ON PLANAR POLYNOMIAL VECTOR FIELDS WITH ELEMENTARY FIRST INTEGRALS

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Abstract. We show that under rather general conditions a polynomial differential system having an elementary first integral already must admit a Darboux first integral, and we explicitly characterize the vector fields in this class. We also investigate some exceptional cases, i.e. equations admitting an elementary first integral but not a Darboux first integral. In particular we provide a rather detailed discussion of exceptional elementary first integrals built from algebraic functions of prime degree.

1. Introduction and survey of results

In the present note we discuss a polynomial vector field

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}, \quad \text{briefly } X = \begin{pmatrix} P \\ Q \end{pmatrix} \tag{1}$$

in $\mathbb{C}^2$, and the associated ordinary differential equation

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y). \end{aligned}$$

Our focus is on systems which admit a first integral that is elementary over the differential field $\mathbb{K} = \mathbb{C}(x, y)$. (The pertinent notions will be introduced below; in particular we collect some basic definitions and facts in an appendix.) Our approach is based on the classical work [12] by Prelle and Singer. We recall some of their main results, specialized to the rational function field.

Theorem 1. (*Prelle and Singer* [12])

(a) *If the polynomial vector field X admits an elementary first integral then there exist an integer $m \geq 0$, algebraic functions $v, u_1, \ldots, u_m$ over $\mathbb{K}$ and nonzero constants $c_1, \ldots, c_m \in \mathbb{C}$ such that*

$$X(v) + \sum_{i=1}^{m} c_i \frac{X(u_i)}{u_i} = X\left(v + \sum_{i=1}^{m} c_i \log(u_i)\right) = 0, \tag{2}$$

but $v + \sum_{i=1}^{m} c_i \log(u_i)$ is not constant. The c_i may be chosen linearly independent over the rational numbers $\mathbb{Q}$.

(b) If the vector field X admits an elementary first integral then it admits an integrating factor of the special form

$$\mu = f_1^{-d_1} \cdots f_r^{-d_r}, \tag{3}$$

with irreducible and pairwise relatively prime polynomials $f_1, \ldots, f_r$, and exponents $d_1, \ldots, d_r \in \mathbb{Q}$.

The second statement includes the case when a rational first integral exists; in the particular case of a polynomial first integral there may be a constant integrating factor.

The existence of an elementary first integral $v + \sum_{i=1}^{m} c_i \log(u_i)$ for (1) is equivalent to the existence of a first integral

$$\exp(v) \cdot \prod_{i=1}^{m} u_i^{c_i}. \tag{4}$$

In the special case when the algebraic functions v and u_i in (4) are actually rational, one speaks of a *Darboux first integral.*
One aim of the present paper is to gain a clearer understanding of the relation between integrability by elementary functions and Darboux integrability. In particular we prove: Whenever v and $u_1, \ldots, u_m$ are algebraic functions such that the trace of v is not constant or the norm of some u_i is not constant then the existence of a first integral (4) already implies the existence of a Darboux first integral. This leads to a precise characterization of the vector fields which admit such a first integral. With regard to the remaining cases we prove some general results about the simplest one (the case of a first integral (2) with $m = 1$), and discuss some examples. Our results indicate that these cases may righfully be called "exceptional".

Before we state the results in detail, we briefly recall the general setting, with a particular focus on integrating factors of type (3). By Singer [16] and Christopher [4], the vector field (1) admits a Liouvillian first integral if it admits an integrating factor of type (3) with complex exponents d_i. Moreover, there exists a Liouvillian first integral if and only if there exists a Darboux integrating factor. (The existence of a Liouvillian first integral is not always associated to the existence of some invariant algebraic curve, see [8].) The special structure of polynomial vector fields admitting an integrating factor of type (3) (with exponents $d_i \in \mathbb{C}$) has been discussed in a number of recent papers, e.g. [2, 10, 5, 6, 7]. We will use some of these results below.

Returning to polynomial vector fields with elementary first integrals, the particular vector fields which even admit a rational first integral form a special class. On the one hand such vector fields are easy to construct by forming the Hamiltonian of a rational function and multiplying it by a suitable factor to obtain a polynomial (with relatively prime entries, if desired). On the other hand it may be problematic to decide for a given polynomial vector field whether it admits a rational first integral or not. Moreover it is known that vector fields which admit no rational first integrals exist (and are in fact abundant; see Remark 1 below). The paper by Bostan et al. [1] contains an overview of known results, as well as an improved algorithm to compute rational first integrals of a given polynomial vector field when degree bounds are given. However, determining such degree bounds a priori is an unsolved problem.

We will frequently require that a vector field under consideration does not admit a rational first integral. This implies in particular (as will be shown in Section 3 below) that we may assume that v is not constant, $m > 0$ and the u_i are not constant in (2).

Our principal result states that under rather weak assumptions the existence of an elementary first integral will already imply the existence of a Darboux first integral. This statement is obvious when a rational first integral is assumed to exist, hence excluding this case in the theorem is unproblematic. Before stating the result, we recall the notions of trace and norm of an algebraic function w: Given the normalized polynomial relation of smallest degree

$$w^d + \sum_{i=1}^{d} g_i w^{d-i} = 0,$$

with all $g_i \in \mathbb{C}(x, y)$, by definition the trace of w is equal to $-g_1$ and the norm of w is equal to $(-1)^d g_d$.

Theorem 2. *Let the vector field X given in* (1) *admit the elementary first integral $v + \sum_{i=1}^{m} c_i \log(u_i)$, where $m > 0$ and v, $u_1, \dots, u_m$ are nonconstant algebraic functions over $\mathbb{K} = \mathbb{C}(x, y)$, and furthermore $c_1, \dots, c_m$ are complex constants linearly independent over $\mathbb{Q}$. If X does not admit a rational first integral then the following hold.*

(a) Whenever some u_j has non-constant norm, or whenever v has non-constant trace, then X admits a Darboux first integral.
(b) Moreover, if (3) *is an integrating factor for X, with some u_j having non-constant norm, then it is uniquely determined (up to a nonzero complex factor) and all d_i are nonnegative integers.*

We will prove this in Section 3. Moreover, given statement (a) of Theorem 2, the corresponding polynomial vector fields can be determined explicitly, up to multiplication by a polynomial function, by invoking a result of Chavarriga et al. [2]; see Theorem 5 and Corollary 1.

Theorem 2 leaves the problem to investigate "exceptional" cases admitting an elementary first integral (2) with all u_i of constant norm, and v of constant trace. In the present work we restrict our attention to the (presumably) simplest case $m = 1$. One may then take $c_1 = 1$ with no further loss of generality. Thus we assume a relation

$$X(v) + \frac{X(u)}{u} = 0, \tag{5}$$

with u and v non-constant algebraic functions over $\mathbb{K}$ (but not both in $\mathbb{K}$), with u of constant norm and v of constant trace. In Section 4 we will prove:

Proposition 1. *Let relation* (5) *be given, and assume that X admits no rational first integral.*

(a) The intersection $\mathbb{K}[u] \cap \mathbb{K}[v]$ is nontrivial, i.e. $\neq \mathbb{K}$.
(b) If the degree of $\mathbb{K}[v]$ over $\mathbb{K}$ is a prime number, then $\mathbb{K}[v] \subseteq \mathbb{K}[u]$. If both degrees are prime numbers then $\mathbb{K}[v] = \mathbb{K}[u]$.

We proceed to discuss the cases when both u and v are contained in an extension $\mathbb{F}$ of $\mathbb{K}$ whose degree is a prime number. For quadratic extensions we give a complete characterization as follows.

Theorem 3. *Assume relation* (5) *with u of constant norm and v of constant trace, both contained in a degree two extension of $\mathbb{K}$, but neither contained in $\mathbb{K}$. Then, with no loss of generality, one may take u and v to satisfy*

$$u^2 + 2g \cdot u + 1 = 0 \text{ and } v = b(g + u)$$

with a non-constant rational function g and a nonzero rational function b.

(a) There is a rational function s such that

$$\begin{aligned} P = &\ -s \cdot \left((g^2 - 1)b_y + (bg - 1)g_y\right), \\ Q = &\ s \cdot \left((g^2 - 1)b_x + (bg - 1)g_x\right), \end{aligned}$$

defines a polynomial vector field. If one requires P and Q to have relatively prime entries then s is unique up to a factor in $\mathbb{C}^*$.
(b) This vector field admits the elementary first integral $v + \log u$ and the integrating factor $(\sqrt{g^2 - 1} \cdot s)^{-1}$.
(c) If this vector field admits a Darboux first integral then it admits a rational first integral.

For the proof see Section 4; we also give some examples which do not admit a rational first integral.

Turning to extensions of prime degree $p > 2$, we will show that the hypothesis of Proposition 1 is never satisfied. We have:

Theorem 4. *Let $\mathbb{F} = \mathbb{K}[u] = \mathbb{K}[v]$ be an extension of prime degree p, with v of constant trace and u of constant norm, and consider a polynomial vector field X admitting the elementary first integral* (5)*. If X admits no rational first integral then $p = 2$.*

The proof will also be given in Section 4. While this result does not provide complete information on the possible elementary first integrals of type (5) (and the corresponding polynomial vector fields), it indicates that such cases are indeed exceptional.

2. Notions and known facts

We start by recalling some notions and facts (see the appendix for more). We sketch some proofs, for easy reference and the reader's convenience.

Definition 1. *We call a function*

$$\exp(R) \cdot \prod S_i^{c_i},$$

with rational functions R and S_i and complex constants c_i, a Darboux function.

The notions of Darboux first integral and Darboux integrating factor are then self-explaining.

Lemma 1. *Let* (1) *be given, and H an analytic first integral of this vector field on an open set U. If the nonzero polynomial vector field*

$$\widetilde{X} = \widetilde{P}\frac{\partial}{\partial x} + \widetilde{Q}\frac{\partial}{\partial y}$$

also admits the first integral H on U then there exists a rational function R such that $\widetilde{X} = R \cdot X$.

Proof. There exist analytic μ and $\widetilde{\mu}$ such that

$$\mu X = X_H = -H_y\, \partial/\partial x + H_x\, \partial/\partial y = \widetilde{\mu}\widetilde{X},$$

see Section 5. Hence one has $\widetilde{X} = R \cdot X$ on a nonempty open subset of $\mathbb{C}^2$, and (e.g.) $R = \widetilde{P}/P$ shows that R is rational. □

The vector fields with non-constant Darboux integrating factor (3) (with fixed polynomials and exponents) form a linear space $\mathcal{F}$. The so-called *trivial vector fields admitting* (3) form a subspace $\mathcal{F}^0 \subseteq \mathcal{F}$. To construct these, start with an arbitrary polynomial g and define

$$Z_g = Z_g^{(d_1,\cdots,d_r)} = \text{ Hamiltonian vector field of } g/\left(f_1^{d_1-1}\cdots f_r^{d_r-1}\right).$$

Then, according to [6], the trivial vector field

$$f_1^{d_1}\cdots f_r^{d_r}\cdot Z_g = f\cdot X_g - \sum_{i=1}^{r}(d_i-1)g\frac{f}{f_i}\cdot X_{f_i} \tag{6}$$

(with X_h denoting the Hamiltonian vector field of a function h) is polynomial and admits (3) by construction.

Proposition 2. *If $d_1, \ldots, d_r$ are rational numbers then every element of $\mathcal{F}^0$ admits a rational (hence elementary) first integral.*

Proof. Some integer power of $g/\left(f_1^{d_1-1}\cdots f_r^{d_r-1}\right)$ is a rational function. □

As was shown in [5, 6], one has $\mathcal{F} = \mathcal{F}^0$ whenever the geometry of the underlying curves (including the line at infinity) is nondegenerate, thus each irreducible component is smooth, all pairwise intersections of irreducible components are transversal, and there are no triple intersections.
In such cases Proposition 2 applies. Next we recall a precise description of polynomial vector fields which admit a Darboux first integral.

Theorem 5. *(See Chavarriga et al.* [2], *and* [10]*.) Let*

$$H(x,y) = f_1^{\lambda_1} \cdots f_r^{\lambda_r} \exp(g/(f_1^{n_1} \cdots f_r^{n_r}))$$

be a Darboux function with $\lambda_1, \cdots, \lambda_r \in \mathbb{C}$*,* $n_1, \cdots, n_r \in \mathbb{N} \cup \{0\}$ *and* $g \in \mathbb{C}[x,y]$ *coprime with* f_i *whenever* $n_i \neq 0$*. Then* H *is a first integral of the polynomial vector field*

$$\widehat{X} = \prod_{k=1}^{r} f_k^{n_k+1} \cdot \left(\sum_{k=1}^{r} \lambda_k X_{f_k}/f_k + Z_g^{(n_1+1,\ldots,n_r+1)} \right)$$

which, in turn, admits the integrating factor $\prod_{k=1}^r f_k^{-(n_k+1)}$*. Moreover, any polynomial vector field admitting the first integral* H *admits a rational integrating factor.*

Sketch of proof. One verifies (as in [10]) directly that

$$\widehat{X} = \left(\prod_{l=1}^{r} f_l^{n_l} \right) \sum_{i=1}^{r} \lambda_i \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j \right) X_{f_i} - g \sum_{i=1}^{r} n_i \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j \right) X_{f_i} + \left(\prod_{j=1}^{r} f_j \right) X_g$$

admits the first integral H. Comparison with equation (6) shows that $\widehat{X}$ admits a representation as stated. The remaining assertions are obvious, resp. follow from Lemma 1. □

Corollary 1. *If the vector field* (1) *admits a Darboux first integral but not a rational first integral then it admits (up to constant multiples) a unique integrating factor, and this integrating factor is rational.*

As for a converse, we have the following result which easily follows from Rosenlicht [15]:

Proposition 3. *If the vector field* (1) *admits a rational integrating factor then it admits a Darboux first integral.*

Proof. There exists a rational function μ such that $\mu \cdot X$ is the Hamiltonian vector field of some analytic function H, hence H_x and H_y are rational. We apply Rosenlicht [15], Theorem 3, for $k = \mathbb{K}$ and the pair of derivations $\{\partial/\partial x,\, \partial/\partial y\}$, letting $\alpha_D = H_x$ for $D = \partial/\partial x$ and $\alpha_D = H_y$ for $D = \partial/\partial y$. By the theorem there exist $u_1, \ldots, u_m$ and v in $\mathbb{K}$, and $c_1, \ldots, c_m \in \mathbb{C}$ such that

$$\begin{array}{rcl} H_x &=& \sum_i c_i u_{i,x}/u_i + v_x \\ H_y &=& \sum_i c_i u_{i,y}/u_i + v_y \end{array}$$

which implies $H = v + \sum c_i \log u_i$; thus $\exp H$ is a Darboux first integral. $\square$

3. Elementary and Darboux first integrals

In this section we will prove Theorem 2. By Theorem 1 we may assume there exists an elementary first integral for X of the form (2), which is built from algebraic functions over $\mathbb{K} = \mathbb{C}(x, y)$, and their logarithms.

Given an algebraic function w, we denote its *minimal polynomial* by

$$M_w(T) := T^d + \sum_{i=1}^{d} g_i T^{d-i} \in \mathbb{K}[T].$$

As mentioned earlier, $-g_1$ is the *trace* of w and $(-1)^d g_d$ is the *norm* of w. (Trace and norm are also being used in Prelle and Singer [12]). The other zeros of M_w (in a suitable extension field $\mathbb{F}$ of $\mathbb{K}$) are called the *conjugates* of w. (For algebraic notions and facts the reader may consult e.g. Lang [9].) The following lemma is well-known; we include a proof for the reader's convenience.

Lemma 2. *If an algebraic function w is a first integral of* (1) *then all nonconstant coefficients of its minimal polynomial are first integrals of* (1). *Hence there exist rational first integrals.*

Proof. Indeed, $X(w) = 0$ implies

$$\begin{aligned} 0 &= X(M_w(w)) \\ &= X(w) \cdot \left(dw^{d-1} + \textstyle\sum_{i=1}^{d-1} (d-i) g_i w^{d-i-1}\right) + \textstyle\sum_{i=1}^{d} X(g_i) w^{d-i} \\ &= \textstyle\sum_{i=1}^{d} X(g_i) w^{d-i}, \end{aligned}$$

which forces all $X(g_i) = 0$ by uniqueness of the minimal polynomial. $\square$

We note a few more useful properties.

Lemma 3. *(a) If the polynomial vector field X admits an elementary first integral* (2) *then it admits an algebraic integrating factor μ in $\mathbb{F} := \mathbb{K}[v, u_1, \ldots, u_m]$.*
(b) Moreover, if X admits no rational first integral and $\mathbb{F} = \mathbb{K}[\mu]$ with $[\mathbb{F} : \mathbb{K}] = d$ then $\mu^d \in \mathbb{K}$.

Proof. As a preliminary step we show that the partial derivatives of v and all u_i are contained in $\mathbb{K}[v, u_1, \ldots, u_m]$. Indeed, let

$$v^e + \sum_{i=1}^{e} h_i v^{e-i} = 0; \quad h_1, \ldots, h_e \in \mathbb{K}$$

be a relation of minimal degree. Differentiate to obtain

$$0 = v_x \cdot (ev^{e-1} + \sum_{i=1}^{e-1} h_i (e-i) v^{e-i-1}) + \sum_{i=1}^{e} h_{i,x} v^{e-i}$$

and hence $v_x \in \mathbb{K}(v) = \mathbb{K}[v]$, see for more details Proposition 1.4 in Ch. V of [9]. The same proof applies to v_y, and by the same arguments we obtain that $u_{i,x}$ and $u_{i,y}$ lie in $\mathbb{K}[u_i]$ for $1 \leq i \leq m$. We now turn to the proofs of the assertions.

(a) Thus, setting

$$H := v + \sum c_i \log(u_i)$$

we see that H_x and H_y are contained in $\mathbb{F}$. Since H is a first integral for X, we have

$$H_y = \mu \cdot P, \quad -H_x = \mu \cdot Q$$

with some analytic function μ. Since H_x, H_y, P, Q are all contained in $\mathbb{F}$, the same must hold for μ.

(b) From the proof of Proposition 2 in Prelle and Singer [12] one sees : If $\widehat{\mathbb{F}}$ is a normal extension of $\mathbb{F}$ with degree $[\widehat{\mathbb{F}} : \mathbb{K}] = n$ then $\mu^n \in \mathbb{K}$. But since $\mathbb{K}$ contains all n^{th} roots of unity, the polynomial $T^n - g$ splits over $\mathbb{F}$, and therefore $\mathbb{F} = \mathbb{K}[\mu]$ itself is normal over $\mathbb{K}$ (see e.g. Lang [9], Ch. V, Thm. 3.3). Now one can invoke the argument from Prelle and Singer again.

□

Remark 1. *We frequently exclude polynomial vector fields admitting rational first integrals from our considerations. Therefore it may be appropriate to indicate how restrictive this condition is.*
Section 2 of Bostan et al. [1] *contains a very good overview of known facts and results concerning rational first integrals, and Chavarriga et al.* [3] *give a detailed study of the cofactor conditions at stationary points when invariant algebraic curves (which are necessary ingredients of rational first integrals) exist. We refer to these sources for more details.*
To see directly that "most" polynomial vector fields do not admit rational first integrals, consider the space of all polynomial vector fields with degree less or equal than some fixed m. By their coefficients such vector fields may be identified with points in $\mathbb{C}^M$, M suitable. From [11]*, Corollary 2 and Corollary 4, one sees: A vector field which admits a stationary point whose linearization has a non-rational eigenvalue ratio cannot admit a rational first integral. Due to this fact the subset of $\mathbb{C}^M$ which corresponds to vector fields admitting a rational first integral has Lebesgue measure zero.*
In this sense, requiring the nonexistence of rational first integrals means to impose a weak condition, and the argument may extend to vector fields satisfying additional conditions, such as admitting a given integrating factor, on a case-by-case basis. But for any given vector field it may be hard to verify the non-existence of a rational first integral.

Since we are interested in vector fields which do not admit a rational first integral, we may assume that the vector field X given in (1) admits an elementary first integral (2) with $m > 0$; equivalently we may require

$$X\left(\exp(v)\cdot\prod_{i=1}^{m}u_i^{c_i}\right)=0$$

with $m > 0$. We may and will assume that the c_i are linearly independent over the rational number field $\mathbb{Q}$.

Proof of Theorem 2. (i) Let $\mathbb{F}$ be the smallest normal extension of $\mathbb{K}$ which contains v and all u_i, and denote its Galois group by G. Since $\mathbb{K}$-automorphisms of $\mathbb{F}$ commute with derivations (see e.g. Rosenlicht [13], Proposition and p. 156), we also have

$$X\left(\sigma(v)\right)+\sum_{i=1}^{m}c_i\frac{X\left(\sigma(u_i)\right)}{\sigma(u_i)}=0,$$

for all $\sigma\in G$, and summation yields

$$X\left(\sum_{\sigma\in G}\sigma(v)\right)+\sum_{i=1}^{m}c_i\frac{X\left(\prod_{\sigma\in G}\sigma(u_i)\right)}{\prod_{\sigma\in G}\sigma(u_i)}=0.$$

Since $R:=\sum_{\sigma\in G}\sigma(v)$ is a positive integer multiple of the trace of v, and $S_i:=\prod_{\sigma\in G}\sigma(u_i)$ is a power of the norm of u_i with a positive integer exponent, $1\le i\le m$, this identity involves only rational functions.[1] In particular, whenever $R+\sum c_i\log S_i\neq\text{const.}$, equivalently $\exp(R)\cdot\prod S_i^{c_i}\neq\text{const.}$, we have a Darboux first integral. Note that whenever all u_i have constant norm then the trace of v must also be constant since there exists no rational first integral by assumption.

(ii) We show in detail that $\exp(R)\cdot\prod S_i^{c_i}\neq\text{const.}$ whenever some u_k has non-constant norm. Thus let T be an irreducible polynomial and m_k a nonzero integer such that $S_k=T^{m_k}\cdot\widetilde{S}_k$, with T dividing neither numerator nor denominator of $\widetilde{S}_k$. Likewise we have $S_i=T^{m_i}\cdot\widetilde{S}_i$, with an integer m_i and T dividing neither numerator nor denominator of $\widetilde{S}_i$, for all $i\neq k$. As a preliminary step we show that $\prod S_i^{c_i}\neq\text{const.}$ Due to the linear independence of the c_i over $\mathbb{Q}$ and $m_k\neq 0$ we have

$$\prod S_i^{c_i}=T^{\sum c_i m_i}\cdot\prod\widetilde{S}_i^{c_i},\quad \sum c_i m_i\neq 0,$$

and it suffices to show that

$$T\cdot S^*:=T\cdot\prod\widetilde{S}_i^{e_i}\neq\text{const.},\quad\text{with } e_i:=c_i/\sum c_j m_j.$$

There exists $(x_0,y_0)\in\mathbb{C}^2$ be such that $T(x_0,y_0)=0$ and S^* is analytic and does not vanish in this point. Thus $T\cdot S^*$ has a zero in (x_0,y_0) but is not identically zero.

[1] The positive integer in question is the degree $\left[\mathbb{F}:\widehat{\mathbb{F}}\right]$, with $\widehat{\mathbb{F}}\subset\mathbb{F}$ denoting the splitting field of v, respectively of u_i.

(iii) We next show the full assertion. Let $R = T^\ell \cdot \widetilde{R}$, with T not dividing numerator or denominator of $\widetilde{R}$. Whenever $\ell \geq 0$ the argument from above still works to prove non-constancy. Thus assume $\ell < 0$ and consider

$$\begin{aligned} R^* &:= T^{-\ell}\,(R + \textstyle\sum c_i \log S_i) \\ &= \widetilde{R} + (\textstyle\sum c_i m_i)\, T^{-\ell} \log T + T^{-\ell} \textstyle\sum c_i \log \widetilde{S}_i. \end{aligned}$$

There exists $(x_0, y_0) \in \mathbb{C}^2$ at which $\widetilde{R}$ and all $\widetilde{S}_i$ are analytic and nonzero but $T(x_0, y_0) = 0$. Let $\gamma(t) = (x_0, y_0) + t \cdot a$, with $a \in \mathbb{C}^2$ parameterize a transversal line segment to the curve defined by $T = 0$. Then standard growth estimates yield

$$\lim_{t \to 0} R^*(t) = \widetilde{R}(0) \neq 0, \text{ thus } |R + \sum c_i \log S_i| \to \infty \text{ as } t \to 0,$$

and non-constancy follows as asserted. Thus part (a) is proven.

(iv) The assertion of part (b) follows by Theorem 5 and Corollary 1.

□

4. The simplest exceptional setting

In this section we will discuss exceptional vector fields which, by definition, admit an elementary first integral but not a Darboux first integral. We restrict attention to the simplest setting, starting from a first integral $v + \log u$ of X as in relation (5), with u and v nonconstant and algebraic over $\mathbb{K}$ (but not both in $\mathbb{K}$), with u of constant norm and v of constant trace. Since u may be changed to $\alpha \cdot u$ with some $\alpha \in \mathbb{C}^*$ and v may be changed to $v + \beta$ with some $\beta \in \mathbb{C}$, we may assume that u has norm one and v has trace zero.

4.1. **Some generalities.** In this subsection we prove that $\mathbb{K}[u] \cap \mathbb{K}[v] \neq \mathbb{K}$ under the assumptions of Proposition 1. We first remark that neither u nor v are elements of $\mathbb{K}$, since this would imply $u \in \mathbb{C}$ (resp. $v \in \mathbb{C}$) and the existence of a rational first integral follows from Lemma 2.

The assertion of Proposition 1(a) is then a direct consequence of the following general fact; note that $X(u)/u$ and $X(v)$ are contained in the intersection due to (5).

Lemma 4. *Let $s \neq 0$ be algebraic over $\mathbb{K}$ with constant norm, Y a derivation of $\mathbb{K}[s]$ extending a derivation of $\mathbb{K}$, and $Y(s)/s \in \mathbb{K}$. Then $Y(s) = 0$, hence s is constant unless Y admits a rational first integral.*

Proof. The minimal polynomial of s yields a relation

$$s^m + g_1 s^{m-1} + \cdots + g_{m-1} s + g_m = 0$$

with all $g_i \in \mathbb{K}$ ($g_0 := 1$) and constant g_m. Applying the derivation yields

$$0 = Y(s) \cdot \sum_{i=1}^{m} g_{i-1}(m - i + 1) s^{m-i} + \sum_{i=1}^{m-1} Y(g_i) s^{m-i},$$

in view of $Y(g_m) = 0$. Letting $Y(s) = h \cdot s$ with $h \in \mathbb{K}$, one finds

$$mhs^m + \sum_{i=1}^{m-1} \left((m-i)hg_i + Y(g_i)\right) s^{m-i} = 0.$$

On the other hand

$$mhs^m + \sum_{i=1}^{m} mhg_i s^{m-i} = 0$$

with the same leading coefficient. By uniqueness of the minimal polynomial, all coefficients must be equal, which implies $mhg_m = 0$, and thus $h = 0$. So $Y(s) = 0$ and consequently $s \in \mathbb{C}$ unless Y admits a first integral in $\mathbb{K}$. □

Part (b) of Proposition 1 is a direct consequence of part (a) and multiplicativity of the degree for field extensions.

Proposition 4. *Let $\mathbb{F} : \mathbb{K}$ be a field extension of prime degree p, and let X admit an elementary first integral* (5) *which is not Darboux. Additionally, assume that X admits no rational first integral. Then the integrating factor μ of X (which is unique up to a nonzero scalar) satisfies $\mathbb{F} = \mathbb{K}[\mu]$ and there exists $g \in \mathbb{K}$ which is not a p^{th} power in $\mathbb{K}$ such that*

$$\mu^p - g = 0. \tag{7}$$

In particular $\mathbb{F} : \mathbb{K}$ is a cyclic Galois extension of degree p.

Proof. This is a direct consequence of Lemma 3. Note that $\mu \notin \mathbb{K}$ by Proposition 3.

□

4.2. **Quadratic extensions.** In this subsection we consider the smallest possible degrees of $\mathbb{K}[u]$ and $\mathbb{K}[v]$. Thus in view of Proposition 1 we assume relation (5) with u and v contained in a degree two extension $\mathbb{F} = \mathbb{K}[u] = \mathbb{K}[v]$ of $\mathbb{K}$. With u of norm one, there exists $g \in \mathbb{K}$ such that

$$u^2 + 2g \cdot u + 1 = 0. \tag{8}$$

This relation may be rewritten in the form

$$(u+g)^2 = g^2 - 1 \tag{9}$$

which in turn implies

$$u_x = -\frac{u}{u+g} g_x, \qquad u_y = -\frac{u}{u+g} g_y. \tag{10}$$

Moreover

$$v = a + bu, \quad \text{with } a, b \in \mathbb{K} \text{ and } b \neq 0 \tag{11}$$

by Proposition 1.

Using (8) and (11) we obtain

$$\begin{aligned} v^2 &= a^2 + 2abu + b^2u^2 \\ &= (a^2 - b^2) + (2a - 2gb)bu \\ &= \left(a^2 - b^2 - 2a(a - gb)\right) + 2(a - gb)v \end{aligned}$$

and trace zero forces $a = gb$, therefore

$$v = b(g + u).$$

Proof of Theorem 3. The assertions regarding u and v have already been shown. We determine the Hamiltonian vector field of $H := b(u + g) + \log u$. With (10) and (9) one finds

$$\begin{aligned} H_x &= b_x(u+g) + b(u_x + g_x) + u_x/u \\ &= b_x(u+g) + bgg_x - g_x/(u+g) \\ &= (u+g)^{-1} \cdot \big(b_x(g^2-1) + g_x(bg-1)\big) \\ &= 1/\sqrt{g^2-1} \cdot \big(b_x(g^2-1) + g_x(bg-1)\big) \end{aligned}$$

and a similar expression for H_y. For the rational vector field

$$\widehat{X} = \begin{pmatrix} -b_y(g^2-1) - g_y(bg-1) \\ b_x(g^2-1) + g_x(bg-1) \end{pmatrix}$$

there obviously exists a unique (up to a scalar factor) rational function s such that $s \cdot \widehat{X}$ is polynomial with relatively prime entries. Thus part (a) holds, and (b) is satisfied by construction.

To prove part (c), assume that the vector field admits a Darboux first integral. Then Corollary 1 yields a contradiction since the integrating factor from part (b) is not rational. □

Theorem 3 describes all polynomial vector fields which admit an exceptional elementary first integral of type (5) with u and v in a quadratic extension of $\mathbb{K}$. One may start with arbitrary rational functions g (subject to $u \notin \mathbb{K}$) and $b \neq 0$. But one has to ascertain that the vector field admits no rational first integral. To verify that there actually exist such vector fields in the given class, we discuss two examples.

Example 1. *This is known from Example 2 of Prelle and Singer* [12]*: Let $g = x$ and $b = y$. The vector field*

$$\widehat{X} = \begin{pmatrix} 1 - x^2 \\ xy - 1 \end{pmatrix}$$

admits the integrating factor $1/\sqrt{1-x^2}$ and the elementary first integral

$$H(x,y) = y \cdot \sqrt{x^2-1} + \log\left(-x + \sqrt{x^2-1}\right).$$

Prelle and Singer [12] *showed by differential-algebraic arguments that no rational first integral exists.*

Example 2. *The polynomial differential system*

$$\begin{aligned} \dot{x} &= 1 - x^2, \\ \dot{y} &= 1 - x^2 - xy, \end{aligned}$$

admits the integrating factor $(x^2-1)^{-3/2}$ and the elementary first integral

$$H = e^{\frac{y}{\sqrt{x^2-1}}} / \left(x + \sqrt{x^2-1}\right)$$

(or equivalently $\log H = \frac{y}{\sqrt{x^2-1}} - \log(x + \sqrt{x^2-1})$*). In order to show that this system has no Darboux first integral, it is again sufficient to prove there is no rational first integral. We proceed here using an argument different from Prelle and Singer.*

Obviously the only stationary points of the system are $(1,0)$ *and* $(-1,0)$*, and the lines defined by* $x \pm 1 = 0$ *are invariant algebraic curves with cofactors* $-x \pm 1$*, respectively.*

We show that no other invariant algebraic curves exist. Thus assume that $f(x,y) = 0$*, with* $f = \sum_{i=0}^{N} \phi_i(x) y^i \in \mathbb{C}[x,y]$*, defines an invariant algebraic curve* C *and let* $(x_0,\ y_0) \in C$ *with* $x_0 \notin \{-1,1\}$ *and* $\phi_N(x_0) \neq 0$*. Then going to the splitting field of* f *over* $\mathbb{C}(x)$*, one gets*

$$f(x,y) = \phi_N(x_0)(y - \psi_1(x)) \cdots (y - \psi_N(x))$$

with $\psi_1, \cdots, \psi_N$ *algebraic functions. We may assume that* C *is defined by* $y = \psi_1(x)$ *in a neighborhood of* $(x_0,\ y_0)$*. On the other hand the first integral* H *implies that*

$$y = \left(c + \log\left(x + \sqrt{x^2-1}\right)\right)\sqrt{x^2-1}$$

near (x_0, y_0)*, with some* $c \in \mathbb{C}$*. Since this is a transcendental function, we have a contradiction.*

To summarize: The only possible rational first integrals have the form $(x-1)^m(x+1)^n$*, with integers* m *and* n*. But any integer linear combination* $m \cdot (-x+1) + n \cdot (-x-1)$ *of the cofactors is nonzero unless* m *and* n *are both zero. Therefore, no rational first integral exists.*

4.3. **Extensions of odd prime degree.** Finally, we will prove Theorem 4. We start with an auxiliary result.

Lemma 5. *Let* p *be a prime number,* $g \in \mathbb{K} \setminus \mathbb{C}(x)$ *not a* p^{th} *power in* $\mathbb{K}$*, and* $\mathbb{F} = \mathbb{K}[\mu]$ *with* $\mu^p = g$*. Then* $\mathbb{C}(x)$ *is algebraically closed in* $\mathbb{F}$*; thus every* $\rho \in \mathbb{F}$ *that is algebraic over* $\mathbb{C}(x)$ *already lies in* $\mathbb{C}(x)$*.*

Proof. By assumption we have $g_y \neq 0$. Moreover we may assume that there is a prime polynomial q such that $g = q^\ell \widetilde{g}$, $q_y \neq 0$, q divides neither numerator nor denominator of $\widetilde{g}$ and $\ell \neq 0$ is not an integer multiple of p.

(i) We note the identity

$$\mu_y = \frac{1}{p}\frac{g_y}{g} \cdot \mu$$

which follows from logarithmic differentiation of $\mu^p = g$.

(ii) We first show that $\rho_y = 0$. Indeed, from the minimal polynomial of ρ over $\mathbb{C}(x)$ one obtains a relation

$$\rho^m + \sum_{i=0}^{m-1} a_i \rho^{m-i} = 0; \quad a_1, \ldots, a_{m-1} \in \mathbb{C}(x),$$

of smallest degree m, and differentiation yields

$$0 = \rho_y \left(m\rho^{m-1} + \sum_{i=0}^{m-1} (m-i) a_i \rho^{m-i-1} \right)$$

with the term in brackets nonzero.

(iii) Now consider the unique representation

$$\rho = \sum_{i=0}^{p-1} b_i \mu^i \text{ with } b_i \in \mathbb{K}.$$

Using part (ii) we find

$$\begin{aligned} 0 = \rho_y &= \sum_{i=0}^{p-1} b_{i,y} \mu^i + \sum_{i=1}^{p-1} i b_i \mu^{i-1} \mu_y \\ &= b_{0,y} + \sum_{i=1}^{p-1} \left(b_{i,y} + \frac{i}{p} \frac{g_y}{g} b_i \right) \mu^i \end{aligned}$$

Since all coefficients in the last sum must be zero, we find that

$$\frac{\partial}{\partial y} \left(b_i^p g^i \right) = 0, \quad 0 \le i < p.$$

Thus $b_0 \in \mathbb{C}(x)$, and moreover there exist $k_i \in \mathbb{C}(x)$ such that

$$b_i(x,y)^p = k_i(x)/g(x,y)^i, \quad 1 \le i < p.$$

(iv) Now assume $k_i \neq 0$ and let $q \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ be as above. Let $b_i = q^{m_i} \cdot \widetilde{b}_i$ with q dividing neither numerator nor denominator of $\widetilde{b}_i$. Comparing exponents of q yields

$$p \cdot |m_i| = i \cdot |\ell|.$$

This is a contradiction, since p divides neither factor on the right-hand side. Therefore all $k_i = 0$, and $\rho \in \mathbb{K}$.

□

Proof of Theorem 4. Assume that the prime number p satisfies $p > 2$, and that $v + \log u$ is an elementary first integral, with $\mathbb{F} = \mathbb{K}[u] = \mathbb{K}[v]$ of degree p over $\mathbb{K}$. Let μ be an integrating factor for system (1). Then $\mu^p = g \in \mathbb{K}$ by Lemma 3, and we have, by the integrating factor condition,

$$v_x + \frac{u_x}{u} = -\mu Q.$$

Denote by ζ a primitive p^{th} root of unity. Applying the automorphism σ defined by $\mu \mapsto \zeta\mu$ to this identity (and recalling that σ commutes with all derivations) we get

$$\begin{aligned} \sigma(v)_x &+ \frac{\sigma(u)_x}{\sigma(u)} &= -\zeta\mu\, Q \\ \Rightarrow \quad \zeta^{-1}\sigma(v)_x &+ \zeta^{-1}\frac{\sigma(u)_x}{\sigma(u)} &= -\mu Q. \end{aligned}$$

Combining these relations, we obtain

$$\big(v - \zeta^{-1}\sigma(v)\big)_x - \zeta^{-1}\frac{\sigma(u)_x}{\sigma(u)} + \frac{u_x}{u} = 0.$$

To this relation one can apply the Theorem from Rosenlicht [14], with the identifications $k := \mathbb{C}(x)$, $K = \mathbb{F}$, $t = y$ and $.' = \frac{\partial}{\partial x}$ (hence $t' = 0 \in k$). The hypotheses of this theorem are satisfied, since by Lemma 5, k is algebraically closed in K, and due to $p > 2$ we have that 1 and ζ^{-1} are linearly independent over the rational number field. Rosenlicht's theorem now implies that $u \in k = \mathbb{C}(x)$. This in turn implies $\mathbb{F} = \mathbb{K}[u] = \mathbb{K}$; a contradiction. □

In the proof of Theorem 4 we did not use all the available information. We collect some facts to conclude the paper, and to point towards further research.

Remark 2. (a) *In the situation of Lemma 3 we have*

$$\mathbb{K} \subseteq \mathbb{K}[\mu] \subseteq \mathbb{F} \subseteq \widehat{\mathbb{F}}$$

and $\mathbb{K}[\mu] : \mathbb{K}$ *is Galois as well as* $\widehat{\mathbb{F}} : \mathbb{K}$. *Therefore, with the Galois group* G *of* $\widehat{\mathbb{F}}$, $\mathbb{K}[\mu]$ *is the fixed point field of a normal subgroup* H *of* G, *and* G/H *is cyclic of order* d. *This shows that strong restrictions exist with regard to the possible field extensions* $\mathbb{F}$ *related to elementary first integrals of the form* (2) *which are not logarithms of Darboux first integrals.*

(b) *In particular, Theorem 4 characterizes every field extension* $\mathbb{F} : \mathbb{K}$ *related to an elementary first integral which has the form* (5)*, but is not a logarithm of a Darboux first integral, with the property that its smallest normal extension* $\widehat{\mathbb{F}}$ *has a simple Galois group.*

5. Appendix

In this section we recall some familiar notions, terminology and facts.

Definition 2. *Let* U *be a nonempty open subset of* $\mathbb{C}^2$.

We say that a nonconstant function $H : U \longrightarrow \mathbb{C}$ *is a* first integral *of the vector field* X *on* U *if and only if the relation* $X(H) = 0$ *holds on* U.

Given some nonconstant analytic function g *on* U, *the analytic vector field*

$$X_g := -\frac{\partial g}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial x}\frac{\partial}{\partial y}$$

is called the Hamiltonian vector field *of* g. *By construction it admits the first integral* g.

We say that a nonconstant function $\mu : U \longrightarrow \mathbb{C}$ *is an* integrating factor *of the vector field* X *on* U *if and only if* $X(\mu) = -(P_x + Q_y)\mu$ *on* U; *in other words* $\operatorname{div}(\mu \cdot X) = 0$.

If H is a first integral of the vector field X, then there is a unique integrating factor μ satisfying

$$\mu P = \frac{\partial H}{\partial y} \qquad \text{and} \qquad \mu Q = -\frac{\partial H}{\partial x}.$$

Next we recall some properties of invariant algebraic curves.

Definition 3. *Let $f \in \mathbb{C}[x, y]$ be an irreducible polynomial. The algebraic curve defined by $f(x, y) = 0$ is an* invariant algebraic curve *of the polynomial vector field* (1) *if there exists a polynomial $K \in \mathbb{C}[x, y]$ (called the* cofactor *of f) such that*

$$Xf = Kf.$$

Such a condition is necessary and sufficient for the curve to be an invariant set, thus a union of trajectories of the vector field (1).

If the vector field (1) admits an integrating factor (3) then the algebraic curves defined by $f_i = 0$ are invariant. Moreover, if K_i is the cofactor of f_i, respectively, then

$$\sum d_i K_i + \operatorname{div} X = 0.$$

We are furthermore concerned with elementary first integrals of (1). Thus consider the differential field $\mathbb{K} = \mathbb{C}(x, y)$ with derivations $\partial/\partial x$ and $\partial/\partial y$. Every derivation of $\mathbb{K}$ has the form $Y = R\frac{\partial}{\partial x} + S\frac{\partial}{\partial y}$ with rational R and S.

Definition 4. *(a) An extension field $\mathbb{L}$ of $\mathbb{K}$ is called* elementary *if there is a finite tower of extension fields $\mathbb{K} = \mathbb{L}_0 \subset \mathbb{L}_1 \subset \cdots \subset \mathbb{L}_n = \mathbb{L}$ such that $\mathbb{L}_{i+1}$ is obtained from $\mathbb{L}_i$ by adjoining an algebraic element, an exponential (i.e. an element w such that $Y(w)/w \in \mathbb{L}_i$ for some derivation Y) or a logarithm (i.e. an element w such that $Z(w) = Z(a)/a$ for some $a \in \mathbb{L}_i$ and some derivation Z).*

(b) An extension field $\mathbb{L}$ is called Liouvillian *over $\mathbb{K}$ if there is a finite tower of extension fields $\mathbb{K} = \mathbb{L}_0 \subset \mathbb{L}_1 \subset \cdots \subset \mathbb{L}_n = \mathbb{L}$ such that $\mathbb{L}_{i+1}$ is obtained from $\mathbb{L}_i$ by adjoining an algebraic element, an integral element (i.e. an element w such that $Z(w) = a$ for some $a \in \mathbb{L}_i$), or an exponential. Every element of $\mathbb{L}$ is called a* Liouvillian function of two variables.

One may regard such field extensions as abstract objects, but in the present work we realize them (locally, on suitable open subsets of $\mathbb{C}^2$) as analytic functions of two variables. The class of Liouvillian functions obviously contains the class of elementary functions, and the inclusion is proper.

Acknowledgements

The second author is partially supported by an AGAUR grant 2014SGR-568. The second and third author are also supported by MINECO-FEDER grant MTM2016-77278-P, and a MINECO grant MTM2013-40998-P. The third author is partially supported by an AGAUR grant 2014SGR-504, and the MECC-FEDER grant MTM2015-65715-P. The fourth author acknowledges the hospitality of the Mathematics Department at UAB when this manuscript was prepared.

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