Analysis and Applications

THREEFOLD SYMMETRIC HAHN-CLASSICAL MULTIPLE ORTHOGONAL POLYNOMIALS

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We characterize all the multiple orthogonal threefold symmetric polynomial sequences whose sequence of derivatives is also multiple orthogonal. Such a property is commonly called the Hahn property and it is an extension of the concept of classical polynomials to the context of multiple orthogonality. The emphasis is on the polynomials whose indices lie on the step line, also known as 2-orthogonal polynomials. We explain the relation of the asymptotic behavior of the recurrence coefficients to that of the largest zero (in absolute value) of the polynomial set. We provide a full characterization of the Hahn-classical orthogonality measures supported on a 3-star in the complex plane containing all the zeros of the polynomials. There are essentially three distinct families, one of them 2-orthogonal with respect to two confluent functions of the second kind. This paper complements earlier research of Douak and Maroni.

Keywords: orthogonal polynomials, multiple orthogonal polynomials, confluent hypergeometric function, Airy function, Hahn classical polynomials, recurrence relation, linear differential equation

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1. Introduction and motivation

In this paper we investigate and characterize all the multiple orthogonal polynomials of type II that are threefold symmetric and are such that the polynomial sequence of its derivatives is also a multiple orthogonal sequence of type II. Within the standard orthogonality context there are only four families satisfying this property, commonly referred to as the *Hahn property*, and they are the Hermite, Laguerre, Jacobi and Bessel, collectively known as the *classical orthogonal polynomials*. These four families of polynomials also share a number of analytic and algebraic properties. Several studies are dedicated to extensions of those properties to the context of multiple orthogonality. However, those extensions give rise to completely different sequences of multiple orthogonal polynomials. For the usual orthogonal polynomials the Hahn property is equivalent with the Bochner characterization (polynomials satisfy a second order differential equation of Sturm-Liouville type) and the existence of a Rodrigues formula. This is no longer true for multiple orthogonal polynomials since there are families of multiple orthogonal polynomials with a Rodrigues type formula [1] that do not satisfy the Hahn property, and various examples of multiple orthogonal polynomials satisfy a higher-order linear recurrence relation, which is not of Sturm-Liouville type [11]. Hence characterizations for multiple orthogonal polynomials based on the Hahn property, the Bochner property or the Rodrigues formula give different families of polynomials.

A sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$ with $\deg P_n = n$ is orthogonal with respect to a Borel measure μ whenever

$$\int P_n(x)x^k \mathrm{d}\mu(x) = 0 \ if \ k = 0, 1, \ldots, n-1, \ n \geq 0$$

and

$$\int P_n(x)x^n \mathrm{d}\mu(x) \neq 0 \ for \ n \geqslant 0.$$

Without loss of generality, often we normalize μ so that it is a probability measure. Obviously, an orthogonal polynomial sequence forms a basis of the vector space of polynomials $\mathcal{P}$. The measures described above can be represented via a linear functional $\mathcal{L}$, defined on $\mathcal{P}'$, the dual space of $\mathcal{P}$, and it is understood that the action of $\mathcal{L}$ over a polynomial f corresponds to $\int f(x)\mathrm{d}\mu(x)$. Throughout, we denote this action as

$$\langle \mathcal{L}, f(x)\rangle := \int f(x)\mathrm{d}\mu(x).$$

The derivative of a function f is denoted by f' or $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$. Properties on $\mathcal{P}'$, such as differentiation or multiplication by a polynomial, can be defined by duality. A detailed explanation can be found in [29] [31]. In particular, given $g \in \mathcal{P}$ and a functional $\mathcal{L} \in \mathcal{P}'$, we define

$$\langle g(x)\mathcal{L}, f(x)\rangle := \langle \mathcal{L}, g(x)f(x)\rangle \quad \text{and} \quad \langle \mathcal{L}', f(x)\rangle := -\langle \mathcal{L}, f'(x)\rangle \tag{1.1}$$

for any polynomial f.

From the definition it is straightforward that an orthogonal polynomial sequence satisfies a second order recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \ n \geqslant 1, \tag{1.2}$$

with $P_0(x) = 1$, $P_1(x) = x - \beta_0$ and $\gamma_n \neq 0$ for all integers $n \geqslant 1$. This relation is often called the three-term recurrence relation, but we will avoid this terminology as we will be dealing with three-term recurrence relations which are of higher order. There is an important converse of this connection between orthogonal polynomials and second order recurrence relations, known as the Shohat-Favard theorem or spectral theorem for orthogonal polynomials. It states that any sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$ with $\deg P_n = n$, satisfying the recurrence relation (1.2) with $\gamma_n \neq 0$ for all $n \geqslant 1$ and initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, is always an orthogonal polynomial sequence with respect to some measure μ and, if, in addition, $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$ for all $n \geqslant 0$, then μ is a positive measure on the real line. So, basically, the orthogonality conditions and the second order recurrence relations are two equivalent ways to characterize an orthogonal polynomial sequence.

Multiple orthogonal polynomials of type II correspond to a sequence of polynomials of a single variable that satisfy multiple orthogonality conditions with respect to $r > 1$ measures. With the multi-index $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, type II multiple orthogonal polynomials correspond to a (multi-index) sequence of monic polynomials $P_{\vec{n}}(x)$ of degree $|\vec{n}| = n_1 + \ldots + n_r$ for which there is a vector of measures $(\mu_0, \ldots, \mu_{r-1})$ such that

$$\int x^k P_{\vec{n}}(x) \mathrm{d}\mu_j(x) = 0, \quad 0 \leqslant k \leqslant n_j - 1 \tag{1.3}$$

holds for every $j = 0, 1, \ldots, r-1$. By setting $r = 1$, we recover the usual orthogonal polynomials. Obviously, (1.3) amounts to the same as saying there exists a vector of r linear functionals $(\mathcal{L}_0, \ldots, \mathcal{L}_{r-1})$ such that

$$\langle \mathcal{L}_j, x^k P_{\vec{n}}(x) \rangle = 0, \quad 0 \leqslant k \leqslant n_j - 1.$$

This polynomial $P_{\vec{n}}$ set may not exist or is not unique unless we impose some extra conditions on the r measures $\mu_0, \ldots, \mu_{r-1}$, but under appropriate conditions the polynomial set satisfies a system of r recurrence relations, relating $P_{\vec{n}}$ with its nearest neighbors $P_{\vec{n}+\vec{e}_k}$ and $P_{\vec{n}-\vec{e}_j}$, where $\vec{e}_k$ consists of the r-dimensional unit vector with zero entries except for the kth component which is 1 (see [37] for further details). The focus of the present work is on the polynomials whose multi-indices lie on the step-line near the diagonal $\vec{n} = (n, n, \ldots, n)$, and are defined by

$$P_{rn+j}(x) = P_{\vec{n}+\vec{s}_j}(x), \quad \text{with} \quad \vec{s}_j = \sum_{\ell=0}^{j} \vec{e}_\ell, \qquad j = 0, 1, \ldots, r-1,$$

that is,

$$P_{rn+j}(x) = P_{(\underbrace{n+1,\ldots,n+1}_{j},\underbrace{n,\ldots,n}_{r-j})}(x), \qquad j = 0, 1, \ldots, r-1.$$

This *step-line* polynomial sequence $\{P_n\}_{n\geqslant 0}$ satisfies a $(r+1)$th order recurrence relation (involving up to $(r+2)$ consecutive terms) and it can be written as

$$\begin{aligned} &xP_n(x) = P_{n+1}(x) + \sum_{j=0}^{r} \zeta_{n,j} P_{n-j}(x), \qquad n \geqslant r, \\ &P_j(x) = x^j, \qquad j = 0, \ldots, r, \end{aligned} \tag{1.4}$$

where $\zeta_{n,r} \neq 0$ for all $n \geqslant r$. Such a step-line polynomial sequence $\{P_n\}_{n\geqslant 0}$ corresponds to so-called *d-orthogonal polynomials* (with $d = r$), see [28,15,32] and Figure

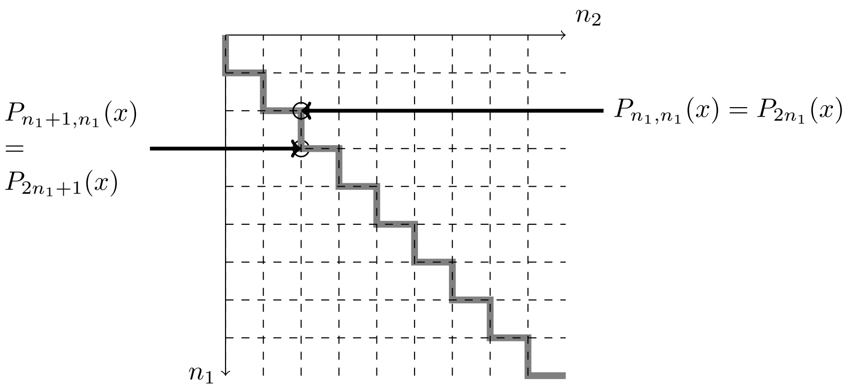

Fig. 1. Step-line for the multi-index (n_1, n_2) when $r = 2$.

1 for the case of $r = 2$. So, basically, if there exists a vector of linear functionals $(\mathcal{L}_0, \ldots, \mathcal{L}_{d-1})$ and a polynomial sequence $\{P_n\}_{n\geqslant 0}$ satisfying

$$\langle \mathcal{L}_j, x^k P_n(x)\rangle = 0 \quad \text{and} \quad \langle \mathcal{L}_j, x^n P_{dn+j}(x)\rangle \neq 0, \quad 0 \leqslant k \leqslant \left\lfloor \frac{n-j}{d} \right\rfloor, \ n \geqslant 0,$$

then the polynomials $\{P_n\}_{n\geqslant 0}$ are related by (1.4). There is a converse result, which is a natural generalization of the Shohat-Favard theorem, in the sense that if a polynomial sequence $\{P_n\}_{n\geqslant 0}$ satisfies (1.4) with $\zeta_{n,d} \neq 0$ for all $n \geqslant d$, then there is a vector of r linear functionals $(\mathcal{L}_0, \ldots, \mathcal{L}_{d-1})$ with respect to which $\{P_n\}_{n\geqslant 0}$ is d-orthogonal. Such a vector of linear functionals is not unique, but we can consider its components to be the first d elements of the dual sequence $\{u_n\}_{n\geqslant 0}$ associated to $\{P_n\}_{n\geqslant 0}$, which always exist and which are defined by

$$\langle u_n, P_m\rangle \coloneqq \delta_{n,m}, \quad m, n \geqslant 0,$$

where $\delta_{n,m}$ represents the *Kronecker symbol.* The remaining elements of the dual sequence of a d-orthogonal polynomial sequence can be generated from the first ones: for each pair of integers (n, j) there exist polynomials $q_{n,j,\nu}(x)$ such that [28]

$$u_{dn+j} \ = \sum_{\nu=0}^{d-1} q_{n,j,\nu}(x) u_\nu, \quad \text{for} \quad 0 \leqslant j \leqslant d-1,$$

where $\deg q_{n,j,j}(x) = n$, $\deg q_{n,j,\nu}(x) \leqslant n$ for $0 \leqslant \nu \leqslant j-1$ if $1 \leqslant j \leqslant d-1$ and $\deg q_{n,j,\nu}(x) \leqslant n-1$ for $j+1 \leqslant \nu \leqslant d-1$ if $0 \leqslant j \leqslant d-2$. Unlike the standard orthogonality (case $d = 1$), this result only provides structural properties for the dual sequence and in practical terms it can be used in that sense.

In this paper we mainly deal with 2-orthogonal polynomial sequences, but our results and ideas can be extended to the d-orthogonal case. Here, we provide a characterization of all the 2-orthogonal polynomial sequences that are *threefold symmetric* and possess the *Hahn property* in the context of 2-orthogonality, i.e., the sequence of monic derivatives is also 2-orthogonal. The notion and properties of *threefold symmetric* 2-orthogonal polynomial sequences are revised and discussed in Section 2, while we also bring a new result (Theorem 2.2) relating the asymptotic behavior of the recurrence coefficient to the asymptotic behavior of the absolute value of the largest zero. Section 3 is dedicated to a detailed characterization of all the threefold symmetric 2-orthogonal polynomials satisfying the Hahn property. We bring together old and new results in a self-contained and complete analysis to fully characterize this type of sequences in terms of their explicit recurrence relations, weight functions and a differential equation of third order. In that regard, we complete the study initiated by Douak and Maroni in [14,15] by providing sufficient conditions and taking the study based only on the properties of the zeros in combination with the weights. We end up this coherent description with a detailed analysis of the four distinct families of threefold symmetric 2-orthogonal polynomials satisfying the Hahn property. The first case, treated in Section 3.1 corresponds to polynomials with the Appell property whose 2-orthogonality weights are supported on the three-starlike set (as in Fig.2) represented via the Airy function and its derivative. These polynomials have been studied in [13], but the explicit representation of the weights supported on a set containing all the zeros has not been considered there. In Sections 3.2 and 3.3 we study the second and third cases where the weights are represented via the confluent hypergeometric Kummer function of 2nd kind. These are, to the best of our knowledge, new. The fourth case is treated in Section 3.4, where the orthogonality weights are expressed via hypergeometric functions and depend on two parameters. For special choices of those parameters we recover some known polynomial families. However, the existent literature on the subject has not taken into account the explicit representation of the weights supported on a set containing all the zeros and this has been accomplished here for all the threefold symmetric polynomials satisfying the *Hahn property* within the context of 2-orthogonality. The orthogonality (1.3) of these polynomials corresponds to an non-hermitian inner product (no conjugation is used) on three-starlike sets. The Airy function, the confluent hypergeometric function and the hypergeometric function are analytic functions, so the paths of integration can be deformed away from the three-starlike sets without loosing orthogonality. However, the choice of the three-starlike sets is convenient because of the positivity of the measures and the location of the zeros. It turns out that the orthogonality measures under analysis are solutions to a second order differential equation and, as such, the recurrence coefficients for the whole set of the corresponding multiple orthogonal polynomials of type II can only be obtained algorithmically via the nearest-neighbor algorithm explained in [37] [19]. Some of the d-orthogonal polynomials that we found appear in the theory of random matrices, in particular in the investigation of singular values of products of Ginibre matrices, which uses multiple orthogonal polynomials with weight functions expressed in terms of Meijer G-functions [24]. For $r = 2$ these Meijer G-functions are hypergeometric or confluent hypergeometric functions.

2. Threefold 2-orthogonal polynomial sequence

A sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$ (with $\deg P_n = n$) is *symmetric* whenever $P_n(-x) = (-1)^n P_n(x)$ for all $n \geqslant 0$. This means that all even degree polynomials are even functions while odd degree polynomials are odd functions. Hermite and Gegenbauer polynomials are examples of symmetric polynomial sets, which also happen to be the only classical orthogonal polynomials that are symmetric. Many other examples of symmetric orthogonal polynomial sequences are around in the literature.

The notion of symmetry of polynomial sequences has been extended and commonly referred to as d-symmetric in works by Maroni [28], [15] and followed by Ben Cheikh and his collaborators [6] [8] [9][25]. The case $d = 1$ would correspond to the usual symmetric case. We believe the name is misleading, as will soon become apparent, and therefore we call it differently as it pictures better the nature of the problem.

Definition 2.1. A polynomial sequence $\{P_n\}_{n\geqslant 0}$ is **m-fold symmetric** if

$$P_n(\omega x) = \omega^n\, P_n(x), \quad \text{for any} \quad n \geqslant 0. \tag{2.1}$$

where $\omega = \mathrm{e}^{\frac{2i\pi}{m}}$.

By induction, property (2.1) corresponds to

$$P_n(\omega^k x) = \omega^{nk}\, P_n(x), \quad \text{for any} \quad n \geqslant 0 \quad \text{and} \quad k = 1, 2, \ldots, m-1.$$

So, an m-fold symmetric sequence is a $(m-1)$-symmetric sequence in [28], [15], [6], [8],[9],[25]. (The symmetric sequences of Hermite and Gegenbauer polynomials are examples of *twofold symmetric* polynomials.) In particular, a *threefold symmetric* sequence $\{P_n\}_{n\geqslant 0}$, is such that

$$P_n(\omega x) = \omega^n P_n(x) \quad \text{and} \quad P_n(\omega^2 x) = \omega^{2n} P_n(x) \quad \text{with} \quad \omega = \mathrm{e}^{\frac{2i\pi}{3}},\ n \geqslant 0,$$

which corresponds to say that there exist three sequences $\{P_n^{[j]}\}_{n\geqslant 0}$ with $j \in \{0, 1, 2\}$ such that

$$P_{3n+j}(x) = x^j P_n^{[j]}(x^3), \quad j = 0, 1, 2. \tag{2.2}$$

Throughout, we will refer to $\{P_n^{[j]}\}_{n\geqslant 0}$ as the *diagonal components of the cubic decomposition* of the threefold symmetric sequence $\{P_n\}_{n\geqslant 0}$, which is line with the terminology adopted in a more general cubic decomposition framework in [33].

In the case of a 2-orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ with respect to a vector linear functional $\mathbf{u} = (u_0, u_1)$, we have, as discussed in Section 1,

$$\langle u_0, x^m P_n\rangle = \begin{cases} 0 & \text{for } n \geqslant 2m+1 \\ N_0(n) \neq 0 & \text{for } n = 2m \end{cases} \tag{2.3}$$

$$\langle u_1, x^m P_n\rangle = \begin{cases} 0 & \text{for } n \geqslant 2m+2 \\ N_1(n) \neq 0 & \text{for } n = 2m+1, \end{cases} \tag{2.4}$$

and there exists a set of coefficients $\{(\beta_n, \alpha_n, \gamma_n)\}_{n\geqslant 0}$ such that $\{P_n\}_{n\geqslant 0}$ satisfies a third order recurrence relation (see [28,39])

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1}P_{n-2}(x) \tag{2.5}$$

with $P_{-2}(x) = P_{-1}(x) = 0$ and $P_0(x) = 1$. Straightforwardly from the definition, one has

$$\gamma_{2n+1} = \frac{\langle u_0, x^{n+1}P_{2n+2}\rangle}{\langle u_0, x^n P_{2n}\rangle} \neq 0, \quad \gamma_{2n+2} = \frac{\langle u_1, x^{n+1}P_{2n+3}\rangle}{\langle u_1, x^n P_{2n+1}\rangle} \neq 0, \quad n \geqslant 0, \tag{2.6}$$

or, equivalently,

$$\langle u_0, x^{n+1}P_{2n+2}\rangle = \prod_{k=0}^{n} \gamma_{2k+1} \quad \text{and} \quad \langle u_1, x^{n+1}P_{2n+3}\rangle = \prod_{k=0}^{n} \gamma_{2k+2}, \quad \text{for} \quad n \geqslant 0.$$

Whenever a 2-orthogonal polynomial sequence is threefold symmetric, the recurrence relation (2.5) reduces to a three-term relation, where the β- and α-coefficients all vanish. For this type of 2-orthogonal polynomial sequences more can be said.

Proposition 2.1. *[14] Let $\{P_n\}_{n\geqslant 0}$ be a 2-orthogonal polynomial sequence with respect to the linear functional $U = (u_0, u_1)$ satisfying* (2.3)-(2.4)*. The following statements are equivalent:*

(a) The sequence $\{P_n\}_{n\geqslant 0}$ is threefold symmetric.
(b) The linear functional is threefold symmetric, that is,

$$(u_\nu)_{3n+j} = 0, \quad \textit{for} \quad \nu = 0, 1 \quad \textit{and} \quad j = 1, 2 \quad \textit{with} \quad j \neq \nu, \tag{2.7}$$

where $(u)_m := \langle u, x^m\rangle$ for $m \geqslant 0$.
(c) The sequence $\{P_n\}_{n\geqslant 0}$ satisfies the third order recurrence relation

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x), \qquad n \geqslant 2, \tag{2.8}$$

with $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = x^2$.

Each of the components of the cubic decomposition are also 2-orthogonal polynomial sequences.

Lemma 2.1. *[28] Let $\{P_n\}_{n\geqslant 0}$ be a threefold symmetric 2-OPS. The three polynomial sequences $\{P_n^{[j]}\}_{n\geqslant 0}$ (with $j = 0, 1, 2$) in the cubic decomposition of $\{P_n\}_{n\geqslant 0}$ described in* (2.2) *are 2-orthogonal polynomial sequences satisfying:*

$$P_{n+1}^{[j]}(x) = (x - \beta_n^{[j]})P_n^{[j]}(x) - \alpha_n^{[j]}P_{n-1}^{[j]}(x) - \gamma_{n-1}^{[j]}P_{n-2}^{[j]}(x), \tag{2.9}$$

where

$$\begin{aligned}
\beta_n^{[j]} &= \gamma_{3n-1+j} + \gamma_{3n+j} + \gamma_{3n+1+j}, \qquad n \geqslant 0,\\
\alpha_n^{[j]} &= \gamma_{3n-2+j}\gamma_{3n+j} + \gamma_{3n-1+j}\gamma_{3n-3+j} + \gamma_{3n-2+j}\gamma_{3n-1+j}, \qquad n \geqslant 1,\\
\gamma_n^{[j]} &= \gamma_{3n-2+j}\gamma_{3n+j}\gamma_{3n+2+j} \neq 0, \qquad n \geqslant 2.
\end{aligned}$$

Moreover, $\{P_n^{[j]}\}_{n\geqslant 0}$ *is 2-orthogonal with respect to the vector functional* $U^{[j]} = (u_0^{[j]}, u_1^{[j]})$ *with*

$$u_\nu^{[j]} = \sigma_3(x^j u_{3\nu+j}) \quad \textit{for each} \quad j = 0,1,2 \quad \textit{and} \quad \nu = 0,1.$$

where $\{u_n\}_{n\geqslant 0}$ *represents the dual sequence of* $\{P_n\}_{n\geqslant 0}$ *and* $\sigma_3 : \mathcal{P}' \longrightarrow \mathcal{P}'$ *is the operator defined by* $\langle \sigma_3(v), f(x)\rangle := \langle v, f(x^3)\rangle$ *for any* $v \in \mathcal{P}'$ *and* $f \in \mathcal{P}$.

The orthogonality measures are supported on a starlike set with three rays.

Theorem 2.1. *[3] If* $\gamma_n > 0$ *for* $n \geqslant 1$ *in* (2.8)*, then there exists a vector of linear functionals* (u_0, u_1) *such that the polynomials* P_n *defined by* (2.8) *satisfy the 2-orthogonal relations* (2.3)*-*(2.4)*. Moreover, the vector of linear functionals* (u_0, u_1) *satisfies* (2.7) *and there exist a vector of two measures* (μ_0, μ_1) *such that*

$$\langle u_0, f(x)\rangle = \int_S f(x)\,d\mu_0(x)$$
$$\langle u_1, f(x)\rangle = \int_S f(x)\,d\mu_1(x)$$

where S *represents the starlike set*

$$S := \bigcup_{k=0}^{2} \Gamma_k \qquad \textit{with} \qquad \Gamma_k = [0, \mathrm{e}^{2\pi i k/3}\infty),$$

and the measures have a common support which is a subset of S *and are invariant under rotations of* $2\pi/3$.

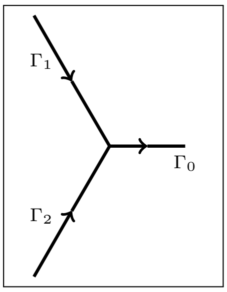

Fig. 2. The three rays Γ_0, Γ_1 and Γ_2.

Regarding the behavior of the zeros of any threefold symmetric 2-orthogonal polynomial sequence$\{P_n\}_{n\geqslant 0}$, it was shown in [9] that between two nonzero consecutive roots of P_{n+2} there is exactly one root of P_n and one root of P_{n+1} and all those roots lie on the starlike set S.

Proposition 2.2. *Let* $\{P_n\}_{n\geqslant 0}$ *be a 2-OPS satisfying* (2.8)*. If* $\gamma_n > 0$*, then the following statements hold:*

(a) *If x is a zero of P_{3n+j}, then $\omega^j x$ are also zeros of P_{3n+j} with $\omega = \mathrm{e}^{2\pi i/3}$.*
(b) *0 is a zero of P_{3n+j} of multiplicity j when $j = 1, 2$.*
(c) *P_{3n+j} has n distinct positive real zeros $\{x^{(j)}_{n,k}\}_{k=1}^{n}$ with $0 < x^{(j)}_{n,1} < \ldots < x^{(j)}_{n,n}$.*
(d) *Between two real zeros of P_{3n+j+3} there exist only one zero of P_{3n+j+2} and only one zero of P_{3n+j+1}, that is, $x^{(j+2)}_{n,k} < x^{(j)}_{n,k+1} < x^{(j+1)}_{n,k+1} < x^{(j+2)}_{n,k+1}$.*

Proof. The result is a consequence of [9, Theorem 2.2] for the case $d = 2$. □

So, P_{3n+j} with $j \in \{0, 1, 2\}$ has n zeros $(x^{(j)}_{n,1}, \ldots, x^{(j)}_{n,n})$ on the positive real line and all the other zeros are obtained by rotations of $2\pi/3$ of $(x^{(j)}_{n,1}, \ldots, x^{(j)}_{n,n})$ and 0 is a single zero for P_{3n+1} and a double zero for P_{3n+2}. The connection between the asymptotic behavior of the γ-recurrence coefficients in (2.8) and the upper bound for the largest zero $x^{(j)}_{n,n}$ is discussed in [2], but for bounded recurrence coefficients. Here, we extend that discussion, by embracing the cases where the recurrence coefficients are unbounded with different asymptotic behavior for even and odd order indices, which will be instrumental in Section 3.

Theorem 2.2. *If γ_n in (2.8) are positive and, additionally, $\gamma_{2n} = c_0 n^\alpha + o(n^\alpha)$ and $\gamma_{2n+1} = c_1 n^\alpha + o(n^\alpha)$ for large n, with $\min\{c_0, c_1\} \geqslant 0$, $c = \max\{c_0, c_1\} > 0$ and $\alpha \geqslant 0$, then largest zero in absolute value $|x_{n,n}|$ behaves as*

$$|x_{n,n}| \leqslant \frac{3}{2^{2/3}} c^{1/3} n^{\alpha/3} + o(n^{\alpha/3}), \qquad n \geqslant 1. \tag{2.10}$$

Proof. Consider the Hessenberg matrix

$$\mathbf{H}_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \gamma_1 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \ddots & & & \ddots & & & \\ & & & \ddots & & & \ddots & & \\ 0 & 0 & \cdots & 0 & \gamma_{n-4} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \gamma_{n-3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{n-2} & 0 & 0 \end{pmatrix}$$

so that the recurrence relation (2.8) can be expressed as

$$\mathbf{H}_n \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-1}(x) \end{pmatrix} = x \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-1}(x) \end{pmatrix} - P_n(x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

and each zero of P_n is an eigenvalue of the matrix $\mathbf{H}_n$. The spectral radius of the matrix $\mathbf{H}_n$,

$$\rho(\mathbf{H}_n) = \max\{|\lambda| : \ \lambda \text{ is an eigenvalue of } \mathbf{H}_n\},$$

is bounded from above by $||\mathbf{H}_n||$ where $||\cdot||$ denotes a matrix norm (see [23, Section 5.6]). We take the matrix norm

$$||\mathbf{H}_n||_S = ||S^{-1}\mathbf{H}_n S||_\infty = \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\left|(S^{-1}\mathbf{H}_n S)_{i,j}\right|\right\},$$

where S corresponds to a non-singular matrix and $(S^{-1}\mathbf{H}_n S)_{i,j}$ denotes the ith row and jth column entry of the product matrix $S^{-1}\mathbf{H}_n S$. In particular if S is an invertible diagonal matrix $S = \mathrm{diag}(d_1,\ldots,d_k,\ldots,d_n)$, then

$$||\mathbf{H}_n||_S = \max\left\{\frac{d_2}{d_1},\frac{d_3}{d_2},\frac{d_4+d_1\gamma_1}{d_3},\ldots,\frac{d_k+d_{k-3}\gamma_{k-3}}{d_{k-1}},\ldots,\frac{d_n+d_{n-3}\gamma_{n-3}}{d_{n-1}},\frac{d_{n-2}\gamma_{n-2}}{d_n}\right\}.$$

Setting $d_k = d^k(k!)^{\alpha/3}\neq 0$, for some positive constant d, gives

$$||\mathbf{H}_n||_S \leqslant 2^{\alpha/3}\left(d+\frac{c}{d^2}\right)n^{\alpha/3}+o(n^{\alpha/3}) \quad \text{as} \quad n\to+\infty.$$

The choice of $d=(2c)^{1/3}$ gives a minimum to $\left(d+\frac{c}{d^2}\right)$, so that

$$||\mathbf{H}_n||_S \leqslant \frac{3}{4^{1/3}}\left(c\,n^{\alpha}\right)^{1/3}+o(n^{\alpha/3}) \quad \text{as} \quad n\to+\infty,$$

which implies the result. □

To summarize, Proposition 2.2 combined with Lemma 2.1 allow us to conclude that each $P_n^{[j]}$ has exactly n real zeros $\{x_{n,k}^{[j]}\}_{k=1}^n$ with $0 < x_{n,1}^{[j]} < \ldots < x_{n,n}^{[j]}$. Moreover, between two consecutive zeros of $P_n^{[j+2]}$ there is exactly one zero of $P_n^{[j]}$ and another of $P_n^{[j+1]}$: $x_{n,k}^{[j+2]} < x_{n,k+1}^{[j]} < x_{n,k+1}^{[j+1]} < x_{n,k+1}^{[j+2]}$. Taking into consideration the asymptotic behavior for the largest zero $|x_{n,n}|$ of P_n described in Theorem 2.2, we then conclude that

$$x_{n,k}^{[j]} \leqslant \frac{27}{4}c\,n^{\alpha}+o(n^{\alpha}).$$

Therefore, for each $j\in\{0,1,2\}$, all the zeros $x_{n,k}^{(j)}$ of P_{3n+j} distinct from 0 correspond to the cubic roots of $x_{n,k}^{[j]}>0$, with $k\in\{1,\ldots,n\}$, and they all lie on the starlike set S and within the disc centred at the origin with radius $b=\frac{27}{4}c\,n^{\alpha}+o(n^{\alpha})$.

Multiple orthogonal polynomials on starlike sets received attention in recent years, under different frameworks. This includes the asymptotic behavior of polynomial sequences generated by recurrence relations of the type (2.8) when further assumptions are taken regarding specific behaviour for the γ-coefficient [2] or for certain type of the 2-orthogonality measures in [12] and [27]. Faber polynomials associated with hypocycloidal domains and stars have also been studied in [21].

Here, we describe all the threefold 2-orthogonal polynomial sequences that are classical in Hahn's sense, and our study includes the representations for the measures supported on a set containing all the zeros of the polynomial sequence. From

Theorem 2.1 and Proposition 2.2, the support lies on a starlike set, that, according to Theorem 2.2 is bounded if the γ-coefficients are bounded, and unbounded otherwise.

3. Threefold symmetric Hahn-classical 2-orthogonal polynomial sequence

The classical orthogonal polynomial sequences of Hermite, Laguerre, Jacobi and Bessel collectively satisfy the so-called *Hahn property*: the sequence of its derivatives is again an orthogonal polynomial sequence. In the context of 2-orthogonality, this algebraic property is portrayed as follows:

Definition 3.1. A monic 2-orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ is "2-Hahn-classical" when the sequence of its derivatives $\{Q_n\}_{n\geqslant 0}$, with $Q_n(x) := \frac{1}{n+1}P'_{n+1}(x)$ is also a 2-orthogonal polynomial sequence.

The study of this type of 2-orthogonal sequences was initiated in the works by Douak and Maroni [14]-[17]. In those works (as well as in [32]) several properties of these polynomials were given, with the main pillars of the study being the structural properties, including the recurrence relations, satisfied by the polynomials. All in all, those studies encompassed the analysis of a nonlinear system of equations fulfilled by the recurrence coefficients. Douak and Maroni treated some special solutions to that system of equations, bringing to light several examples of these threefold symmetric "2-Hahn-classical" polynomials: see [13,16,17]. However, for those cases the support of the corresponding orthogonality measures that they found consisted of the positive real axis, which does not contain all the zeros. Here, we base our analysis on the properties of the orthogonality measures and deduce the properties of the recurrence coefficients. We incorporate the works by Douak and Maroni and go beyond that by fully describing all the threefold "2-Hahn-classical" polynomials and bringing up explicitly the orthogonality measures along with the asymptotic behavior of the largest zero in absolute value as well as a Bochner type result for the polynomials (i.e., characterizing these polynomials via a third order differential equation).

We start by observing that the threefold symmetry of $\{P_n\}_{n\geqslant 0}$ readily implies the threefold symmetry of $\{Q_n(x) := \frac{1}{n+1}P'_{n+1}(x)\}_{n\geqslant 0}$. This is a straightforward consequence of Definition 2.1, as it suffices to take single differentiation of relation (2.1). Such property is valid for any polynomial sequence, regardless any orthogonality properties.

Regarding threefold symmetric 2-orthogonal polynomial sequences possessing Hahn's property, more can be said. The next result summarizes a characterization of the orthogonality measures along side with the recurrence relations of the original sequence $\{P_n(x)\}_{n\geqslant 0}$ and the sequence of derivatives $\{Q_n(x) := \frac{1}{n+1}P'_{n+1}(x)\}_{n\geqslant 0}$. This characterization can be found in the works [14,15,32]. Nonetheless, we revisit these results for a matter of completion while bringing different approaches to the original proofs, highlighting that all the structural properties for the polynomials can be derived from the corresponding 2-orthogonality measures.

Theorem 3.1. *Let* $\{P_n\}_{n\geqslant 0}$ *be a threefold symmetric 2-orthogonal polynomial sequence for* (u_0, u_1) *satisfying the recurrence relation* (2.8) *and let* $\{Q_n(x) := \frac{1}{n+1}P'_{n+1}(x)\}_{n\geqslant 0}$. *The following statements are equivalent:*

(a) $\{Q_n(x)\}_{n\geqslant 0}$ *is a threefold symmetric 2-orthogonal polynomial sequence, satisfying the third-order recurrence relation*

$$Q_{n+1}(x) = xQ_n(x) - \widetilde{\gamma}_{n-1}Q_{n-2}(x), \tag{3.1}$$

with initial conditions $Q_k(x) = x^k$ *for* $k \in \{0, 1, 2\}$.

(b) *The vector functional* (u_0, u_1) *satisfies the matrix differential equation*

$$\left(\mathbf{\Phi}\begin{bmatrix}u_0\\u_1\end{bmatrix}\right)' + \mathbf{\Psi}\begin{bmatrix}u_0\\u_1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \tag{3.2a}$$

where

$$\Phi = \begin{bmatrix}\vartheta_1 & (1-\vartheta_1)x\\ \dfrac{2}{\gamma_1}(1-\vartheta_2)\,x^2 & 2\vartheta_2 - 1\end{bmatrix} \quad \text{and} \quad \mathbf{\Psi} = \begin{bmatrix}0 & 1\\ \frac{2}{\gamma_1}x & 0\end{bmatrix} \tag{3.2b}$$

for some constants ϑ_1 *and* ϑ_2 *such that*

$$\vartheta_1, \vartheta_2 \neq \frac{n-1}{n}, \quad \text{for all } \ n \geqslant 1. \tag{3.2c}$$

(c) *there are coefficients* $\vartheta_1, \vartheta_2 \neq \frac{n-1}{n}$, *such that* $\mathbf{U} = (u_0, u_1)$ *satisfies*

$$\Big(\phi(x)u_0\Big)'' + \left(\frac{2}{\gamma_1}(\vartheta_2 + \vartheta_1 - 2)x^2 u_0\right)' + \frac{2}{\gamma_1}\,(\vartheta_1 - 2)\,xu_0 = 0 \tag{3.3}$$

and

$$\begin{cases} (\vartheta_1 - 2)\,(2\vartheta_2 - 1)\,u_1 \\ \quad = \phi(x)u_0' - \dfrac{2}{\gamma_1}\,(\vartheta_1 - 1)\,(2\vartheta_2 - 3)\,x^2 u_0, & \text{if} \quad \vartheta_1 \neq 2, \quad (3.4a) \\ x\,u_1' = 2u_0', & \text{if} \quad \vartheta_1 = 2, \quad (3.4b) \end{cases}$$

where

$$\phi(x) = \vartheta_1\,(2\vartheta_2 - 1) - \frac{2}{\gamma_1}\,(\vartheta_1 - 1)\,(\vartheta_2 - 1)\,x^3. \tag{3.5}$$

(d) *There exists a sequence of numbers* $\{\widetilde{\gamma}_{n+1}\}_{n\geqslant 0}$ *such that*

$$P_{n+3}(x) = Q_{n+3}(x) + \Big((n+1)\gamma_{n+2} - (n+3)\widetilde{\gamma}_{n+1}\Big)Q_n(x), \tag{3.6}$$

with initial conditions $P_0(x) = Q_0(x) = 1$, $P_1(x) = Q_1(x) = x$ *and* $P_2(x) = Q_2(x) = x^2$.

Proof. We prove that (a) $\Rightarrow$ (d) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (a).

In order to see (a) implies (d), we differentiate the recurrence relation satisfied by P_n to then replace the differentiated terms by its definition of Q_n. A substitution of the term $xQ_n(x)$ by the expression provided in (3.1) finally gives (3.6).

Now we prove that (d) implies (b). Any linear functional w in $\mathcal{P}'$ can be written as

$$w = \sum_{k=0}^{\infty} \langle w, P_k \rangle u_k.$$

Based on this, the relation $Q_n(x) := \frac{1}{n+1} P'_{n+1}(x)$ gives a differential relation between the dual sequence $\{v_n\}_{n\geqslant 0}$ of $\{Q_n\}_{n\geqslant 0}$ in terms of $\{u_n\}_{n\geqslant 0}$ (the dual sequence of $\{P_n\}_{n\geqslant 0}$), which is

$$v'_n = -(n+1)u_{n+1}, \qquad n \geqslant 0, \tag{3.7}$$

whilst (3.6) leads to

$$v_n = u_n + \left((n+1)\gamma_{n+2} - (n+3)\widetilde{\gamma}_{n+1}\right) u_{n+3}, \qquad n \geqslant 0. \tag{3.8}$$

The choice of $n = 0$ and $n = 1$ in both (3.7) and (3.8) respectively gives

$$\begin{bmatrix} v'_0 \\ v'_1 \end{bmatrix} = -\begin{bmatrix} u_1 \\ 2\,u_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_0 + (\gamma_2 - 3\widetilde{\gamma}_1)\, u_3 \\ u_1 + (2\gamma_3 - 4\widetilde{\gamma}_2)\, u_4 \end{bmatrix}. \tag{3.9}$$

As explained in [32] (and, alternatively, in [14], [15] and [28]), the elements of the dual sequence $\{u_n\}_{n\geqslant 0}$ of the 2-orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ can be written as

$$\begin{aligned} u_{2n} &= E_n(x)u_0 + a_{n-1}(x)u_1, \\ u_{2n+1} &= b_n(x)u_0 + F_n(x)u_1, \end{aligned}$$

where E_n and F_n are polynomials of degree n, while a_n and b_n are polynomials of degree less than or equal to n under the assumption that $a_{-1}(x) = 0$, $E_0 = 1$, $F_0 = 1$ and $b_0 = 0$. The recurrence relations (2.8) fulfilled by $\{P_n\}_{n\geqslant 0}$ yield (see [32, Lemma 6.1])

$$\begin{aligned} &\gamma_{2n+2}F_{n+1}(x) - xF_n(x) = -a_{n-1}(x), \qquad \gamma_{2n+3}a_{n+1}(x) - xa_n(x) = -F_n(x), \\ &\gamma_{2n+2}b_{n+1}(x) - xb_n(x) = -E_n(x) \quad \text{and} \quad \gamma_{2n+3}E_{n+2}(x) - xE_{n+1}(x) = -b_n(x), \end{aligned}$$

with initial conditions $a_0(x) = 0$ and $E_1(x) = \frac{1}{\gamma_1}x$. In particular, we obtain:

$$b_1(x) = -\frac{1}{\gamma_2}, \quad F_1(x) = \frac{1}{\gamma_2}x, \quad E_2(x) = \frac{1}{\gamma_1\gamma_3}x^2, \quad \text{and} \quad a_1(x) = -\frac{1}{\gamma_3},$$

so that

$$u_2 = \frac{1}{\gamma_1}x\,u_0, \quad u_3 = -\frac{1}{\gamma_2}u_0 + \frac{1}{\gamma_2}x\,u_1 \quad \text{and} \quad u_4 = \frac{1}{\gamma_1\gamma_3}x^2\,u_0 - \frac{1}{\gamma_3}\,u_1. \tag{3.10}$$

Consequently, the first identity in (3.9) reads as

$$\begin{bmatrix} v'_0 \\ v'_1 \end{bmatrix} = -\mathbf{\Psi}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad \text{with} \quad \mathbf{\Psi} = \begin{bmatrix} 0 & 1 \\ \frac{2}{\gamma_1}x & 0 \end{bmatrix}. \tag{3.11a}$$

Using (3.10) in the second identity in (3.9) leads to

$$\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \boldsymbol{\Phi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad \text{with} \quad \boldsymbol{\Phi} = \begin{bmatrix} \phi_{0,0} & \phi_{0,1} \\ \phi_{1,0} & \phi_{1,1} \end{bmatrix}, \tag{3.11b}$$

where (*c.f.* [32, Eq. (6.17)])

$$\phi_{00}(x) = \frac{3\widetilde{\gamma}_1}{\gamma_2}, \quad \phi_{01}(x) = \left(1 - \frac{3\widetilde{\gamma}_1}{\gamma_2}\right) x,$$
$$\phi_{10}(x) = 2\left(\frac{1}{\gamma_1} - \frac{2\widetilde{\gamma}_2}{\gamma_3}\right) x^2 \quad \text{and} \quad \phi_{11}(x) = -1 + \frac{4\widetilde{\gamma}_2}{\gamma_3}.$$

We set $\vartheta_1 = \frac{3\widetilde{\gamma}_1}{\gamma_2} \neq 0$ and $\vartheta_2 = \frac{2\widetilde{\gamma}_2}{\gamma_3} \neq 0$, and now (3.2a)-(3.2b) follows after differentiating both sides of the equation in (3.11b) and then compare with (3.11a), which is a system of two functional equations in (u_0, u_1):

$$\begin{cases} \vartheta_1 u_0' + (1-\vartheta_1)x\, u_1' + (2-\vartheta_1)u_1 = 0 \\ \frac{2}{\gamma_1}(1-\vartheta_2)x^2\, u_0' + \frac{2}{\gamma_1}(3-2\vartheta_2)x\, u_0 - (1-2\vartheta_2)u_1' = 0. \end{cases} \tag{3.12}$$

The action of each of these equations over the monomials gives

$$\begin{cases} n\vartheta_1 (u_0)_{n-1} + ((n-1) - n\vartheta_1)\,(u_1)_n = 0, \\ \dfrac{1}{\gamma_1}\,((n-1) - n\vartheta_2)\,(u_0)_{n+1} + (2\vartheta_2 - 1)n(u_1)_{n-1} = 0, \qquad n \geqslant 0, \end{cases}$$

under the assumption that $(u_0)_{-1} = (u_1)_{-1} = 0$. All the moments are well defined provided that

$$((n-1) - n\vartheta_1)\,((n-1) - n\vartheta_2) \neq 0, \qquad n \geqslant 0,$$

which corresponds to condition (3.2c). Under this assumption, the threefold symmetry implies that

$$(u_0)_n = (u_1)_{n+1} = 0 \quad \text{for} \quad n \not\equiv 0 \bmod 3,$$

whilst for $n \equiv 0 \bmod 3$, we have $(u_0)_n\ (u_1)_{n+1} \neq 0$, and these are recursively defined by

$$\frac{2}{\gamma_1}\,(n+1-(n+2)\vartheta_2)\,(n-(n+1)\vartheta_1)\,(u_0)_{n+3} = \vartheta_1(2\vartheta_2 - 1)(n+1)(n+2)(u_0)_n\ ,$$
$$(n+1-(n+2)\vartheta_2)\,(n+3-(n+4)\vartheta_1)\,(u_1)_{n+4} = \vartheta_1(2\vartheta_2 - 1)(n+2)(n+4)(u_1)_{n+1}\ ,$$

where $(u_0)_0 = (u_1)_1 = 1$.

Concerning the proof of (b) $\Rightarrow$ (c), we start by noting that the system of equations (3.2a) can be written as in (3.12) which reads as

$$C \begin{bmatrix} u_1' \\ u_1 \end{bmatrix} = -D \begin{bmatrix} u_0' \\ u_0 \end{bmatrix}, \tag{3.13}$$

with

$$C = \begin{bmatrix} (1-\vartheta_1)\,x & 2-\vartheta_1 \\ 2\vartheta_2 - 1 & 0 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} \vartheta_1 & 0 \\ \frac{2}{\gamma_1}(1-\vartheta_2)x^2 & \frac{2}{\gamma_1}(3-2\vartheta_2)x \end{bmatrix}.$$

Observe that $\det C = -(2-\vartheta_1)(2\vartheta_2 - 1)$ is constant. The matrix C is nonsingular if $\vartheta_1 \neq 2$. For this case, a left multiplication by its adjoint matrix C^* gives

$$\begin{cases} (\theta_1 - 2)\,(2\theta_2 - 1)\,u_1' \\ \quad = -\dfrac{2}{\gamma_1}\,(\vartheta_1 - 2)\,(1-\vartheta_2)\,x^2 u_0' + \dfrac{2}{\gamma_1}\,(\vartheta_1 - 2)\,(2\vartheta_2 - 3)\,x u_0, & \text{(3.14a)} \\ (\theta_1 - 2)\,(2\theta_2 - 1)\,u_1 = \phi(x) u_0' - \dfrac{2}{\gamma_1}\,(\vartheta_1 - 1)\,(2\vartheta_2 - 3)\,x^2 u_0, & \text{(3.14b)} \end{cases}$$

where $\phi(x) = \det \Phi$ which is given in (3.5). We differentiate (3.14b) once and compare with (3.14a) to obtain (3.3) and (3.4a).

When $\vartheta_1 = 2$, the functional equations (3.12) (or equivalently (3.13)) become

$$\begin{cases} 2u_0' = x\,u_1', \\ \frac{2}{\gamma_1}(1-\vartheta_2)x^2\,u_0' + \frac{2}{\gamma_1}(3-2\vartheta_2)x\,u_0 - (1-2\vartheta_2)u_1' = 0, \end{cases}$$

and this gives (3.3) and (3.4b).

The implication (c) $\Rightarrow$ (b) can be trivially obtained by performing reciprocal operations to the ones just described.

The proof of (b) $\Rightarrow$ (a) consists of showing that $\{Q_n\}_{n\geqslant 0}$ is 2-orthogonal for (v_0, v_1) given by (3.11b) based on (3.2a)-(3.2b) together with the 2-orthogonality of $\{P_n\}_{n\geqslant 0}$ with respect to (u_0, u_1) and on account of conditions (3.2c). We will omit the details which can be followed in [32, Proposition 6.2, p.324], with the obvious adaptations to the threefold symmetric case. □

The vector functional $\mathbf{U} = (u_0, u_1)$ satisfies a matrix equation of Pearson type (3.2a). This somehow mimics the properties of the classical orthogonal polynomials. However, we refrain ourselves from calling such polynomials 2-orthogonal classical, since other characteristic properties of the classical orthogonal polynomials can be interpreted in the context of 2-orthogonality (or multiple orthogonality) and give rise to completely different sets of polynomials.

As a straightforward consequence of statement (c) in the latter result, we derive equations for the moments of the *Hahn-classical* vector linear functional (u_0, u_1):

Corollary 3.1. *Under the same assumptions of Theorem 3.1, the moments of* u_0 *and* u_1 *satisfy*

$$\begin{cases} \dfrac{2}{\gamma_1}\Big[(3n+2)\Big((3n+1)\,(\vartheta_1 - 1)\,(\vartheta_2 - 1) - (\vartheta_2 + \vartheta_1 - 2)\Big) - (\vartheta_1 - 2)\Big](u_0)_{3n+3} \\ \qquad\qquad = (3n+2)(3n+1)\vartheta_1\,(2\vartheta_2 - 1)\,(u_0)_{3n}\ , \\ (u_0)_{3n+1} = (u_0)_{3n+2} = 0 \quad \text{and} \quad (u_0)_0 = 1, \end{cases} \tag{3.15}$$

while

$$\begin{cases} (u_1)_{3n+1} = \dfrac{2\,(\vartheta_1-1)\,((3n+2)\vartheta_2-(3n+1))}{\gamma_1\,(\vartheta_1-2)\,(2\vartheta_2-1)}(u_0)_{3n+3} \\ \qquad\qquad -\,(3n+1)\vartheta_1\,(2\vartheta_2-1)\,(u_0)_{3n}, & \text{if } \vartheta_1\neq 2 \\ (u_1)_{3n+1} = \frac{2(3n+1)}{3n+2}(u_0)_{3n} & \text{if } \vartheta_1 = 2 \\ (u_1)_{3n} = (u_1)_{3n+2} = 0 \quad \text{and} \quad (u_1)_1 = 1. \end{cases} \tag{3.16}$$

Proof. From (3.3)-(3.4) it follows

$$\langle\Big(\phi(x)u_0\Big)'' + \left(\frac{2}{\gamma_1}(\vartheta_2+\vartheta_1-2)x^2u_0\right)' + \frac{2}{\gamma_1}\,(\vartheta_1-2)\,xu_0, x^n\rangle = 0, \qquad n\geqslant 0,$$

and

$$\begin{cases} \langle(\vartheta_1-2)\,(2\vartheta_2-1)\,u_1, x^n\rangle = \langle\phi(x)u_0' - \frac{2}{\gamma_1}\,(\vartheta_1-1)\,(2\vartheta_2-3)\,x^2u_0, x^n\rangle, & \text{if } \vartheta_1\neq 2, \\ x\,u_1' = 2\langle u_0', x^n\rangle, & \text{if } \vartheta_1 = 2, \end{cases}$$

which, on account of (1.1), leads to (3.15).

Similarly, by taking into account the operations defined in (3.43), we deduce that relations (3.4) imply (3.16). □

The "2-Hahn-classical" polynomials satisfy a third order recurrence relation (2.5) and the γ-recurrence coefficients have a specific rational structure. Here, we show that such expression for the γ-coefficients actually characterizes the three-fold symmetric "2-Hahn-classical" polynomials, by proving the reciprocal condition found in the work by Douak and Maroni [14]. For a matter of completion, we obtain the γ-coefficients directly from the functional equations (3.2a), rather than from algebraic manipulations on the recurrence relations.

Theorem 3.2. *Let $\{P_n\}_{n\geqslant 0}$ be a monic 2-orthogonal polynomial sequence. Then $\{P_n\}_{n\geqslant 0}$ satisfies* (2.8) *with*

$$\gamma_{n+2} = \frac{n+3}{n+1}\,\frac{n(\vartheta_n-1)+1}{(n+4)(\vartheta_{n+1}-1)+1}\gamma_{n+1}, \tag{3.17}$$

where

$$\vartheta_n = \left(\frac{1-(-1)^n}{2}\right)\frac{1-\frac{n+1}{2}(1-\vartheta_1)}{1-\frac{n-1}{2}(1-\vartheta_1)} + \left(\frac{1+(-1)^n}{2}\right)\frac{1-\frac{n}{2}(1-\vartheta_2)}{1-(\frac{n}{2}-1)(1-\vartheta_2)}, \qquad n\geqslant 1, \tag{3.18}$$

with ϑ_1, ϑ_2 subject to (3.2c), *if and only if the sequence $\{Q_n(x) := \frac{1}{n+1}P'_{n+1}(x)\}_{n\geqslant 0}$ is 2-orthogonal satisfying the recurrence relation* (3.1) *with*

$$\widetilde{\gamma}_n = \frac{n}{n+2}\vartheta_n\gamma_{n+1}, \quad \text{for} \quad n\geqslant 1. \tag{3.19}$$

Proof. We start by proving that if $\{Q_n\}_{n\geqslant 0}$ satisfies the recurrence relation (3.1) with the $\widetilde{\gamma}$-coefficients given by (3.19), then the γ-coefficients in the recurrence relation of $\{P_n\}_{n\geqslant 0}$ are given by (3.17)-(3.18). The assumption means that $\{Q_n\}_{n\geqslant 0}$ is 2-orthogonal and therefore $\{P_n\}_{n\geqslant 0}$ is Hahn-classical. According to Theorem 3.1, $\{P_n\}_{n\geqslant 0}$ is 2-orthogonal with respect to a vector functional (u_0, u_1) satisfying the differential equation (3.2a) and $\{Q_n\}_{n\geqslant 0}$ is 2-orthogonal for $\mathbf{V} = (v_0, v_1)$ given by (3.11a)-(3.11b). The relation $Q_n(x) = \frac{1}{n+1}P'_{n+1}(x)$ combined with the properties (1.1) implies

$$\langle v_0, x^n Q_{2n}\rangle = -\frac{1}{2n+1}\langle (x^n v_0)', P_{2n+1}\rangle$$

and

$$\langle v_1, x^n Q_{2n+1}\rangle = -\frac{1}{2n+2}\langle (x^n v_1)', P_{2n+2}\rangle$$

for any $n \geqslant 0$. In the latter identities we replace (v_0', v_1') and (v_0, v_1) by the respective expressions given in (3.11a) and (3.11b), to obtain

$$\begin{aligned}
&\langle v_0, x^n Q_{2n}\rangle = (2n+1)^{-1}(1 - n\phi'_{0,1}(0))\langle u_1, x^n P_{2n+1}\rangle, \quad \text{for} \quad n \geqslant 0,\\
&\langle v_1, x^n Q_{2n+1}\rangle = (2n+2)^{-1}\left(\psi'(0) - n\frac{\phi''_{1,0}(0)}{2}\right)\langle u_0, x^{n+1} P_{2n+2}\rangle, \quad \text{for} \quad n \geqslant 0,
\end{aligned}$$

which are the same as

$$\begin{aligned}
&\langle v_0, x^n Q_{2n}\rangle = (2n+1)^{-1}(1 - (1-\vartheta_1)n)\langle u_1, x^n P_{2n+1}\rangle, \quad \text{for} \quad n \geqslant 0,\\
&\langle v_1, x^n Q_{2n+1}\rangle = (2n+2)^{-1}\frac{2}{\gamma_1}\left(1 - (1-\vartheta_2)n\right)\langle u_0, x^{n+1} P_{2n+2}\rangle, \quad \text{for} \quad n \geqslant 0.
\end{aligned}$$

Taking into account (2.6), from the latter we obtain (3.19) if we set ϑ_n as in (3.18). Observe that

$$\vartheta_{2n+1} = \frac{(n+1)\vartheta_1 - n}{n\vartheta_1 - (n-1)} \quad \text{and} \quad \vartheta_{2n+2} = \frac{(n+1)\vartheta_2 - n}{n\vartheta_2 - (n-1)}, \qquad n \geqslant 0,$$

and from this, we can clearly see that ϑ_n is actually a solution to the Riccati equation

$$\vartheta_{n+3} + \frac{1}{\vartheta_{n+1}} = 2, \quad \text{for} \quad n \geqslant 0. \tag{3.20}$$

Conversely, if $\{P_n\}_{n\geqslant 0}$ satisfies (2.8) with the γ-coefficients given by (3.17) where ϑ_n is given by (3.18), then we prove that $\{Q_n\}_{n\geqslant 0}$ satisfies a third order recurrence relation (3.1) with $\widetilde{\gamma}$-coefficients as in (3.19). We differentiate the recurrence relation (2.8) once and then take into account the definition of $Q_n(x) := \frac{1}{n+1}P'_{n+1}(x)$ obtain the structural relation

$$P_n(x) = (n+1)Q_n(x) - nxQ_{n-1}(x) + (n-2)\gamma_{n-1}Q_{n-3}.$$

In (2.8), we replace P_{n+1}, P_n and P_{n-2} by the expressions provided in the latter identity to obtain

$$\begin{aligned}(n+2)Q_{n+1}(x) &= 2(n+1)xQ_n(x) - nx^2Q_{n-1}(x) + 2(n-2)\gamma_{n-1}xQ_{n-3}(x)\\ &\quad - (n+1)(\gamma_{n-1}+\gamma_n)Q_{n-2}(x) - (n-4)\gamma_{n-3}\gamma_{n-1}Q_{n-5}(x),\end{aligned} \tag{3.21}$$

which is valid for any $n \geqslant 0$, under the assumption that $Q_{-n}(x) = 0$. As $\{Q_n\}_{n\geqslant 0}$ is a basis for $\mathcal{P}$, there are coefficients $\xi_{n+1,\nu}$ such that

$$xQ_n(x) = \sum_{\nu=0}^{n+1} \xi_{n+1,\nu} Q_\nu(x), \tag{3.22}$$

where $\xi_{n+1,n+1+\nu} = 0$ for any $\nu \geqslant 0$ and $\xi_{n+1,n+1} = 1$ because Q_n is monic. Based on this, we can also write

$$x^2Q_n(x) = \sum_{\nu=1}^{n+2}\sum_{\sigma=\nu-1}^{n+1} \xi_{n+1,\sigma}\xi_{\sigma+1,\nu}Q_\nu(x) + \sum_{\sigma=0}^{n+1}\xi_{n+1,\sigma}\xi_{\sigma+1,0}Q_0(x). \tag{3.23}$$

The threefold symmetry of $\{P_n\}_{n\geqslant 0}$ readily implies that of $\{Q_n\}_{n\geqslant 0}$ and this means that $\xi_{n,\nu} = 0$ whenever $n+\nu \neq 0 \bmod 3$, so that $\xi_{n,n-1}$, $\xi_{n,n-2}$, $\xi_{n,n-4}$, $\xi_{n,n-5}, \dots$ are all equal zero. We replace the terms xQ_n, xQ_{n-3} and x^2Q_{n-2} by the respective expressions given by (3.22) and (3.23) in the relation (3.21) to obtain

$$\begin{aligned}(n+2)Q_{n+1}(x) &= 2(n+1)\sum_{\nu=0}^{n+1}\xi_{n+1,\nu}Q_\nu(x) + 2(n-2)\gamma_{n-1}\sum_{\nu=0}^{n-2}\xi_{n-2,\nu}Q_\nu(x)\\ &\quad - n\left(\sum_{\nu=1}^{n+1}\sum_{\sigma=\nu-1}^{n}\xi_{n,\sigma}\xi_{\sigma+1,\nu}Q_\nu(x) + \sum_{\sigma=0}^{n}\xi_{n,\sigma}\xi_{\sigma+1,0}Q_0(x)\right)\\ &\quad - (n-1)(\gamma_{n-1}+\gamma_n)Q_{n-2}(x) - (n-4)\gamma_{n-3}\gamma_{n-1}Q_{n-5}(x).\end{aligned} \tag{3.24}$$

For $n \geqslant 2$, we equate the coefficients of Q_{n-2}, giving

$$0 = 2(n+1)\xi_{n+1,n-2} + 2(n-2)\gamma_{n-1} - n\xi_{n,n-3} - (n-1)(\gamma_{n-1}+\gamma_n). \tag{3.25}$$

In particular, for $n = 2$ the latter becomes

$$0 = 4\xi_{3,0} - \gamma_1 - \gamma_2,$$

and, because of (3.17), we have $\gamma_1 = \frac{1}{3}(4\vartheta_1 - 3)\gamma_2$ so that

$$\xi_{3,0} = \frac{\vartheta_1}{3}\gamma_2.$$

If we assume that for some $n \geqslant 3$,

$$\xi_{n,n-3} = \frac{n-2}{n}\vartheta_{n-2}\gamma_{n-1},$$

then (3.25) implies

$$(n+2)\xi_{n+1,n-2} = \big((n-2)\vartheta_{n-2} - (n-3)\big)\gamma_{n-1} + (n-1)\gamma_n,$$

which, after writing γ_{n-1} in terms of γ_n via (3.17), reads as

$$\xi_{n+1,n-2} = \frac{n-1}{n+1}\vartheta_{n-1}\gamma_n,$$

and therefore we conclude that

$$\xi_{n+3,n} = \frac{n+1}{n+3}\vartheta_{n+1}\gamma_{n+2}, \qquad \text{for all} \quad n \geqslant 0. \tag{3.26}$$

Equating the coefficients of Q_{n-5} in (3.24) leads to

$$\begin{aligned} 0 = (n+2)\xi_{n+1,n-5} + \left(2(n-2)\gamma_{n-1} - n\xi_{n,n-3}\right)\xi_{n-2,n-5} \\ - n\xi_{n,n-6} - (n-4)\gamma_{n-3}\gamma_{n-1}, \end{aligned} \tag{3.27}$$

for $n \geq 5$. The particular choice of $n = 5$ in the latter identity becomes

$$0 = 7\xi_{6,0} + \left(2 - \vartheta_3\right)\vartheta_1\gamma_2\gamma_4 - \gamma_2\gamma_4,$$

after replacing $\xi_{5,2}$ and $\xi_{3,0}$ by the expressions provided by (3.26). Using the identity (3.20) for $n = 4$, we then conclude that $\xi_{6,0} = 0$. Now, suppose that $\xi_{n,n-6} = 0$ for some $n \geqslant 6$. Identity (3.27) tells that

$$(n+2)\xi_{n+1,n-5} = -\left(2(n-2)\gamma_{n-1} - n\xi_{n,n-3}\right)\xi_{n-2,n-5} + (n-4)\gamma_{n-3}\gamma_{n-1},$$

which, because of (3.26), becomes

$$(n+2)\xi_{n+1,n-5} = -(n-4)\left(2 - \vartheta_{n-2}\right)\vartheta_{n-4}\gamma_{n-1}\gamma_{n-3} + (n-4)\gamma_{n-3}\gamma_{n-1},$$

and hence, due to (3.20), we conclude

$$\xi_{n+6,n} = 0, \qquad \text{for all } n \geqslant 0.$$

Now, equating the coefficients of Q_{n+1-3j} for $j \geqslant 3$ in (3.24) gives

$$\begin{aligned} 0 = (n+2)\xi_{n+1,n+1-3j} + \left(2(n-2)\gamma_{n-1} - n\xi_{n,n-3}\right)\xi_{n-2,n+1-3j} \\ - n\sum_{\sigma=n-3j}^{n-6}\xi_{n,\sigma}\xi_{\sigma+1,n+1-3j}, \end{aligned}$$

which is also

$$\begin{aligned} 0 = (n+2)\xi_{n+1,n+1-3j} + \left(2(n-2)\gamma_{n-1} - n\xi_{n,n-3}\right)\xi_{n-2,n+1-3j} \\ - n\sum_{\sigma=2}^{j}\xi_{n,n-3\sigma}\xi_{n+1-3\sigma,n+1-3j}. \end{aligned}$$

Suppose that $\xi_{n,n-3k} = 0$ for each $k = 2, 3, \ldots, j-1$ and $n \geqslant 3k$, then the latter becomes

$$(n+2)\xi_{n+1,n+1-3j} = n\xi_{n,n-3j},$$

which yields

$$\xi_{n+1,n+1-3j} = \frac{3j(3j+1)}{(n+2)(n+1)}\xi_{3j,0}. \tag{3.28a}$$

In particular this implies that

$$\xi_{3j+1,1} = \frac{3j}{(3j+2)}\xi_{3j,0} \quad \text{and} \quad \xi_{3j+2,2} = \frac{3j+1}{(3j+3)}\xi_{3j,0}. \tag{3.28b}$$

Finally we compare the coefficients of Q_0 in (3.24) to obtain

$$0 = (n+2)\xi_{n+1,0} + 2(n-2)\gamma_{n-1}\xi_{n-2,0} - n\left(\sum_{\sigma=0}^{n-1}\xi_{n,\sigma}\xi_{\sigma+1,0}\right),$$

for $n \geq 6$. Recall that $\xi_{k,0} = 0$ for $k \neq 0 \bmod 3$, so that the latter identity simplifies to

$$0 = (3n+3)\xi_{3n+3,0} + 6n\gamma_{3n}\xi_{3n,0} - (3n+2)\left(\sum_{\sigma=1}^{n-2}\xi_{3n+2,3\sigma+2}\xi_{3\sigma+3,0}\right) - n\xi_{3n+2,2}\xi_{3,0}, \tag{3.29}$$

for $n \geq 6$. We have already seen that $\xi_{6,0} = 0$. Proceeding by induction, we promptly observe that if $\xi_{3j,0} = 0$ for $j = 0, 1, \ldots, n$, then identities (3.28a)–(3.28b) allow us to conclude from (3.29) that $\xi_{3(n+1),0} = 0$ and this implies

$$\xi_{n,k} = 0, \qquad \text{for } \ 0 \leqslant k \leqslant n-4.$$

As a result, we conclude that

$$xQ_n = Q_{n+1} + \xi_{n+1,n-2}Q_{n-2}, \quad \text{with} \quad \xi_{n+1,n-2} = \frac{n-1}{n+1}\vartheta_{n-1}\gamma_n \neq 0, \qquad \text{for } n \geqslant 2,$$

with $Q_j(x) = x^j$ for $j = 0, 1, 2$. This means that $\{Q_n\}_{n\geqslant 0}$ is 2-orthogonal and threefold symmetric. □

In [17] Douak and Maroni have highlighted several properties of threefold-symmetric (therein referred to as "2-symmetric") 2-classical polynomials including a differential equation of third order. Here we show that such differential equation actually characterizes these polynomials, bringing to the theory a Bochner-type characterization.

Proposition 3.1. *Let $\{P_n\}_{n\geqslant 0}$ be a threefold symmetric 2-orthogonal polynomial sequence satisfying* (2.8). *The sequence $\{P_n\}_{n\geqslant 0}$ is Hahn-classical if and only if each P_n satisfies*

$$(a_n x^3 - b_n)P_n''' + c_n x^2 P_n'' + d_n x P_n' = e_n P_n, \tag{3.30}$$

where

$$a_n = (\vartheta_n - 1)(\vartheta_{n+1} - 1) \tag{3.31a}$$

$$b_n = \frac{\gamma_n\left((n-1)\vartheta_{n-1} - n + 2\right)\left(n\vartheta_n - n + 1\right)\left((n+1)\vartheta_{n+1} - n\right)}{n(n+1)} \tag{3.31b}$$

$$c_n = \vartheta_n\vartheta_{n+1} - 1 - (n-3)(\vartheta_n - 1)(\vartheta_{n+1} - 1) \tag{3.31c}$$

$$d_n = n\vartheta_{n+1} - (n-1)\vartheta_n(2\vartheta_{n+1} - 1) \tag{3.31d}$$

$$e_n = n\vartheta_{n+1}, \tag{3.31e}$$

for all $n \geq 1$, *and* ϑ_n *given in* (3.18) *with initial values* $\vartheta_1 = \frac{3(\gamma_1+\gamma_2)}{4\gamma_2}$ *and* $\vartheta_2 = \frac{3(\gamma_1+\gamma_2)}{10\gamma_3} + \frac{4}{5}$ *subject to* $\vartheta_1, \vartheta_2 \notin \{\frac{n-1}{n} : n \geqslant 1\}$ *.*

Proof. The necessary condition was proved in [17, Proposition 3.2], where Douak and Maroni have shown that threefold-symmetric (therein referred to as "2-symmetric") 2-orthogonal polynomials satisfying Hahn's property are solutions to (3.30) under the definitions (3.31a) and (3.31c)-(3.31e) with

$$b_n = \frac{\gamma_{n+3}\left((n+3)\vartheta_n - n + 2\right)\left((n+4)\vartheta_{n+1} - (n+3)\right)\left((n+5)\vartheta_{n+2} - (n+3)\right)}{(n+3)(n+4)},$$

where ϑ_n is given in (3.18). It turns out that b_n can be written as in (3.31b) because, according to Theorem 3.2, the γ-recurrence coefficients are recursively given by (3.17) which implies

$$\gamma_{n+3} = \frac{(n+4)(n+3)}{n(n+1)} \times \frac{\Big((n+1)(\vartheta_{n+1}-1)+1\Big)\Big(n(\vartheta_n-1)+1\Big)\Big((n-1)(\vartheta_{n-1}-1)+1\Big)}{\Big((n+5)(\vartheta_{n+2}-1)+1\Big)\Big((n+4)(\vartheta_{n+1}-1)+1\Big)\Big((n+3)(\vartheta_n-1)+1\Big)}\,\gamma_n.$$

Conversely, suppose that the 2-orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ fulfils (3.30) under the definitions (3.31a)-(3.31e) with ϑ_n given in (3.18). Observe that ϑ_n satisfies the Riccati equation (3.20) with initial conditions $\vartheta_1 = \frac{3(\gamma_1+\gamma_2)}{4\gamma_2}$ and $\vartheta_2 = \frac{3(\gamma_1+\gamma_2)}{10\gamma_3} + \frac{4}{5}$, both assumed to be different from $\frac{k-1}{k}$ for all integers $k \geqslant 1$. For values of $n \geqslant 3$, we consider the expansion $P_n(x) = x^n + \lambda_n x^{n-3} + \dots$ and insert it in the differential equation (3.30). We equate the coefficients of x^n and this gives

$$\lambda_n = -\frac{(n-2)(n-1)nb_n}{3\left((n-3)\vartheta_n - n + 4\right)\left((n-2)\vartheta_{n+1} - n + 3\right)}, \qquad n \geqslant 0.$$

On the other hand, from the recurrence relation (2.8) we deduce

$$\gamma_n = \lambda_{n+1} - \lambda_{n+2}, \quad n \geqslant 0.$$

After combining the latter two expressions we obtain

$$\gamma_n = -\frac{(n-1)n(n+1)b_{n+1}}{3\left((n-2)\vartheta_{n+1} - n + 3\right)\left((n-1)\vartheta_{n+2} - n + 2\right)} + \frac{n(n+1)(n+2)b_{n+2}}{3\left((n-1)\vartheta_{n+2} - n + 2\right)\left(n\vartheta_{n+3} - n + 1\right)},$$

where b_n is given in (3.31b). Based on (3.20), we consider the following substitutions in the latter expression

$$((n-2)\vartheta_{n+1} - n + 3) = \frac{((n-1)\vartheta_{n-1} - (n-2))}{\vartheta_{n-1}}$$

and

$$(n\vartheta_{n+3} - (n-1)) = \frac{((n+1)\vartheta_{n+1} - n)}{\vartheta_{n+1}},$$

to find

$$\gamma_n = n\vartheta_n\left((n+2)\vartheta_{n+2}-(n+1)\right)\Big(-\gamma_{n+1}\frac{(n-1)\vartheta_{n-1}\left((n+1)\vartheta_{n+1}-n\right)}{3(n+2)\left((n-1)\vartheta_{n-1}-(n-2)\right)} + \gamma_{n+2}\frac{(n+1)\vartheta_{n+1}\left((n+3)\vartheta_{n+3}-(n+2)\right)}{3(n+3)\left(n\vartheta_n-(n-1)\right)}\Big),$$

which is the same as

$$\frac{\gamma_n}{n\left((n+3)\vartheta_n-(n+2)\right)} = \gamma_{n+2}\frac{(n+1)\left((n+4)\vartheta_{n+1}-(n+3)\right)}{3(n+3)\left(n\vartheta_n-(n-1)\right)} - \gamma_{n+1}\frac{(n-1)\vartheta_{n-1}\left((n+1)\vartheta_{n+1}-n\right)}{3(n+2)\left((n-1)\vartheta_{n-1}-(n-2)\right)} \tag{3.32}$$

after taking the following substitutions (derived from (3.20))

$$\vartheta_n\left((n+2)\vartheta_{n+2}-(n+1)\right) = \left((n+3)\vartheta_n-(n+2)\right),$$

and

$$\vartheta_{n+1}\left((n+3)\vartheta_{n+3}-(n+2)\right) = \left((n+4)\vartheta_{n+1}-(n+3)\right).$$

We subtract $\frac{\gamma_{n+1}}{(n+2)((n-1)\vartheta_{n-1}-(n-2))}$ from both sides of (3.32) and this leads to

$$\frac{-3\gamma_n}{n\left((n+3)\vartheta_n-(n+2)\right)\gamma_{n+1}}\left(1-\frac{\gamma_{n+1}}{\gamma_n}\frac{n\left((n+3)\vartheta_n-(n+2)\right)}{(n+2)\left((n-1)\vartheta_{n-1}-(n-2)\right)}\right) = \left(1-\frac{\gamma_{n+2}}{\gamma_{n+1}}\frac{(n+1)\left((n+4)\vartheta_{n+1}-(n+3)\right)}{(n+3)\left(n\vartheta_n-(n-1)\right)}\right),$$

which implies

$$\left(\prod_{\ell=1}^{n}\frac{(-3)\gamma_\ell}{\ell\left((\ell+3)\vartheta_\ell-(\ell+2)\right)\gamma_{\ell+1}}\right)\left(1-\frac{\gamma_2}{\gamma_1}\frac{(4\vartheta_1-3)}{3}\right) = \left(1-\frac{\gamma_{n+2}}{\gamma_{n+1}}\frac{(n+1)\left((n+4)\vartheta_{n+1}-(n+3)\right)}{(n+3)\left(n\vartheta_n-(n-1)\right)}\right).$$

The assumption on the initial value for ϑ_1 readily implies the left-hand side of the latter equality to be zero and therefore we conclude

$$0 = \left(1-\frac{\gamma_{n+2}}{\gamma_{n+1}}\frac{(n+1)\left((n+4)\vartheta_{n+1}-(n+3)\right)}{(n+3)\left(n\vartheta_n-(n-1)\right)}\right) \text{ for all } n \geqslant 1.$$

Now, Theorem 3.2 ensures the 2-orthogonality of the sequence $\{Q_n(x) := \frac{1}{n+1}P'_{n+1}(x)\}_{n\geqslant 0}$, which means that $\{P_n\}_{n\geqslant 0}$ is Hahn-classical. □

Remark 3.1. After a single differentiation of (3.30), we obtain a differential equation for the polynomials Q_n and these satisfy:

$$(a_n x^3 - b_n)Q_n''' + (c_n+3a_n)x^2 Q_n'' + (d_n+2c_n)xQ_n' = (e_n-d_n)Q_n, \qquad n \geqslant 0,$$

with a_n, b_n, c_n, d_n and e_n given by (3.31a)–(3.31e).

Theorem 3.2 shows that a threefold symmetric 2-orthogonal polynomial sequence is *Hahn-classical* if and only if the γ-recurrence coefficients in (2.8) can be written as (3.17), provided that $\vartheta_1, \vartheta_2 \neq \frac{n}{n+1}$ for all positive integers n.

If $\vartheta_1, \vartheta_2 \geqslant 1$, then γ_n and $\widetilde{\gamma}_n$ are both positive for all integers $n \geqslant 1$. Furthermore, from (3.17)-(3.19) we readily see that γ_n and $\widetilde{\gamma}_n$ are two rational functions in n, both having the same asymptotic behavior:

$$\gamma_n = cn^\alpha + o(n^\alpha) \quad \text{as} \quad n \to +\infty.$$

Hence, as a result of Theorem 2.1 together with Theorem 2.2 and Theorem 3.1 the two linear functionals associated with a Hahn-classical threefold symmetric 2-orthogonal polynomial sequence admit the following integral representation.

Theorem 3.3. *Let $\{P_n\}_{n\geqslant 0}$ be a threefold symmetric and 2-orthogonal polynomial sequence with respect to the vector linear functional (u_0, u_1) fulfilling (3.3)-(3.4). If $\vartheta_1, \vartheta_2 \geqslant 1$, then $\{P_n\}_{n\geqslant 0}$ satisfies the recurrence relation (2.8) with $\gamma_n > 0$ and u_0 and u_1 admit the integral representation*

$$\langle u_k, f\rangle = \frac{1}{3}\left(\int_0^b f(x)\mathcal{U}_k(x)\,\mathrm{d}x + \omega^{2k-1}\int_0^{b\omega} f(x)\mathcal{U}_k(\omega^2 x)\,\mathrm{d}x + \omega^{1-2k}\int_0^{b\omega^2} f(x)\mathcal{U}_k(\omega x)\,\mathrm{d}x\right), \tag{3.33}$$

(with $k = 0, 1$) for any polynomial f, with $\omega = \mathrm{e}^{2\pi i/3}$ and $b = \lim_{n\to\infty}\left(\frac{27}{4}\gamma_n\right)$, provided that there exist two twice differentiable functions $\mathcal{U}_0$ and $\mathcal{U}_1$ mapping $[0, b)$ to $\mathbb{R}$ such that $\mathcal{U}_0$ is solution to

$$\Big(\phi(x)\mathcal{U}_0(x)\Big)'' + \left(\frac{2(\vartheta_2+\vartheta_1-2)}{\gamma_1}x^2\mathcal{U}_0(x)\right)' + \frac{2\,(\vartheta_1-2)}{\gamma_1}x\mathcal{U}_0(x) = \lambda_0 g_0(x)\ , \tag{3.34}$$

and $\mathcal{U}_1$ is given by

$$\begin{cases} (\vartheta_1-2)\,(2\vartheta_2-1)\,\mathcal{U}_1(x) \\ \quad = \phi(x)\mathcal{U}_0'(x) - \dfrac{2\,(\vartheta_1-1)\,(2\vartheta_2-3)\,x^2}{\gamma_1}\mathcal{U}_0(x) + \lambda_1 g_1(x)\ , & \text{if } \vartheta_1 \neq 2, \quad (3.35\text{a}) \\ x\mathcal{U}_1'(x) = 2\,\mathcal{U}_0'(x) + \lambda_1 g_1(x)\ , & \text{if } \vartheta_1 = 2, \quad (3.35\text{b}) \end{cases}$$

where $\phi(x) = \vartheta_1\,(2\vartheta_2-1) - \frac{2(\vartheta_1-1)(\vartheta_2-1)}{\gamma_1}x^3$, and satisfying

$$\lim_{x\to b} f(x)\frac{\mathrm{d}^l}{\mathrm{d}x^l}\mathcal{U}_0(x) = 0, \quad \text{for any } l \in \{0,1\} \quad \text{and} \quad f \in \mathcal{P}, \tag{3.36}$$

$$\int_0^b \mathcal{U}_0(x)\,\mathrm{d}x = 1,$$

where λ_k is a complex constant (possibly zero) and g_k a function whose moments vanish identically on the support of $\mathcal{U}_k$.

Proof. According to Theorem 2.1, the threefold symmetry of a 2-orthogonal polynomial sequence ensures the existence of two orthogonality measures μ_0 and μ_1 (respectively defined by the vector functional (u_0, u_1)) supported on a starlike set S on the three rays of the complex plane. On the other hand, Theorem 3.1 tells that u_0 satisfies (3.3) and u_1 fulfils (3.4a)-(3.4b). Based on these equations, we seek an integral representation for both u_0 and u_1 via a weight function $\mathcal{U}$ such that (3.33) holds for any polynomial f. Such a representation readily ensures the threefold symmetry of (u_0, u_1). Identity (3.3) is the same as

$$\left\langle (\phi(x)u_0)'' + \left(\frac{2(\vartheta_2+\vartheta_1-2)}{\gamma_1}x^2u_0\right)' + \frac{2(\vartheta_1-2)}{\gamma_1}xu_0\ ,\ f\right\rangle = 0, \qquad \forall f\in\mathcal{P},$$

which, because of (1.1), reads as

$$\left\langle u_0, \phi(x)f''(x) - \frac{2(\vartheta_2+\vartheta_1-2)}{\gamma_1}x^2f'(x) + \frac{2(\vartheta_1-2)}{\gamma_1}xf(x)\right\rangle = 0, \qquad \forall f\in\mathcal{P}, \tag{3.37}$$

with

$$\phi(x) = \vartheta_1(2\vartheta_2-1) - \frac{2(\vartheta_1-1)(\vartheta_2-1)}{\gamma_1}x^3.$$

We seek a function $\mathcal{W}_0$, at least twice differentiable, defined on an open set D containing the piecewise differentiable curve $\mathcal{C}$ containing all the zeros of $\{P_n\}_{n\geqslant 0}$ that is a subset of $S = \Gamma_0\cup\Gamma_1\cup\Gamma_2$ (represented in Fig.2) and such that

$$\langle u_0, f\rangle = \int_{\mathcal{C}} f(x)\mathcal{W}_0(x)\,\mathrm{d}x \tag{3.38}$$

holds for every polynomial f. In the light of Theorem 2.2, the support of the weight function is $\mathcal{C} = \widetilde{\Gamma}_0\cup\widetilde{\Gamma}_1\cup\widetilde{\Gamma}_2$ where $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ are the two straight lines starting at some point $b\omega$ and $b\omega^2$ (respectively) and ending at the origin, while $\widetilde{\Gamma}_0$ corresponds to the straight line on the positive real axis starting at the origin and ending at b. Here b is a positive real number or can represent a point at infinity. Thus, from (3.37), the weight function $\mathcal{W}_0$ we seek must be such that

$$\int_{\mathcal{C}} \mathcal{W}_0(x)\left(\phi(x)f''(x) - \frac{2(\vartheta_2+\vartheta_1-2)}{\gamma_1}x^2f'(x) + \frac{2(\vartheta_1-2)}{\gamma_1}xf(x)\right)\mathrm{d}x = 0,\ \forall f\in\mathcal{P}.$$

We use (complex) integration by parts to deduce

$$\begin{aligned}&\int_{\mathcal{C}}\left((\phi(x)\mathcal{W}_0(x))'' + \left(\frac{2(\vartheta_2+\vartheta_1-2)}{\gamma_1}x^2\mathcal{W}_0(x)\right)' + \frac{2(\vartheta_1-2)}{\gamma_1}x\mathcal{W}_0(x)\right)f(x)\,\mathrm{d}x\\ &+\sum_{j\in\{1,2\}}\left(f'(x)\phi(x)\mathcal{W}_0(x) + f(x)\left(\frac{2(\vartheta_2+\vartheta_1-2)}{\gamma_1}x^2\mathcal{W}_0(x) - (\phi(x)\mathcal{W}_0(x))'\right)\right)\Bigg|_{\omega^j b}^{b}\\ &=0,\end{aligned}$$

which must hold for every polynomial f. This condition is fulfilled if each of the following conditions hold:

(a) $\mathcal{W}_0$ is a solution of the second order differential equation

$$(\phi(x)\mathcal{W}_0(x))'' + \left(\frac{2(\vartheta_2+\vartheta_1-2)}{\gamma_1}x^2\mathcal{W}_0(x)\right)' + \frac{2(\vartheta_1-2)}{\gamma_1}x\mathcal{W}_0(x) = \lambda_0 g_0(x), \tag{3.39}$$

for some constant λ_0 (possibly zero) and a function g_0 representing the null linear functional on the vector space of polynomials supported on $\mathcal{C}$, that is

$$\int_{\mathcal{C}} g_0(x)f(x)\,\mathrm{d}x = 0, \qquad \text{for every polynomial } \ f.$$

(b) The solution $\mathcal{W}_0$ of (3.39) satisfies the boundary conditions

$$\left(f'(x)\phi(x)\mathcal{W}_0(x) + f(x)\left(\frac{2(\vartheta_2+\vartheta_1-2)}{\gamma_1}x^2\mathcal{W}_0(x) - (\phi(x)\mathcal{W}_0(x))'\right)\right)\Bigg|_{\omega^j b}^{b} = 0, \tag{3.40}$$

with $j=1,2$, for every polynomial f.

(c) All the moments of $\mathcal{W}_0$ coincide with those of u_0, that is,

$$(u_0)_n = \int_S \mathcal{W}_0(x)x^n\,\mathrm{d}x, \qquad n \geqslant 0.$$

This corresponds essentially to (3.38), insofar as $\{x^n\}_{n\geqslant 0}$ forms a basis for $\mathcal{P}$. In particular, and in the light of Proposition 2.1, the threefold symmetry of the linear functional u_0 means that $(u_0)_n = 0$ if $n \neq 0 \bmod 3$.

Observe that if y is a solution of (3.39), then so are the functions $\omega^j\, y(\omega^j x)$ for $j=1,2$. Taking into consideration the threefold symmetry of u_0, it follows that

$$\mathcal{W}_0(x) = \begin{cases} \frac{1}{3}\mathcal{U}_0(x) & \text{if } x \in \widetilde{\Gamma}_0 = [0,b), \\ -\frac{1}{3}\omega^2\mathcal{U}_0(\omega^2 x) & \text{if } x \in \widetilde{\Gamma}_1, \\ -\frac{1}{3}\omega\mathcal{U}_0(\omega x) & \text{if } x \in \widetilde{\Gamma}_2, \\ 0 & \text{if } x \notin \mathcal{C} = \widetilde{\Gamma}_0 \cup \widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_2. \end{cases} \tag{3.41}$$

where $\mathcal{U}_0 : [0,b) \to \mathbb{R}$ is an at least twice differentiable function that is a solution of the differential equation (3.39) satisfying the conditions (3.36), so that (3.40) holds. Here, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ are the two straight lines starting at some point $b\omega$ and $b\omega^2$ (respectively), with $\omega = \mathrm{e}^{2\pi i/3}$, and ending at the origin, while $\widetilde{\Gamma}_0$ corresponds to the straight line on the positive real axis starting at the origin and ending at b. Hence (3.33) is proved for $k=0$.

Similarly, an integral representation for the linear functional u_1 can be obtained from (3.4) by seeking a differentiable function $\mathcal{W}_1$ defined on $\mathcal{C}$ such that

$$\langle u_1, f(x)\rangle = \int_{\mathcal{C}} f(x)\mathcal{W}_1(x)\,\mathrm{d}x, \ \text{for every polynomial } f. \tag{3.42}$$

If $\vartheta_1 \neq 2$, then u_1 satisfies (3.4a) and this gives

$$\langle u_1, f\rangle = (\vartheta_1-2)^{-1}(2\vartheta_2-1)^{-1}\langle\phi(x)u_0' - \frac{2(\vartheta_1-1)(2\vartheta_2-3)}{\gamma_1}x^2u_0, f\rangle.$$

Based on the properties (1.1), the latter becomes

$$\langle u_1, f\rangle = (\vartheta_1-2)^{-1}(2\vartheta_2-1)^{-1}\langle u_0, -(\phi f)' - \frac{2(\vartheta_1-1)(2\vartheta_2-3)}{\gamma_1}x^2 f\rangle. \tag{3.43}$$

Taking into consideration (3.38), we then have

$$\begin{aligned}\langle u_1, f\rangle &= -(\vartheta_1-2)^{-1}(2\vartheta_2-1)^{-1}\\ &\times \int_{\mathcal{C}} \mathcal{W}_0(x)\left((\phi(x)f(x))' - \frac{2(\vartheta_1-1)(2\vartheta_2-3)}{\gamma_1}x^2 f(x)\right)\mathrm{d}x.\end{aligned}$$

We perform integration by parts on the first term of the integral, to obtain

$$\begin{aligned}&(\vartheta_1-2)(2\vartheta_2-1)\langle u_1, f\rangle\\ &= \sum_{j\in 1,2}\left(\phi(x)\mathcal{W}_0(x)f(x)\Big|_{b\omega^j}^{b}\right)\\ &+ \int_{\mathcal{C}} f(x)\left(\phi(x)\mathcal{W}_0'(x) - \frac{2(\vartheta_1-1)(2\vartheta_2-3)}{\gamma_1}x^2\mathcal{W}_0(x)\right)\mathrm{d}x.\end{aligned}$$

The vanishing conditions (3.36) at the end points of the contour readily imply that the integrated term vanishes identically for every polynomial f considered. As a consequence, a weight function associated with u_1 corresponds to

$$\begin{aligned}\mathcal{W}_1(x) = (\vartheta_1-2)^{-1}(2\vartheta_2-1)^{-1}\left(\phi(x)\mathcal{U}'(x) - \frac{2(\vartheta_1-1)(2\vartheta_2-3)}{\gamma_1}x^2\mathcal{W}_0(x)\right)\\ + \lambda_1 g_1(x),\end{aligned}$$

where λ_1 represents a complex constant (possibly zero) and g_1 a function representing the null linear functional. By construction, this choice for the weight function guarantees that (3.42) is well defined and, in particular, all the moments satisfy the threefold symmetry property described in Proposition 2.1. So, we conclude that (3.33) also holds for $k = 1$ with $\mathcal{U}_1$ given by (3.35a) when $\vartheta_1 \neq 2$.

If $\vartheta_1 = 2$, then using an entirely analogous approach we deduce (3.35b) from (3.4b). □

Remark 3.2. An alternative to the representation (3.33) stated in Theorem 3.3 is

$$\langle u_k, f\rangle = \int_{\mathcal{C}} f(x)\mathcal{W}_k(x)\,\mathrm{d}x, \qquad (k=0,1),$$

where $\mathcal{W}_k(x)$ is given in (3.41).

The weight functions $\mathcal{W}_0$ and $\mathcal{W}_1$ are *threefold symmetric* in the sense that they satisfy the following rotational invariant property:

$$\omega^j\mathcal{W}_k(\omega^j x) = \mathcal{W}_k(x), \qquad j = 0, \pm1, \pm2, \ldots, \text{ for each } k = 0, 1.$$

We consider the following *equivalence relation* between two polynomial sequences $\{P_n\}_{n\geqslant 0}$ and $\{B_n\}_{n\geqslant 0}$:

$$\{P_n\}_{n\geqslant0} \sim \{B_n\}_{n\geqslant0} \text{ iff } \exists\, a \in \mathbb{C}\backslash\{0\},\, b\in\mathbb{C} \text{ such that } B_n(x) = a^{-n}P_n(ax+b), \tag{3.44}$$

for all $n \geqslant 0$. In this case, if $\{P_n\}_{n\geqslant 0}$ satisfies (2.5) then $\{B_n\}_{n\geqslant 0}$ satisfies

$$B_{n+1}(x) = \left(x - \frac{\beta_n - b}{a}\right) B_n(x) - \frac{\alpha_n}{a^2} B_{n-1}(x) - \frac{\gamma_{n-1}}{a^3} B_{n-2}(x), \qquad n \geqslant 1,$$

with initial conditions $B_{-2}(x) = B_{-1}(x) = 0$ and $B_0(x) = 1$.

Observe that ϑ_n is a solution of a Riccati equation (3.20) for which $\vartheta_n = 1$ is a trivial solution. Depending on the initial conditions ϑ_1 and ϑ_2, there are four sets of independent solutions which in fact give rise to four equivalence classes of the threefold symmetric 2-Hahn-classical polynomials. In other words, up to a linear transformation of the variable, there are at most four distinct families of threefold symmetric 2-orthogonal Hahn-classical polynomials, which we single out:

Case A: $\vartheta_1 = 1 = \vartheta_2$. This implies that $\vartheta_n = 1$ for all $n \geqslant 0$.
Case B$_1$: $\vartheta_1 \neq 1$ but $\vartheta_2 = 1$ so that by setting $\vartheta_1 = \frac{\mu+2}{\mu+1}$ it follows that

$$\vartheta_{2n-1} = \frac{n+\mu+1}{n+\mu} \quad \text{and} \quad \vartheta_{2n} = 1 \ , \qquad n \geqslant 1.$$

Case B$_2$: $\vartheta_1 = 1$ but $\vartheta_2 \neq 1$ so that by setting $\vartheta_2 = \frac{\rho+2}{\rho+1}$ it follows that

$$\vartheta_{2n-1} = 1 \quad \text{and} \quad \vartheta_{2n} = \frac{n+\rho+1}{n+\rho} \ , \qquad n \geqslant 1.$$

Case C: $\vartheta_1 \neq 1$ and $\vartheta_2 \neq 1$ and hence by setting $\vartheta_1 = \frac{\mu+2}{\mu+1}$ and $\vartheta_2 = \frac{\rho+2}{\rho+1}$ it follows that

$$\vartheta_{2n-1} = \frac{n+\mu+1}{n+\mu} \quad \text{and} \quad \vartheta_{2n} = \frac{n+\rho+1}{n+\rho} \ , \qquad n \geqslant 1.$$

All these cases were highlighted in [14], where expressions for the recurrence coefficients for the three components of the cubic decomposition of each of these threefold symmetric Hahn-classical polynomials were deduced. Only some of these polynomials were studied in detail: those in case A in [13] and a few subcases of Case C in [16,17]. The representations for the orthogonality measures that we describe in the next sections are new. The support of these orthogonality measures contain all the zeros of the polynomial sequences. Even for particular cases that already appeared in the literature, the integral representations for the orthogonality measures were either not given or given on the positive real line (with oscillating terms), which is only part of the starlike set S. In the next subsections we fully describe all these cases in detail.

As observed in [14], the following limiting relations take place

$$\text{case C} \xrightarrow[\rho\to\infty]{} \mathrm{B}_1 \xrightarrow[\mu\to\infty]{} \text{case A} \ ,$$

and also

$$\text{case C} \xrightarrow[\mu\to\infty]{} \mathrm{B}_2 \xrightarrow[\rho\to\infty]{} \text{case A} \ .$$

It turns out that cases B_1 and B_2 are related to each other by differentiation, as explained in Section 3.2 and Section 3.3.

3.1. *Case A*

In the light of Theorem 3.2, for this choice of initial conditions one has $\vartheta_n = 1$ for all integers $n \geqslant 1$, so that the γ-recurrence coefficients are given by $\gamma_{n+1} = (n+1)(n+2)\frac{\gamma_1}{2}$ and $\widetilde{\gamma}_{n+1} = \gamma_{n+1}$ for $n \geqslant 0$. As a consequence, $Q_n(x) = P_n(x)$ for all $n \geqslant 0$ and, for this reason, the 2-orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ is an *Appell sequence*.

Since the 2-orthogonality property is invariant under any linear transformation, we can set $\gamma_1 = 2$, and, with this choice, recalling (3.17)-(3.19) it follows that

$$\gamma_{n+1} = \widetilde{\gamma}_{n+1} = (n+2)(n+1), \qquad n \geqslant 0,$$

so that

$$\begin{aligned} &P_{n+1}(x) = xP_n(x) - n(n-1)P_{n-2}(x), \qquad n \geqslant 2, \\ &P_0(x) = 1,\ P_1(x) = x \ \text{ and } \ P_2(x) = x^2. \end{aligned} \tag{3.45}$$

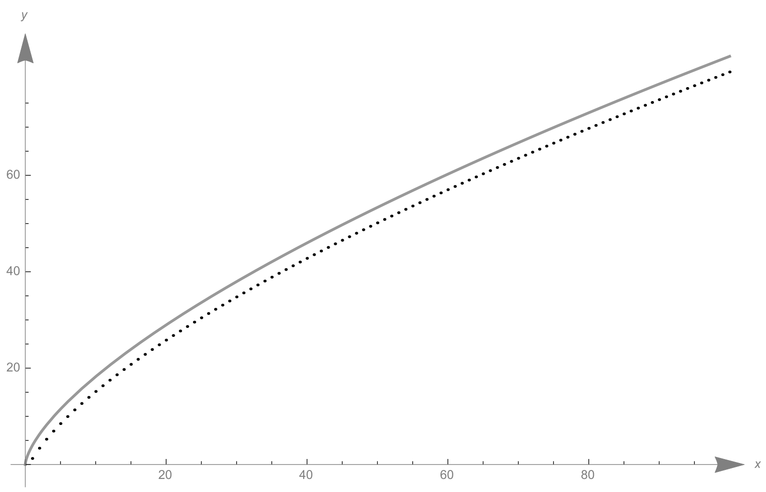

Fig. 3. Case A: Plot of the largest zero in absolute value of P_{3n} against the curve $y = \frac{3^{5/3}}{2^{2/3}} x^{2/3}$.

Figure 3.1 illustrates the behavior of the largest zero in absolute value (plot generated in *Mathematica*): bounded from above by the curve $y = \frac{3}{2^{2/3}} n^{2/3}$ suggested by Theorem 2.2.

Regarding the integral representation of the corresponding orthogonality measures, we have:

Proposition 3.2. *The threefold symmetric polynomial sequence* $\{P_n\}_{n\geqslant 0}$ *defined by the recurrence relation* (3.45) *is 2-orthogonal with respect to* (u_0, u_1) *admitting the integral representation* (3.33)*, where*

$$\mathcal{U}_0(x) = \mathrm{Ai}(x) \ \ \textit{and} \ \ \mathcal{U}_1(x) = -\mathrm{Ai}'(x),$$

and $b = +\infty$. Moreover, the sequence $\{Q_n(x) \coloneqq \frac{1}{n+1}P'_{n+1}(x)\}_{n\geqslant 0}$ coincides with $\{P_n\}_{n\geqslant 0}$ (i.e., $\{P_n\}_{n\geqslant 0}$ is an Appell sequence).

Here and in what follows Ai and Bi are the *Airy functions* of the first and second kind (see [34, §9]), respectively.

Proof. Under the assumptions, by virtue of Theorem 3.2 and Theorem 3.1, this polynomial sequence is 2-orthogonal with respect to (u_0, u_1) satisfying (3.3)-(3.4) which reads as

$$\begin{cases} u_0'' - x\; u_0 = 0, \\ u_1 = -u_0', \end{cases}$$

from which we conclude that the corresponding sequence of the moments $\{(u_0)_n\}_{n\geqslant 0}$ and $\{(u_1)_n\}_{n\geqslant 0}$ satisfy

$$(u_0)_{n+3} = (n+1)(n+2)(u_0)_n \quad \text{and} \quad (u_1)_n = n(u_0)_{n-1},$$

with initial conditions $(u_0)_0 = 1$ and $(u_0)_1 = (u_0)_2 = (u_1)_0 = 0$. This implies

$$\begin{aligned} (u_0)_{3n} &= \tfrac{(3n)!}{3^n (n!)} \quad \text{and} \quad (u_0)_{3n+1} = (u_0)_{3n+2} = 0, \\ (u_1)_{3n+1} &= \tfrac{(3n+1)!}{3^n (n!)} \quad \text{and} \quad (u_1)_{3n} = (u_1)_{3n+2} = 0, \qquad n \geqslant 0. \end{aligned} \tag{3.46}$$

According to Theorem 3.3, u_0 and u_1 admit the representation (3.33) provided that there exist two twice differentiable functions $\mathcal{U}_0$ and $\mathcal{U}_1$ from $\mathbb{R}$ to $\mathbb{R}$ that are solutions to

$$\begin{cases} \mathcal{U}_0''(x) - x\mathcal{U}_0(x) = \lambda_0 g_0(x), \\ -\mathcal{U}_1(x) = \mathcal{U}_0'(x) + \lambda_1 g_1(x)\ , \end{cases} \tag{3.47}$$

and satisfying

$$\lim_{x\to\omega^j\infty} f(x)\frac{\mathrm{d}^l}{\mathrm{d}x^l}\mathcal{U}_0(x) = 0, \quad \text{for } j,l \in \{0,1,2\} \;\text{ and }\; f \in \mathcal{P}, \tag{3.48}$$

$$\int_0^{\infty} \mathcal{U}_0(x)\,\mathrm{d}x = 1,$$

where $\lambda_k \in \mathbb{C}$ (possibly zero) and $g_k(x) \neq 0$ are rapidly decreasing functions, locally integrable, representing the null functional ($k = 0, 1$). For $\lambda_0 = 0$, the general solution of the first equation in (3.47) can be written as

$$y(x) = c_1 \mathrm{Ai}(x) + c_2 \mathrm{Bi}(x),$$

for arbitrary constants c_1, c_2. Observe that (3.48) is realized if we take $c_2 = 0$ (see [34, (9.7.5)-(9.7.8)]). Since [34, (9.10.17)]

$$\int_0^{+\infty} x^n \mathrm{Ai}(x)\,\mathrm{d}x = \frac{\Gamma(n+1)}{3^{\frac{n}{3}+1}\Gamma(\frac{n}{3}+1)} < +\infty, \qquad \text{for all } n \geqslant 0,$$

the result now follows if we take $c_1 = 1$ and $\lambda_1 = 0$. □

3.1.1. *The differential equation*

Under these assumptions (3.30) becomes

$$-P_n'''(x) + xP_n'(x) = nP_n(x), \qquad n \geqslant 0,$$

whose general solution is given by

$$P_n(x) = c_0\ {}_1F_2\left(\begin{matrix} -\frac{n}{3} \\ \frac{1}{3}, \frac{2}{3} \end{matrix}; \frac{x^3}{9}\right) + c_1\ x\ {}_1F_2\left(\begin{matrix} -\frac{n-1}{3} \\ \frac{2}{3}, \frac{4}{3} \end{matrix}; \frac{x^3}{9}\right) + c_2\ x^2\ {}_1F_2\left(\begin{matrix} -\frac{n-2}{3} \\ \frac{4}{3}, \frac{5}{3} \end{matrix}; \frac{x^3}{9}\right),$$

with integration constants $c_0,\ c_1$ and c_2. Observe that for each n there is one polynomial solution and we have

$$\begin{aligned} P_{3n}(x) &= P_n^{[0]}(x^3) \quad \text{with} \quad P_n^{[0]}(x) = (-9)^n (1/3)_n (2/3)_n\ {}_1F_2\left(\begin{matrix} -n \\ \frac{1}{3}, \frac{2}{3} \end{matrix}; \frac{x}{9}\right), \\ P_{3n+1}(x) &= x\ P_n^{[1]}(x^3) \ \text{with} \quad P_n^{[1]}(x) = (-9)^n (2/3)_n (4/3)_n\ {}_1F_2\left(\begin{matrix} -n \\ \frac{2}{3}, \frac{4}{3} \end{matrix}; \frac{x}{9}\right), \\ P_{3n+2}(x) &= x^2\ P_n^{[2]}(x^3) \ \text{with} \quad P_n^{[2]}(x) = (-9)^n (4/3)_n (5/3)_n\ {}_1F_2\left(\begin{matrix} -n \\ \frac{4}{3}, \frac{5}{3} \end{matrix}; \frac{x}{9}\right). \end{aligned}$$

3.1.2. *The cubic decomposition*

In [7, §5] the cubic decomposition of an Appell 2-orthogonal sequence has been highlighted. This also happens to be threefold symmetric, and the corresponding polynomials were called Hermite-type 2-orthogonal polynomials. The three 2-orthogonal polynomial sequences $\{P_n^{[j]}\}_{n\geqslant 0}$ $(j = 0, 1, 2)$ in the cubic decomposition of $\{P_n\}_{n\geqslant 0}$ are 2-orthogonal with respect to weights involving modified Bessel functions of the second kind, studied in [7,38].

Following Lemma 2.1, each of the three polynomial sequences $\{P_n^{[j]}\}_{n\geqslant 0}$, with $j \in \{0, 1, 2\}$, is 2-orthogonal and satisfies the third order recurrence relation (2.9), where

$$\begin{aligned} \beta_n^{[j]} &= 3\left(j^2 + 6jn + j + 9n^2\right) + 9n + 2, \qquad n \geqslant 0, \\ \alpha_n^{[j]} &= 3(j + 3n - 2)(j + 3n - 1)^2 (j + 3n), \qquad n \geqslant 1, \\ \gamma_n^{[j]} &= (j + 3n - 2)(j + 3n - 1)(j + 3n)(j + 3n + 1)(j + 3n + 2)(j + 3n + 3), \qquad n \geqslant 2. \end{aligned}$$

It should be noted that the Appell polynomials $\{P_n\}_{n\geqslant 0}$ and the components in the cubic decomposition were treated in [7] under a different normalisation, namely by considering $2/\gamma_1 = 9$. However the integral representation provided in [7] is supported on the positive real axis, and therefore different from the one given here. In [25, Cor. 5,7] the authors have also studied these polynomials, where the focus was put on algebraic properties, including generating functions.

Remark 3.3. This Appell sequence $\{P_n\}_{n\geqslant 0}$ has already appeared in the literature in other contexts, often not recognized as 2-orthogonal polynomial sequences. For instance, it is linked to the Vorob'ev-Yablonski polynomials associated with rational solutions of the second Painlevé equations [10]. Furthermore, in [40] Widder studied the so-called Airy transform defined as follows

$$u(x,t) = \int_{-\infty}^{+\infty} f(y) \frac{1}{(3t)^{1/3}} \mathrm{Ai}\left(\frac{y-x}{3t^{1/3}}\right) \mathrm{d}y.$$

Up to a scalling, it maps the sequence of monomials to the Appell 2-orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ [40, §8]. For this reason, in [36, §4.2.3] (and also [4]) they have been referred to as "Airy polynomials".

Figure 3.1.2 corroborates the statement in Proposition 2.2: the zeros on the positive real axis of three consecutive polynomials interlace and all the other non-zero zeros are rotations of $2\pi/3$ of them.

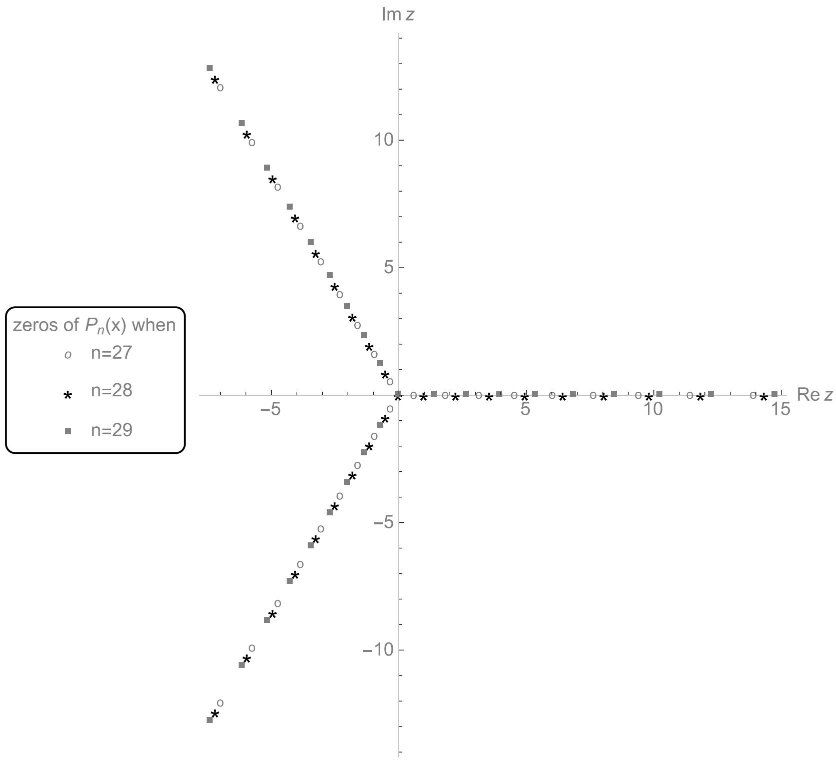

Fig. 4. Case A: Zeros of P_{27} (empty circles), P_{28} (stars) and P_{29} (solid squares).

3.2. *Case B_1*

In this case, $\vartheta_{2n} = 1$ and $\vartheta_{2n-1} = \frac{n+\mu+1}{n+\mu}$, under the constraint that $\mu \neq -n$, for all $n \geqslant 1$. Therefore, the threefold symmetric sequence $\{P_n(\cdot;\mu;\gamma_1)\}_{n\geqslant 0}$ satisfies the recurrence relation (2.8), where

$$\gamma_{2n+1} = \frac{(n+1)(2n+1)(\mu+2)}{(3n+\mu+2)}\gamma_1,$$
$$\gamma_{2n+2} = \frac{(n+1)(2n+3)(n+\mu+1)(\mu+2)}{(3n+\mu+2)(3n+\mu+5)}\gamma_1, \quad n \geqslant 0.$$

Observe that γ_1 is a mere scaling factor and, from this point forth, we set $\gamma_1 = \frac{2}{3(\mu+2)}$. The discussion is therefore about on $\{P_n(\cdot;\mu) := P_n(\cdot;\mu, \frac{2}{3(\mu+2)})\}_{n\geqslant 0}$. The general case $\{P_n(\cdot;\mu;\gamma_1)\}_{n\geqslant 0}$ can be deduced after a linear transformation of the variable:

$$P_n(x;\mu;\gamma_1) = a^{-n} P_n\left(ax;\mu\right), \quad \text{with } a = \left(\tfrac{2}{3(\mu+2)\gamma_1}\right)^{1/3}, \qquad n \geqslant 0.$$

The two sequences are equivalent, under (3.44).

Hence, $\{P_n(\cdot;\mu)\}_{n\geqslant 0}$ satisfies (2.8) with $\gamma_{n+1} := \gamma_{n+1}(\mu)$ given by

$$\gamma_{2n+1} = \frac{2}{3}\frac{(n+1)(2n+1)}{(3n+\mu+2)}, \quad \gamma_{2n+2} = \frac{2}{3}\frac{(n+1)(2n+3)(n+\mu+1)}{(3n+\mu+2)(3n+\mu+5)}, \qquad n \geqslant 0, \tag{3.49}$$

and it is 2-orthogonal for (u_0, u_1) for which

$$\frac{1}{3}u_0'' + x^2 u_0' - (\mu-2)x u_0 = 0 \quad \text{and} \quad \begin{cases} u_1 = -\frac{(\mu+2)}{\mu}\left(u_0' + 3x^2 u_0\right) & \text{if } \mu \neq 0, \\ x u_1' = 2u_0' & \text{if } \mu = 0. \end{cases} \tag{3.50}$$

Consequently,

$$(u_0)_{3n} = \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{\left(\frac{\mu+2}{3}\right)_n} \quad \text{and} \quad (u_0)_{3n+1} = (u_0)_{3n+2} = 0, \tag{3.51}$$

and

$$(u_1)_{3n+1} = \frac{\left(\frac{2}{3}\right)_n\left(\frac{4}{3}\right)_n}{\left(\frac{\mu+5}{3}\right)_n} \quad \text{and} \quad (u_1)_{3n} = (u_1)_{3n+2} = 0. \tag{3.52}$$

Observe that in order to have $\gamma_{n+1} > 0$ for all $n \geqslant 0$, the constraint $\mu > -1$ needs to be imposed.

Particular choices of the parameter μ make the γ-coefficients linear functions in n, namely:

$$\text{for } \mu = -1/2: \quad \gamma_{2n}(-1/2) = \frac{4n}{27}, \quad \gamma_{2n+1}(-1/2) = \frac{4(n+1)}{9},$$

whilst

$$\text{for } \mu = 1: \quad \gamma_{2n}(1) = \frac{2(2n+1)}{27}, \quad \gamma_{2n+1}(1) = \frac{2(2n+1)}{9}.$$

For each $\mu > -1$, we have

$$\gamma_{2n+1}(\mu) = \frac{4n}{9} + \frac{2(5-2\mu)}{27}\mathcal{O}(1) \quad \text{whilst} \quad \gamma_{2n+2}(\mu) = \frac{4n}{27} + \frac{2(7+2\mu)}{81}\mathcal{O}(1) \qquad n \to +\infty.$$

In the light of Theorem 2.2, an upper bound for the absolute value of the largest zero $|x_{n,n}|$ of P_n is given by (2.10) with $c = \frac{4}{9}$ and $\alpha = 1$ so that we obtain

$$|x_{n,n}| \leqslant 3^{1/3} n^{1/3} + o(n^{1/3}), \qquad n \geqslant 1.$$

The accuracy of the result is illustrated in Figure 5, where we have only plotted the positive zeros of $P_{3n}(x)$, but similar results hold for the zeros of $P_{3n+1}(x)$ and $P_{3n+2}(x)$, which are just rotations of the positive zeros.

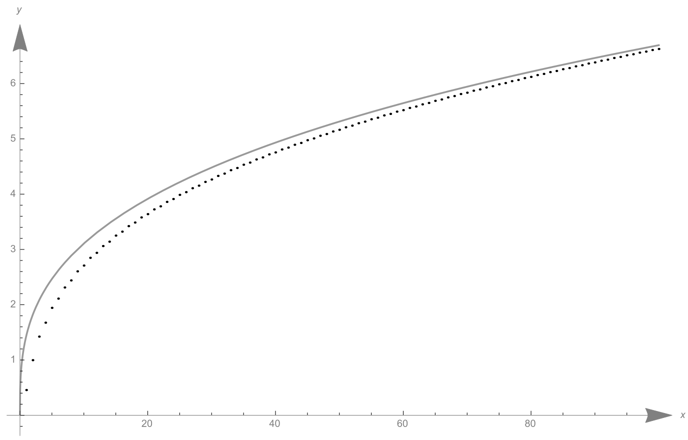

Fig. 5. Plot of of the curve $y = 3^{1/3}x^{1/3}$ (grey solid line) and the largest zeros in absolute value (black dots) of $P_{3n}(x; 3)$ with n ranging from 0 to 100.

The zeros of three consecutive polynomials interlace and lie on the three-starlike set S : Figure 6 illustrates this.

In the light of Theorem 2.1, the two orthogonality weights can be expressed via the *confluent hypergeometric function of the second kind* $\mathbf{U}(a, b; x)$, which admits the following integral representation [34, (13.4.4)]

$$\mathbf{U}(a, b; x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(t+1)^{-a+b-1} \mathrm{e}^{-tx}\, \mathrm{d}t,$$

provided that $\Re(a) > 0$ and $|\arg(x)| < \pi/2$, whilst

$$\mathbf{U}(0, b; x) = 1,$$

and one has the identity $\mathbf{U}(a, b; x) = x^{1-b}\mathbf{U}(a - b + 1, 2 - b; x)$.

Proposition 3.3. *Let* $\mu > -1$. *The threefold symmetric polynomial sequence* $\{P_n(\cdot : \mu)\}_{n \geqslant 0}$ *defined by the recurrence relation* (2.8), *with* γ_n *given by* (3.49), *is 2-orthogonal with respect to* (u_0, u_1) *admitting the integral representation* (3.33)

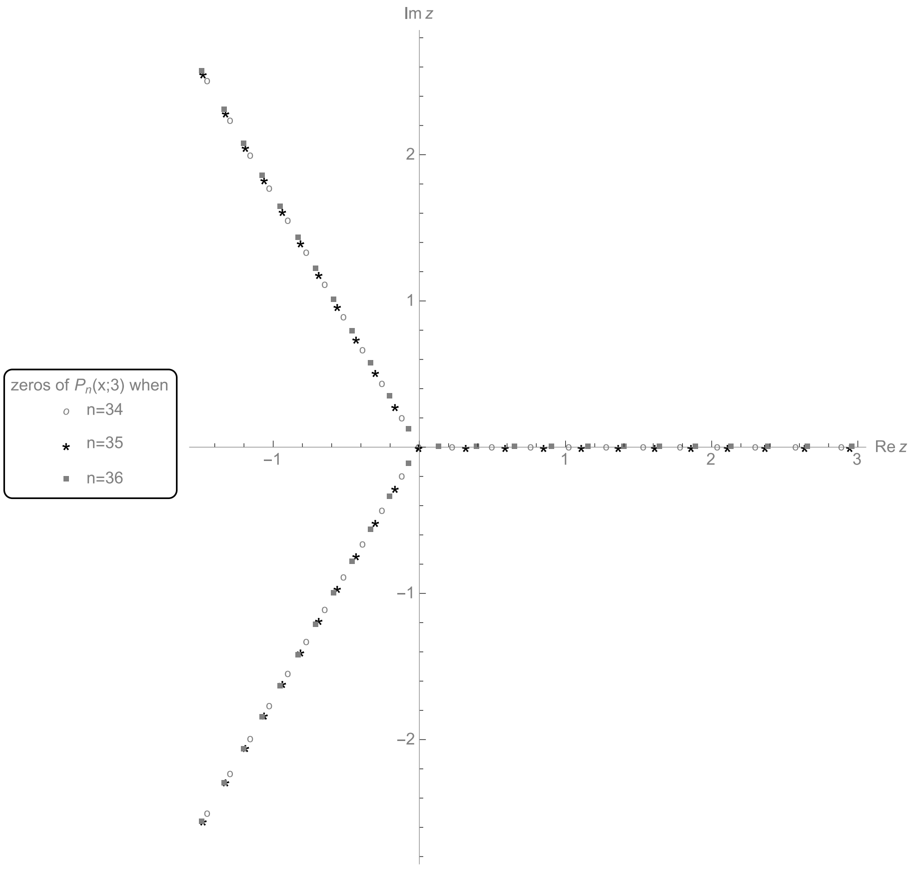

Fig. 6. Zeros of $P_{34}(x;\mu)$ (circle), $P_{35}(x;\mu)$ (star) and $P_{36}(x;\mu)$ (square) with $\mu = 3$, where $P_n(x;\mu)$ is the 2-orthogonal polynomial sequence studied in case B1.

where

$$\mathcal{U}_0(x) \coloneqq \mathcal{U}_0(x;\mu) \ = \ \frac{3\Gamma(\frac{\mu+2}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} e^{-x^3} \mathbf{U}(\tfrac{\mu}{3}, \tfrac{2}{3}; x^3), \tag{3.53}$$

$$\mathcal{U}_1(x) \coloneqq \mathcal{U}_1(x;\mu) \ = \ \frac{9\Gamma\left(\frac{\mu+5}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} x^2 \ e^{-x^3} \ \mathbf{U}\left(\tfrac{\mu}{3}+1, \tfrac{5}{3}, x^3\right), \tag{3.54}$$

and $b = +\infty$*, where* $\mathbf{U}$ *represents the Kummer (confluent hypergeometric) function of second kind.*

Proof. Under the assumptions, it follows that u_0 and u_1 satisfy (3.50) and $\{P_n(\cdot : \mu)\}_{n\geqslant 0}$ satisfies the recurrence relation (2.8) with $\gamma_n > 0$ for all $n \geqslant 1$. In the light of Theorem 2.1, there exists a function $\mathcal{W}_0$ and a domain $\mathcal{C}$ containing all the zeros of $\{P_n\}_{n\geqslant 0}$ so that

$$\langle u_0, f(x)\rangle = \int_{\mathcal{C}} f(x)\mathcal{W}_0(x)\,\mathrm{d}x,$$

is valid for every polynomial f. By virtue of Theorem 2.2 and the asymptotic behavior of γ_n for large n, the curve that contains all the zeros of $\{P_n\}_{n\geqslant 0}$ corresponds to the starlike set S in Fig. 2. According to Theorem 3.3, u_0 and u_1 admit the representation (3.33), provided that there exist two twice differentiable functions $\mathcal{U}_0$ and $\mathcal{U}_1$ from $\mathbb{R}$ to $\mathbb{R}$ that are solutions to

$$\begin{cases} \frac{1}{3}\mathcal{U}_0''(x) + x^2\mathcal{U}_0'(x) - (\mu-2)x\mathcal{U}_0(x) = \lambda_0 g_0(x) & \text{if} \quad \mu > -1, \\ \mathcal{U}_1(x) = \frac{2}{3}\mathcal{U}_0'(x) + 2x^2\mathcal{U}_0(x) + \lambda_1 g_1(x) & \text{if} \quad \mu \neq 0 \text{ and } \mu > -1, \\ x\mathcal{U}_1'(x) = 2\mathcal{U}_0'(x) + \lambda_1 g_1(x) & \text{if} \quad \mu = 0, \end{cases} \tag{3.55}$$

and satisfying

$$\lim_{x\to\omega^j\infty} f(x)\frac{\mathrm{d}^l}{\mathrm{d}x^l}\mathcal{U}_0(x) = 0, \quad \text{for } j,l \in \{0,1,2\} \text{ and } f\in\mathcal{P}, \tag{3.56}$$

$$\int_0^\infty \mathcal{U}_0(x)\mathrm{d}x = 1, \tag{3.57}$$

where $\lambda_k \in \mathbb{C}$ (possibly zero) and $g_k \neq 0$ are rapidly decreasing functions, locally integrable, representing the null functional ($k = 0,1$). For $\lambda_0 = 0$, the general solution of the first equation in (3.55) can be written as

$$y(x) = c_1\ {}_1F_1\left(\begin{matrix}\frac{2-\mu}{3} \\ \frac{2}{3}\end{matrix}; -x^3\right) + c_2\ x\ {}_1F_1\left(\begin{matrix}1-\frac{\mu}{3} \\ \frac{4}{3}\end{matrix}; -x^3\right), \tag{3.58}$$

with c_1, c_2 two arbitrary constants. Based on [34, Eqs. (13.2.39) and (13.2.42)] we have

$$\mathrm{e}^{-x^3}\mathbf{U}\left(\tfrac{\mu}{3}, \tfrac{2}{3}, x^3\right) = \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{\mu+1}{3})}{}_1F_1\left(\begin{matrix}\frac{2-\mu}{3} \\ \frac{2}{3}\end{matrix}; -x^3\right) + \frac{\mu\Gamma(-\frac{1}{3})}{3\Gamma(\frac{\mu}{3}+1)}\ x\ {}_1F_1\left(\begin{matrix}1-\frac{\mu}{3} \\ \frac{4}{3}\end{matrix}; -x^3\right),$$

which is valid when b is not an integer. Observe that (see [34, (13.7.3)])

$$G(x;\mu) = \mathrm{e}^{-x^3}\mathbf{U}\left(\tfrac{\mu}{3}, \tfrac{2}{3}, x^3\right) \sim \mathrm{e}^{-x^3}x^{-\mu}\sum_{s=0}^{\infty}\frac{\left(\frac{\mu}{3}\right)_s\left(\frac{\mu+1}{3}\right)_s}{s!}(-z)^{-s},$$

and also that [34, (13.3.27)]

$$\frac{\mathrm{d}}{\mathrm{d}x}G(x;\mu) = -3x^2\mathrm{e}^{-x^3}\mathbf{U}(\tfrac{\mu}{3}, \tfrac{5}{3}, x^3),$$

so that, for both $l = 0,1$, we obtain $\lim_{x\to\infty} f(x)\dfrac{\mathrm{d}^l G(x;\mu)}{\mathrm{d}x^l} = 0$ for every polynomial f.

According to [20, Eq. (7.621.6)], the following identity

$$\int_0^{+\infty} t^{b-1}\mathbf{U}(a,c;t)\,\mathrm{e}^{-t}\,\mathrm{d}t = \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)}, \tag{3.59}$$

holds, provided that $\Re(b) > \max(0, \Re(c)-1)$. Therefore

$$\int_0^{+\infty} x^n \mathrm{e}^{-x^3} \mathbf{U}(\tfrac{\mu}{3}, \tfrac{2}{3}; x^3)\, \mathrm{d}x = \frac{1}{3}\int_0^{+\infty} t^{\frac{n+1}{3}-1}\ \mathrm{e}^{-t} \mathbf{U}(\tfrac{\mu}{3}, \tfrac{2}{3}; t)\, \mathrm{d}t = \frac{1}{3}\frac{\Gamma(\frac{n+1}{3})\Gamma(\frac{n+2}{3})}{\Gamma(\frac{n+\mu+2}{3})},$$

for $n \geqslant 0$, which, upon the substitution $n \to 3n$, becomes

$$\int_0^{+\infty} x^{3n} \mathrm{e}^{-x^3} \mathbf{U}(\tfrac{\mu}{3}, \tfrac{2}{3}; x^3)\, \mathrm{d}x = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{3\Gamma(\frac{\mu+2}{3})}\ \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{(\frac{\mu+2}{3})_n}, \qquad n \geqslant 0.$$

Consequently, the particular choices of

$$c_1 = \frac{3\Gamma(\frac{\mu+2}{3})}{\Gamma(\frac{2}{3})\Gamma(\frac{\mu+1}{3})} \quad \text{and} \quad c_2 = \frac{-3\mu\Gamma(\frac{\mu+2}{3})}{\Gamma(\frac{\mu}{3}+1)\Gamma(\frac{1}{3})}$$

in the general solution (3.58) gives the function $\mathcal{U}_0(x;\mu)$ in (3.53), which meets the requirements (3.56)-(3.57). Furthermore, we have

$$(u_0)_n = \left(\int_{\Gamma_0} \mathrm{d}x - \omega^2 \int_{\Gamma_1} \mathrm{d}x - \omega \int_{\Gamma_2} \mathrm{d}x\right) \left(x^n\ \frac{\Gamma(\frac{\mu+2}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \mathrm{e}^{-x^3} \mathbf{U}(\tfrac{\mu}{3}, \tfrac{2}{3}; x^3)\right),$$

which matches with (3.51).

Now, for the integral representation of the linear functional u_1 we consider (3.55) with $\lambda_1 = 0$ and this gives

$$\begin{cases} \mathcal{U}_1(x;\mu) = -\dfrac{(\mu+2)}{\mu}\Big(\mathcal{U}_0'(x) + 3x^2\mathcal{U}_0(x)\Big) \\ \qquad\qquad\ = \dfrac{9\Gamma\left(\frac{\mu+5}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} x^2\ e^{-x^3}\ \mathbf{U}\left(\tfrac{\mu}{3}+1, \tfrac{5}{3}, x^3\right) & \text{if} \quad \mu \neq 0, \\ \\ \mathcal{U}_1(x;\mu) = \frac{9\Gamma(\frac{5}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}\Gamma\left(\tfrac{2}{3}, x^3\right) & \text{if} \quad \mu = 0, \end{cases}$$

where, for the first line of the latter identity we have used [34, (13.3.27)] as well as the contiguous relation [34, (13.3.10)]. The second line of the latter identity corresponds to the first line after taking $\mu = 0$ because of [34, Eq. (13.6.6)], so that we obtain (3.54). Here, $\Gamma(\alpha, z)$ is the incomplete Gamma function: $\Gamma(\alpha, z) = \displaystyle\int_z^{+\infty} t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t$ provided that $\alpha > 0$. Moreover, based on (3.59), we have

$$\int_0^{+\infty} x^{3n+1}\mathcal{U}_1(x;\mu)\,\mathrm{d}x = \frac{(2/3)_n(4/3)_n}{\left(\frac{\mu+5}{3}\right)_n}, \qquad \mu > -1,$$

which coincides with the moments of u_1 given in (3.52). □

3.2.1. *Particular cases*

For the following choices of the parameter μ the expressions (3.53)-(3.54) relate to other functions. Namely we have:

- $\mu = -1/2$, then

$$\mathcal{U}_0(x;-1/2) = \frac{3}{2}\sqrt{\frac{3}{\pi}}e^{-x^3}U\left(-\frac{1}{6},\frac{2}{3},x^3\right) \quad \text{and} \quad \mathcal{U}_1(x;-1/2) = \frac{9\sqrt{3}e^{-\frac{x^3}{2}}xK_{\frac{1}{3}}\left(\frac{x^3}{2}\right)}{4\pi},$$

where K_ν is the modified Bessel function of the second kind (see [34, Ch. 10.25]).

- $\mu = 0$, then

$$\mathcal{U}_0(x;0) \;=\; \frac{3}{\Gamma(\frac{1}{3})}\mathrm{e}^{-x^3} \quad \text{and} \quad \mathcal{U}_1(x;0) \;=\; \frac{9\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\Gamma\left(\frac{2}{3};x^3\right).$$

- $\mu = 1$, then

$$\mathcal{U}_0(x;1) = \frac{3\sqrt{3}e^{-\frac{x^3}{2}}\sqrt{x}K_{\frac{1}{6}}\left(\frac{x^3}{2}\right)}{2\pi^{3/2}} \quad \text{and} \quad \mathcal{U}_1(x;1) = \frac{9\sqrt{3}e^{-x^3}U\left(\frac{2}{3},\frac{1}{3},x^3\right)}{2\pi}.$$

- $\mu = 2$, then

$$\mathcal{U}_0(x;2) = \frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)}{2\pi}\Gamma\left(\frac{1}{3},x^3\right) \quad \text{and} \quad \mathcal{U}_1(x;2) = \frac{2\sqrt{3}\Gamma\left(\frac{1}{3}\right)}{\pi}\Gamma\left(\frac{2}{3},x^3\right).$$

3.2.2. *The differential equation*

Following Lemma 3.1, the polynomial $P_n(x;\mu)$ is a solution of the differential equation

$$-\frac{2}{3}y'''(x)+2x^2y''(x)+2x\left(\mu+\frac{3}{4}\left((-1)^n+3\right)-\frac{n}{2}\right)y'(x) = 2n\left(\mu+\frac{n}{2}+\frac{3(-1)^n}{4}+\frac{5}{4}\right)y(x),$$

whose general solution can be written as

$$y(x) = c_1\,{}_2F_2\left(\begin{matrix}-\frac{n}{3},\frac{n}{6}+\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{5}{12}\\ \frac{1}{3},\frac{2}{3}\end{matrix};x^3\right) + c_2\;x\;{}_2F_2\left(\begin{matrix}\frac{1}{3}-\frac{n}{3},\frac{n}{6}+\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{3}{4}\\ \frac{2}{3},\frac{4}{3}\end{matrix};x^3\right)$$
$$+c_3\;x^2\;{}_2F_2\left(\begin{matrix}\frac{2}{3}-\frac{n}{3},\frac{n}{6}+\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{13}{12}\\ \frac{4}{3},\frac{5}{3}\end{matrix};x^3\right).$$

For each positive integer n, there is only one (monic) polynomial solution, but it is not always the same: it depends on whether n equals $0,1$ or $2 \bmod 3$. To be precise, we have

$$
\begin{aligned}
P_{3n}(x;\mu) &\coloneqq P_n^{[0]}(x^3;\mu) \\
&= \frac{(-1)^n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{\left(\frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{5}{12}\right)_n} {}_2F_2\left(\begin{matrix}-n, \frac{n}{2}+\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{5}{12} \\ \frac{1}{3}, \frac{2}{3}\end{matrix}; x^3\right), \\
P_{3n+1}(x;\mu) &\coloneqq xP_n^{[1]}(x^3;\mu) \\
&= x\, \frac{(-1)^n \left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n}{\left(\frac{n}{2}+\frac{1}{4}(-1)^{n+1}+\frac{\mu}{3}+\frac{11}{12}\right)_n} {}_2F_2\left(\begin{matrix}-n, \frac{\mu}{3}+\frac{n}{2}+\frac{1}{4}(-1)^{n+1}+\frac{11}{12} \\ \frac{2}{3}, \frac{4}{3}\end{matrix}; x^3\right), \\
P_{3n+2}(x;\mu) &\coloneqq x^2P_n^{[2]}(x^3;\mu) \\
&= x^2\, \frac{(-1)^n \left(\frac{4}{3}\right)_n \left(\frac{5}{3}\right)_n}{\left(\frac{n}{2}+\frac{1}{4}(-1)^{n+2}+\frac{\mu}{3}+\frac{17}{12}\right)_n} {}_2F_2\left(\begin{matrix}-n, \frac{\mu}{3}+\frac{n}{2}+\frac{1}{4}(-1)^{n+2}+\frac{17}{12} \\ \frac{4}{3}, \frac{5}{3}\end{matrix}; x^3\right).
\end{aligned}
$$

These polynomial sequences $\{P_n^{[k]}(\cdot;\mu)\}_{n\geqslant 0}$, with $k \in \{0,1,2\}$, are precisely the 2-orthogonal polynomial sequences in the cubic decomposition of $\{P_n(\cdot;\mu)\}_{n\geqslant 0}$. In fact, from Lemma 2.1, these three 2-orthogonal polynomial sequences are not threefold symmetric and satisfy the recurrence relation (2.9), whose recurrence coefficients are given in the Appendix for completeness. These coefficients have been computed in [14, Tableau 4, 8 and 12 - Case C], for a different choice of the "free" parameter γ_1. We have included them here for a matter of completeness.

3.2.3. *The sequence of derivatives*

Following Theorem 3.1 along with Theorem 3.2, $\{Q_n(\cdot;\mu) \coloneqq \frac{1}{n+1}P'_{n+1}(x;\mu)\}_{n\geqslant 0}$ is 2-orthogonal for the vector functional (v_0, v_1), which admits an integral representation via two weight functions $(\widetilde{\mathcal{W}}_0(\cdot;\mu), \widetilde{\mathcal{W}}_1(\cdot;\mu))$ with support on the three-starlike set S which are given by

$$
\begin{bmatrix}\widetilde{\mathcal{W}}_0(x;\mu) \\ \widetilde{\mathcal{W}}_1(x;\mu)\end{bmatrix} = \begin{bmatrix}\frac{\mu+2}{\mu+1} & -\frac{x}{\mu+1} \\ 0 & 1\end{bmatrix} \begin{bmatrix}\mathcal{W}_0(x;\mu) \\ \mathcal{W}_1(x;\mu)\end{bmatrix}.
$$

Proposition 3.3 allows us to conclude

$$
\langle v_k, f\rangle = \left(\int_{\Gamma_0} \mathrm{d}x - \omega^2 \int_{\Gamma_1} \mathrm{d}x - \omega \int_{\Gamma_2} \mathrm{d}x\right) \left(f(x)\ \mathcal{V}_k(x;\mu)\right), \qquad \forall f \in \mathcal{P},\ k = 0, 1,
$$

where

$$
\begin{aligned}
\mathcal{V}_0(x;\mu) &= \frac{\mu+2}{\mu+1}\mathcal{U}_0(x;\mu) - \frac{x}{\mu+1}\mathcal{U}_1(x;\mu) \;=\; \mathcal{U}_0(x;\mu+3), \\
\mathcal{V}_1(x;\mu) &= \mathcal{U}_1(x;\mu).
\end{aligned}
$$

Furthermore, $\{Q_n(\cdot;\mu)\}_{n\geqslant 0}$ satisfies the second order recurrence relation (3.1) where

$$\begin{aligned}\widetilde{\gamma}_{2n} &\coloneqq \widetilde{\gamma}_{2n}(\mu) = \frac{2}{3}\frac{n(2n+1)}{(3n+2+\mu)}, \qquad n\geqslant 1,\\ \widetilde{\gamma}_{2n+1} &\coloneqq \widetilde{\gamma}_{2n+1}(\mu) = \frac{2}{3}\frac{(n+1)(2n+1)(n+2+\mu)}{(3n+2+\mu)(3n+5+\mu)}, \qquad n\geqslant 0.\end{aligned} \tag{3.60}$$

For each integer $n\geqslant 0$, the polynomial $Q_n(x;\mu)$ is a solution to the differential equation

$$-\frac{2}{3}y^{(3)}(x)+2x^2y''(x)-\frac{1}{2}x\left(-4\mu+3\left((-1)^n-5\right)+2n\right)y'(x)=\frac{1}{2}n\left(4\mu+2n-3(-1)^n+11\right)y(x),$$

from which we deduce

$$\begin{aligned}Q_{3n}(x;\mu) &= \frac{(-1)^n(\frac{1}{3})_n(\frac{2}{3})_n}{(\frac{n}{2}-\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{11}{12})_n}\,{}_2F_2\left(\begin{matrix}-n,\frac{n}{2}-\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{11}{12}\\ \frac{1}{3},\frac{2}{3}\end{matrix};x^3\right),\\ Q_{3n+1}(x;\mu) &= x\,\frac{(-1)^n(\frac{2}{3})_n(\frac{4}{3})_n}{(\frac{\mu}{3}+\frac{n}{2}+\frac{(-1)^n}{4}+\frac{17}{12})_n}\,{}_2F_2\left(\begin{matrix}-n,\frac{\mu}{3}+\frac{n}{2}+\frac{(-1)^n}{4}+\frac{17}{12}\\ \frac{2}{3},\frac{4}{3}\end{matrix};x^3\right),\\ Q_{3n+2}(x;\mu) &= x^2\,\frac{(-1)^n(\frac{4}{3})_n(\frac{5}{3})_n}{(\frac{\mu}{3}+\frac{n}{2}-\frac{(-1)^n}{4}+\frac{23}{12})_n}\,{}_2F_2\left(\begin{matrix}-n,\frac{\mu}{3}+\frac{n}{2}-\frac{(-1)^n}{4}+\frac{23}{12}\\ \frac{4}{3},\frac{5}{3}\end{matrix};x^3\right), \qquad n\geqslant 0.\end{aligned}$$

It turns out that the forthcoming case B_2 is closely connected to this case B_1: the sequence of derivatives in case B_1 belongs to case B_2.

3.3. *Case* B_2

For this family $\vartheta_{2n+1}=1$ and we can write $\vartheta_{2n}=\frac{n+\rho+1}{n+\rho}$, under the relation $\vartheta_2=\frac{\rho+2}{\rho+1}$ with $\rho\neq -n$. As a result, the threefold symmetric Hahn-classical sequence $\{P_n(\cdot;\rho;\gamma_1)\}_{n\geqslant 0}$ satisfies the recurrence (2.8) with

$$\begin{aligned}\gamma_{2n} &= \frac{n(2n+1)(\rho+3)}{(3n+\rho)}\gamma_1, \qquad n\geqslant 1,\\ \gamma_{2n+1} &= \frac{(n+1)(2n+1)(n+\rho)(\rho+3)}{(3n+\rho)(3n+\rho+3)}\gamma_1, \qquad n\geqslant 0.\end{aligned}$$

Analogously to the former cases, the parameter γ_1 is redundant for the study, and therefore we can choose a representative value for γ_1. Here we set $\gamma_1=\frac{2}{3(\rho+3)}$. Hence the 2-orthogonal polynomial sequence $\{P_n(\cdot;\rho)\coloneqq P_n(\cdot;\rho;\frac{2}{3(\rho+3)})\}_{n\geqslant 0}$ satisfies (2.8) with

$$\begin{aligned}\gamma_{2n} &\coloneqq \gamma_{2n}(\rho) = \frac{2n(2n+1)}{3(3n+\rho)}, \qquad n\geqslant 1,\\ \gamma_{2n+1} &\coloneqq \gamma_{2n+1}(\rho) = \frac{2(n+1)(2n+1)(n+\rho)}{3(3n+\rho)(3n+\rho+3)}, \qquad n\geqslant 0.\end{aligned} \tag{3.61}$$

Furthermore, the 2-orthogonal polynomial sequence $\{Q_n(x;\rho) := \frac{1}{n+1}P'_{n+1}(x;\rho)\}_{n\geqslant 0}$ satisfies (3.1) with

$$\begin{aligned} \widetilde{\gamma}_{2n} &:= \widetilde{\gamma}_{2n}(\rho) = \frac{2n(2n+1)(n+\rho+1)}{3(3n+\rho)(3n+\rho+3)}, \qquad n \geqslant 1, \\ \widetilde{\gamma}_{2n+1} &:= \widetilde{\gamma}_{2n}(\rho) = \frac{2(n+1)(2n+1)}{3(3n+\rho+3)}, \qquad n \geqslant 0. \end{aligned} \tag{3.62}$$

Observe that the recurrence coefficients uniquely determine a 2-orthogonal polynomial sequence. A comparison between (3.49) and (3.62) and between (3.60) and (3.61) shows that

$$Q_n^{\text{case B}_2}(x;\mu) = P_n^{\text{case B}_1}(x;\mu+1), \qquad \text{for all } n \geqslant 0,$$

while

$$Q_n^{\text{case B}_1}(x;\mu) = P_n^{\text{case B}_2}(x;\mu+2), \qquad \text{for all } n \geqslant 0,$$

where the notation $P_n^{\text{case B}_1}$ and $P_n^{\text{case B}_2}$ is used for the threefold 2-orthogonal polynomial sequences defined by the recurrence coefficients (3.49), in case B_1, and (3.61), in case B_2, respectively. Likewise, $Q_n^{\text{case B}_1}$ and $Q_n^{\text{case B}_1}$ denote the monic derivatives of $P_n^{\text{case B}_1}$ and $P_n^{\text{case B}_2}$, respectively.

From this, we conclude that the the polynomial sequences $\{Q_n^{\text{case B}_1}(x;\mu) := \frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}P_{n+1}^{\text{case B}_1}(x;\mu)\}_{n\geqslant 0}$ and $\{Q_n^{\text{case B}_2}(x;\rho) := \frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}P_{n+1}^{\text{case B}_2}(x;\rho)\}_{n\geqslant 0}$ are also Hahn-classical. Furthermore, we have

$$\frac{1}{(n+2)(n+1)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_{n+2}(x;\mu) = P_n(x;\mu+3),$$

in both cases B_1 and B_2. These observations also serve as an alternative to the proof of the following result:

Proposition 3.4. *Let $\rho > 0$. The threefold symmetric polynomial sequence $\{P_n(\cdot : \rho)\}_{n\geqslant 0}$ defined by the recurrence relation* (2.8) *with γ_n given by* (3.61) *is 2-orthogonal with respect to (u_0, u_1) admitting the integral representation* (3.33) *where*

$$\begin{aligned} \mathcal{U}_0(x) &:= \mathcal{U}_0(x;\rho) = \frac{3\Gamma\left(\frac{\rho+3}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\mathrm{e}^{-x^3}\mathbf{U}\left(\frac{\rho+1}{3},\frac{2}{3},x^3\right), \\ \mathcal{U}_1(x) &:= \mathcal{U}_1(x;\rho) = \frac{9\Gamma\left(\frac{\rho+3}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}x^2\mathrm{e}^{-x^3}\mathbf{U}\left(\frac{\rho+1}{3},\frac{5}{3},x^3\right), \end{aligned}$$

and $b = +\infty$, where $\mathbf{U}$ represents the Kummer (confluent hypergeometric) function of second kind.

Proof. The proof is entirely analogous to the proof of Proposition 3.3. □

In fact, we have

$$\mathcal{U}_0^{\text{case B}_2}(x;\rho) = \mathcal{U}_0^{\text{case B}_1}(x;\rho+1) \quad \text{and} \quad \mathcal{U}_1^{\text{case B}_2}(x;\rho) = \mathcal{U}_1^{\text{case B}_1}(x;\rho-2),$$

provided that $\rho > 1$.

3.4. *Case C*

Here we have

$$\vartheta_{2n-1} = \frac{n+\mu+1}{n+\mu} \quad \text{and} \quad \vartheta_{2n} = \frac{n+\rho+1}{n+\rho}\ , \qquad n \geqslant 1.$$

so that (3.17) reads as

$$\gamma_{2n} = \frac{2n(2n+1)(n+\mu)}{(3n+\mu-1)(3n+\mu+2)(3n+\rho)} \frac{(\mu+2)(\rho+3)\gamma_1}{2}, \qquad n \geqslant 1,$$
$$\gamma_{2n+1} = \frac{2(n+1)(2n+1)(n+\rho)}{(3n+\mu+2)(3n+\rho)(3n+\rho+3)} \frac{(\mu+2)(\rho+3)\gamma_1}{2}, \qquad n \geqslant 0.$$

The Hahn-classical polynomial sequence obtained under these assumptions clearly depends on the pair of parameters (μ,ρ) and on γ_1, so it makes sense to incorporate this information, and therefore we shall refer to it as $\{P_n(\cdot;\mu,\rho;\gamma_1)\}_{n\geqslant 0}$. Similar to the precedent cases, without loss of generality, it suffices to study the sequence for a particular choice of γ_1. A scaling of the variable would then reproduce all the other sequences $\{P_n(\cdot;\mu,\rho;\gamma_1)\}_{n\geqslant 0}$ within this equivalence class. Hence, we set $\gamma_1 = \dfrac{2}{(\mu+2)(\rho+3)}$, and the analysis is on

$$\left\{P_n(\cdot;\mu,\rho) := P_n\left(\cdot;\mu,\rho;\tfrac{2}{(\mu+2)(\rho+3)}\right)\right\}_{n\geqslant 0},$$

which satisfies the recurrence relation (2.8) where the γ-coefficients are given by

$$\gamma_{2n} := \gamma_{2n}(\mu,\rho) = \frac{2n(2n+1)(n+\mu)}{(3n+\mu-1)(3n+\mu+2)(3n+\rho)}, \qquad n \geqslant 1,$$
$$\gamma_{2n+1} := \gamma_{2n}(\mu,\rho) = \frac{2(n+1)(2n+1)(n+\rho)}{(3n+\mu+2)(3n+\rho)(3n+\rho+3)}, \qquad n \geqslant 0. \tag{3.63}$$

Observe that $\gamma_{n+1} > 0$ for all $n \geqslant 0$ provided that $\mu+1, \rho > 0$ and that

$$\gamma_n = \frac{4}{27} + o(1), \quad \text{as} \quad n \to +\infty.$$

Theorem 2.2 ensures that the largest zero of $P_n(x;\mu,\rho)$ in absolute value is always less than or equal to 1. Figure 7 illustrates the curve defined by these zeros.

Regarding the orthogonality measures for the polynomial sequence $\{P_n(\cdot;\mu,\rho)\}_{n\geqslant 0}$ we have:

Proposition 3.5. *Let $\mu > -1$ and $\rho > 0$. The threefold symmetric polynomial sequence $\{P_n(\cdot;\mu,\rho)\}_{n\geqslant 0}$ defined by the recurrence relation (2.8) with γ_n given by (3.63) is 2-orthogonal with respect to (u_0,u_1), admitting the integral representation*

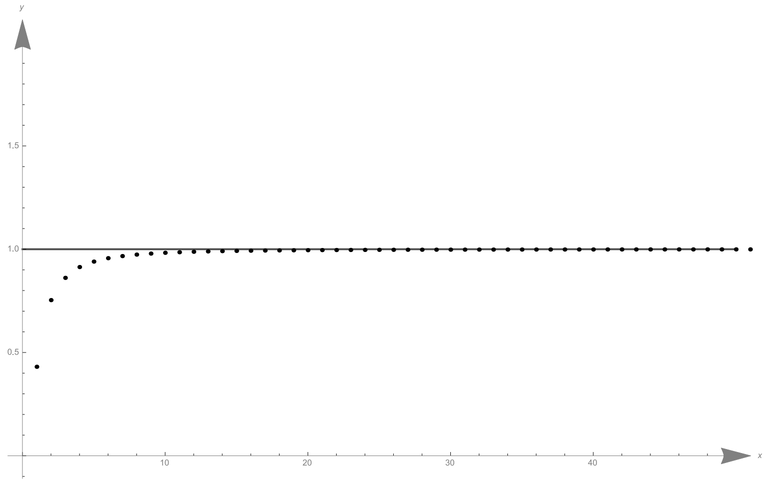

Fig. 7. Plot of the largest zero in absolute value of $P_{3n}(x;\mu,\rho)$ for each $n = 1,\ldots,100$ when $\mu = 2, \rho = 3$, against the curve $y = 1$.

(3.33) *with* $b = 1$ *and*

$$\mathcal{U}_0(x) := \mathcal{U}_0(x;\mu,\rho) = \frac{3\Gamma\left(\frac{\mu+2}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\mu+\rho+2}{3}\right)}(1-x^3)^{\frac{\mu+\rho-1}{3}}\ {}_2F_1\left(\begin{matrix}\frac{\mu}{3}, \frac{\rho+1}{3}\\ \frac{\mu+\rho+2}{3}\end{matrix}; 1-x^3\right), \quad (3.64)$$

$$\mathcal{U}_1(x) := \mathcal{U}_1(x;\mu,\rho) = \frac{3\Gamma\left(\frac{\mu+5}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{\mu+\rho+2}{3}\right)}x^2(1-x^3)^{\frac{\mu+\rho-1}{3}}\ {}_2F_1\left(\begin{matrix}\frac{\mu}{3}+1, \frac{\rho+1}{3}\\ \frac{\mu+\rho+2}{3}\end{matrix}; 1-x^3\right) \quad (3.65)$$

Proof. According to Theorem 3.1, $\{P_n(\cdot;\mu,\rho)\}_{n\geqslant 0}$ is 2-orthogonal with respect to the vector of linear functionals (u_0, u_1) fulfilling the distributional equations

$$\left(1-x^3\right)u_0'' + x^2(\mu+\rho-4)u_0' - (\mu-2)(\rho-1)xu_0 = 0,$$

$$\begin{cases} \frac{\mu}{(\mu+2)}u_1 = \left(x^3-1\right)u_0' - (\rho-1)x^2\ u_0 & \text{if} \quad \mu \neq 0, \\ xu_1' = 2u_0' & \text{if} \quad \mu = 0, \end{cases}$$

which yields

$$(u_0)_{3n} = \frac{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{\left(\frac{\mu+2}{3}\right)_n\left(\frac{\rho+3}{3}\right)_n}, \quad (u_0)_{3n+1} = (u_0)_{3n+2} = 0, \qquad n \geqslant 0,$$

as well as

$$(u_1)_{3n+1} = \frac{\left(\frac{2}{3}\right)_n\left(\frac{4}{3}\right)_n}{\left(\frac{\mu+5}{3}\right)_n\left(\frac{\rho+3}{3}\right)_n}, \quad (u_1)_{3n} = (u_1)_{3n+2} = 0, \qquad n \geqslant 0,$$

because $(u_0)_0 = (u_1)_1 = 1$ and $(u_0)_1 = (u_0)_2 = (u_1)_0 = (u_1)_2 = 0$.

Since $\lim_{n\to\infty}\gamma_n = \frac{4}{27}$, then by virtue of Theorem 3.3, u_0 and u_1 admit the integral representation (3.33) with $b = 1$, provided that there exists a pair of functions

$\mathcal{U}_0$ and $\mathcal{U}_1$ with support on $[0,1)$ satisfying (3.34)-(3.35) and subject to the boundary condition (3.36). To be precise, under the assumptions, we seek $\mathcal{U}_0(x;\mu,\rho)$ and $\mathcal{U}_1(x;\mu,\rho)$ defined on defined on $[0,1)$ such that

$$\left(1-x^3\right)\mathcal{U}_0''(x;\mu,\rho)+x^2(\mu+\rho-4)\mathcal{U}_0'(x;\mu,\rho)-(\mu-2)(\rho-1)x\mathcal{U}_0(x;\mu,\rho)=\lambda_0 g_0(x), \tag{3.66}$$

$$\begin{cases} \dfrac{\mu}{(\mu+2)}\mathcal{U}_1(x;\mu,\rho)=\left(x^3-1\right)\mathcal{U}_0'(x;\mu,\rho)-(\rho-1)x^2\mathcal{U}_0(x;\mu,\rho)+\lambda_1 g_1(x) & \text{if} \quad \mu\neq 0, \\ x\mathcal{U}_1'(x;\mu,\rho)=2\mathcal{U}_0'(x;\mu,\rho)+\lambda_1 g_1(x) & \text{if} \quad \mu=0, \end{cases} \tag{3.67}$$

where λ_j, for $j\in\{0,1\}$, are constants (possibly zero) and g_j are functions representing the null linear functional, i.e.,

$$\int_{\mathcal{C}} g_j(x)x^n\,\mathrm{d}x=0, \qquad n\geqslant 0,$$

and, in addition,

$$\lim_{x\to 1} f(x)\frac{\mathrm{d}^l}{\mathrm{d}x^l}\mathcal{U}_0(x;\mu,\rho)=0, \quad \text{for any } l\in\{0,1\} \text{ and } f\in\mathcal{P}, \tag{3.68}$$

$$\int_0^1 x^{3n}\mathcal{U}_0(x;\mu,\rho)\,\mathrm{d}x=\frac{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{\left(\frac{\mu+2}{3}\right)_n\left(\frac{\rho+3}{3}\right)_n}, \qquad n\geqslant 0. \tag{3.69}$$

For $\lambda_0=0$, the differential equation (3.66) becomes

$$\left(1-x^3\right)\mathcal{U}_0''(x;\mu,\rho)+x^2(\mu+\rho-4)\mathcal{U}_0'(x;\mu,\rho)-(\mu-2)(\rho-1)x\mathcal{U}_0(x;\mu,\rho)=0,$$

whose general solution can be written as

$$\mathcal{U}_0(x;\mu,\rho)=c_1\ {}_2F_1\left(\begin{matrix}\frac{2-\mu}{3},\frac{1-\rho}{3}\\ \frac{2}{3}\end{matrix};x^3\right)+c_2\ x\ {}_2F_1\left(\begin{matrix}1-\frac{\mu}{3},\frac{2-\rho}{3}\\ \frac{4}{3}\end{matrix};x^3\right),$$

with integration constants c_1 and c_2 which must be chosen such that (3.68)-(3.69) hold. By taking

$$c_2=-\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\mu+1}{3}\right)\Gamma\left(\frac{\rho+2}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{\mu}{3}\right)\Gamma\left(\frac{\rho+1}{3}\right)}c_1,$$

we have

$$\begin{aligned}\mathcal{U}_0(x;\mu,\rho)&=c_1\frac{\Gamma\left(\frac{\mu+1}{3}\right)\Gamma\left(\frac{\rho+2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\\ &\quad\left(\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{\mu+1}{3}\right)\Gamma\left(\frac{\rho+2}{3}\right)}\ {}_2F_1\left(\begin{matrix}\frac{2-\mu}{3},\frac{1-\rho}{3}\\ \frac{2}{3}\end{matrix};x^3\right)-\frac{\Gamma\left(\frac{2}{3}\right)\ x}{\Gamma\left(\frac{\mu}{3}\right)\Gamma\left(\frac{\rho+1}{3}\right)}\ {}_2F_1\left(\begin{matrix}1-\frac{\mu}{3},\frac{2-\rho}{3}\\ \frac{4}{3}\end{matrix};x^3\right)\right)\\ &=c_1\frac{\Gamma\left(\frac{\mu+1}{3}\right)\Gamma\left(\frac{\rho+2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\left(1-x^3\right)^{\frac{\mu+\rho-1}{3}}\ {}_2F_1\left(\begin{matrix}\frac{\mu}{3},\frac{\rho+1}{3}\\ \frac{\mu+\rho+2}{3}\end{matrix};1-x^3\right),\end{aligned}$$

where, for the latter identity, we have used [34, (15.10.18)]. In fact, for this choice of c_2, it follows that $\lim\limits_{x\to 1^-} \mathcal{U}_0(x;\mu,\rho)=0$, because

$$ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

is valid whenever $\Re(c-a-b)>0$. Moreover, condition (3.68) is fulfilled, and, in addition, we successively have

$$ \begin{aligned} \int_0^1 x^{3n}\mathcal{U}_0(x;\mu,\rho)\,\mathrm{d}x &= c_1\frac{\Gamma\left(\frac{\mu+1}{3}\right)\Gamma\left(\frac{\rho+2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\int_0^1 x^{3n}(1-x^3)^{\frac{\mu+\rho-1}{3}}\ {}_2F_1\left(\begin{matrix} \frac{\mu}{3},\frac{\rho+1}{3} \\ \frac{\mu+\rho+2}{3} \end{matrix};1-x^3\right)\mathrm{d}x \\ &= c_1\frac{\Gamma\left(\frac{\mu+1}{3}\right)\Gamma\left(\frac{\rho+2}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)}\int_0^1 (1-x)^{n-\frac{2}{3}}x^{\frac{\mu+\rho-1}{3}}\ {}_2F_1\left(\begin{matrix} \frac{\mu}{3},\frac{\rho+1}{3} \\ \frac{\mu+\rho+2}{3} \end{matrix};x\right),\mathrm{d}x \\ &= c_1\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\mu+1}{3}\right)\Gamma\left(\frac{\rho+2}{3}\right)\Gamma\left(\frac{\mu+\rho+2}{3}\right)}{3\Gamma\left(\frac{\mu+2}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}\frac{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{\left(\frac{\mu+2}{3}\right)_n\left(\frac{\rho}{3}+1\right)_n}, \end{aligned} $$

where, for the last identity, we have used [20, (7.512.4)]. If we take

$$ c_1=\frac{3\Gamma\left(\frac{\mu+2}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\mu+1}{3}\right)\Gamma\left(\frac{\rho+2}{3}\right)\Gamma\left(\frac{\mu+\rho+2}{3}\right)} $$

then (3.69) is fulfilled. As a result, the first orthogonality measure can be represented as in (3.33) (case $k=0$) where $\mathcal{U}_0$ is given by (3.64).

Now, in order to obtain the second orthogonality measure, we take $\lambda_1=0$ in (3.67), which involves a derivative of $\mathcal{U}_0(x;\mu,\rho)$, that, according to [34, (15.5.4)], corresponds to

$$ \begin{aligned} \mathcal{U}_0'(x;\mu,\rho) = -\frac{3\Gamma\left(\frac{\mu+2}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\mu+\rho+2}{3}\right)}&(\mu+\rho-1)\,x^2(1-x^3)^{\frac{\mu+\rho+2}{3}-2} \\ &\times {}_2F_1\left(\begin{matrix} \frac{\mu}{3},\frac{\rho+1}{3} \\ \frac{\mu+\rho+2}{3}-1 \end{matrix};1-x^3\right). \end{aligned} $$

For $\mu\neq 0$, it follows from (3.67) that

$$ \begin{aligned} \frac{\mu}{(\mu+2)}\mathcal{U}_1(x;\mu,\rho) =&\frac{9\Gamma\left(\frac{\mu+2}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\mu+\rho+2}{3}\right)}x^2(1-x^3)^{\frac{\mu+\rho-1}{3}} \\ &\left(\left(\frac{\mu+\rho-1}{3}\right)\ {}_2F_1\left(\begin{matrix} \frac{\mu}{3},\frac{\rho+1}{3} \\ \frac{\mu+\rho-1}{3} \end{matrix};1-x^3\right)-\left(\frac{\rho-1}{3}\right){}_2F_1\left(\begin{matrix} \frac{\mu}{3},\frac{\rho+1}{3} \\ \frac{\mu+\rho+2}{3} \end{matrix};1-x^3\right)\right) \\ =&\frac{3\mu\Gamma\left(\frac{\mu+2}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\mu+\rho+2}{3}\right)}x^2(1-x^3)^{\frac{\mu+\rho-1}{3}}{}_2F_1\left(\begin{matrix} \frac{\mu}{3}+1,\frac{\rho+1}{3} \\ \frac{\mu+\rho+2}{3} \end{matrix};1-x^3\right), \end{aligned} $$

where, for the last identity we have used [34, (15.5.15)]

$$ (c-1)\,{}_2F_1\left(\begin{matrix} a,b \\ c-1 \end{matrix};1-z\right)-(-a+c-1)\,{}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};1-z\right)=a\ {}_2F_1\left(\begin{matrix} a+1,b \\ c \end{matrix};1-z\right). $$

As a result, we obtain the expression (3.65) for $\mathcal{U}_1(x;\mu,\rho)$ and we have

$$\int_0^1 x^{3n+1}\mathcal{U}_1(x;\mu,\rho)\,\mathrm{d}x = \frac{\left(\frac{2}{3}\right)_n\left(\frac{4}{3}\right)_n}{\left(\frac{\mu+5}{3}\right)_n\left(\frac{\rho+3}{3}\right)_n} = (u_1)_{3n+1}, \qquad n\geqslant 0.$$

When $\mu=0$, then (3.67) reads as

$$\mathcal{U}_1'(x;0,\rho) = -\frac{18\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{\rho-1}{3}\right)}x\left(1-x^3\right)^{\frac{\rho-4}{3}},$$

so that

$$\mathcal{U}_1(x;0,\rho) = K - \frac{9x^2\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{\rho-1}{3}\right)}{}_2F_1\left(\begin{matrix}\frac{2}{3},\frac{4-\rho}{3}\\ \frac{5}{3}\end{matrix};x^3\right),$$

for some integration constant K. We set $K = \frac{6\Gamma\left(\frac{\rho}{3}+1\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{\rho+1}{3}\right)}$, use identity [34, (15.10.18)] to obtain

$$\mathcal{U}_1(x;0,\rho) = \frac{6\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{\rho+2}{3}\right)}x^2(1-x^3)^{\frac{\rho-1}{3}}{}_2F_1\left(\begin{matrix}1,\frac{\rho+1}{3}\\ \frac{\rho+2}{3}\end{matrix};1-x^3\right),$$

which is the same as (3.65) with $\mu=0$. □

3.4.1. *Differential equation and the cubic decomposition of* $\{P_n(x;\mu,\rho)\}_{n\geqslant 0}$

Let us assume $\mu>-1$ and $\rho>0$. For each positive integer n, the polynomial $P_n(x;\mu,\rho)$ satisfies the differential equation

$$\begin{aligned}&\left(x^3-1\right)y^{(3)}(x)+x^2(\mu+\rho+5)y''(x)\\ &+\tfrac{1}{8}x\left(8\mu\rho+14\mu-6n^2-4n(\mu+\rho+2)-3(-1)^n(2\mu-2\rho+1)+18\rho+27\right)y'(x)\\ &=\tfrac{1}{16}n\left(4\mu+2n+3(-1)^n+5\right)\left(2n-3(-1)^n+4\rho+3\right)y(x).\end{aligned}$$

The general solution of the latter equation can be written as

$$\begin{aligned}P_n(x;\mu,\rho) &= c_1\,{}_3F_2\left(\begin{matrix}-\frac{n}{3},\frac{n}{6}+\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{5}{12},\frac{n}{6}-\frac{(-1)^n}{4}+\frac{\rho}{3}+\frac{1}{4}\\ \frac{1}{3},\frac{2}{3}\end{matrix};x^3\right)\\ &+c_2x\,{}_3F_2\left(\begin{matrix}\frac{1}{3}-\frac{n}{3},\frac{n}{6}+\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{3}{4},\frac{n}{6}-\frac{(-1)^n}{4}+\frac{\rho}{3}+\frac{7}{12}\\ \frac{2}{3},\frac{4}{3}\end{matrix};x^3\right)\\ &+c_3x^2\,{}_3F_2\left(\begin{matrix}\frac{2}{3}-\frac{n}{3},\frac{n}{6}+\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{13}{12},\frac{n}{6}-\frac{(-1)^n}{4}+\frac{\rho}{3}+\frac{11}{12}\\ \frac{4}{3},\frac{5}{3}\end{matrix};x^3\right).\end{aligned}$$

which has only one polynomial solution for each $n\geqslant 0$. Similar to the precedent cases, for each positive integer n, there is only one (monic) polynomial solution and it differs depending on whether n equals $0, 1$ or 2 mod 3. To be precise, we have

$$\begin{aligned}
P_{3n}(x;\mu,\rho) &\coloneqq P_n^{[0]}(x^3;\mu,\rho)\\
&= \frac{(-1)^n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{\left(\frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{5}{12}\right)_n\left(\frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{1}{4}\right)_n}\\
&\qquad {}_3F_2\left(\begin{matrix}-n,\frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{5}{12},\frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{1}{4}\\ \frac{1}{3},\frac{2}{3}\end{matrix};x^3\right),\\
P_{3n+1}(x;\mu,\rho) &\coloneqq xP_n^{[1]}(x^3;\mu,\rho)\\
&= x\,\frac{(-1)^n\left(\frac{2}{3}\right)_n\left(\frac{4}{3}\right)_n}{\left(\frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{11}{12}\right)_n\left(\frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{3}{4}\right)_n}\\
&\qquad {}_3F_2\left(\begin{matrix}-n,\frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{11}{12},\frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{3}{4}\\ \frac{2}{3},\frac{4}{3}\end{matrix};x^3\right),\\
P_{3n+2}(x;\mu,\rho) &\coloneqq x^2P_n^{[2]}(x^3;\mu,\rho)\\
&= x^2\,\frac{(-1)^n\left(\frac{4}{3}\right)_n\left(\frac{5}{3}\right)_n}{\left(\frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{17}{12}\right)_n\left(\frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{5}{4}\right)_n}\\
&\qquad {}_3F_2\left(\begin{matrix}-n,\frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{17}{12},\frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{5}{4}\\ \frac{4}{3},\frac{5}{3}\end{matrix};x^3\right).
\end{aligned}$$

Here, the sequences $\{P_n^{[k]}(\cdot;\mu,\rho)\}_{n\geqslant 0}$, with $k\in\{0,1,2\}$, are precisely the 2-orthogonal polynomial sequences in the cubic decomposition of $\{P_n(\cdot;\mu,\rho)\}_{n\geqslant 0}$. From Lemma 2.1, these three 2-orthogonal polynomial sequences are not threefold symmetric and satisfy the recurrence relation (2.9). These coefficients have been computed in [14, Tableau 1, 5 and 9 - Case A], for a different choice of the "free" parameter γ_1. We have included them in the Appendix for completeness, where we have used the software *Mathematica*.

3.4.2. *The sequence of derivatives*

The 2-orthogonal polynomial sequence $\{Q_n(x;\mu,\rho)\coloneqq \frac{1}{n+1}P'_{n+1}(x;\mu,\rho)\}_{n\geqslant 0}$ satisfies the relation (3.1) with

$$\begin{aligned}
\widetilde{\gamma}_{2n} \coloneqq \widetilde{\gamma}_{2n}(\mu,\rho) &= \frac{2n(2n+1)(n+\rho+1)}{(\mu+3n+2)(3n+\rho)(3n+\rho+3)}, \qquad n\geqslant 1,\\
\widetilde{\gamma}_{2n+1} \coloneqq \widetilde{\gamma}_{2n+1}(\mu,\rho) &= \frac{2(n+1)(2n+1)(\mu+n+2)}{(\mu+3n+2)(\mu+3n+5)(3n+\rho+3)}, \qquad n\geqslant 0,
\end{aligned} \tag{3.70}$$

whose expressions are derived from (3.17) and (3.63). A straightforward comparison between (3.63) and (3.70) readily shows that

$$\widetilde{\gamma}_{n+1}(\mu,\rho) = \gamma_{n+1}(\rho+1,\mu+2), \qquad n\geqslant 0.$$

This, combined with the uniqueness of a polynomial sequence defined by the recurrence relation (3.1) implies $Q_n(x;\mu,\rho) = P_n(x;\rho+1,\mu+2)$ which means

$$\frac{1}{n+1}P'_{n+1}(x;\mu,\rho) = P_n(x;\rho+1,\mu+2), \qquad n \geqslant 0. \tag{3.71}$$

3.4.3. *Particular cases*

The so called *Humbert polynomials* introduced in [22] correspond to $\{P_n(x;\frac{3\nu-1}{2},\frac{3\nu}{2})\}_{n\geqslant 0}$, up to a scaling of the variable. In fact, by setting $\mu = \frac{3\nu-1}{2}$ and $\rho = \frac{3\nu}{2}$, this 2-orthogonal polynomial sequence satisfies

$$\begin{aligned} &P_{n+2}(x;\tfrac{3\nu-1}{2},\tfrac{3\nu}{2}) \\ &\quad = xP_{n+1}(x;\tfrac{3\nu-1}{2},\tfrac{3\nu}{2}) - \frac{4}{27}\frac{n(n+1)(3\nu+n-1)}{(\nu+n-1)(\nu+n)(\nu+n+1)}P_{n-1}(x;\tfrac{3\nu-1}{2},\tfrac{3\nu}{2}), \end{aligned}$$

with initial conditions $P_0 = 1$, $P_1(x) = x$ and $P_2(x) = x^2$. Baker [5, p.60] gave explicit expressions for this particular case via ${}_3F_2$ hypergeometric series and these coincide with those obtained above. *Pincherle polynomials*, introduced in [35], are a particular case obtained with $\nu = 1/2$, which received special attention. For instance, they were discussed in [30] and several properties were further analyzed in [26] with the focus of the study on the algebraic properties, including generating functions, as well as the analysis of the components arising in the cubic decomposition, but integral representations for the orthogonality measures were not included. Prior to this work, Douak and Maroni [16] analyzed in detail the case where $\nu = 1$ (and therefore $\mu = 1$ and $\nu = 3/2$), which gives a threefold symmetric sequence with constant γ-coefficients, namely

$$P_{n+2}(x;1,\tfrac{3}{2}) = xP_{n+1}(x;1,\tfrac{3}{2}) - \tfrac{4}{27}P_{n-1}(x;1,\tfrac{3}{2}),$$

with the same initial conditions. In other words, this means that the associated sequence coincides with the original one, and, regarding the nature of the problem, they referred to this polynomial sequence as *Chebyshev type polynomials*. Therein, an integral representation for the orthogonality functionals was given, with support on an interval on the real line (which obviously does not contain all the zeros of the polynomial sequence [16, Theorem 4.1]), which is, from our point of view, a bit artificial. Following Proposition 3.5, the polynomial sequence $\{P_n(x;1,\frac{3}{2})\}_{n\geqslant 0}$ is 2-orthogonal with respect to (u_0,u_1) which admit the integral representation (3.33) with $b=1$ and

$$\begin{aligned} \mathcal{U}_0(x;1,\tfrac{3}{2}) &= \frac{9\sqrt{3}}{4\pi}\left(\left(1+\sqrt{1-x^3}\right)^{1/3} - \left(1-\sqrt{1-x^3}\right)^{1/3}\right), \\ \mathcal{U}_1(x;1,\tfrac{3}{2}) &= \frac{27\sqrt{3}}{8\pi}\left(\left(\sqrt{1-x^3}+1\right)^{2/3} - \left(1-\sqrt{1-x^3}\right)^{2/3}\right), \end{aligned}$$

after using [34, Eq. (15.4.9)].

Another particular case that made its appearance in the literature corresponds to the case where $\mu = -1/2$ and $\rho = 0$, and therefore $\gamma_{n+2} = \widetilde{\gamma}_n = 4/27$ and $\gamma_1 = 4/9$. In fact, $\{P_n(x; \frac{-1}{2}, 0)\}_{n\geqslant 0}$ is a particular case of *Faber polynomials*, named after the author of [18], and this polynomial sequence with almost constant coefficients has been studied in [17] and [21]. The latter paper was essentially devoted to the study of the zeros, while the former was mainly dedicated to the algebraic properties as well as a representation of the measure on an interval on the real line. Here, we combine the two approaches in a more general setting. So, $\{P_n(x; \frac{-1}{2}, 0)\}_{n\geqslant 0}$ is 2-orthogonal with respect to (u_0, u_1) which admit the integral representation (3.33) with $b = 1$ and

$$\mathcal{U}_0(x; -\tfrac{1}{2}, 0) = \frac{3\sqrt{3}\left(\left(1-\sqrt{1-x^3}\right)^{1/3} + \left(1+\sqrt{1-x^3}\right)^{1/3}\right)}{4\pi\sqrt{1-x^3}},$$
$$\mathcal{U}_1(x; -\tfrac{1}{2}, 0) = \frac{9\sqrt{3}\left(\left(1-\sqrt{1-x^3}\right)^{2/3} + \left(1+\sqrt{1-x^3}\right)^{2/3}\right)}{8\pi\sqrt{1-x^3}}.$$

The penultimate and the latter particular cases are related to each other: if we recall (3.71) we readily see that

$$\frac{1}{n+1} P'_{n+1}(x; -\tfrac{1}{2}, 0) = P_n(x; 1, \tfrac{3}{2}), \qquad n \geqslant 0.$$

It turns out that, the sequence $\{R_n(x; -\frac{1}{2}, 0)\}_{n\geqslant 0}$ of the associated polynomials of $\{P_n(x; -\frac{1}{2}, 0)\}_{n\geqslant 0}$, defined by

$$R_n(x; -\tfrac{1}{2}, 0) := \left\langle u_0, \frac{P_n(x; -\frac{1}{2}, 0) - P_n(t; -\frac{1}{2}, 0)}{x - t} \right\rangle$$

actually coincides with $\{P_n(x; 1, \frac{3}{2})\}_{n\geqslant 0}$, as discussed in [17].

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Appendix

As discussed in Lemma 2.1, the polynomial sequences $\{P_n^{[j]}(x)\}_{n\geqslant 0}$, with $j = 0, 1, 2$, arising from the cubic decomposition of threefold-symmetric 2-Hahn classical polynomials $P_n(x)$ are 2-orthogonal polynomials. The expressions for the corresponding recurrence coefficients can be obtained from the expressions of the γ-coefficients. So, in the light of Lemma 2.1, we computed the expressions of the recurrence coefficients for each of the cubic components of the polynomial sequences discussed in Cases B_1 and C, and this is detailed below. We have included them here for a matter of completion, as these polynomial sequences $\{P_n^{[j]}(x)\}_{n\geqslant 0}$ were already described in [14] up to a linear change of variable.

Case B$_1$

$$\beta_{2n}^{[0]}(\mu) = \frac{2(\mu + 3n(3\mu + 9n(2\mu + 14n + 3) - 2) - 1)}{3(\mu + 9n - 1)(\mu + 9n + 2)}$$

$$\beta_{2n+1}^{[0]}(\mu) = \frac{2(19\mu + 3n(21\mu + 9n(2\mu + 10n + 17) + 77) + 32)}{3(\mu + 9n + 2)(\mu + 9n + 8)}$$

$$\alpha_{2n}^{[0]}(\mu) = \frac{4n(3n-1)(6n-1)\left(-3\mu^2 - 5\mu + 702n^3 + 27(8\mu - 9)n^2 + 3(6(\mu - 3)\mu - 17)n + 8\right)}{3(\mu + 9n - 4)(\mu + 9n - 1)^2(\mu + 9n + 2)}$$

$$\alpha_{2n+1}^{[0]}(\mu) = \frac{4(2n+1)(3n+1)(6n+1)\left(2\mu + 3\left(\mu^2 + 117n^3 + 9(4\mu + 9)n^2 + (3\mu(\mu + 6) + 5)n\right) - 5\right)}{3(\mu + 9n - 1)(\mu + 9n + 2)^2(\mu + 9n + 5)}$$

$$\gamma_{2n}^{[0]}(\mu) = \frac{8n(2n+1)(3n-1)(3n+1)(6n-1)(6n+1)(\mu + 3n - 1)(\mu + 3n)(\mu + 3n + 1)}{3(\mu + 9n - 4)(\mu + 9n - 1)^2(\mu + 9n + 2)^2(\mu + 9n + 5)}$$

$$\gamma_{2n+1}^{[0]}(\mu) = \frac{8(n+1)(2n+1)(3n+1)(3n+2)(6n+1)(6n+5)}{3(\mu + 9n + 2)(\mu + 9n + 5)(\mu + 9n + 8)}$$

$$\beta_{2n}^{[1]}(\mu) = \frac{8(\mu - 1) + 6n(9\mu + 9n(2\mu + 10n + 7) + 5)}{3(\mu + 9n - 1)(\mu + 9n + 5)}$$

$$\beta_{2n+1}^{[1]}(\mu) = \frac{2(31\mu + 3n(27\mu + 9n(2\mu + 14n + 31) + 202) + 143)}{3(\mu + 9n + 5)(\mu + 9n + 8)}$$

$$\alpha_{2n}^{[1]}(\mu) = \frac{4n\left(36n^2 - 1\right)\left(-4\mu + 3n\left(3(\mu - 2)\mu + 117n^2 + 36(\mu - 1)n - 10\right) + 4\right)}{3(\mu + 9n - 4)(\mu + 9n - 1)^2(\mu + 9n + 2)}$$

$$\alpha_{2n+1}^{[1]}(\mu) = \frac{4(2n+1)(3n+1)(3n+2)}{3(\mu + 9n + 2)(\mu + 9n + 5)^2(\mu + 9n + 8)} \left((\mu + 2)(9\mu + 37) + 702n^3 + 27(8\mu + 43)n^2 + 3(6\mu(\mu + 13) + 187)n\right)$$

$$\gamma_{2n}^{[1]}(\mu) = \frac{8n(2n+1)(3n+1)(3n+2)(6n-1)(6n+1)}{3(\mu + 9n - 1)(\mu + 9n + 2)(\mu + 9n + 5)}$$

$$\gamma_{2n+1}^{[1]}(\mu) = \frac{8(n+1)(2n+1)(3n+1)(3n+2)(6n+5)(6n+7)(\mu + 3n + 1)(\mu + 3n + 2)(\mu + 3n + 3)}{3(\mu + 9n + 2)(\mu + 9n + 5)^2(\mu + 9n + 8)^2(\mu + 9n + 11)}$$

$$\beta^{[2]}_{2n}(\mu) = \frac{20(\mu+2)+6n(15\mu+9n(2\mu+14n+17)+58)}{3(\mu+9n+2)(\mu+9n+5)}$$

$$\beta^{[2]}_{2n+1}(\mu) = \frac{2(46\mu+3n(33\mu+9n(2\mu+10n+27)+209)+170)}{3(\mu+9n+5)(\mu+9n+11)}$$

$$\alpha^{[2]}_{2n}(\mu) = \frac{4n(3n+1)(6n+1)\left(\mu+3\left(\mu^2+234n^3+9(8\mu+17)n^2+(6\mu(\mu+5)+7)n\right)-10\right)}{3(\mu+9n-1)(\mu+9n+2)^2(\mu+9n+5)}$$

$$\alpha^{[2]}_{2n+1}(\mu) = \frac{4(2n+1)(3n+2)(6n+5)}{3(\mu+9n+2)(\mu+9n+5)^2(\mu+9n+8)} \left(2(\mu+2)(3\mu+10)+351n^3+54(2\mu+11)n^2+3(3\mu(\mu+14)+98)n\right)$$

$$\gamma^{[2]}_{2n}(\mu) = \frac{8n(2n+1)(3n+1)(3n+2)(6n+1)(6n+5)(\mu+3n)(\mu+3n+1)(\mu+3n+2)}{3(\mu+9n-1)(\mu+9n+2)^2(\mu+9n+5)^2(\mu+9n+8)}$$

$$\gamma^{[2]}_{2n+1}(\mu) = \frac{8(n+1)(2n+1)(3n+2)(3n+4)(6n+5)(6n+7)}{3(\mu+9n+5)(\mu+9n+8)(\mu+9n+11)}$$

Case C

$$\beta^{[0]}_{2n}(\mu,\rho) = \frac{2(2n+1)(3n+1)(6n+1)}{(9n+\mu+2)(9n+\rho+3)} - \frac{4n(3n-1)(6n-1)}{(9n+\mu-1)(9n+\rho-3)}$$

$$\beta^{[0]}_{2n+1}(\mu,\rho) = \frac{4(n+1)(3n+2)(6n+5)}{(9n+\mu+8)(9n+\rho+6)} - \frac{2(2n+1)(3n+1)(6n+1)}{(9n+\mu+2)(9n+\rho+3)}$$

$$\alpha^{[0]}_{2n}(\mu,\rho) = \frac{6n(6n-2)(6n-1)}{(9n+\mu-4)(9n+\mu-1)^2(9n+\rho-3)^2(9n+\rho)}\Bigg((6n-1)(3n+\mu-1)(3n+\rho-1) + \frac{(6n-3)(9n+\mu-1)(3n+\rho-2)(3n+\rho-1)}{9n+\rho-6} + \frac{(6n+1)(3n+\mu-1)(3n+\mu)(9n+\rho-3)}{9n+\mu+2}\Bigg)$$

$$\alpha^{[0]}_{2n+1}(\mu,\rho) = \frac{6(2n+1)(6n+1)}{(9n+\mu+2)^2(9n+\mu+5)(9n+\rho)(9n+\rho+3)^2}\Bigg(2(3n+\mu+1)(3n+\rho)(3n+1)^2 + \frac{6n(3n+\mu)(3n+\mu+1)(9n+\rho+3)(3n+1)}{9n+\mu-1} + \frac{(3n+2)(6n+2)(9n+\mu+2)(3n+\rho)(3n+\rho+1)}{9n+\rho+6}\Bigg)$$

$$\gamma^{[0]}_{2n}(\mu,\rho) = \frac{6n(6n-2)(6n-1)(6n+1)(6n+2)(6n+3)(\mu+3n-1)(\mu+3n)(\mu+3n+1)}{(\mu+9n-4)(\mu+9n-1)^2(\mu+9n+2)^2(\mu+9n+5)(9n+\rho-3)(9n+\rho)(9n+\rho+3)}$$

$$\gamma^{[0]}_{2n+1}(\mu,\rho) = \frac{6n(6n-2)(6n-1)(6n+1)(6n+2)(6n+3)(\mu+3n-1)(\mu+3n)(\mu+3n+1)}{(\mu+9n-4)(\mu+9n-1)^2(\mu+9n+2)^2(\mu+9n+5)(9n+\rho-3)(9n+\rho)(9n+\rho+3)}$$

$$\beta_{2n}^{[1]}(\mu,\rho) = \frac{4(2n+1)(3n+1)(3n+2)}{(9n+\mu+5)(9n+\rho+3)} + \frac{2n-72n^3}{(9n+\mu-1)(9n+\rho)}$$

$$\beta_{2n+1}^{[1]}(\mu,\rho) = \frac{2(n+1)(6n+5)(6n+7)}{(9n+\mu+8)(9n+\rho+9)} - \frac{4(2n+1)(3n+1)(3n+2)}{(9n+\mu+5)(9n+\rho+3)}$$

$$\alpha_{2n}^{[1]}(\mu,\rho) = \frac{6n(6n-1)(6n+1)}{(9n+\mu-1)^2(9n+\mu+2)(9n+\rho-3)(9n+\rho)^2}\Bigg(6n(3n+\mu)(3n+\rho-1) + \frac{(6n+2)(9n+\mu-1)(3n+\rho)(3n+\rho-1)}{9n+\rho+3} + \frac{(6n-2)(3n+\mu-1)(3n+\mu)(9n+\rho)}{9n+\mu-4}\Bigg)$$

$$\alpha_{2n+1}^{[1]}(\mu,\rho) = \frac{6(2n+1)}{(9n+\mu+2)(9n+\mu+5)^2(9n+\rho+3)^2(9n+\rho+6)} \times \Bigg(6(2n+1)(3n+1)(3n+2)(3n+\mu+1)(3n+\rho+1) + \frac{(3n+2)(6n+1)(6n+2)(9n+\mu+5)(3n+\rho)(3n+\rho+1)}{9n+\rho} + \frac{(3n+1)(6n+4)(6n+5)(3n+\mu+1)(3n+\mu+2)(9n+\rho+3)}{9n+\mu+8}\Bigg)$$

$$\gamma_{2n}^{[1]}(\mu,\rho) = \frac{6n(6n-1)(6n+1)(6n+2)(6n+3)(6n+4)(3n+\rho-1)(3n+\rho)(3n+\rho+1)}{(\mu+9n-1)(\mu+9n+2)(\mu+9n+5)(9n+\rho-3)(9n+\rho)^2(9n+\rho+3)^2(9n+\rho+6)}$$

$$\gamma_{2n+1}^{[1]}(\mu,\rho) = \frac{6n(6n-1)(6n+1)(6n+2)(6n+3)(6n+4)(3n+\rho-1)(3n+\rho)(3n+\rho+1)}{(\mu+9n-1)(\mu+9n+2)(\mu+9n+5)(9n+\rho-3)(9n+\rho)^2(9n+\rho+3)^2(9n+\rho+6)}$$

$$\beta_{2n}^{[2]}(\mu,\rho) = \frac{2(2n+1)(3n+2)(6n+5)}{(9n+\mu+5)(9n+\rho+6)} - \frac{4n(3n+1)(6n+1)}{(9n+\mu+2)(9n+\rho)}$$

$$\beta_{2n+1}^{[2]}(\mu,\rho) = \frac{4(n+1)(3n+4)(6n+7)}{(9n+\mu+11)(9n+\rho+9)} - \frac{2(2n+1)(3n+2)(6n+5)}{(9n+\mu+5)(9n+\rho+6)}$$

$$\alpha_{2n}^{[2]}(\mu,\rho) = \frac{6n(6n+1)(6n+2)}{(9n+\mu-1)(9n+\mu+2)^2(9n+\rho)^2(9n+\rho+3)} \Bigg((6n+1)(3n+\mu)(3n+\rho) + \frac{(6n-1)(9n+\mu+2)(3n+\rho-1)(3n+\rho)}{9n+\rho-3} + \frac{(6n+3)(3n+\mu)(3n+\mu+1)(9n+\rho)}{9n+\mu+5} \Bigg)$$

$$\alpha_{2n+1}^{[2]}(\mu,\rho) = \frac{6(2n+1)(6n+5)}{(9n+\mu+5)^2(9n+\mu+8)(9n+\rho+3)(9n+\rho+6)^2} \Bigg(2(3n+\mu+2)(3n+\rho+1)(3n+2)^2 + \frac{6(n+1)(9n+\mu+5)(3n+\rho+1)(3n+\rho+2)(3n+2)}{9n+\rho+9} + \frac{(3n+1)(6n+4)(3n+\mu+1)(3n+\mu+2)(9n+\rho+6)}{9n+\mu+2} \Bigg)$$

$$\gamma_{2n}^{[2]}(\mu,\rho) = \frac{6n(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(\mu+3n)(\mu+3n+1)(\mu+3n+2)}{(\mu+9n-1)(\mu+9n+2)^2(\mu+9n+5)^2(\mu+9n+8)(9n+\rho)(9n+\rho+3)(9n+\rho+6)}$$

$$\gamma_{2n+1}^{[2]}(\mu,\rho) = \frac{6n(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(\mu+3n)(\mu+3n+1)(\mu+3n+2)}{(\mu+9n-1)(\mu+9n+2)^2(\mu+9n+5)^2(\mu+9n+8)(9n+\rho)(9n+\rho+3)(9n+\rho+6)}$$