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Nearest Common Root of a set of polynomials: A structured singular value approach

Olga Limantseva, George Halikias, Nicos Karcanias

Control Engineering Research Centre, School of Engineering and Mathematical Sciences, City University, Northampton Square, London EC1V 0HB, UK

Abstract

The paper considers the problem of calculating the nearest common root of a polynomial set under perturbations in their coefficients. In particular, we seek the minimum-magnitude perturbation in the coefficients of the polynomial set such that the perturbed polynomials have a common root. It is shown that the problem is equivalent to the solution of a structured singular value (μ) problem arising in robust control for which numerous techniques are available. It is also shown that the method can be extended to the calculation of an "approximate GCD" of fixed degree by introducing the notion of the generalized structured singular value of a matrix. The work generalizes previous results by the authors involving the calculation of the "approximate GCD" of two polynomials, although the general case considered here is considerably harder and relies on a matrix-dilation approach and several preliminary transformations.

Keywords:
structured singular value, Sylvester resultant matrix, approximate GCD, distance to singularity, almost common root.

1. Introduction

The study of the Greatest Common Divisor (GCD) of a set of polynomials has received considerable interest in recent years. The notion of GCD of polynomial sets has several applications in Control Theory (e.g. algebraic

Email address: olga.limantseva@city.ac.uk (Olga Limantseva)

control methods, determinantal assignment problems, distance to controllability or observability), Robust Control (stability of dynamic systems subject to structured perturbations), Linear Systems, Numerical Analysis and other Engineering fields. Many systems properties depend on the notion of zeros and the GCD of a set of polynomials. Computation of the GCD is a non-generic problem: Generically any set of polynomials is coprime and hence the question of "coprimeness" needs to be reinterpreted from a binary notion (with a yes or no answer) to one involving a continuously-valued function, a notion which is more suitable for finite-precision calculations. Thus, it is natural to define a set of polynomials as "approximately coprime" if its distance from the nearest non-trivial common divisor (different from one) is "sufficiently large" in an appropriate sense. The concept of "almost zero" was first introduced in [1] as the complex number that is closest to a set of coprime polynomials. Subsequently, this has led to the introduction of the notion of "approximate GCD", see [2], [3], [4], [5], [6], [7] and references therein. Again, the definition of this notion is based on the relaxation of the conditions defining the exact GCD. The approximate GCD of a set of coprime polynomials is a polynomial which in some sense is closest to the set.

The problem of finding the GCD of many polynomials has been a subject of interest for a long time in mathematics, numerical analysis and control theory. The origins go back to Euclid's algorithms for two polynomials and more recently on the work based on generalized resultants [8], [9], [10], [11]. Euclid's algorithm has provided an algebraic framework for two polynomials and its iterative use provides extension for many polynomials. In the later case a matrix method based on the invariance property of GCD under row equivalence and shifting was introduced in [12]. An alternative method reducing GCD computation to matrix pencils was introduced in [3]. Several numerical methods for GCD computation based on relaxation of the exact methods have also been developed. Numerical techniques developed for GCD computations were presented in [13], [14], [15], [3], [16], [6], [17], [2], [7], [18], [19] and references therein.

The importance of defining the notion of an "almost common factor" for a set of polynomials has been highlighted in [20], [21], [22] and references therein. This is based on the relaxation of the conditions involved in exact GCD computations. Alongside the classical computational framework of GCD various strategies have been considered: The invariance property of GCD leads to the numerical method of ERES methodologies [13], [2]; [7]

considers a variety of approaches using the Euclidian algorithm; [23], [20] rely on a matrix pencil technique, while [15] applies an augmented optimization methodology based on factorisation techniques. The majority of the proposed methods in the literature analyse characteristics of the set of polynomials based on the Sylvester matrix representation and Generalized Resultants [10], [14], [5], [20], [21], as these formulate the GCD problem in linear-algebraic terms which can be implemented via reliable numerical algorithms. The ERES methodology [2] differs from the Resultant Approaches since it is based on the invariance property of GCD under Gaussian transformations and shifting [12]. The current work proposes a method for calculating the distance of a set of monic co-prime polynomials to the set of polynomials sharing a common root and, more generally to the set of polynomials with a GCD of fixed degree k, $k \geq 1$. From this viewpoint the work provides an alternative characterisation of the notion of "almost zero" [1].

The proposed technique is based on singular values to define and solve approximate GCD problems. Consider for simplicity the case of two polynomials. These are coprime if and only if the corresponding Sylvester Resultant matrix, $S_{\mathcal{P}}$ say, is nonsingular. Thus, it is reasonable to associate proximity to singularity with the smallest singular value of $S_{\mathcal{P}}$. This follows from the fact that the smallest singular value of a matrix A is equal to the minimum norm perturbation Δ such that $A + \Delta$ is singular (here the norm of Δ, $\|\Delta\|$, indicates the spectral norm, i.e. the largest singular value). Unfortunately, as a measure of proximity to singularity, the smallest singular value of $S_{\mathcal{P}}$ can be conservative. The reason is that perturbations in $S_{\mathcal{P}}$ arise from the perturbations in the polynomials' coefficients which enter $S_{\mathcal{P}}$ in a highly structured way.

Some limitations of the use of the singular values of $S_{\mathcal{P}}$ as indicators of the existence of an approximate GCD of a specific degree (for the case of two polynomials) have been highlighted in [5]. In this reference the notion of the *generalised structured singular value* (μ-value) was introduced to quantify the distance of the corresponding Sylvester resultant matrix from the set of matrices with nullity at least k. This corresponds to the distance from the set of polynomials with a GCD of degree at least k. In the case $k = 1$ this notion reduces to the well-known *structured singular value* which is a well-researched tool in the area of robust control.

The present work generalises the results of [5] to the general case of an arbitrary number of polynomials $h \geq 2$. We proceed by calculating the distance of the Sylvester resultant matrix to a matrix of reduced rank under

appropriate structured perturbations; this corresponds to the computation of the closest (minimum distance) nontrivial GCD, i.e. a GCD with degree at least one. It is shown that the problem is still equivalent to the calculation of a structured singular value, however in this case the solution is significantly more intricate compared to the case $h = 2$. Part of the technical difficulties arise due to the fact that the derived equivalent structured approximation problem no longer involves square matrices and also that the structured approximation sets need to be redefined during the solution procedure. It is also shown that the notion of generalized structured singular value can be extended to general structured rank-approximation problems of non-square matrices, in which approximate GCD problems of general degree can be formulated in a natural way (although their solution is more complicated).

The structure of the paper is as follows: Section 2 summarises some fundamental results related to the GCD of a set of polynomials and its relation to the generalised Sylvester resultant matrix. Section 3 shows that the problem of calculating the closest common root of a set of polynomials subject to minimal magnitude perturbations in the polynomials' coefficients is equivalent to the calculation of a structured singular value of a matrix. It is also shown that the calculation of the optimal approximate GCD of degree $k > 1$ is associated with the solution of a generalised structured singular value problem. Computational techniques for solving structured singular value problems of the type arising in this work are described in Section 4. Finally, the main conclusions of the paper and suggestions for further research are included in Section 5.

Throughout the paper standard notation is used. $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices and $\mathbb{C}^{n\times m}$ the set of $n \times m$ complex matrices. The set of non-negative (positive) numbers is denoted as $\mathbb{R}_{+0}$ ($\mathbb{R}_{+}$, respectively. The sets $\mathbb{R}_{-0}$ and $\mathbb{R}_{-}$ are defined analogously. $O_{n,m}$ denotes the $n\times m$ zero matrix and 1_n the (column) vector of ones in $\mathbb{R}^n$. $\mathbb{R}[s]$ defines the set of polynomials with real coefficients. If $d(s) \in \mathbb{R}[s]$, then the degree of the polynomial is $\deg\{d(s)\}$. A' is the transpose of matrix $A \in \mathbb{R}^{n\times m}$, while A^* is the complex conjugate transpose of $A \in \mathbb{C}^{n\times m}$. The spectrum of $A \in \mathbb{R}^{n\times n}$ is denoted by $\lambda(A)$. $\mathcal{N}(A)$, $\mathcal{R}(A)$ correspond to the right null-space (kernel) and the range (column span) of $A \in \mathbb{R}^{n\times m}$ respectively. If $A \in \mathbb{R}^{n\times m}$, then according to the rank-nullity theorem $\text{rank}(A) + \text{null}(A) = m$, where $\text{null}(A) := \dim(\mathcal{N}(A))$ and $\text{rank}(A) := \dim(\mathcal{R}(A))$. The set of singular values of $A \in \mathbb{R}^{n\times m}$ is defined as $\sigma(A) = \{\sigma_i(A), i = 1, \ldots, min(n, m)\}$, listed in the non-increasing order $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{min(n,m)} \geq 0$. The spectral norm of a matrix A is its

largest singular value, i.e. $\|A\| = \sigma_1(A)$. If $f : \Omega \to \mathbb{R}$ where Ω is an open subset of $\mathbb{R}^q$ is differentiable at $d \in \Omega$ then

$$\nabla f(d) = \begin{pmatrix} \frac{\partial f(d)}{\partial d_1} & \frac{\partial f(d)}{\partial d_2} & \dots & \frac{\partial f(d)}{\partial d_q} \end{pmatrix}' \text{ and } \langle \nabla f(d), x \rangle$$

denote, respectively, the gradient of f and the directional derivative of f along direction $x \in \mathbb{R}^q$, $\|x\| = 1$. If $A \in \mathbb{R}^{m \times n}$, then $\mathrm{vec}(A) : \mathbb{R}^{m \times n} \to \mathbb{R}^{mn}$ denotes the vectorization operation of stacking the columns of A into a vector and $\mathrm{vec}^{-1}(\cdot)$ denotes the inverse operator. Similarly if $m = n$ and $A = A'$ then $\overline{\mathrm{vec}}(A) \in \mathbb{R}^r$, $r = \frac{n(n+1)}{2}$, denotes the vector $\mathrm{vec}(A)$ with all the elements of A below the main diagonal eliminated and $\overline{\mathrm{vec}}^{-1}(\cdot)$ denotes the inverse operation. The Kronecker product of two matrices A and B is denoted as $A \otimes B$. Additional notation is introduced in the sequel as needed.

2. The generalised Sylvester resultant for many polynomials

Consider a set of polynomials $\mathcal{P}_{h+1,n}$:

$$\begin{aligned} \mathcal{P}_{h+1,n} = \{ & a(s), b_i(s) \in \mathbb{R}[s], \;\; i = 1, \dots, h; \\ & n = deg\{a(s)\}, \;\; t = deg\{b_i(s)\}, \;\; i = 1, 2, \dots, h, \;\; n \geq t \} \end{aligned} \tag{1}$$

The notation is occasionally simplified to $\mathcal{P}$ if the integers n and h can be inferred from the context. Let $\phi(s)$ be the GCD of the polynomial set $\mathcal{P}_{h+1,n}$. If the polynomials are coprime, i.e. $\phi(s) = 1$, we denote the polynomial set as $\mathcal{P}^0_{h+1,n}$. The Sylvester matrix background is considered next [9], [10], [13], [14],[16], [11]:

Definition 2.1. Consider the set of monic polynomials $\mathcal{P}_{h+1,n} = \{a(s), b_i(s) \in \mathbb{R}[s], \;\; i = 1, \dots, h\}$, where $a(s)$ and $b_i(s)$, $\forall i = 1, \dots, h$, are given as:

$$\begin{aligned} a(s) &= s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_0, \\ b_i(s) &= s^t + \beta_{t-1,i} s^{t-1} + \dots + \beta_{0,i}, \;\; i = 1, \dots, h \end{aligned} \tag{2}$$

We assume with no loss of generality that the polynomials are monic, i.e. $\alpha_n = \beta_{t,i} = 1$, $i = 1, 2, \dots, h$.

(i) Let $S_0 \in \mathbb{R}^{t \times (n+t)}$ be the Sylvester Resultant associated with $a(s)$:

$$S_0 = \begin{pmatrix} 1 & \alpha_{n-1} & \alpha_{n-2} & \dots & \dots & \alpha_0 & 0 & \dots & 0 \\ 0 & 1 & \alpha_{n-1} & \dots & \dots & \alpha_1 & \alpha_0 & \dots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & \alpha_{t-1} & \dots & \dots & \alpha_0 \end{pmatrix} \tag{3}$$

(ii) Corresponding to $b_i(s)$ for each $i = 1, \ldots, h$ the resultant matrices $S_i \in \mathbb{R}^{n\times(n+t)}$ are structured as:

$$S_i = \begin{pmatrix} 1 & \beta_{i,t-1} & \beta_{i,t-2} & \ldots & \ldots & \beta_{i,0} & 0 & \ldots & 0 \\ 0 & 1 & \beta_{i,t-1} & \ldots & \ldots & \beta_{i,1} & \beta_{i,0} & \ldots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & \ldots & \beta_{i,n-1} & \ldots & \ldots & \beta_{i,0} \end{pmatrix} \tag{4}$$

(iii) The Generalized Resultant for the set $\mathcal{P}_{h+1,n}$ is defined as:

$$S_{\mathcal{P}} = \begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_h \end{pmatrix} \in \mathbb{R}^{(t+hn)\times(n+t)} \tag{5}$$

The Sylvester resultant matrix holds important properties for GCD computations. Moreover, the concept of "approximate GCD" and the distance of the polynomials to the GCD variety can be analysed with respect to the singular values of the corresponding Sylvester resultant. In previous research papers [13], [4], [6] "approximate co-primeness" is studied alongside the Sylvester matrix representation leading to some important notions that are considered next.

Theorem 2.1. [14]: Let $S_{\mathcal{P}}$ be a Sylvester resultant matrix of the set of polynomials $\mathcal{P}_{h+1,n}$

$$S_{\mathcal{P}} = \begin{pmatrix} 1 & \alpha_{n-1} & \alpha_{n-2} & \ldots & \ldots & \alpha_0 & 0 & \ldots & 0 \\ 0 & 1 & \alpha_{n-1} & \ldots & \ldots & \alpha_1 & \alpha_0 & \ldots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & \ldots & \alpha_{t-1} & \ldots & \ldots & \alpha_0 \\ 1 & \beta_{1,t-1} & \beta_{1,t-2} & \ldots & \ldots & \beta_{1,0} & 0 & \ldots & 0 \\ 0 & 1 & \beta_{1,t-1} & \ldots & \ldots & \beta_{1,1} & \beta_{1,0} & \ldots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & \ldots & \beta_{1,n-1} & \ldots & \ldots & \beta_{1,0} \\ \vdots & & \vdots & \vdots & & \vdots & & & \vdots \\ 1 & \beta_{h,t-1} & \beta_{h,t-2} & \ldots & \ldots & \beta_{h,0} & 0 & \ldots & 0 \\ 0 & 1 & \beta_{h,t-1} & \ldots & \ldots & \beta_{h,1} & \beta_{h,0} & \ldots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & \ldots & \beta_{h,n-1} & \ldots & \ldots & \beta_{h,0} \end{pmatrix}. \tag{6}$$

If we denote $\phi(s)$ as the GCD of the corresponding polynomial set $\mathcal{P}_{h+1,n} = \{a(s), b_i(s), \forall i = 1, \ldots, h\}$ than the following properties hold true:

1. For the polynomial set to be coprime it is necessary and sufficient that the Sylvester Resultant has full rank, i.e. $\text{rank}(S_{\mathcal{P}}) = n + t$;

2. If $\mathcal{P}_{h+1,n}$ has non-trivial GCD ($\deg(\phi(s)) \geq 1$), then

$$\text{rank}(S_{\mathcal{P}}) = n + t - deg\{\phi(s)\}; \tag{7}$$

3. Since GCD of $S_{\mathcal{P}}$ stays invariant under elementary row operations, then by reducing it to the row-echelon form the last non-vanishing row will provide the coefficients of $\phi(s)$.

For a proof of these properties see [14].

Remark 1. The degree of the "approximate GCD" can be determined from the values of the smaller singular values of the Sylvester resultant matrix. For example, the matrix pencil technique [3], [16] defines the degree of the approximate GCD with respect to a specified tolerance level *tol*, i.e. as the number of singular values that are less than or equal to *tol*. This approach is refined in this paper by replacing "singular values" by "structured singular values" which take explicitly into account the structure of the resultant matrix.

3. Distance to the closest common root of a set of polynomials, "approximate GCD" and the structured singular value

Structured singular values [24], [25] are a powerful tool for the analysis and synthesis of robust control systems. They can be used to model uncertainty in system dynamics arising from multiple sources, e.g. parametric uncertainty in the coefficients of the differential or difference equations, unstructured norm-bounded uncertainty arising due to unmodelled high-frequency dynamics, or combinations of these two types. Structured singular values can be employed to establish non-conservative conditions for robust-stability and robust-performance analysis and in combination with $\mathcal{H}_\infty$ optimal control they can provide a systematic framework for robust control system design [24], [26], [27].

The computation of the structured singular value (μ) of a matrix is an NP-hard problem. Convex relaxation methods are normally employed to calculate upper bounds of μ; techniques for reducing the duality gap have also been developed. In the present work, structured uncertainty arises from the perturbations in the coefficients of the polynomials entering the generalized resultant matrix and will be represented as a diagonal matrix of repeated scalar perturbations. The definition of the appropriate uncertainty structure requires a sequence of preliminary transformations which are introduced later in the section.

Recently, a new approach was introduced for estimating the distance of *two* polynomials to the set of polynomials with a common root [5]. It was argued that, since the Sylvester resultant matrix is highly structured in terms of the polynomial coefficients (and hence also their perturbations), general approaches which use smallest singular value of the Sylvester resultant as an indicator of numerical singularity tend to underestimate the actual distance as they are more applicable to matrix-distance approximation problems involving unstructured perturbations. As an alternative, the structured singular value (μ-value) methodology was proposed. The effectiveness of this method was demonstrated with a simple example. The technique is strictly applicable to the case of two polynomials (corresponding to a square Sylvester resultant matrix); generalization to the general case of an arbitrary number of polynomials addressed in this paper is a more complex problem.

The definition of the structured singular value of a matrix is given next. Note that the underlying perturbation structure is not the most general possible but is adequate for the purposes of this work:

Definition 3.1. [24] Let $M \in \mathbb{R}^{n\times n}$ and consider the structured set of uncertainties as

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}) : \delta_i \in \mathbb{R}, i = 1, \ldots, s\} \subseteq \mathbb{R}^{n\times n}, \tag{8}$$

where r_i are positive integers corresponding to the block-structure of $\mathbf{\Delta}$, i.e. $\sum_{i=1}^{s} r_i = n$. If there exists $\Delta \in \mathbf{\Delta}$, such that $\det(I_n - M\Delta) = 0$, then the structured singular value of M is:

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\min\{\|\Delta\| : \Delta \in \mathbf{\Delta},\ \det(I_n - M\Delta) = 0\}}, \tag{9}$$

If for all $\Delta \in \mathbf{\Delta}$ we have $\det(I_n - M\Delta) \neq 0$ then $\mu_{\mathbf{\Delta}}(M) = 0$.

The first problem considered is as follows:

Problem 1. (Distance to non-coprimeness). Consider the set $\mathcal{P}^0_{h+1,n}$ of coprime polynomials

$$\begin{aligned} a_0(s) &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0, \\ b_{0,i}(s) &= s^t + \beta_{t-1,i}s^{t-1} + \cdots + \beta_{0,i}, \quad i = 1, \ldots, h \end{aligned} \tag{10}$$

with corresponding Sylvester resultant $S_{\mathcal{P}_0}$ and define the set of perturbed polynomials $\mathcal{P}_{h+1,n}$:

$$\begin{aligned} a(s) &= s^n + (\alpha_{n-1} + \delta_{n-1})s^{n-1} + \cdots + (\alpha_0 + \delta_0), \\ b_i(s) &= s^t + (\beta_{t-1,i} + \epsilon_{t-1,i})s^{t-1} + \cdots + (\beta_{0,i} + \epsilon_{0,i}), \quad i = 1, \ldots, h \end{aligned} \tag{11}$$

and the corresponding Sylvester resultant $S_{\mathcal{P}}$. Then, what is the minimal absolute value perturbation in the coefficients of the nominal polynomials $\mathcal{P}^0_{h+1,n}$ so that the perturbed polynomials $\mathcal{P}_{h+1,n}$ have a common root? Formally define:

$$\gamma = \max\{|\delta_0|, \ldots, |\delta_{n-1}|, |\epsilon_{0,1}|, \ldots, |\epsilon_{t-1,1}|, \ldots, |\epsilon_{0,h}|, \ldots, |\epsilon_{t-1,h}|\}$$

We seek to minimise γ so that the perturbed polynomials (11) have a common root. Introduce the Sylvester resultant of the perturbed polynomials $S_{\mathcal{P}} = S_{\mathcal{P}_0} + E$ where:

$$E = \begin{pmatrix} 0 & \delta_{n-1} & \delta_{n-2} & \ldots & \ldots & \delta_0 & 0 & \ldots & 0 \\ 0 & 0 & \delta_{n-1} & \ldots & \ldots & \delta_1 & \delta_0 & \ldots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & \delta_{t-1} & \ldots & \ldots & \delta_0 \\ 0 & \epsilon_{1,t-1} & \epsilon_{1,t-2} & \ldots & \ldots & \epsilon_{1,0} & 0 & \ldots & 0 \\ 0 & 0 & \epsilon_{1,t-1} & \ldots & \ldots & \epsilon_{1,1} & \epsilon_{1,0} & \ldots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & \epsilon_{1,n-1} & \ldots & \ldots & \epsilon_{1,0} \\ \vdots & & \vdots & \vdots & & \vdots & & & \vdots \\ 0 & \epsilon_{h,t-1} & \epsilon_{h,t-2} & \ldots & \ldots & \epsilon_{h,0} & 0 & \ldots & 0 \\ 0 & 0 & \epsilon_{h,t-1} & \ldots & \ldots & \epsilon_{h,1} & \epsilon_{h,0} & \ldots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & \epsilon_{h,n-1} & \ldots & \ldots & \epsilon_{h,0} \end{pmatrix}. \tag{12}$$

Then, in view of Theorem 2.1 the problem can be formulated as:

$$\inf\{\gamma : \text{null}(S_{\mathcal{P}}) \geq 1\}.$$

Remark 2. For compatibility with previous notation and the framework of [3], [28], [21] the h polynomials $b_i(s), i = 1, \ldots, h$, are considered to be of the same degree t. Note, however, that all results can be easily amended to the general case (i.e. any set of $h+1$ polynomials $\{a(s), b_i(s)\}, i = 1, 2, \ldots, h$, of arbitrary degrees $\deg\{a(s), b_i(s)\} \geq 1, i = 1, 2, \ldots, h$) with only minor modifications.

Problem 1 can be generalized as follows:

Problem 2. (Approximate GCD of degree k) Let all variables be defined as in Problem 1 above. Here we seek to minimise γ so that the perturbed polynomials (11) have a GCD $\phi(s)$ with $\deg(\phi(s)) \geq k$. Equivalently the problem can be formulated as: $\inf\{\gamma : \text{null}(S_{\mathcal{P}}) \geq k\}$.

In the remaining part of this section it is shown that Problem 1 is equivalent to the computation of a structured singular value with respect to a diagonal set of repeated perturbations, while Problem 2 involves the computation of a "generalized structured singular value", defined over a similar diagonal set. These generalize the results in [5] to the case of multiple polynomials and rely on the following Lemma:

Lemma 3.1. Let $M \in \mathbb{R}^{n \times m}$, $n \geq m$ and define:

$$A = \begin{pmatrix} I_n & M \\ M' & 0_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)} \tag{13}$$

Then $\text{null}(M) = \text{null}(A)$. In particular M has full column rank if and only if matrix A is nonsingular.

Proof. Straightforward and therefore omitted. □

Considering Lemma 3.1 and the properties of the Sylvester matrix listed in Theorem 2.1, Problem 1 can be reformulated as follows:

Theorem 3.2. Let $S_{\mathcal{P}_0}$ be the Sylvester Resultant matrix of the set of the $h+1$ coprime polynomials $\mathcal{P}^0_{h+1,n} = \{a_0(s), b_{0,i}(s),\ \ \forall i = 1, \ldots, h\}$. Introduce perturbations $\{\delta_i\}_{i=0,1,\ldots,n-1}$ and $\{\epsilon_{i,j}\}_{i=1,2,\ldots,h}^{j=0,1,\ldots,t-1}$ in the polynomials' coefficients and define γ as in Problem 1. Then the minimum-magnitude perturbation γ^* in the coefficients of $\mathcal{P}^0_{h+1,n}$ such that the perturbed polynomials in $\mathcal{P}_{h+1,n}$ have a common root is given by:

$$\gamma^* = \min\{\gamma : \text{null}(S_{\mathcal{P}_0} + \Theta\Delta Z) \geq 1, \Delta \in \mathbf{\Delta}\}$$

where $\mathbf{\Delta}$ is the structured set:

$$\mathbf{\Delta} = \{\text{diag}(\Delta_1, \Delta_2)\} \subseteq \mathbb{R}^{nt(h+1)\times nt(h+1)} \tag{14}$$

where:

$$\Delta_1 = \text{diag}(\delta_{n-1}I_t, \delta_{n-2}I_t, \ldots, \delta_0 I_t) \in \mathbb{R}^{nt\times nt} \tag{15}$$

and

$$\Delta_2 = \text{diag}(\epsilon_{1,t-1}I_n, \ldots, \epsilon_{1,0}I_n, \ldots, \epsilon_{h,t-1}I_n, \ldots, \epsilon_{h,0}I_n) \in \mathbb{R}^{nth\times nth} \tag{16}$$

in which $\delta_i \in \mathbb{R}, \epsilon_{i,j} \in \mathbb{R}$. Matrix $\Theta \in \mathbb{R}^{(t+nh)\times nt(h+1)}$ is defined as:

$$\Theta = \text{diag}(1'_n \otimes I_t, 1'_t \otimes I_n, \ldots, 1'_t \otimes I_n)$$

which can be also written in expanded form as:

$$\Theta = \left(\begin{array}{ccc|ccc|c|ccc} I_t & \ldots & I_t & O_{t,n} & \ldots & O_{t,n} & \ldots & O_{t,n} & \ldots & O_{t,n} \\ \hline O_{n,t} & \ldots & O_{n,t} & I_n & \ldots & I_n & \ldots & O_{n,n} & \ldots & O_{n,n} \\ \hline \vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ \hline O_{n,t} & \ldots & O_{n,t} & O_{n,n} & \ldots & O_{n,n} & \ldots & I_n & \ldots & I_n \end{array}\right) \tag{17}$$

Matrix Z is defined as:

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_2 \end{pmatrix} = \begin{pmatrix} Z_1 \\ 1_h \otimes Z_2 \end{pmatrix} \in \mathbb{R}^{nt(h+1)\times(n+t)}$$

where

$$Z_1 = \begin{pmatrix} Z^0_{t,n} \\ Z^1_{t,n} \\ \vdots \\ Z^{n-1}_{t,n} \end{pmatrix} \in \mathbb{R}^{nt\times(n+t)} \ \text{ and } \ Z_2 = \begin{pmatrix} Z^0_{n,t} \\ Z^1_{n,t} \\ \vdots \\ Z^{t-1}_{n,t} \end{pmatrix} \in \mathbb{R}^{nt\times(n+t)}$$

in which

$$Z_{t,n}^k = \left(\begin{array}{c|c|c} O_{t,k+1} & I_t & O_{t,n-k-1} \end{array} \right) \in \mathbb{R}^{t\times(n+t)}, \quad k = 0, 1, \ldots, n-1$$

and

$$Z_{n,t}^k = \left(\begin{array}{c|c|c} O_{n,k+1} & I_n & O_{n,t-k-1} \end{array} \right) \in \mathbb{R}^{n\times(n+t)}, \quad k = 0, 1, \ldots, t-1$$

Proof. According to Theorem 2.1 polynomials $\mathcal{P}_{h+1,n}$ have a common root if and only if $S_{\mathcal{P}} = S_{\mathcal{P}_0} + E$ loses rank. By "pulling out the uncertainty" it is straightforward to verify that $S_{\mathcal{P}_0} + E = S_{\mathcal{P}_0} + \Theta\Delta Z$ where $\Delta \in \mathbf{\Delta}$. This follows by writing the Sylvester perturbation matrix in (12) as:

$$E = \begin{pmatrix} \sum_{\lambda=0}^{n-1} \delta_\lambda Z_{t,n}^{n-\lambda-1} \\ \sum_{\lambda=0}^{t-1} \epsilon_{1,\lambda} Z_{n,t}^{t-\lambda-1} \\ \vdots \\ \sum_{\lambda=0}^{t-1} \epsilon_{h,\lambda} Z_{n,t}^{t-\lambda-1} \end{pmatrix} = \begin{pmatrix} \delta_{n-1} Z_{t,n}^0 + \delta_{n-2} Z_{t,n}^1 + \ldots + \delta_0 Z_{t,n}^{n-1} \\ \epsilon_{1,t-1} Z_{n,t}^0 + \epsilon_{1,t-2} Z_{n,t}^1 + \ldots + \epsilon_{1,0} Z_{n,t}^{t-1} \\ \vdots \\ \epsilon_{h,t-1} Z_{n,t}^0 + \epsilon_{h,t-2} Z_{n,t}^1 + \ldots + \epsilon_{h,0} Z_{n,t}^{t-1} \end{pmatrix}$$

which reveals the displacement structure of the Sylvester resultant matrix. Further note that $\gamma = \|\Delta\|$ and hence:

$$\gamma^* = \inf\{\|\Delta\| : \text{null}(S_{\mathcal{P}_0} + \Theta\Delta Z) \geq 1, \Delta \in \mathbf{\Delta}\} \tag{18}$$

Consider the perturbations: $\delta_0 = -\alpha_0$, $\epsilon_{i,0} = -\beta_{i,0}$, $i = 1, 2, \ldots, h$. Then all perturbed polynomials $\mathcal{P}_{h+1,n}$ have a common root at the origin and hence $\gamma^* \leq \hat{\gamma} := \max\{|\alpha_0|, |\beta_{1,0}|, \ldots, |\beta_{h,0}|\}$. Hence we can restrict the constraint set in (18) to the compact set:

$$\{\Delta : \text{null}(S_{\mathcal{P}_0} + \Theta\Delta Z) \geq 1, \Delta \in \mathbf{\Delta}, \|\Delta\| \leq \hat{\gamma}\}$$

Since the function $\Delta \to \|\Delta\|$ is continuous the infimum in (18) is attained. □

The following result shows that the solution of Problem 1 is equivalent to the calculation of a structured singular value.

Theorem 3.3. Let all variables be defined as in Theorem 3.2. Then the minimum-magnitude perturbation γ^* in the coefficients of $\mathcal{P}_{h+1,n}^0$ such that the perturbed polynomials $\mathcal{P}_{h+1,n}$ have a common root is $\gamma^* = \mu_{\tilde{\mathbf{\Delta}}}^{-1}(\tilde{M})$ where

$$\tilde{M} = -P' \begin{pmatrix} \Theta' S_{\mathcal{P}_0}(S_{\mathcal{P}_0}' S_{\mathcal{P}_0})^{-1} Z' & \Theta'(I - S_{\mathcal{P}_0}(S_{\mathcal{P}_0}' S_{\mathcal{P}_0})^{-1} S_{\mathcal{P}_0}')\Theta \\ -Z(S_{\mathcal{P}_0}' S_{\mathcal{P}_0})^{-1} Z' & Z(S_{\mathcal{P}_0}' S_{\mathcal{P}_0})^{-1} S_{\mathcal{P}_0}' \Theta \end{pmatrix} P$$

and P is a permutation matrix such that $\mathrm{diag}(\mathbf{\Delta}, \mathbf{\Delta}) = P\tilde{\mathbf{\Delta}}P'$ in which

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_{n-1}I_t, \ldots, \delta_0 I_t, \epsilon_{1,t-1}I_n, \ldots, \epsilon_{1,0}I_n, \ldots, \epsilon_{h,t-1}I_n, \ldots, \epsilon_{h,0}I_n)\} \tag{19}$$

and

$$\tilde{\mathbf{\Delta}} = \{\mathrm{diag}(\delta_{n-1}I_{2t}, \ldots, \delta_0 I_{2t}, \epsilon_{1,t-1}I_{2n}, \ldots, \epsilon_{1,0}I_{2n}, \ldots, \epsilon_{h,t-1}I_{2n}, \ldots, \epsilon_{h,0}I_{2n})\} \tag{20}$$

Proof. According to Theorem 3.2

$$\gamma^* = \min\{\|\Delta\| : \mathrm{null}(S_{\mathcal{P}_0} + \Theta\Delta Z) \geq 1, \Delta \in \mathbf{\Delta}\} \tag{21}$$

From Lemma 3.1:

$$\mathrm{null}(S_{\mathcal{P}_0} + \Theta\Delta Z) \geq 1 \quad \Leftrightarrow \quad \det\begin{pmatrix} I & S_{\mathcal{P}_0} + \Theta\Delta Z \\ S'_{\mathcal{P}_0} + Z'\Delta\Theta & 0 \end{pmatrix} = 0$$

which is also equivalent to condition:

$$\det\left\{\begin{pmatrix} I & S_{\mathcal{P}_0} \\ S'_{\mathcal{P}_0} & 0 \end{pmatrix} + \begin{pmatrix} \Theta & 0 \\ 0 & Z' \end{pmatrix}\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} 0 & Z \\ \Theta' & 0 \end{pmatrix}\right\} = 0 \tag{22}$$

Since the set of polynomials $\mathcal{P}_{h+1,n}$ are assumed coprime the Sylvester resultant $S_{\mathcal{P}_0}$ has full column rank and hence the first matrix in equation (22) is nonsingular (see Lemma 3.1). Next let $\Delta \in \mathbf{\Delta}$ and introduce permutation P so that $\mathrm{diag}(\Delta, \Delta) = P\tilde{\Delta}P'$, $\tilde{\Delta} \in \tilde{\mathbf{\Delta}}$ and note that:

$$\Delta \in \mathbf{\Delta} \quad \Leftrightarrow \quad \tilde{\Delta} \in \tilde{\mathbf{\Delta}}$$

Thus condition 22 is equivalent to: $\det(I + \tilde{M}\tilde{\Delta}) = 0$ where

$$\tilde{M} = P'\begin{pmatrix} \Theta' & 0 \\ 0 & Z \end{pmatrix}\begin{pmatrix} I & S_{\mathcal{P}_0} \\ S'_{\mathcal{P}_0} & 0 \end{pmatrix}^{-1}\begin{pmatrix} 0 & \Theta \\ Z' & 0 \end{pmatrix}P$$

Thus from equation (21) it follows that:

$$\gamma^* = \min\{\|\tilde{\Delta}\| : \det(I + \tilde{M}\tilde{\Delta}) = 0, \tilde{\Delta} \in \tilde{\mathbf{\Delta}}\} = \mu^{-1}_{\tilde{\mathbf{\Delta}}}(\tilde{M}) \tag{23}$$

as required. The form of $\tilde{M}$ given in the statement of the Theorem follows on noting that:

$$\begin{pmatrix} I & S_{\mathcal{P}_0} \\ S'_{\mathcal{P}_0} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} I - S_{\mathcal{P}_0}(S'_{\mathcal{P}_0}S_{\mathcal{P}_0})^{-1}S'_{\mathcal{P}_0} & S_{\mathcal{P}_0}(S'_{\mathcal{P}_0}S_{\mathcal{P}_0})^{-1} \\ (S'_{\mathcal{P}_0}S_{\mathcal{P}_0})^{-1}S'_{\mathcal{P}_0} & -(S'_{\mathcal{P}_0}S_{\mathcal{P}_0})^{-1} \end{pmatrix}$$

after some algebra. □

Remark 3. The proposed method can be easily generalised to the case, when the nominal coefficients of the polynomials are complex and the perturbations are real. First, Lemma 3.1 can be generalised to the complex case by replacing M' by M^* in equation (13). This leads to the computation of the real structured singular value of a complex matrix (relative to a real diagonal structure with repeated elements). This can be calculated by separating real and imaginary parts (see [24] for details).

In order to illustrate the structured singular value method for many polynomials a simple numerical example is presented below.

Example 1. Consider the set of coprime polynomials $\mathcal{P}_{3,2}$:

$$\begin{cases} p_1(s) = s^2 + s - 2 \\ p_2(s) = s^2 + 6.0020s - 6.9860 \\ p_3(s) = s^2 + 4s - 5 \end{cases}$$

The corresponding resultant matrix is:

$$S_{\mathcal{P}_0} = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 6.0020 & -6.9860 & 0 \\ 0 & 1 & 6.0020 & -6.9860 \\ 1 & 4 & -5 & 0 \\ 0 & 1 & 4 & -5 \end{pmatrix}$$

with singular values $\sigma(S_{\mathcal{P}_0}) = \{13.6359, 8.9945, 0.9044, 0.0067\}$. Since the smallest singular value is almost zero the numerical rank of $S_{\mathcal{P}_0}$ is 3, indicating an "approximate GCD" of degree one.

As $S_{\mathcal{P}_0} \in \mathbb{R}^{6\times 4}$ the generalised approach of Theorem 3.2 can be applied, leading to an augmented matrix:

$$
M = \left(\begin{array}{cccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 6.0020 & -6.9860 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 6.0020 & -6.9860 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 4 & -5 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 4 & -5 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 6.0020 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\
-2 & 1 & -6.9860 & 6.0020 & -5 & 4 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & -6.9860 & 0 & -5 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Structured singular value calculations were performed with MATLAB's μ-analysis and synthesis toolbox [29]. The minimum distance to singularity (i.e. the minimum norm perturbation in the coefficients of the three polynomials such that the perturbed polynomials have a common root) is obtained (with an accuracy of four decimal places) as $\gamma^* = 0.0035$. This is the exact μ-value as the lower and upper bounds, obtained via MATLAB, coincide (within the specified tolerance):

$$
\begin{cases}
p_1 = s^2 + 1.0017s - 1.9983 \\
p_2 = s^2 + 5.9986s - 6.9894 \\
p_3 = s^2 + 4.0034s - 4.9965
\end{cases}
$$

which have GCD $\phi(s) = s - 0.9989$.

In the last part of this section we turn our attention to the solution of Problem 2 which involves the calculation of the numerical GCD (of arbitrary degree) of a set of polynomials $\mathcal{P}^0_{h+1,n}$ (assumed coprime without loss of generality). One possible approach is to develop an approximate solution to this problem in the form of an iterative algorithm. This extracts sequentially approximate common factors $\phi_i(s)$, by calculating the corresponding structured singular value $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{M})$ and a corresponding minimum-norm singularising matrix perturbation by repeated application of Theorem 3.2. After the extraction of each factor, the quotient $a_{i+1}(s) = a_i(s)/\phi_i(s)$ and $b_{i+1,j}(s) = b_{i,j}(s)/\phi_i(s)$ are calculated, ignoring possible (small) remainder terms of the divisions. The procedure is initialised by setting $a_0(s) = a(s)$,

$b_{0,j}(s) = b_j(s),\ j = 1, 2, \ldots, h$, and iterates by constructing at each step of the algorithm the reduced-dimension Sylvester matrix corresponding to the polynomials $(a_{i+1}(s), \{b_{i+1,j}(s)\}_{j=1,2,\ldots,h})$; this is followed by calculating the new structured singular value $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{M})$ and a corresponding $\tilde{\Delta}_0 \in \tilde{\mathbf{\Delta}}$, which in turn leads to the extraction of the new approximate factor $\phi_{i+1}(s)$. The whole process is repeated until a tolerance condition is met, at which stage the approximate GCD $\phi(s)$ can be constructed by accumulating the extracted common factors $\phi_i(s)$. Special care is needed to ensure that any complex roots in $\phi(s)$ appear in conjugate pairs.

Compared to this procedure a more elegant (and exact) approach is to extract the approximate GCD of the polynomial set $\mathcal{P}^0_{h+1,n}$ by solving a single optimization problem. This involves the following generalization of the notion of the structured singular value of a matrix:

Definition 3.2. Let $M \in \mathbb{R}^{n\times n}$ and define the "structured" set:

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1 I_{r_1},\ \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s})\ :\ \delta_i \in \mathbb{R},\ i = 1, 2, \ldots, s\} \tag{24}$$

where the r_i are positive integers such that $\sum_{i=1}^{s} r_i = n$. (Note that $\mathbf{\Delta}$ is a subspace of $\mathbb{R}^{n\times n}$). The *generalised structured singular value* of M relative to "structure" $\mathbf{\Delta}$ and for a non-negative integer k is defined as:

$$\hat{\mu}_{\mathbf{\Delta},k}(M) = \frac{1}{\min\{\|\Delta\|\ :\ \Delta \in \mathbf{\Delta},\ \mathrm{null}(I_n - M\Delta) > k\}} \tag{25}$$

unless there does not exist a $\Delta \in \mathbf{\Delta}$ such that $\mathrm{null}(I_n - M\Delta) > k$, in which case $\hat{\mu}_{\mathbf{\Delta},k}(M) = 0$.

It follows immediately from the definition that $\hat{\mu}_{\mathbf{\Delta},0}(M) = \mu_{\mathbf{\Delta}}(M)$ and that $\hat{\mu}_{\mathbf{\Delta},k}(M) \geq \hat{\mu}_{\mathbf{\Delta},k+1}(M)$ for each integer $k \geq 0$. Further if for some integer k, $\hat{\mu}_{\mathbf{\Delta},k}(M) > 0$ and $\hat{\mu}_{\mathbf{\Delta},k+1}(M) = 0$, then any Δ_0 that minimises the denominator in (25) has $\mathrm{null}(I_n - M\Delta) = k+1$. We can now state and prove the following Theorem:

Theorem 3.4. Consider Problem 2 and let all variables be defined as in Theorem 3.2. Then the minimum-magnitude perturbation γ^* in the coefficients of the polynomial set $\mathcal{P}^0_{h+1,n}$ such that the perturbed polynomials $\mathcal{P}_{h+1,n}$ have a GCD of degree at least k $(1 \leq k \leq t)$ is $\gamma^* = \hat{\mu}^{-1}_{\tilde{\mathbf{\Delta}},k-1}(\tilde{M})$ where $\tilde{M}$ and $\tilde{\mathbf{\Delta}}$ are as defined in Theorem 3.3.

Proof. This is a simple generalization of the proof of Theorem 3.3 based on the general conditions given in (13). Note that the inverse of $\hat{\mu}_{\tilde{\mathbf{\Delta}},k-1}(\tilde{M})$, $k = 1, 2, \ldots, t$ always exists since t is the minimum degree of all polynomials in set $\mathcal{P}^0_{h+1,n}$ and there are always perturbations in the coefficients of $\mathcal{P}^0_{h+1,n}$ such that the perturbed polynomials $\mathcal{P}_{h+1,n}$ have at least k common roots. For example the perturbations:

$$\begin{aligned}
&\delta_j = -\alpha_j, \;\; \epsilon_{i,j} = -\beta_{i,j}, \;\; i = 1, 2, \ldots, h, \;\; j = 0, 1, \ldots, k-1 \\
&\delta_j = 0, \;\; \epsilon_{i,j} = 0, \;\; i = 1, 2, \ldots, h, \;\; j \geq k
\end{aligned}$$

result in a perturbed set of polynomials $\mathcal{P}_{h+1,n}$ which have at least k common roots at the origin and hence

$$\hat{\mu}(\tilde{M}) \geq \frac{1}{\max\{|\alpha_j|, |\beta_{i,j}|\}_{i=1,2,\ldots,h}^{j=0,1,\ldots,k-1}} > 0$$

since the polynomials $\mathcal{P}^0_{h+1,n}$ have been assumed to be coprime. □

Theorem 3.4 suggests that the GCD of the polynomial set $\mathcal{P}^0_{h+1,n}$ can be obtained by calculating successively $\hat{\mu}_{\tilde{\mathbf{\Delta}},k}(\tilde{M})$ for $k = 0, 1, \ldots, t-1$. The procedure terminates when either $k = t-1$ is reached, or when the inverse of the generalised structured singular value falls below a pre-specified tolerance level.

The calculation of $\hat{\mu}_{\tilde{\mathbf{\Delta}},k}(\tilde{M})$ is a nonconvex optimization problem. An upper bound can be obtained as:

$$\hat{\mu}_{\tilde{\mathbf{\Delta}},k}(\tilde{M}) \leq \inf_{D \in \mathbf{D}} \sigma_{k+1}(D\tilde{M}D^{-1}) \tag{26}$$

where $\mathbf{D}$ is the set of all positive definite matrices which commute with $\tilde{\mathbf{\Delta}}$. See [5], [25] for details. The efficient calculation of $\hat{\mu}_{\tilde{\mathbf{\Delta}},k}(\tilde{M})$ is a challenging problem for which (to our knowledge) so solution is currently available. In the following section we review the literature for calculating the (standard) structured singular value and propose an algorithm for estimating the upper bound of the generalized structured singular value given in in equation (26).

4. Computational algorithms for μ and generalized μ problems

The computation of the structured singular value of a matrix M is an NP-hard problem [30]. Thus, given any algorithm to compute $\mu(M)$, there will be "worst-case problems" for which the algorithm will fail to find the answer in polynomial-time. Although the structured singular value can be obtained by maximizing the spectral radius of a matrix over a set of scaling matrices, the objective function is non-concave and the corresponding algorithms rarely converge to the optimal solution. In practice, upper bounds are often sought by applying the so-called D-iteration procedure, which solves a convex minimization problem and is equivalent to a Linear Matrix Inequality (LMI). It can be shown that for certain simple perturbation structures the gap between μ and its convex upper bound is zero. The same is also true for certain other problems of special structure (rank-1 matrices [31], reciprocal matrices [32], etc). These problem classes, however, are typically too small for most practical applications. In general, the gap between μ and its convex upper bound can be arbitrarily large (but grows no faster than linearly in the number of uncertainty blocks) [30].

A systematic investigation of the gap between μ and its convex upper bound was presented in [33]. It was shown that the gap can be breached by solving an eigenvalue problem, provided a sufficiently tight bound can be obtained for an auxiliary reduced-rank μ-problem, defined from the optimal scaling matrices of the D-iteration procedure. In many cases the complexity of the auxiliary problem is significantly reduced and breaching the convex upper bound is feasible. Several other optimization methods for the general μ problem or various of its specialized versions have been reported in the literature [29], [34], [35], [24], [25] as well as the more recent one [36].

In the remaining of the section we outline a method for minimizing the bound in (26) using gradient descent algorithms. Note that the problem is in general nonconvex, so convergence to the global optimum cannot be guaranteed. However, the approach can prove useful in practice if a good starting point for the descent algorithm is available.

Consider the minimization problem: $\inf_{D\in\mathbf{D}} \sigma_{k+1}(DMD^{-1})$ in which $M \in \mathbb{R}^{n\times n}$ and

$$\mathbf{D} = \{\mathrm{diag}(D_1, D_2, \ldots, D_s) : D_i \in \mathbb{R}^{r_i\times r_i},\ D_i = D_i' > 0\},\ \sum_{i=1}^{s} r_i = n$$

Note that matrices in $\mathbf{D}$ commute with matrices in the underlying perturba-

tion set $\mathbf{\Delta}$ defined in (8). Let

$$d \in \mathbb{R}^q,\ q = \frac{1}{2}\sum_{i=1}^{s} r_i(r_i+1)$$

be the vector of the (non-repeated) variables of $D \in \mathbf{D}$. Further assume that Ω is an open subset of $\mathbb{R}^q$ and let $A(d) = DMD^{-1} : \Omega \to \mathbb{R}^{n\times n}$ be a real matrix function of $d \in \Omega$. Let $\Sigma(d) = \mathrm{diag}(\sigma_1(d), \sigma_2(d), \ldots, \sigma_n(d)), \sigma_1(d) \geq \sigma_2(d) \geq \cdots \geq \sigma_n(d) \geq 0$ be the singular values of $A(d)$. Assume that $\sigma_{k+1}(d)$ is non-repeated for every $d \in \Omega$. This assumption is made for simplicity and ensures the differentiability of $\sigma_{k+1}(d)$ in Ω; if it fails at some $d \in \Omega$ the gradient of $\sigma_{k+1}(d)$ is not defined at that point and the descent-direction algorithm may need to be modified using subgradient techniques [37], [38].

Define the matrix:

$$H(d) = \begin{pmatrix} 0 & A(d) \\ A'(d) & 0 \end{pmatrix} \in \mathbb{R}^{2n\times 2n} \tag{27}$$

The eigenvalues of $H(d)$ (arranged in non-increasing order) are related to the singular values of $A(d)$ as follows:

$$\sigma_1(d) \geq \cdots \geq \sigma_n(d) \geq 0 \geq -\sigma_n(d) \geq \cdots \geq -\sigma_1(d),\ \ \forall d \in \Omega \tag{28}$$

Thus the sensitivity of the singular values of $A(d)$ can be inferred from the sensitivity of the eigenvalues of $H(d)$ [38].

Next fix $d_0 \in \Omega$ and obtain the spectral decomposition of $H(d_0)$:

$$H(d_0) = W \begin{pmatrix} \Sigma(d_0) & 0 \\ 0 & -\Sigma(d_0) \end{pmatrix} W' \tag{29}$$

in which the eigenvector matrix $W \in \mathbb{R}^{2n\times 2n}$ is orthogonal. Denote by W_{k+1} the $(k+1)$-th column of W ($(k+1)$-th eigenvector of $H(d_0)$). Then, for each vector $x \in \mathbb{R}^q$, $\|x\| = 1$ the directional derivative of $\sigma_{k+1}(DMD^{-1})$ at d_0 along direction x is given as:

$$\langle \nabla\sigma_{k+1}(d_0), x\rangle = W'_{k+1}\left\{\sum_{i=1}^{q} x_i \begin{pmatrix} 0 & \frac{\partial(DMD^{-1})}{\partial d_i}(d_0) \\ \frac{\partial(DMD^{-1})}{\partial d_i}(d_0) & 0 \end{pmatrix}\right\} W_{k+1} \tag{30}$$

where x_i is the i-th component of x. Equivalently, if W_{k+1} is partitioned as:

$$W_{k+1} = \begin{pmatrix} U_{k+1} \\ V_{k+1} \end{pmatrix}, \; U_{k+1} \in \mathbb{R}^n, \;\; V_{k+1} \in \mathbb{R}^n \tag{31}$$

then:

$$\langle \nabla \sigma_{k+1}(d_0), x \rangle = 2U'_{k+1} \left(\sum_{i=1}^{q} x_i \frac{\partial (DMD^{-1})}{\partial d_i}(d_0) \right) V_{k+1} \tag{32}$$

Note that from the chain rule:

$$\frac{\partial (DMD^{-1})}{\partial d_i} = \frac{\partial D}{\partial d_i} MD^{-1} - DMD^{-1} \frac{\partial D}{\partial d_i} D^{-1}, \;\; i = 1, 2 \ldots, q \tag{33}$$

The following steepest-descent algorithm can now be applied for solving the optimization problem: $\inf_{D \in \mathbf{D}} \sigma_{k+1}(DMD^{-1})$:

Algorithm:

Initialise: $j = 0$, $D_j = I_n$ and $d_j = \overline{\text{vec}}(D_j) \in \mathbb{R}^r$ where $r = \frac{n(n+1)}{2}$. Set tolerance parameters $\epsilon_1 > 0$ and $\epsilon_2 > 0$.

Step 1: Define $A(d_j) = D_j M D_j^{-1}$, $H(d_j)$ as in equation (27) and perform the spectral decoposition (29) to obtain $\Sigma(d_j)$ and $W(d_j)$. Set $W_{k+1}(d_j)$ as the $(k+1)$-th column of $W(d_j)$ and decompose it as in equation (31) to obtain the two Schmidt vectors $U_{k+1}(d_j)$ and $V_{k+1}(d_j)$.

Step 2: Using equation (32) calculate $\nabla \sigma_{k+1}(d_j)$ by setting

$$(\nabla \sigma_{k+1}(d_j))_i = \langle \nabla \sigma_{k+1}(d_j), e_i \rangle, \;\; i = 1, 2, \ldots, q$$

where e_i is the i-th column of I_q.

Step 3: If $\|\nabla \sigma_{k+1}(d_j)\| \leq \epsilon_1$ stop and exit.

Step 4: Set

$$\Psi_j = \overline{\text{vec}}^{-1}(\nabla \sigma_{k+1}(d_j)), \;\; \Phi_j(t) = D_j - t\Psi_j, \;\; t \geq 0$$

and define the function:

$$f_j : I_j \to \mathbb{R}_+, f_j(t) = \sigma_{k+1}\left(\Phi_j(t) M \Phi_j^{-1}(t)\right)$$

whose domain $I_j \subseteq \mathbb{R}_{0+}$ is defined as follows: Solve the generalized (symmetric) eigenvalue problem $\det(D_j - t\Psi_j) = 0$ and let $\lambda(D_j, \Psi_j)$ be the set of eigenvalues. If $\lambda(D_j, \Psi_j) \subseteq \mathbb{R}_{0-}$ set $I_j = \mathbb{R}_{0+}$, otherwise set $I_j = [0, \gamma_j)$ where γ_j is the smallest positive eigenvalue.

Step 5: Find the optimal step length t_j^*, $t > 0$, and the optimal scaling matrix D_{j+1} such that

$$t_j^* \in \operatorname{argmin}\{f_j(t) : t \in I_j\}, \quad D_{j+1} = D_j - t_j^* \Psi_j$$

Step 6: If $\|D_{j+1} - D_j\| \leq \epsilon_2$ stop and exit.

Step 7: Set $d_{j+1} = \overline{\mathrm{vec}}(D_{j+1})$.

Step 8: Set $D_j \leftarrow D_{j+1}$, $d_j \leftarrow d_{j+1}$, $j \leftarrow j+1$ and go to step 1.

Remark 4. (1) We stress again that the steepest descent method implemented in the algorithm above guarantees convergence to a local minimum only. Since the objective function is not in general convex this may not correspond to the globally optimal solution. (2) Restricting the optimization in step 5 to the interval $[0, \gamma_j)$ ensures that the optimal solution of the scaling matrix remains always within the positive-definite cone so that f_j remains bounded. (3) Ideally the optimal steplength at every iteration has to be selected as the global minimum of f_j in I_j, which it typically estimated numerically. If this is too expensive, approximate methods can be used. In our programme implementation we estimate the optimal steplength by griding the interval $[0, \eta\gamma_j)$ where η is a fixed parameter such that $0 < \eta < 1$ (typically we take $\eta = 0.95$).

The following examples illustrate our algorithm:

Example 2. Consider two coprime polynomials

$$a_0(s) = s^2 + \alpha_1 s + \alpha_0, \quad b_0(s) = s^2 + \epsilon_1 s + \epsilon_0$$

and the corresponding Sylvester resultant matrix:

$$S_0 = \begin{pmatrix} 1 & \alpha_1 & \alpha_0 & 0 \\ 0 & 1 & \alpha_1 & \alpha_0 \\ 1 & \beta_1 & \beta_0 & 0 \\ 0 & 1 & \beta_1 & \beta_0 \end{pmatrix},$$

in which $\alpha_1 = -1.3026$, $\alpha_0 = -0.4218$, $\beta_1 = -1.0026$ and $\beta_0 = -0.3218$. The singular value set of S_0 is $\sigma(S_0) = \{2.5323, 1.8778, 0.1667, 0.0140\}$ indicating a numerical rank of 2 (or 3, depending on the required tolerance), hence

identifying the approximate GCD degree of the corresponding polynomials as two (or one, again depending on the tolerance level). In this example the exact analytical solution can be obtained for $\hat{\mu}_{\mathbf{\Delta},1}(M)$, where

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1 I_2, \delta_0 I_2, \epsilon_1 I_2, \epsilon_0 I_2)\} \subseteq \mathbb{R}^{8\times 8}$$

and $M = -ZS_0^{-1}\Theta$, where Z, Θ are as specified in Theorem 3.2. Since the two perturbed (monic) polynomials

$$a(s) = s^2 + (\alpha_1 + \delta_1)s + (\alpha_0 + \delta_0), \quad b(s) = s^2 + (\beta_1 + \epsilon_1)s + (\beta_0 + \epsilon_0)$$

have two common roots if and only if they are *identical* we have that:

$$\hat{\mu}_{\mathbf{\Delta},1}^{-1} = \max\left\{\frac{|\alpha_1 - \beta_1|}{2}, \frac{|\alpha_0 - \beta_0|}{2}\right\}$$

For the selected numerical values of the coefficients $\hat{\mu}_{\mathbf{\Delta},1}^{-1} = 0.15$. The upper bound of $\hat{\mu}_{\mathbf{\Delta},1}$ given in (26) was calculated via the steepest descent algorithm implemented in MATLAB. The local minimum was achieved at the 35-th iteration as illustrated in figure 1. The value of $\sigma_2(DMD^{-1})$ obtained at convergence results in the bound:

$$\hat{\mu}_{\mathbf{\Delta},1}(M) \leq 6.9867$$

This says that no structured perturbation $\Delta \in \mathbf{\Delta}$ with norm $\|\Delta\| \leq 6.9867^{-1} = 0.1431$ can give rise to a pair of perturbed polynomials with two common roots, which is consistent with the exact value of $\hat{\mu}_{\mathbf{\Delta},1}^{-1} = 0.15$ obtained above.

Next we applied our algorithm to bound the distance of the Sylvester matrix to singularity (which corresponds to at least one common root for the pair of perturbed polynomials), given by

$$\hat{\mu}_{\mathbf{\Delta},0}^{-1}(M) = \mu_{\mathbf{\Delta}}^{-1}(M)$$

In this case the steepest-descent algorithm produced the bound:

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\gamma^*} \leq 119.1796$$

which is in fact the exact value of $\mu_{\mathbf{\Delta}}(M)$ (up to the 9-th decimal point) as calculated by Matlab's μ-Control toolbox.

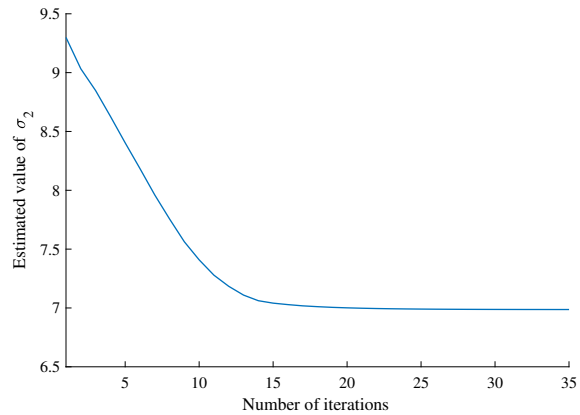

Figure 1: Steepest-descent method for $\hat{\mu}_{\mathbf{\Delta},1}(M)$ upper bound (Example 2)

Example 3. Consider now a case of three polynomials,

$$\begin{cases} a(s) = s^3 - 6s^2 + 11s - 6 = (s-1)(s-2)(s-3) \\ b(s) = s^2 - 3s + 2.09 = (s-1.1)(s-1.9) \\ c(s) = s^2 - 2.9s + 1.68 = (s-0.8)(s-2.1) \end{cases}$$

that have an approximate common divisor of degree 2. As it is numerically hard to obtain an exact solution $\hat{\mu}_{\mathbf{\Delta},1}(M)$ can be evaluated iteratively as specified in Theorem 3.4. Such an approach does not guarantee the exact solution, but can be used as an approximation.

At first, with the Sylvester resultant

$$S_0 = \begin{pmatrix} 1 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\ 0 & 1 & \alpha_2 & \alpha_1 & \alpha_0 \\ 1 & \beta_1 & \beta_0 & 0 & 0 \\ 0 & 1 & \beta_1 & \beta_0 & 0 \\ 0 & 0 & 1 & \beta_1 & \beta_0 \\ 1 & \theta_1 & \theta_0 & 0 & 0 \\ 0 & 1 & \theta_1 & \theta_0 & 0 \\ 0 & 0 & 1 & \theta_1 & \theta_0 \end{pmatrix} = \begin{pmatrix} 1 & -6 & 11 & -6 & 0 \\ 0 & 1 & -6 & 11 & -6 \\ 1 & -3 & 2.09 & 0 & 0 \\ 0 & 1 & -3 & 2.09 & 0 \\ 0 & 0 & 1 & -3 & 2.09 \\ 1 & -2.9 & 1.68 & 0 & 0 \\ 0 & 1 & -2.9 & 1.68 & 0 \\ 0 & 0 & 1 & -2.9 & 1.68 \end{pmatrix}$$

we structure an augmented matrix M based on Theorem 3.2 and proceed with calculating the minimum norm perturbations, corresponding to a bound

of $\mu_{\mathbf{\Delta}}^{(1)}(M) = 29.3311$. From the perturbed polynomials evaluated at the first iteration we form a reduced Sylvester matrix, S_1, by factoring out the nearest common root from the derived polynomials. Then, the structure of M, based on the reduced Sylvester matrix is used for the computation of the structured singular values at the second iteration corresponding to a bound on $\mu_{\mathbf{\Delta}}^{(2)}(M) = 11.7145$.

Merging the results achieved an estimate of $\hat{\mu}_{\mathbf{\Delta},1}(M) = 4.8538$. This corresponds to the maximal absolute value of the perturbations in the coefficients of the original polynomials. The final generalised structured singular value estimate obtained as a result of this factorisation scheme is suboptimal in general. However, for the purpose of our analysis it can be compared with the upper bound obtained via the proposed steepest-descent algorithm which also gives an approximation of $\hat{\mu}_{\mathbf{\Delta},1}(M)$. In this case this is obtained at the 8-th iteration as

$$\hat{\mu}_{\mathbf{\Delta},1}(M) \leq 8.7760.$$

Example 4. Assume that polynomials from the Example 3 are modified as follows

$$\begin{cases} a(s) = (s-1)(s-2)(s-3) \\ b(s) = (s-1.1+\epsilon)(s-1.9-\delta) \\ c(s) = (s-0.8-2\epsilon)(s-2.1+\delta) \end{cases}$$

where δ and ϵ are given perturbations, i.e. $(\delta, \epsilon) \in \{0.02, 0.04, 0.06, 0.08\}^2$. Applying the steepest-descent algorithm for all the combinations of δ and ϵ we obtain the results of $\hat{\mu}_{\mathbf{\Delta},1}(M)$ upper bound as presented in the Table 1. It can be observed that within the specified tolerance the smaller the gap between the common roots in the given polynomials, the greater is the value of the bound.

5. Conclusion

In this paper we propose a novel method for calculating the distance of a set of co-prime polynomials to the set of polynomials with a common root. The problem is motivated by our work in algebraic control theory and has several important applications in Numerical Analysis, Robust Control, Linear Systems and other Engineering fields. The approach seeks to identify the

Parameters	$\delta_1 = 0.02$	$\delta_2 = 0.04$	$\delta_3 = 0.06$	$\delta_4 = 0.08$
$\epsilon_1 = 0.02$	11.0057	9.8397	8.8328	7.9497
$\epsilon_2 = 0.04$	17.1778	14.7157	12.6656	10.9659
$\epsilon_3 = 0.06$	33.2325	27.9176	22.1343	17.6500
$\epsilon_4 = 0.08$	34.0382	53.1205	67.4566	44.388

Table 1: $\hat{\mu}_{\mathbf{\Delta},1}(M)$ upper bound for the different values of δ and ϵ

minimum-magnitude perturbations in the coefficients of the polynomials so that the perturbed polynomials have a common root. It is demonstrated that the problem is equivalent to the calculation of a structured singular value of a matrix which is extensively studied in Robust Control. Our approach generalises previous results in [5], however, unlike the case of two polynomials, the problem is significantly harder due to its non-square nature and the fact that the uncertainty structure is redefined during the solution process. In the last part of the paper our method is generalized and applied to the calculation of the approximate GCD of an arbitrary set of polynomials. This leads naturally to the concept of the "generalised structured singular value" which involves the solution of a structured approximation problem with rank constraints. Although in this case an upper bound can be obtained via an optimal pair of positive-definite scaling matrices which commute with the uncertainty structure, the resulting optimization problem is non-convex and convergence to the global optimum cannot be guaranteed. A steepest descent algorithm is proposed as a possible approach for tackling the problem which is shown to perform well for problems of small complexity. However further work is required to assess its numerical properties and its applicability to problems of higher complexity.

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