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SELF-ADJOINT BOUNDARY-VALUE PROBLEMS ON TIME-SCALES

FORDYCE A. DAVIDSON, BRYAN P. RYNNE

Abstract. In this paper we consider a second order, Sturm-Liouville-type boundary-value operator of the form

$$Lu := -[pu^{\nabla}]^{\Delta} + qu,$$

on an arbitrary, bounded time-scale $\mathbb{T}$, for suitable functions p, q, together with suitable boundary conditions. We show that, with a suitable choice of domain, this operator can be formulated in the Hilbert space $L^2(\mathbb{T}_\kappa)$, in such a way that the resulting operator is self-adjoint, with compact resolvent (here, 'self-adjoint' means in the standard functional analytic meaning of this term). Previous discussions of operators of this, and similar, form have described them as 'self-adjoint', but have not demonstrated self-adjointness in the standard functional analytic sense.

1. Introduction

Over the past decade a large number of papers on second order, Sturm-Liouville-type boundary value problems on bounded time-scales $\mathbb{T}$ have appeared. Most of these deal with a $\Delta\Delta$ formulation of the corresponding differential operator, viz.

$$Lu := -(pu^{\Delta})^{\Delta} + qu^{\sigma}, \quad u \in D(L), \tag{1.1}$$

for suitable functions p, q, on a suitable domain $D(L)$ (the specification of the domain $D(L)$ includes suitable boundary conditions on u; in this introductory section we omit details of spaces and domains). Much of the basic theory of such operators is described in, for example, [3, Chapter 4]. Such operators have often been termed 'self-adjoint'. However, it was shown in [6] that expressions of this form do not, in general, yield self-adjoint operators, in the standard functional-analytic meaning of the term 'self-adjoint'. Indeed, it is shown in [6] that a fundamental property of self-adjoint operators can fail for operators of the form (1.1), so that the standard theory of self-adjoint operators cannot readily be applied to such operators.

More recently, in an attempt to obtain self-adjointness, differential operators in the following $\nabla\Delta$ form

$$Lu := -(pu^{\nabla})^{\Delta} + qu, \quad u \in D(L) \tag{1.2}$$

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have been considered, see for example, [2, 9] and the references therein. Such 'mixed' operators result in a symmetric Green's function, which is taken to indicate that the corresponding operators possess some of the features of self-adjoint operators. However, the operators constructed in these papers map between (different) Banach spaces of continuously differentiable functions on $\mathbb{T}$, whereas, in the standard functional-analytic definition, a self-adjoint operator is defined on a subspace of a Hilbert space H, and maps this subspace of H into H itself. This Hilbert space formulation is necessary to obtain many of the desirable properties of such operators.

In this paper our goal is to formulate the $\nabla\Delta$ operator in (1.2) in the setting of the Hilbert space $L^2(\mathbb{T}_\kappa)$ defined in [11]. This formulation is based on the Sobolev-type spaces defined in [11] consisting of functions on $\mathbb{T}$ having L^2-type generalised derivatives. We then show that the resulting operator L, in $L^2(\mathbb{T}_\kappa)$, is an unbounded, self-adjoint operator, with compact resolvent (in the standard functional-analytic sense). The extensive functional-analytic theory of such operators is then available for this operator, although, for brevity, we will not discuss any applications of this general theory to this operator.

Remark 1.1. We consider the $\nabla\Delta$ operator in (1.2), but operators involving $\Delta\nabla$ combinations (see e.g. [2, 4, 9]) could be treated similarly, there is no essential difference in these formulations. Using the $\nabla\Delta$ form allows us to apply the results in [11] (based on a Lebesgue-type 'Δ-integral') unaltered. A corresponding Lebesgue-type '∇-integral' could be constructed using the methods in [11], and this would then allow $\Delta\nabla$ operators to be considered in a similar manner.

2. Preliminaries

Papers on time-scales usually go through a set of standard definitions of integration and differentiation on time-scales. For brevity we will omit this and simply refer to [11, Section 2] for this standard material (which is, of course, also discussed in most other time-scales papers). In particular, we will use the Lebesgue-type Δ-integral defined in [11]. A similar Lebesgue-type ∇-integral could readily be defined, but will not be required here. However, we will need to use spaces of ∇-differentiable functions, in addition to the spaces of Δ–differentiable functions discussed in [11]. To distinguish between these spaces will require some slight modifications to the notation used for various spaces and norms in [11], so we briefly discuss time-scale differentiation, and the notation we will use.

Recall that a function $u : \mathbb{T} \to \mathbb{R}$ is ∇-*differentiable* on $\mathbb{T}$ if, at each $t \in \mathbb{T}_\kappa$, there exists $u^\nabla(t)$ such that, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$s \in \mathbb{T} \text{ and } |t-s| < \delta \implies |u(\rho(t)) - u(s) - u^\nabla(t)(\rho(t)-s)| \le \epsilon|\rho(t)-s|,$$

see, for example, [4, Ch. 3]; the Δ-derivative is defined similarly, by replacing $\rho(t)$ with $\sigma(t)$ throughout.

We let $C^0(\mathbb{T})$ (respectively $C^0_{\rm rd}(\mathbb{T})$, $C^0_{\rm ld}(\mathbb{T})$) denote the set of continuous (respectively rd-continuous, ld-continuous) functions on $\mathbb{T}$; with the norm

$$|u|_\mathbb{T} := \sup_{t\in\mathbb{T}} |u(t)|, \quad u \in C_{\rm rd}(\mathbb{T}) \cup C_{\rm ld}(\mathbb{T}),$$

all these spaces are Banach spaces. We now let $C^1(\mathbb{T},\Delta)$ (respectively $C^1_{\rm rd}(\mathbb{T},\Delta)$) denote the set of functions $u \in C^0(\mathbb{T})$ which are Δ-differentiable and for which

$u^\Delta \in C^0(\mathbb{T}^\kappa)$ (respectively $u^\Delta \in C^0_{\rm rd}(\mathbb{T}^\kappa)$); with the norm

$$|u|_{\mathbb{T},\Delta} := |u|_{\mathbb{T}} + |u^\Delta|_{\mathbb{T}^\kappa}, \quad u \in C^1_{\rm rd}(\mathbb{T},\Delta),$$

these spaces are Banach spaces. Similarly, we define the Banach spaces $C^1(\mathbb{T},\nabla)$ and $C^1_{\rm ld}(\mathbb{T},\nabla)$ with norm

$$|u|_{\mathbb{T},\nabla} := |u|_{\mathbb{T}} + |u^\nabla|_{\mathbb{T}_\kappa}, \quad u \in C^1_{\rm ld}(\mathbb{T},\nabla).$$

The spaces $C^1(\mathbb{T},\nabla)$ and $C^1(\mathbb{T},\Delta)$ need not be equal. For example, let $\mathbb{T} = [-1,0] \cup [1,2]$ and define the function $u \equiv 0$ on $[-1,0]$, $u(t) = t$ on $[1,2]$. It can be verified that $u \in C^1(\mathbb{T},\nabla)$, but $u \notin C^1(\mathbb{T},\Delta)$. However, the following result gives a simple relationship between these spaces

Lemma 2.1 ([9, Theorem 6]). *$C^1(\mathbb{T},\nabla) \subset C^1_{\rm rd}(\mathbb{T},\Delta)$. If $u \in C^1(\mathbb{T},\nabla)$ then $u^\Delta = (u^\nabla)^\sigma$.*

It will also be necessary to ∇-differentiate indefinite Δ-integrals, for which we will require the following lemma.

Lemma 2.2 ([1, Theorem 2.10]). *If $u \in C^0(\mathbb{T})$, $t_0 \in \mathbb{T}$, and*

$$U_{t_0}(t) := \int_{t_0}^t u\,\Delta, \quad t \in \mathbb{T},$$

then $U_{t_0} \in C^1_{\rm ld}(\mathbb{T},\nabla)$ and $U_{t_0}^\nabla = u^\rho$ on $\mathbb{T}_\kappa$.

We will also require the Sobolev-type space of functions with generalised Δ-derivatives defined in [11], which we will denote here by $H^1(\mathbb{T},\Delta)$ with associated norm

$$\|u\|_{\mathbb{T},\Delta} := \|u\|_{\mathbb{T}} + \|u^\Delta\|_{\mathbb{T}}, \quad u \in H^1(\mathbb{T},\Delta),$$

where

$$\|u\|^2_{\mathbb{T}} := \int_{\mathbb{T}} |u|^2 \Delta, \quad u \in L^2(\mathbb{T}).$$

Note that the integral used here is the Lebesgue-type Δ-integral constructed in [11]. We also note that [11, Lemma 3.5] shows that $C^1_{\rm rd}(\mathbb{T},\Delta) \subset H^1(\mathbb{T},\Delta)$, so Lemma 2.1 has the following simple corollary, which will be required below.

Corollary 2.3. $C^1(\mathbb{T},\nabla) \subset H^1(\mathbb{T},\Delta)$.

Finally, in this preliminary section, we recall some basic functional-analytic definitions, see for example [10, Ch. 13]. Let $T : D(T) \subset H \to H$ be a linear operator in a Hilbert space H, with inner product $\langle \cdot\,,\cdot \rangle$. Then T is *symmetric* if

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \quad \forall\, x,\, y \in D(T),$$

and T is *self-adjoint* if $D(T)$ is dense in H and

$$\langle Tx, y\rangle = \langle x, z\rangle, \quad \forall\, x \in D(T) \implies y \in D(T) \text{ and } z = Ty.$$

3. A boundary value linear operator

3.1. **Definition of L.** Let $a = \inf \mathbb{T}$, $b = \sup \mathbb{T}$. We are interested in the class of functions defined on $\mathbb{T}$ which satisfy the boundary conditions

$$u^\nabla(\sigma(a)) = \gamma_a u(\sigma(a)), \quad u^\nabla(b) = -\gamma_b u(b), \tag{3.1}$$

with arbitrary constants $\gamma_a \in (-\infty, \infty]$, $\gamma_b \in (-\infty, \infty]$, and we define the following set of functions

$$\mathcal{D} := \{u \in C^1(\mathbb{T}, \nabla) : u^\nabla \in H^1(\mathbb{T}_\kappa, \Delta) \text{ and } u \text{ satisfies (3.1)}\},$$
$$D(L) := \{w \in L^2(\mathbb{T}_\kappa) : w = u|_{\mathbb{T}_\kappa^\kappa} \text{ for some } u \in \mathcal{D}\}$$

(in the definition of $D(L)$, $w = u|_{\mathbb{T}_\kappa^\kappa}$ denotes the restriction of u to the set $\mathbb{T}_\kappa^\kappa$, and we recall from [11] that the point b has $\mu_\mathbb{T}$-measure zero, so in the setting of equivalence classes of $L^2(\mathbb{T}_\kappa)$ functions, the value $u(b)$ is not well-defined).

Throughout, we impose the following additional assumption on γ_a, γ_b.

Assumption 3.1. *(i) If a is right-scattered then $\gamma_a < \infty$. (ii) If b is left-scattered then $1 + \gamma_b(b - \rho(b)) \neq 0$.*

These constructions require some further explanation and remarks.

(a) In the above notation the cases $\gamma_a = \infty$ or $\gamma_b = \infty$ are taken to mean the conditions $u(\sigma(a)) = 0$ or $u(b) = 0$, and in these cases it is the latter form that would be used in the calculations below. Furthermore, if a is right-dense, these cases correspond to the Dirichlet-type conditions $u(a) = 0$ or $u(b) = 0$. It will be seen in Remark 3.1 below that when a is right-scattered the Dirichlet-type condition at a arises from a different value of γ_a.

(b) Assumption 3.1 precludes the boundary conditions $u(\sigma(a)) = 0$ (when a is right-scattered) or $u(\rho(b)) = 0$ (when b is left-scattered). Either of these conditions lead to certain pathological properties of the operator L which we wish to avoid.

(c) If a is right-scattered it is natural, in view of the definition of the ∇-derivative, to formulate the first boundary condition in (3.1) in terms of $u^\nabla(\sigma(a))$, but the use of $u(\sigma(a))$, rather than $u(a)$, may seem slightly strange. This formulation is chosen, primarily, to simplify certain formulae arising from various integrations by parts below. The following remark shows that $u(a)$ could be used in (3.1) simply by changing the value of γ_a.

(d) If a is right-scattered or b is left-scattered, the corresponding boundary conditions in (3.1) can be rewritten in the alternative forms

$$\begin{aligned} u(a) - \big(1 + (\sigma(a) - a)\gamma_a\big)u(\sigma(a)) = 0, \\ u(\rho(b)) - \big(1 + (b - \rho(b))\gamma_b\big)u(b) = 0. \end{aligned} \tag{3.2}$$

Hence, by (3.2) and Assumption 3.1, if $u \in \mathcal{D}$ then $u(a)$ and $u(b)$ are determined by $u(\sigma(a))$ and $u(\rho(b))$, that is, u is determined entirely by its restriction $w = u|_{\mathbb{T}_\kappa^\kappa} \in D(L)$. Conversely, by using (3.2), any function $w \in D(L)$ can be extended to $\mathbb{T}$ to yield a function $u \in \mathcal{D}$. Thus, the sets $\mathcal{D}$ and $D(L)$ are (algebraically) isomorphic, and can be naturally identified with each other. In our discussion of L below we will make use of this identification, and we will generally use the symbol u interchangeably for an element of either $\mathcal{D}$ or $D(L)$.

(e) If a is right-scattered then it follows from (3.2) that we obtain the Dirichlet-type condition $u(a) = 0$ by choosing γ_a such that $1 + (\sigma(a) - a)\gamma_a = 0$.

Having dealt with the boundary conditions, we now define the desired differential operator L. Suppose that $p \in H^1(\mathbb{T}_\kappa, \Delta)$, $q \in L^2(\mathbb{T}_\kappa)$, with

$$p_{\min} := \min\{p(t) : t \in \mathbb{T}_\kappa\} > 0,$$

and define the linear operator $L : D(L) \subset L^2(\mathbb{T}_\kappa) \to L^2(\mathbb{T}_\kappa)$ by

$$Lu := -[pu^\nabla]^{\Delta_g} + qu, \quad u \in D(L),$$

where Δ_g is the generalised $\Delta-$derivative constructed in [11]. The definition L also requires some further explanation and remarks.

(a) The set $\mathcal{D}$ would be a natural domain for the operator L. However, to obtain a self-adjoint operator it is necessary that the domain and range of L lie in the same Hilbert space (which we take to be $L^2(\mathbb{T}_\kappa)$). For this reason we introduce the domain $D(L) \subset L^2(\mathbb{T}_\kappa)$, isomorphic to $\mathcal{D}$.

(b) In light of the identification of $\mathcal{D}$ and $D(L)$ described in Remark 3.1 above, we regard the calculation of $Lu \in L^2(\mathbb{T}_\kappa)$ from $u \in D(L)$ as proceeding in the following manner: use (3.2) to extend u from the set $\mathbb{T}_\kappa^\kappa$ to $\mathbb{T}$ (yielding an element of $\mathcal{D}$, which we still write as u), and then construct $u^\nabla \in H^1(\mathbb{T}_\kappa, \Delta)$ and $(u^\nabla)^{\Delta_g} \in L^2(\mathbb{T}_\kappa)$ in the usual manner (by the definition of $\mathcal{D}$, these are well-defined for $u \in \mathcal{D}$).

(c) The operator $Lu = -[pu^\Delta]^{\Delta_g} + qu^\sigma$, on a similar domain, was considered in [11]. However, we will see that the above operator is self-adjoint, (in the functional-analytic sense), whereas the operator in [11] is not. Despite this difference, the comments in [11, Remarks 5.1 and 5.2] regarding the definition of L there apply equally well to the above operator.

3.2. **Properties of** L**.** We now obtain various basic properties of L.

Lemma 3.2. *The operator L is symmetric with respect to the inner product $\langle \cdot\, , \cdot \rangle_{\mathbb{T}_\kappa}$ on $L^2(\mathbb{T}_\kappa)$, that is,*

$$\langle Lu, v\rangle_{\mathbb{T}_\kappa} = \langle u, Lv\rangle_{\mathbb{T}_\kappa}, \quad u,\, v \in D(L). \tag{3.3}$$

Proof. By definition, we can regard u, v as belonging to $\mathcal{D}$, that is u, $v \in C^1(\mathbb{T}, \nabla)$. Thus, by Lemma 2.1, u, $v \in C^1_{\rm rd}(\mathbb{T}, \Delta)$, and hence, by [11, Corollary 4.6 (f)],

$$\begin{aligned} \langle Lu, v\rangle_{\mathbb{T}_\kappa} &= \int_{\sigma(a)}^b (pu^\nabla)^\sigma v^\Delta \,\Delta - [pu^\nabla v]_{\sigma(a)}^b + \int_{\sigma(a)}^b quv\,\Delta \\ &= \int_{\sigma(a)}^b \big(p^\sigma u^\Delta v^\Delta + quv\big)\,\Delta + B(u,v) \end{aligned} \tag{3.4}$$

where, by (3.1),

$$B(u,v) = \gamma_a p(\sigma(a))u(\sigma(a))v(\sigma(a)) + \gamma_b p(b)u(b)v(b)$$

(if $\gamma_a = \infty$ or $\gamma_b = \infty$ then we omit the corresponding term in this formula). The result now follows from the symmetry in u and v of the right hand side of (3.4). □

Lemma 3.3. *There exists a constant C_L such that*

$$\langle Lu, u\rangle_{\mathbb{T}_\kappa} \ge \tfrac{1}{2} p_{\min}\|u\|^2_{\mathbb{T}_\kappa,\Delta} + C_L\|u\|^2_{\mathbb{T}_\kappa}, \quad u \in D(L). \tag{3.5}$$

Remark 3.4. The constant C_L in Lemma 3.3 need not be positive.

Proof. Suppose that $u \in D(L)$. Then we can regard u as belonging to $\mathcal{D}$, and it follows from (3.4) and the Cauchy-Schwarz inequality that

$$\langle Lu, u\rangle_{\mathbb{T}_\kappa} \geq p_{\min}\|u^\Delta\|^2_{\mathbb{T}_\kappa} - C_1|u|^2_{\mathbb{T}_\kappa}, \tag{3.6}$$

for some constant $C_1 \geq 0$ (independent of u), and by (3.2) and Assumption 3.1,

$$|u|^2_{\mathbb{T}_\kappa} \leq C_2|u|^2_{\mathbb{T}^\kappa_\kappa}, \tag{3.7}$$

for some constant $C_2 > 0$. Also, a straightforward modification of the proof of [11, Theorem 4.16] shows that for any $\epsilon > 0$ there exists a constant $C_3(\epsilon) > 0$ such that

$$|w|_{\mathbb{T}^\kappa_\kappa} \leq \epsilon\|w^\Delta\|_{\mathbb{T}_\kappa} + C_3(\epsilon)\|w\|_{\mathbb{T}_\kappa}, \quad w \in H^1(\mathbb{T}_\kappa, \Delta). \tag{3.8}$$

By Corollary 2.3 and the definition of $\mathcal{D}$, $u \in H^1(\mathbb{T}_\kappa, \Delta)$, so putting ϵ sufficiently small and $w = u$ in (3.8) and combining this with (3.6) and (3.7) yields (3.5). □

Invertibility of L will be important below, and it will be seen that invertibility follows from injectivity of L, so we now consider this. In general, L need not be injective, but the following result shows that we can obtain injectivity by adding to L a sufficiently large scalar multiple of the identity operator $I : L^2(\mathbb{T}_\kappa) \to L^2(\mathbb{T}_\kappa)$. In many situations, if L itself is not injective then it is possible, with no loss of generality, to replace L with the injective operator L_c given by the following result.

Theorem 3.5. *If $c + C_L > 0$ then the operator $L_c := L + cI$ is injective.*

Proof. It follows from (3.5) that

$$\langle L_c u, u\rangle_{\mathbb{T}_\kappa} \geq \tfrac{1}{2}p_{\min}\|u\|^2_{\mathbb{T}_\kappa,\Delta} + (c + C_L)\|u\|^2_{\mathbb{T}_\kappa} > 0, \quad 0 \neq u \in D(L),$$

which proves that L_c is injective. □

The following result gives simple criteria under which L itself is injective.

Theorem 3.6. *Suppose that $q \geq 0$ on $\mathbb{T}_\kappa$ and $\gamma_a, \gamma_b \geq 0$. Then L is injective under either of the hypotheses*:

(i) $\gamma_a + \gamma_b > 0$;
(ii) $\|q\|_{\mathbb{T}_\kappa} > 0$.

Proof. We consider hypothesis (i), a similar proof holds for hypothesis (ii). Suppose that $0 \neq u \in D(L)$ and $Lu = 0$. It follows from this and (3.4) that

$$0 = \langle Lu, u\rangle_{\mathbb{T}_\kappa} \geq p_{\min}\|u^\Delta\|^2_{\mathbb{T}_\kappa} + B(u, u),$$

and hence, by [11, Corollary 4.6], $u \equiv 0$ on $\mathbb{T}_\kappa$. □

We will also need the following result regarding solutions of the corresponding initial value problem. This result can be proved in a similar manner to that of [11, Theorem 5.8].

Theorem 3.7. *For any $h \in L^2(\mathbb{T}_\kappa)$ and $\tau \in \mathbb{T}_\kappa$, $\eta_1, \eta_2 \in \mathbb{R}$, the initial value problem*

$$\begin{gathered} -(pu^\nabla)^{\Delta_g} + qu = h, \\ u(\tau) = \eta_1, \quad u^\nabla(\tau) = \eta_2, \end{gathered} \tag{3.9}$$

has a unique solution $u \in C^1(\mathbb{T}, \nabla)$, with $u^\nabla \in H^1(\mathbb{T}_\kappa, \Delta)$.

Let ϕ, ψ be the solutions of (3.9) given by Theorem 3.7, with $h = 0$ and the 'initial' conditions

$$\begin{aligned} \phi(\sigma(a)) = 1, \quad & \phi^{\nabla}(\sigma(a)) = \gamma_a, \\ \psi(b) = 1, \quad & \psi^{\nabla}(b) = -\gamma_b, \end{aligned} \tag{3.10}$$

with the obvious modification, here and below, when $\gamma_a = \infty$ or $\gamma_b = \infty$. Also, let

$$W := p\big(\phi^{\nabla}\psi - \psi^{\nabla}\phi\big) \in H^1(\mathbb{T}_\kappa, \Delta)$$

(it follows from the properties of ϕ, ψ given by Theorem 3.7, together with Corollary 2.3 and [11, Corollary 4.6], that $W \in H^1(\mathbb{T}_\kappa, \Delta)$).

Lemma 3.8. *W is constant on $\mathbb{T}_\kappa$. The operator L is injective if and only if $W \neq 0$.*

Proof. From the definitions of ϕ, ψ, Corollary 2.3 and [11, Corollary 4.6],

$$\begin{aligned} W^{\Delta_g} &= (p\phi^{\nabla})^{\Delta_g}\psi + (p\phi^{\nabla})^{\sigma}\psi^{\Delta} - (p\psi^{\nabla})^{\Delta_g}\phi - (p\psi^{\nabla})^{\sigma}\phi^{\Delta} \\ &= q\phi\psi + (p\phi^{\nabla}\psi^{\nabla})^{\sigma} - q\psi\phi - (p\psi^{\nabla}\phi^{\nabla})^{\sigma} = 0, \end{aligned}$$

so by [11, Corollary 4.6], $W \equiv \text{const}$. Moreover, by (3.10),

$$W = p(b)\big(\phi^{\nabla}(b) + \gamma_b\phi(b)\big),$$

so $W = 0$ if and only if ϕ satisfies the boundary conditions (3.1). Clearly, if ϕ satisfies (3.1) then L is not injective, and the converse follows immediately from linearity and the uniqueness of the solution of the initial value problem for ϕ. □

We can now begin the construction of the inverse of L (when L is injective). Equivalently, we construct a solution of the boundary value problem

$$Lu = h, \quad h \in L^2(\mathbb{T}_\kappa), \quad u \in D(L), \tag{3.11}$$

for any $h \in L^2(\mathbb{T}_\kappa)$.

Definition 3.9. Suppose that L is injective. For $(t, s) \in \mathbb{T} \times \mathbb{T}$ let

$$g(t,s) := \begin{cases} W^{-1}\psi(t)\phi(s), & \text{if } t \geq s, \\ W^{-1}\phi(t)\psi(s), & \text{if } t \leq s. \end{cases}$$

Clearly, g is continuous on $\mathbb{T} \times \mathbb{T}$. For any $h \in L^2(\mathbb{T}_\kappa)$, let

$$Gh(t) := \int_{\sigma(a)}^{b} g(t,\cdot)h\,\Delta, \quad t \in \mathbb{T}_\kappa^\kappa. \tag{3.12}$$

Theorem 3.10. *Suppose that L is injective. Then:*

(i) *for any $h \in L^2(\mathbb{T}_\kappa)$ the function $u = Gh \in D(L)$, and u is the unique solution of* (3.11);
(ii) *the operators $L : D(L) \subset L^2(\mathbb{T}_\kappa) \to L^2(\mathbb{T}_\kappa)$, $G : L^2(\mathbb{T}_\kappa) \to D(L) \subset L^2(\mathbb{T}_\kappa)$, are invertible, linear operators and $L^{-1} = G$, $G^{-1} = L$. The operator G is compact, while if $\dim L^2(\mathbb{T}_\kappa) = \infty$ then L is unbounded.*

Remark 3.11. We call g the *Green's function* and G the *Green's operator* for the operator L.

Proof. The uniqueness follows immediately from the injectivity of L. Now suppose that $h \in C^0(\mathbb{T}_\kappa)$. To simplify the following calculations we will suppose that $p \equiv 1$; the general proof is similar.

It is clear that the formula for $u = Gh$ in (3.12) can be extended to define a function on the whole of $\mathbb{T}$, which we continue to denote by u. Now suppose that a is right-dense. Then by direct calculation (using Lemma 2.2 above, the product rule for nabla derivatives, see [3], and for generalised derivatives, see Corollary 4.6 in [11], and (3.3) in [11]),

$$\begin{aligned}
Wu(t) &= \psi(t)\int_{\sigma(a)}^{t} \phi h\,\Delta + \phi(t)\int_{t}^{b} \psi h\,\Delta, \quad t \in \mathbb{T},\\
Wu^{\nabla}(t) &= \psi^{\rho}(t)\phi^{\rho}(t)h^{\rho}(t) + \psi^{\nabla}(t)\int_{\sigma(a)}^{t} \phi h\,\Delta\\
&\quad - \phi^{\rho}(t)\psi^{\rho}(t)h^{\rho}(t) + \phi^{\nabla}(t)\int_{t}^{b} \psi h\,\Delta\\
&= \psi^{\nabla}(t)\int_{\sigma(a)}^{t} \phi h\,\Delta + \phi^{\nabla}(t)\int_{t}^{b} \psi h\,\Delta, \quad t \in \mathbb{T},\\
W(u^{\nabla})^{\Delta_g}(t) &= -Wh(t) + (\psi^{\nabla})^{\Delta_g}(t)\int_{\sigma(a)}^{\sigma(t)} \phi h\,\Delta + (\phi^{\nabla})^{\Delta_g}(t)\int_{\sigma(t)}^{b} \psi h\Delta\\
&= -Wh(t) + q(t)\Big\{\psi(t)\int_{\sigma(a)}^{\sigma(t)} \phi h\,\Delta + \phi(t)\int_{\sigma(t)}^{b} \psi h\,\Delta\Big\}\\
&= -Wh(t) + q(t)\Big\{\psi(t)\int_{\sigma(a)}^{t} \phi h\,\Delta + \phi(t)\int_{t}^{b} \psi h\,\Delta\Big\}\\
&= -Wh(t) + q(t)Wu(t), \quad \mu_{\mathbb{T}}\text{-a.e. } t \in \mathbb{T}.
\end{aligned}$$

It follows directly from these formulae and (3.10) that $u = Gh \in \mathcal{D}$, with

$$\|Gh\|_{\mathcal{D}} = \|u\|_{\mathcal{D}} := |u|_{\nabla,\mathbb{T}} + \|u^{\nabla}\|_{\mathbb{T}_\kappa,\Delta} \le C\|h\|_{\mathbb{T}_\kappa}, \quad h \in C^0(\mathbb{T}_\kappa), \tag{3.13}$$

for some constant C. On the other hand, if a is right-scattered the calculation of $Wu(a)$ and $Wu^{\nabla}(\sigma(a))$ needs to be amended slightly to avoid reference to the (undefined) value $h(a)$, in the following manner. Directly from (3.12),

$$Wu(a) = \phi(a)\int_{\sigma(a)}^{b} \psi h\,\Delta, \quad Wu(\sigma(a)) = \int_{\sigma(a)}^{b} \psi h\,\Delta,$$

$$Wu^{\nabla}(\sigma(a)) = W\frac{u(\sigma(a)) - u(a)}{\sigma(a) - a} = \phi^{\nabla}(\sigma(a))\int_{\sigma(a)}^{b} \psi h\,\Delta = W\gamma_a u(\sigma(a)).$$

Thus the boundary condition at $\sigma(a)$ also holds in this case, and it follows from the preceding formulae that if $\sigma(a)$ is right-dense then $\lim_{t\to\sigma(a)^+} u(t) = u(\sigma(a))$, $\lim_{t\to\sigma(a)^+} u^{\nabla}(t) = u^{\nabla}(\sigma(a))$, (here, $t \to \sigma(a)^+$ through points in $\mathbb{T}_\kappa$), so that $u \in C^1(\mathbb{T}, \nabla)$, that is we again have $u \in \mathcal{D}$. Clearly, the inequality (3.13) also holds in this case.

Now, since $C^0(\mathbb{T}_\kappa)$ is dense in $L^2(\mathbb{T}_\kappa)$ (see Lemma 3.5 in [11]), and $\mathcal{D}$ is a Banach space (with respect to the above norm $\|\cdot\|_{\mathcal{D}}$), the inequality (3.13) extends from the set $C^0(\mathbb{T}_\kappa)$ to the whole of $L^2(\mathbb{T}_\kappa)$ by continuity. Hence, in particular, G is bounded as an operator from $L^2(\mathbb{T}_\kappa)$ into $\mathcal{D}$. The assertions about the operators L and G

now follow immediately (the compactness of G follows from (3.13), Lemma 2.1 and the compactness of the embedding $C^1_{\rm rd}(\mathbb{T},\Delta) \to C^0(\mathbb{T})$, see [6, Lemma 2.2]). □

We can now prove that L is self-adjoint (irrespective of injectivity of L).

Theorem 3.12. *The domain $D(L)$ is dense in $L^2(\mathbb{T}_\kappa)$, and the operator L is self-adjoint with respect to the inner product $\langle \cdot\,,\cdot\rangle_{\mathbb{T}_\kappa}$ on $L^2(\mathbb{T}_\kappa)$.*

Proof. It suffices to prove the result for L_c, for arbitrary $c \in \mathbb{R}$, so without loss of generality we suppose that $c = 0$ and L is injective. If $D(L)$ is not dense in $L^2(\mathbb{T}_\kappa)$ then there exists $0 \neq w \in L^2(\mathbb{T}_\kappa)$ such that

$$\langle u, w\rangle_{\mathbb{T}_\kappa} = 0, \quad \forall u \in D(L).$$

Since $R(L) = L^2(\mathbb{T}_\kappa)$, we have $w = Lz$ for some $z \in D(L)$, so by Lemma 3.2,

$$0 = \langle u, Lz\rangle_{\mathbb{T}_\kappa} = \langle Lu, z\rangle_{\mathbb{T}_\kappa}, \quad \forall u \in D(L),$$

and hence $z = 0$ (again, since $R(L) = L^2(\mathbb{T}_\kappa)$). However, this implies that $w = 0$, which contradicts the choice of w, and so proves that $D(L)$ is dense in $L^2(\mathbb{T}_\kappa)$.

Now suppose that

$$\langle Lu, v\rangle_{\mathbb{T}_\kappa} = \langle u, w\rangle_{\mathbb{T}_\kappa}, \quad \forall u \in D(L), \tag{3.14}$$

for some v, $w \in L^2(\mathbb{T}_\kappa)$. We again have $w = Lz$, for some $z \in D(L)$, and so from (3.14),

$$\langle Lu, v - z\rangle_{\mathbb{T}_\kappa} = 0, \quad \forall u \in D(L),$$

and hence $v = z$. That is, $v \in D(L)$ and $w = Lv$, which proves that L is self-adjoint. □

Corollary 3.13. *Suppose that L is injective. Then the operator G is self-adjoint with respect to the inner product $\langle \cdot\,,\cdot\rangle_{\mathbb{T}_\kappa}$ on $L^2(\mathbb{T}_\kappa)$.*

Proof. Let u, $v \in L^2(\mathbb{T}_\kappa)$ be arbitrary. Then by Theorem 3.10, $u = Lx$, $v = Ly$, for some x, $y \in D(L)$. Hence, by Lemma 3.2 and Theorem 3.10,

$$\langle Gu, v\rangle_{\mathbb{T}_\kappa} = \langle x, Ly\rangle_{\mathbb{T}_\kappa} = \langle Lx, y\rangle_{\mathbb{T}_\kappa} = \langle u, Gv\rangle_{\mathbb{T}_\kappa},$$

which proves that G is self-adjoint. □

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