(i) = h(k)$ and $\mu(k) = h(i)$. Thus, in the reduced problem with $I' = \{i, k\}$ and $H' = \{h(i), h(k)\}$, by *Pareto-optimality* both agents obtain the same allotments ($\mu'(i) = h(k)$ and $\mu'(k) = h(i)$).

Case 4. We consider the following subcases:

(a) according to R_j, j ranks $h(i)$ above $h(k)$ and h_0,

(b) according to R_j, j ranks $h(k)$ above $h(i)$ and h_0,

(c) according to R_j, j ranks h_0 above $h(i)$ and $h(k)$.

Case 4(a). In this case, in the TC-OB algorithm for (I, H, R), either broker j points to $h(i)$, or j points to $h(\ell)$ and, as agent i or agent k ranks $h(\ell)$ top, is forced to downgrade his pointing and points to $h(i)$ afterwards. In either case, we have $\mu(j) = h(i)$. Thus, in the reduced problem, we either have $I' = \{i, j\}$ and $H' = \{h(i), h(\ell)\}$ or $I' = \{j, k\}$ and $H' = \{h(i), h(k)\}$.

If $I' = \{i, j\}$ and $H' = \{h(i), h(\ell)\}$, then in the TC-OB algorithm for (I', H', R') agent i points to $h(\ell)$ and, since he has ownership according to c for (I', H'), obtains $h(\ell)$. Hence, in this case both i and j obtain the same allotments.

If $I' = \{j, k\}$ and $H' = \{h(i), h(k)\}$, then in the TC-OB algorithm for (I', H', R') agent j points to $h(i)$, and since he has ownership according to c for (I', H'), obtains $h(i)$. Hence, in this case both j and k obtain the same allotments.

Case 4(b). This case can be handled with a completely analogous argument to Case 4(a) by switching the roles of agents i and k.

Case 4(c). In this case, in the TC-OB algorithm for (I, H, R), either broker j points to h_0, or j points to $h(\ell)$ and, as agent i or agent k ranks $h(\ell)$ top, is forced to downgrade his pointing and points to h_0 afterwards. Thus, j and h_0 form a simple trading cycle. After this cycle is resolved ($\mu(j) = h_0$), $h(\ell)$ points to k. We consider two cases, (i) according to R_k, agent k ranks $h(\ell)$ top and we have $\mu(k) = h(\ell)$, or (ii) according to R_k, agent k ranks $h(i)$ top and we have $\mu(i) = h(\ell)$ and $\mu(k) = h(i)$.

Case 4(ci). According to R_k, agent k ranks $h(\ell)$ top and we have $\mu(k) = h(\ell)$. We consider the cases $I' = \{i, j\}$ and $H' = \{h(i), h(k)\}$, $I' = \{i, k\}$ and $H' = \{h(i), h(k), h(\ell)\}$, and $I' = \{j, k\}$ and $H' = \{h(k), h(\ell)\}$.

If $I' = \{i, j\}$, by *Pareto-optimality* we have $\mu'(j) = h_0$ and i obtains his favorite house among $h(i)$ and $h(k)$, which is also his allotment under μ.

If $I' = \{i, k\}$, then k owns $h(\ell)$ according to c at (I', H') and thus $\mu'(k) = h(\ell)$ and i obtains his favorite house among $h(i)$ and $h(k)$, which is also his allotment under μ.

If $I' = \{j, k\}$, then k owns $h(\ell)$ at (I', H') and thus $\mu'(k) = h(\ell)$ and j obtains h_0 which is also his allotment under μ.

Case 4(cii). According to R_k, agent k ranks $h(i)$ top and we have $\mu(i) = h(\ell)$ and $\mu(k) = h(i)$. We consider the cases $I' = \{i, k\}$ and $H' = \{h(i), h(k), h(\ell)\}$ and $I' = \{j, k\}$ and $H' = \{h(i), h(k)\}$.

If $I' = \{i, k\}$, then by *Pareto-optimality* we have $\mu'(i) = h(\ell) = \mu(i)$ and $\mu'(k) = h(i) = \mu(k)$.

If $I' = \{j, k\}$, then by *Pareto-optimality* we have $\mu'(j) = h_0 = \mu(j)$ and $\mu'(k) = h(i) = \mu(k)$. □

Example 8 (**Case (ii)**)**.** Consider $\mathcal{I}' = \{i, j, k\}$, $\mathcal{I}'_T = \{i, k\}$, and $\mathcal{H}' = \{h(i), h(k), h\}$. We use the structure of control rights c and the TC-OB rule ϕ^c from Example 7 to construct a rule ϕ' as follows. For any problem (I, H, R) with $I \subseteq \mathcal{I}'$, $H \subseteq \mathcal{H}'$ we let $\phi'(I, H, R) = \phi^c(I, H, R)$ with the interpretation that $h = h(\ell)$. *Individual-rationality*, *Pareto-optimality*, *group-strategy-proofness*, and *consistency* of ϕ' follows from the previous analysis of Example 7. □

Example 9 (**Case (iii)**)**.** Consider $\mathcal{I}' = \{i, j, k, \ell\}$, $\mathcal{I}'_T = \{i, \ell\}$, and $\mathcal{H}' = \{h(i), h(\ell)\}$. We use the structure of control rights c and the TC-OB rule ϕ^c from Example 7 to construct a rule ϕ' as follows. Recall that $\mathcal{I} = \{i, j, k, \ell\}$, $\mathcal{I}_T = \{i, k, \ell\}$, and $\mathcal{H} = \{h(i), h(k), h(\ell)\}$.

First, we map any problem $(I', H', R') \in \mathcal{I}' \times \mathcal{H}' \times \mathcal{R}(I', H')$ into a problem $(I, H, R) \in \mathcal{I} \times \mathcal{H} \times \mathcal{R}(I, H)$ such that $I = I'$, $H = H' \cup \{h(k)\}$, and R is such that $R^{H'} = R'$, agents i and j and ℓ rank $h(k)$ last, and agent k ranks $h(k)$ just above the null house.

Second, we define rule ϕ' such that for each problem $(I', H', R') \in \mathcal{I}' \times \mathcal{H}' \times \mathcal{R}(I', H')$, $\phi'(I', H', R') = \phi^c(I, H, R)$ with the modification that whenever at $\phi^c(I, H, R)$ agent k received house $h(k)$ he in fact receives the null house.

Individual-rationality, *Pareto-optimality*, *group-strategy-proofness*, and *consistency* of ϕ' all now follow rather immediately from the fact that ϕ^c satisfies these properties. □

Example 10 (**Case (iv)**)**.** Consider $\mathcal{I}' = \{i, j, k\}$, $\mathcal{I}'_T = \{i\}$, and $\mathcal{H}' = \{h(i), h\}$. We use the structure of control rights c and the TC-OB rule ϕ^c from Example 9 to construct a rule ϕ' as follows. For any problem (I, H, R) with $I \subseteq \mathcal{I}'$, $H \subseteq \mathcal{H}'$ we let $\phi'(I, H, R) = \phi^c(I, H, R)$ with the interpretation that $h = h(\ell)$. *Individual-rationality*, *Pareto-optimality*, *group-strategy-proofness*, and *consistency* of ϕ' follows from the previous discussion. □

Finally, we show that our examples (Examples 7 - 10) of TC-OB rules are not equivalent to TTC rules. For Case (i), let $I = \{i, j, k\}$, $H = \mathcal{H}$ (with $H_V = \{h(\ell)\}$), and consider the following preferences

- $R_i : h(\ell)\ P_i\ h(i)\ P_i \ldots,$
- $R_j : h(\ell)\ P_j\ h(i)\ P_j\ h(k)\ P_j\ h_0,$
- $R_k : h(\ell)\ P_k\ h(k)\ P_k \ldots,$
- $R'_j : h(\ell)\ P'_j\ h(k)\ P'_j\ h(i)\ P'_j\ h_0.$

Let $R = (R_i, R_j, R_k) \in \mathcal{R}(I, H)$ and $R' = (R_i, R'_j, R_k) \in \mathcal{R}(I, H)$. Note that at preference profiles R and R' the house $h(\ell)$ is the top-ranked house by all agents in I. Hence, each TTC rule assigns house $h(\ell)$ to the same agent at profiles R and R'. However, the TC-OB rule according to c assigns house $h(\ell)$ to different agents at profiles R and R', i.e., $\phi_i^c(I, H, R) = h(\ell)$ and $\phi_k^c(I, H, R') = h(\ell)$. Similar examples can easily be constructed for the remaining Cases (ii), (iii), and (iv).

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