A FULL INVESTIGATION OF THE DIRECTIONAL CONGESTION IN DATA ENVELOPMENT ANALYSIS

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Abstract. One of interesting subjects in Data Envelopment Analysis (DEA) is estimation of congestion of Decision Making Units (DMUs). Congestion is evidenced when decreases (increases) in some inputs result in increases (decreases) in some outputs without worsening (improving) any other input/output. Most of the existing methods for measuring the congestion of DMUs utilize the traditional definition of congestion and assume that inputs and outputs change with the same proportion. Therefore, the important question that arises is whether congestion will occur or not if the decision maker (DM) increases or decreases the inputs dis-proportionally. This means that, the traditional definition of congestion in DEA may be unable to measure the congestion of units with multiple inputs and outputs. This paper focuses on the directional congestion and proposes methods for recognizing the directional congestion using DEA models. To do this, we consider two different scenarios: (i) just the input direction is available. (ii) none of the input and output directions are available. For each scenario, we propose a method consists in systems of inequalities or linear programming problems for estimation of the directional congestion. The validity of the proposed methods are demonstrated utilizing two numerical examples.

Keywords: Data Envelopment Analysis; Directional Congestion; Decision making Units.

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1. Introduction

In recent years, DEA has been applied as a powerful tool for performance evaluation. Since then, a large amount of studies has been done in DEA theory and practice. Emrouznejad and Yang [9] reported DEA studies from 1978 to end of 2016. The concept of congestion is an important subject in DEA and management. It occurs when the reduction (increment) in some inputs results in the maximum possible increase (decrease) in some outputs without worsening (improving) other input/output (Cooper et al. [5]). The DM can apply congestion to decide about increasing or decreasing the size of a particular DMU. The problem of the congestion estimation of decision making units has attracted attentions of several scholars. Färe and Sevensson [10] formulated a linear programming model to define the concept of congestion in a production technology with a single output. Färe et al. [11] emphasized on the efficiency evaluation and proposed a radial model to estimate the congestion. Because of the congestion is a kind of inefficiency, therefore, their model may be unable to recognize the congestion in some situations. Later, Cooper et al. [6] proposed a slack-based method to measure the congestion of DMUs. The advantage of their method over previous methods is that it not only distinguishes the congested inputs, but also measures the amount of congestion of each input. Cooper et al. [7] developed an additive model for identifying the congestion of units.

Regarding the traditional definition of congestion, the congestion is evidenced when the increase in some inputs results in the decrease in some outputs, hence, it can identify the shortfall of outputs. In this respect, Wei and Yan [23] and Tone and Sahoo [21] considered the congestion in outputs. Wei and Yan [24] considered the output oriented DEA models to recognize the necessary and sufficient conditions for the existence of congestion. Sueyoshi and Sekitani [20] dealt with the situation that there exist alternative optimal solutions in models for measuring the congestion and proposed an approach to measure the congestion of decision making units. Ghomashi and Abbasi [12] proposed a linear inequality and equality system to estimate the congestion of decision making units. Kheirollahi et al. [15] developed an input relaxation model to identify the input congestion of units in data envelopment analysis with fuzzy data. Khoveyni et al. [16] proposed mixed integer programming (MIP) models to determine the strongly and weakly most congested units in the presence of negative data. Mehdiloozad et al. [18] introduced the concept of Max-projection for the congestion of inefficient units and proposed a linear programming model to identify the Max-projection and also developed a single-stage LP model to estimate the congestion of units. Ebrahimzade Adimi et al. [8] introduced the concept of the congestion hyperplane and applied it to measure the congestion of DMUs. For more studies about congestion see Jahanshahloo and Khodabakhshi [13], Kao [14], Noura et al. [19], Khoveyni et al. [17] and Wu et al. [27], Wu et al. [25], [26].

The conventional DEA models assume that all inputs and outputs are deterministic. However, this assumption may not always hold true due to existence of uncertainty for example inputs and outputs might be stochastic or fuzzy data. See

Wanke et al. [22] for more studies about the comparison between DEA and fuzzy DEA models.

There are two basic terms in the congestion literature: strong and weak congestion. If decreases in all (some) inputs of the DMU result in increases in all (some) outputs of it, then this decision making unit has strong (weak) congestion. There exist main drawbacks with the definitions of strong and weak congestion. First, strong and weak congestion only consider the situation that the decreases in inputs can be associated with the increases in outputs and the case that the increases in some inputs result in the decreases in some outputs without improving any other input or output is not considered in these definitions. Second, in case of strong or weak congestion, we cannot recognize the precise direction along which the congestion occurs. This means that we do not know whether congestion will occur or not if the DM increases (or decreases) inputs and outputs dis-proportionally. Therefore, we may be unable to estimate the congestion of DMUs with multiple inputs and outputs. In this regard, Yang [28] proposed the definition of the directional congestion along certain input and output directions. He proposed two methods with different perspectives to estimate the directional congestion. Both methods assume that input and output directions are predetermined by the DM.

The methods of Yang [28] only consider the existence of congestion along the certain input/output directions. In other words, if a DMU has no congestion along these certain directions, no information about the existence or absence of the congestion along other directions can be obtained. Therefore, we consider the situation in which at least one input or output direction is not specified. Two different scenarios are considered in this paper: (i) a scenario in which only the input direction is specified. We present two methods to estimate the directional congestion for this scenario. If there exists the congestion along a certain input direction, both methods can find an output direction along which congestion occurs. (ii) a scenario based on the assumption that both input and output directions are not specified. We propose a system of inequalities to find input and output directions along which the congestion occurs in this scenario. This study addresses the relationship between our defined directional congestion and the classical definition of strong and weak congestion.

The rest of this paper is organized as follows: Section 1 reviews preliminaries and basic definitions. In section 2, we present three methods to estimate the directional congestion. Numerical examples are provided in section 3. Section 4 concludes the paper.

2. Preliminaries and basic definitions

Suppose that there exist n decision making units, $DMU_j, j = 1, ..., n$, and each DMU consumes m inputs to produce s outputs. The i^{th} input and r^{th} output for DMU_j are denoted by x_{ij} and y_{rj}, respectively for $i = 1, ...m$ and $r = 1, ..., s$. We assume that all input and output values are non-negative, and at least one of each is non-zero. Let $DMU_o = (x_o, y_o)$ be the unit under assessment. The production

Possibility Set (PPS) with variable returns to scale (VRS) defined by Banker et al. [1] is as follows:

$$T_v = \{(x, y)|x \geq \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, ..., n\} \tag{1}$$

The output-oriented BCC model proposed by Banker et al. [1] for evaluating the efficiency score of DMU_o is as follows:

$$\begin{aligned}
&\psi^* = \max \rho + \epsilon(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+) \\
&s.t. \ \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, \qquad i = 1, ..., m, \qquad (2)\\
&\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \rho \ y_{ro}, \qquad r = 1, ..., s, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \geq 0, \qquad j = 1, ..., n, \\
&s_i^- \geq 0, \qquad i = 1, ..., m, \\
&s_r^+ \geq 0, \qquad r = 1, ..., s.
\end{aligned}$$

where ϵ is non-Archimedean.
Banker et al. [1] presented the following definition:

Definition 1. Suppose that $(\rho^*, s^{-^*}, s^{+^*}, \lambda^*)$ is an optimal solution for model (2) evaluating DMU_o. If $\rho^* = 1$ then DMU_o is called technically efficient. Furthermore, if $\psi^* = 1$, DMU_o is called strongly efficient.

Cooper et al. [3] and Brockett et al. [2] presented the classical definition of congestion as follows:

Definition 2. The unit $DMU_o = (x_o, y_o)$ has congestion if the decreases (increases) in some inputs result in the increases (decreases) in some outputs without worsening (improving) other inputs/outputs.

There are several approaches to recognize the congestion. For example, Tone and Sahoo [21] proposed TS method which is summarized as follows:

2.1. TS method

Tone and Sahoo [21] defined the PPS accepting all assumptions to build T_v except one assumption, strong disposal. They considered weak disposal instead, which was defined as follows:

Definition 3. The PPS satisfies weak disposal assumption if for each $(\bar{x}, \bar{y})$ belonging to the PPS and vector (x, y) where $x = \bar{x}$ and $y \leq \bar{y}$, (x, y) belongs to the PPS.

Therefore, Tone and Sahoo [21] presented the following PPS:

$$P_{convex} = \{(x,y)|x = \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, ..., n\} \qquad (3)$$

They proposed the following model to evaluate the efficiency score of DMU_o, with respect to P_{convex} :

$$\begin{aligned}
\Phi^* = \max\ & \Phi + \epsilon(\sum_{r=1}^{s} s_r^+) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io}, && i = 1, ..., m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \Phi y_{ro}, && r = 1, ..., s, \\
& \sum_{j=1}^{n} \lambda_j = 1, && \\
& \lambda_j \geq 0, && j = 1, ..., n, \\
& s_r^+ \geq 0, && r = 1, ..., s.
\end{aligned} \qquad (4)$$

It is clear that the target unit for DMU_o, located on the strongly efficient frontier of P_{convex} is as follows:

$$\begin{aligned}
\hat{x}_{io} &= x_{io}, && i = 1, ..., m, \\
\hat{y}_{ro} &= \Phi^* y_{ro} + s_r^{+*}, && r = 1, ..., s.
\end{aligned} \qquad (5)$$

Tone and Sahoo [21] presented the following definitions for strongly efficient unit, strong congestion and weak congestion, respectively:

Definition 4. The unit $DMU_o = (x_o, y_o)$ is strongly efficient unit with respect to P_{convex}, if $\Phi^* = 1$.

Definition 5. Suppose that $DMU_o = (x_o, y_o)$ is strongly efficient unit with respect to P_{convex}. DMU_o has strong congestion if there exists $(\bar{x}_o, \bar{y}_o) \in P_{convex}$ such that $\bar{x}_o = \alpha x_o (0 < \alpha < 1)$ and $\bar{y}_o \geq \beta y_o (\beta > 1)$.

Definition 6. Suppose that $DMU_o = (x_o, y_o)$ is strongly efficient unit with respect to P_{convex}. DMU_o has weak congestion if there exists an activity in P_{convex} that uses less resources in some components of the input vector to produce more products in some components of the output vector.

Tone and Sahoo [21] assumed that DMU_o is strongly efficient unit with respect to P_{convex}. If not, they projected DMU_o on to the strongly efficient frontier of P_{convex} and then applied the following method to recognize the strongly and weakly congested units:

Step 1. Solve model (2). Suppose that $(\rho^*, s^{-*}, s^{+*}, \lambda^*)$ is an optimal solution for this model:

(a) If $\rho^* = 1, s^{-*} = 0, s^{+*} = 0$, then $DMU_o = (x_o, y_o)$ is BCC-efficient and not congested.

(b) If $\rho^* = 1, s^{-*} \neq 0, s^{+*} = 0$, then $DMU_o = (x_o, y_o)$ is technically inefficient.

(c) If $\rho^* = 1, s^{+*} \neq 0$ or $\rho^* > 1$, then $DMU_o = (x_o, y_o)$ has congestion. Go Step 2.

Step 2. Solve model (6):

$$\begin{aligned}
\tilde{u} = \max\ & u_o \\
s.t.\quad & \sum_{r=1}^{s} u_r y_{ro} = 1, && (6) \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \leq 0, && j = 1, ..., n, j \neq o, \\
& \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + u_o = 0, && \\
& u_r \geq 0, && r = 1, ..., s, \\
& v_i \ \textit{free}, && i = 1, ..., m, \\
& u_o \ \textit{free}. &&
\end{aligned}$$

suppose that $\tilde{u}$ is the optimal value of model (6) and $\tilde{\rho} = 1 + \tilde{u}$. If $\tilde{\rho} < 0$ then DMU_o has strong congestion, otherwise it has weak congestion.

Step 3. Termination.

The definition of strong and weak congestion has main drawbacks. First, strong and weak congestion only consider the situation that decreases in inputs can be associated with increases in outputs and the case that increases in inputs result in decreases in outputs without improving any other input or output, is not considered in these definitions. Second, in case of strong or weak congestion, we cannot recognize the precise direction along which the congestion occurs. This means that we do not know whether congestion will occur or not if the DM increases (or decreases) inputs dis-proportionally. Therefore, we may be unable to estimate the congestion of DMUs with multiple inputs and outputs. In this regard, Yang [28] proposed the definition of directional congestion along certain input and output directions. He proposed two methods with different perspectives to estimate the directional congestion. Both methods assume that input and output directions are predetermined.

The methods of Yang [28] only consider the existence of congestion along the certain input/output directions. In other words, if a DMU has no congestion along these certain directions, no information about the existence or absence of the congestion along other directions can be obtained. Therefore, we consider the situation in which at least one input or output direction is not specified. Two different scenarios are considered in this paper: (i) a scenario in which only the input direction is specified. We present two methods to estimate the directional congestion for this scenario. If there exists the congestion along a certain input direction, both methods can find an output direction along which congestion occurs. (ii) a scenario based on the assumption that both input and output directions are not specified. We propose a system of inequalities to find input and output directions along which the congestion occurs in this scenario. This study addresses the relationship between our defined directional congestion and the classical definition of strong and weak congestion.

3. Our proposed methods for determining the directional congestion

In this section, we consider two different scenarios to recognize the directional congestion. In the first scenario only the input direction is specified. We present two methods to estimate the directional congestion for this scenario. One of them is based on solving linear programming problems and the other one is based on solving systems of inequalities. If a unit is directionally congested along a certain input direction, then our methods find an output direction along which congestion occurs. In the second scenario both input and output directions are not specified. We propose systems of inequalities to find the input and output directions along which congestion occurs in this scenario.

We consider P_{convex} in both scenarios and we investigate the congestion for the strong efficient unit with respect to P_{convex}.

3.1. The first scenario

Assume that the input direction vector $\vec{w} = (w_1, w_2, ..., w_m) \geq 0$ is specified by the DM along which the components of the input vector of DMU_o should be changed. First, the components of the input vector of DMU_o are decreased along $\vec{w}$ and we find an output direction vector $(\vec{\delta_o})$ along which the output components of DMU_o are increased. If there exists such an output direction vector, then DMU_o is directionally congested from the left along $\vec{w}$. Then, the components of the input vector of DMU_o are increased along $\vec{w}$ and we find an output direction vector $(\vec{\delta_o'})$ along which the output components of DMU_o are decreased. If there exists such an output direction vector, then DMU_o is directionally congested from the right along $\vec{w}$. If DMU_o is directionally congested from the left and right, then this unit is directionally congested along $\vec{w}$. Otherwise DMU_o is not directionally congested along the input direction vector $\vec{w}$.

We propose two methods in sections 2.1.1 and 2.1.2 to determine whether DMU_o is directionally congested along $\vec{w}$ or not.

In the first proposed method for the first scenario, we determine the output direction vectors along which the congestion occurs from the left and right by solving the systems of inequalities. In the second proposed method for the first scenario, we determine the output direction vectors along which the congestion occurs from the left and right by solving linear programming problems.

3.1.1. *The first method for the first scenario*

In this method, firstly, we consider the reduction in the components of the input vector of DMU_o along $\vec{w}$ and the increment in the components of the output vector of DMU_o along an output direction vector $(\vec{\delta_o})$ obtained by solving a system of inequalities. Afterwards, we consider the increment in the components of the input vector of DMU_o along $\vec{w}$ and the reduction in the components of the output vector of this unit along an output direction vector $(\vec{\delta_o'})$ obtained by solving another system of inequalities. In the following, we present Algorithm I to recognize the directional congestion in the first scenario.

Algorithm I

Step 1. Solve model (7) to know whether we can decrease the components of the input vector of DMU_o, along the input direction vector $\vec{w} = (w_1, w_2, ..., w_m)$ or not.

$$
\begin{aligned}
&\theta_o^* = \max \theta \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - \theta w_i x_{io}, && i = 1, ..., m, \qquad (7) \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geq (1+\beta) y_{ro}, && r = 1, ..., s, \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1, \\
&\qquad \lambda_j \geq 0, && j = 1, ..., n, \\
&\qquad \theta \geq 0, \quad \beta \text{ free}
\end{aligned}
$$

Theorem 1. If $\theta_o^* = 0$ at the optimality of model (7), then DMU_o is not directionally congested along $\vec{w} = (w_1, w_2, ..., w_m)$.

Proof. If $\theta_o^* = 0$, then the maximum possible decrease in the components of the input vector of DMU_o along $\vec{w} = (w_1, w_2, ..., w_m)$ is zero. This means that we cannot decrease inputs along $\vec{w}$. Hence, DMU_o is not directionally congested from the left along $\vec{w}$. Therefore, this unit is not directionally congested along $\vec{w} = (w_1, w_2, ..., w_m)$. □

If $\theta_o^* > 0$, then go to step 2.

Step 2. Consider the following system of inequalities:

$$
\begin{aligned}
&\sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - \alpha w_i x_{io}, && i = 1, ..., m, \quad (8)\\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \delta_{ro} y_{ro}, && r = 1, ..., s,\\
&\sum_{j=1}^{n} \lambda_j = 1,\\
&\sum_{r=1}^{s} \delta_{ro} = 1,\\
&\lambda_j \geq 0, && j = 1, ..., n,\\
&\delta_{ro} \geq 0, && r = 1, ..., s,\\
&0 \leq \alpha \leq \theta_o^*.
\end{aligned}
$$

If system (8) has no solution, then DMU_o is not directionally congested from the left along $\vec{w} = (w_1, w_2, ..., w_m)$ because there exists no direction vector along which the output components of DMU_o increase. Otherwise, there exists an output direction vector $\vec{\delta_o} = (\delta_{1o}, \delta_{2o}, ..., \delta_{so})$, obtained from system (8), such that the output components of DMU_o increase along it. This means that DMU_o is directionally congested from the left along the input direction $\vec{w} = (w_1, w_2, ..., w_m)$ and the output direction $\vec{\delta_o} = (\delta_{1o}, \delta_{2o}, ..., \delta_{so})$.

Go step 3 to determine the right-hand congestion:

Step 3. Solve model (9) to determine whether we can increase the components of the input vector of DMU_o, along the input direction vector $\vec{w} = (w_1, w_2, ..., w_m)$ or not.

$$
\begin{aligned}
&\theta_o^{'*} = \max \theta^{'}\\
&s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} + \theta^{'} w_i x_{io}, && i = 1, ..., m, \quad (9)\\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geq (1 + \beta) y_{ro}, && r = 1, ..., s,
\end{aligned}
$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$
$$\lambda_j \geq 0, \qquad j = 1, ..., n,$$
$$\theta^{'} \geq 0, \quad \beta \text{ free.}$$

Theorem 2. If $\theta_o^{'*} = 0$ at the optimality of model (9), then DMU_o is not directionally congested along $\vec{w} = (w_1, w_2, ..., w_m)$.

Proof. If $\theta_o^{'*} = 0$ then the maximum possible increase in the components of the input vector of DMU_o along $\vec{w} = (w_1, w_2, ..., w_m)$ is zero. This means that we cannot increase inputs along $\vec{w}$. Hence, DMU_o is not directionally congested from the right along $\vec{w}$. Therefore, this unit is not directionally congested along $\vec{w} = (w_1, w_2, ..., w_m)$.

□

If $\theta_o^{'*} > 0$, then go step 4.

Step 4. Consider the following system of inequalities:

$$\begin{aligned}
&\sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} + \alpha w_i x_{io}, && i = 1, ..., m, \qquad (10)\\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} - \delta_{ro}^{'} y_{ro}, && r = 1, ..., s,\\
&\sum_{j=1}^{n} \lambda_j = 1,\\
&\sum_{r=1}^{s} \delta_{ro}^{'} = 1,\\
&\lambda_j \geq 0, && j = 1, ..., n,\\
&\delta_{ro}^{'} \geq 0, && r = 1, ..., s,\\
&0 \leq \alpha \leq \theta_o^{'*}.
\end{aligned}$$

If system (10) has no solution, then DMU_o is not directionally congested from the right along $\vec{w} = (w_1, w_2, ..., w_m)$ because there exists no direction vector along which the output components of DMU_o decrease. Otherwise, there exists an output direction vector $\vec{\delta_o^{'}} = (\delta_{1o}^{'}, \delta_{2o}^{'}, ..., \delta_{so}^{'})$, obtained from system (10), such that the output components of DMU_o decrease along it. This means that DMU_o is directionally congested from the right along the input direction $\vec{w} = (w_1, w_2, ..., w_m)$ and the output direction vector $\vec{\delta_o^{'}} = (\delta_{1o}^{'}, \delta_{2o}^{'}, ..., \delta_{so}^{'})$. If DMU_o is directionally congested from the both sides, then, this unit is directionally congested along the input direction $\vec{w} = (w_1, w_2, ..., w_m)$.

In what follows, we propose the second method.

3.1.2. *The second method for the first scenario*

In the following, we present Algorithm II to recognize the directional congestion in the first scenario.

Algorithm II

Step 1. Solve model (7) to determine whether we can decrease the components of the input vector of DMU_o, along the input direction vector $\vec{w} = (w_1, w_2, ..., w_m)$ or not. As we said in Algorithm I, if $\theta_o^* = 0$ at the optimality of model (7), then DMU_o is not directionally congested from the left along $\vec{w} = (w_1, w_2, ..., w_m)$. Otherwise, go to step 2.

Step 2. Solve model (11) or the linearized form of it, namely model (12), to find the output direction vector $\vec{\delta_o} = (\delta_{1o}, ..., \delta_{so})$ along which the output components of DMU_o have the maximum possible increase:

$$
\begin{aligned}
\tau_o^* = \max \ & \big(\min_{1 \le r \le s} \delta_{ro}\big) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - \alpha w_i x_{io}, && i = 1, ..., m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} + \delta_{ro} y_{ro}, && r = 1, ..., s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \sum_{r=1}^{s} \delta_{ro} = 1, \\
& \lambda_j \ge 0, && j = 1, ..., n, \\
& \delta_{ro} \ge 0, && r = 1, ..., s, \\
& 0 \le \alpha \le \theta_o^*.
\end{aligned} \tag{11}
$$

Model (11) can be transformed into linear programming model (12) by introducing the variable $\tau = \min_{1 \le r \le s} \delta_{ro}$:

$$
\begin{aligned}
\tau_o^* = \max \ & \tau \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - \alpha w_i x_{io}, && i = 1, ..., m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} + \delta_{ro} y_{ro}, && r = 1, ..., s, \\
& \sum_{j=1}^{n} \lambda_j = 1,
\end{aligned} \tag{12}
$$

$$\sum_{r=1}^{s} \delta_{ro} = 1,$$
$$\tau \leq \delta_{ro}, \qquad r = 1, ..., s,$$
$$\lambda_j \geq 0, \qquad j = 1, ..., n,$$
$$\delta_{ro} \geq 0, \qquad r = 1, ..., s,$$
$$\tau \geq 0,$$
$$0 \leq \alpha \leq \theta_o^*.$$

The feasible set of model (12) is as follows:

$$S_1 = \{(\alpha, \tau, \vec{\lambda}, \vec{\delta_o}) | \vec{\lambda} X \leq x_o - \alpha \vec{w} x_o, \vec{\lambda} Y \geq y_o + \vec{\delta_o} y_o, \vec{1}.\vec{\lambda} = 1, \vec{1}.\vec{\delta_o} = 1, \vec{\lambda} \in R^n_{\geq}, \vec{\delta_o} \in R^s_{\geq}, \alpha \leq \theta_o^*, \vec{\tau} \leq \vec{\delta_o}\}. \qquad (13)$$

where $\vec{\tau} = (\tau, ..., \tau) \in R^s$ and also $X = [x_1, x_2, ..., x_n]_{m \times n}$ and $Y = [y_1, y_2, ..., y_n]_{s \times n}$ are input and output matrices, respectively.

Therefore, there are two the following cases:

1) If $S_1 = \emptyset$, then there exists no output direction vector along which the components of the output vector of DMU_o increase. Therefore DMU_o is not directionally congested from the left along $\vec{w} = (w_1, w_2, ..., w_m)$, therefore it is not directionally congested along $\vec{w}$.

2) Let $S_1 \neq \emptyset$ and $(\alpha^*, \tau^*, \vec{\lambda^*}, \vec{\delta_o^*})$ be an optimal solution for model (12). Therefore $(\delta^*_{1o}, \delta^*_{2o}, ..., \delta^*_{so})$ is the output direction vector along which the components of the output vector of DMU_o have the maximum possible increase. This means that DMU_o is directionally congested from the left along $\vec{w} = (w_1, w_2, ..., w_m)$, and go to step 3.

Step 3. Solve model (9) to determine whether we can increase the components of the input vector of DMU_o, along the input direction vector $\vec{w} = (w_1, w_2, ..., w_m)$ or not. As we said in Algorithm I, if $\theta_o^{'*} = 0$ at the optimality of model (9), then DMU_o is not directionally congested from the right along $\vec{w} = (w_1, w_2, ..., w_m)$. Otherwise, go to step 4.

Step 4. Solve model (14) or the linearized form of it, namely model (15), to find the output direction vector $\vec{\delta_o'} = (\delta'_{1o}, ..., \delta'_{so})$ along which the output components of DMU_o have the maximum possible decrease:

$$\tau_o^{'*} = \max(\min_{1 \leq r \leq s} \delta'_{ro})$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} + \alpha w_i x_{io}, \qquad i = 1, ..., m, \qquad (14)$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} - \delta'_{ro} y_{ro}, \qquad r = 1, ..., s,$$
$$\sum_{j=1}^{n} \lambda_j = 1,$$
$$\sum_{r=1}^{s} \delta'_{ro} = 1,$$
$$\lambda_j \geq 0, \qquad j = 1, ..., n,$$
$$\delta'_{ro} \geq 0, \qquad r = 1, ..., s,$$
$$0 \leq \alpha \leq \theta'^{*}_{o}.$$

Model (14) can be transformed into linear programming model (15) by introducing the variable $\tau' = \min_{1 \leq r \leq s} \delta'_{ro}$:

$$\tau'^{*}_{o} = \max \tau'$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} + \alpha w_i x_{io}, \qquad i = 1, ..., m, \qquad (15)$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} - \delta'_{ro} y_{ro}, \qquad r = 1, ..., s,$$
$$\sum_{j=1}^{n} \lambda_j = 1,$$
$$\sum_{r=1}^{s} \delta'_{ro} = 1,$$
$$\tau' \leq \delta'_{ro}, \qquad r = 1, ..., s,$$
$$\tau' \geq 0,$$
$$\lambda_j \geq 0, \qquad j = 1, ..., n,$$
$$\delta'_{ro} \geq 0, \qquad r = 1, ..., s,$$
$$0 \leq \alpha \leq \theta'^{*}_{o}.$$

The feasible set of model (15) is as follows:

$$S_2 = \{(\alpha, \vec{\tau'}, \vec{\lambda}, \vec{\delta'_o}) | \vec{\lambda} X \leq x_o + \alpha \vec{w} x_o, \vec{\lambda} Y \geq y_o - \vec{\delta'_o} y_o, \vec{1}.\vec{\lambda} = 1, \vec{1}.\vec{\delta'_o} = 1, \vec{\lambda} \in R^n_{\geq}, \vec{\delta'_o} \in R^s_{\geq}, \alpha \leq \theta'^{*}_{o}, \vec{\tau'} \leq \vec{\delta'_o}\}. \qquad (16)$$

where $\vec{\tau'} = (\tau', ..., \tau')$.
Therefore, there are two the following cases:

1) If $S_2 = \emptyset$, then there exists no output direction vector along which the components of the output vector of DMU_o decrease. Therefore, DMU_o is not directionally congested from the right along $\vec{w} = (w_1, w_2, ..., w_m)$ and it is not

directionally congested along $\vec{w}$.

2) Let $S_2 \neq \emptyset$ and $(\alpha^*, \tau^{'*}, \vec{\lambda^*}, \vec{\delta_o^{'}})$ be an optimal solution for model (15). Therefore $(\delta_{1o}^{'*}, \delta_{2o}^{'*}, ..., \delta_{so}^{'*})$ is the output direction vector along which the components of the output vector of DMU_o have the maximum possible decrease. This means that DMU_o is directionally congested from the right along $\vec{w} = (w_1, w_2, ..., w_m)$.

Now, we show the relationship between our defined directionally congestion and the classical definitions for the strong and weak congestion.

Theorem 3. If DMU_o is directionally congested from the left along the input direction vector $\vec{w} = (w_1, w_2, ..., w_m)$, it is weakly congested.

Proof. If DMU_o is directionally congested from the left along $\vec{w} = (w_1, w_2, ..., w_m)$ then the components of the input vector of DMU_o decrease along $\vec{w}$ and there exists an output direction vector $\vec{\delta_o} = (\delta_{1o}, \delta_{2o}, ..., \delta_{so})$ along which the components of the output vector of DMU_o are increased. Since, some components of $\vec{w} = (w_1, w_2, ..., w_m)$ and $\vec{\delta_o} = (\delta_{1o}, \delta_{2o}, ..., \delta_{so})$ may be zero, there exists at least one activity in P_{convex} that uses less resources in some inputs to make more products in some outputs. In other words, DMU_o is weakly congested.

□

Theorem 4. Suppose that DMU_o is directionally congested from the left along the input direction vector $\vec{w} = (w_1, w_2, ..., w_m) > 0$. If the components of the output direction vector, obtained from system (8) or model (12), are positive, i.e., $\vec{\delta_o} = (\delta_{1o}, \delta_{2o}, ..., \delta_{so}) > 0$, then DMU_o is strongly congested.

Proof. Since $\vec{w} = (w_1, w_2, ..., w_m) > 0$ and DMU_o is directionally congested from the left, therefore, all input components are reduced along $\vec{w}$. Suppose that the components of the output direction vector, obtained by Algorithm I and Algorithm II, are positive, i.e., $\vec{\delta_o} = (\delta_{1o}, \delta_{2o}, ..., \delta_{so}) > 0$. This means that, all output components are increased along $\vec{\delta_o}$. In other words, by decreasing all input components of DMU_o along $\vec{w}$, all output components of this unit are increased along the output direction vector $\vec{\delta_o} = (\delta_{1o}, \delta_{2o}, ..., \delta_{so})$. Therefore, DMU_o is strongly congested.

□

3.2. The second scenario

Now, we consider the scenario where none of input or output directions are specified. At the first step, we try to find an input direction vector $\vec{w_o} = (w_{1o}, w_{2o}, ..., w_{mo})$ and an output direction vector $\vec{\delta_o} = (\delta_{1o}, \delta_{2o}, ..., \delta_{so})$ such that the input components of DMU_o decrease along $\vec{w_o}$, and the output components of DMU_o increase along $\vec{\delta_o}$. If such vectors exist, then DMU_o is directionally congested from the left. At the second step, we try to find an input direction vector

$\vec{w_o'} = (w_{1o}', w_{2o}', ..., w_{mo}')$ and an output direction vector $\vec{\delta_o'} = (\delta_{1o}', \delta_{2o}', ..., \delta_{so}')$ such that the input components of DMU_o increase along $\vec{w_o'}$ and the output components of DMU_o decrease along $\vec{\delta_o'}$. If such vectors exist, then DMU_o is directionally congested from the right. Finally, if DMU_o is directionally congested from both sides, then this unit is directionally congested. The following algorithm recognizes the directional congestion.

Algorithm III

Step 1: Solve system (17):

$$\begin{aligned}
&\sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - w_{io}x_{io}, && i = 1, ..., m, && (17)\\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \delta_{ro}y_{ro}, && r = 1, ..., s,\\
&\sum_{j=1}^{n} \lambda_j = 1,\\
&\sum_{r=1}^{s} \delta_{ro} = 1,\\
&\sum_{i=1}^{m} w_{io} = 1,\\
&\lambda_j \geq 0, && j = 1, ..., n,\\
&\delta_{ro} \geq 0, && r = 1, ..., s,\\
&w_{io} \geq 0, && i = 1, ..., m.
\end{aligned}$$

If system (17) has no solution, then DMU_o is not directionally congested from the left and consequently it is not directionally congested along any input and output directions. Otherwise, suppose that $\vec{w_o} = (w_{1o}, w_{2o}, ..., w_{mo})$ and $\vec{\delta_o} = (\delta_{1o}, \delta_{2o}, ..., \delta_{so})$ are the solutions of system (17). Therefore, the components of the output vector of DMU_o can not decrease along $\vec{\delta_o}$ and the components of the input vector of DMU_o can not increase along $\vec{w_o}$. In other words, there exists an activity belongings to P_{convex} that uses less resources in one or more inputs to make more products in one or more outputs. Therefore DMU_o is directionally congested from the left along the directions $\vec{w_o} = (w_{1o}, w_{2o}, ..., w_{mo})$ and $\vec{\delta_o} = (\delta_{1o}, \delta_{2o}, ..., \delta_{so})$

Step 2: Solve system (18):

$$\sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} + w_{io}'x_{io}, \qquad i = 1, ..., m, \qquad (18)$$

$$\sum_{j=1}^{n}\lambda_j y_{rj} \geq y_{ro} - \delta^{'}_{ro} y_{ro}, \qquad r = 1,...,s,$$
$$\sum_{j=1}^{n}\lambda_j = 1,$$
$$\sum_{r=1}^{s}\delta^{'}_{ro} = 1,$$
$$\sum_{i=1}^{m} w^{'}_{io} = 1,$$
$$\lambda_j \geq 0, \qquad j = 1,...,n,$$
$$\delta^{'}_{ro} \geq 0, \qquad r = 1,...,s,$$
$$w^{'}_{io} \geq 0, \qquad i = 1,...,m.$$

If system (18) has no solution, then DMU_o is not directionally congested from the right and consequently it is not directionally congested along any input and output directions. Otherwise, suppose that $\vec{w^{'}_o} = (w^{'}_{1o}, w^{'}_{2o}, ..., w^{'}_{mo})$ and $\vec{\delta^{'}_o} = (\delta^{'}_{1o}, \delta^{'}_{2o}, ..., \delta^{'}_{so})$ are the solutions of system (18). Therefore, the components of the output vector of DMU_o can not increase along $\vec{\delta^{'}_o}$ and the components of the input vector of DMU_o can not decrease along $\vec{w^{'}_o}$. Therefore DMU_o is directionally congested from the right along the directions $\vec{w^{'}_o} = (w^{'}_{1o}, w^{'}_{2o}, ..., w^{'}_{mo})$ and $\vec{\delta^{'}_o} = (\delta^{'}_{1o}, \delta^{'}_{2o}, ..., \delta^{'}_{so})$. If DMU_o is directionally congested on both sides then this unit is directionally congested.

In the next section, we illustrate our proposed algorithm for the various scenarios in the case studies from Tone and Sahoo [21] and Yang [28].

4. Numerical examples

This section uses two numerical examples taken from DEA literature to compare our proposed method for identifying the directional congestion of units with the existing methods.

Example 1. In this example, the results of applying the proposed approach to the data set of chain stores in Japan for a period of 27 years from 1957 trough 2001, reported in Tone and Sahoo [21], are presented. This dataset has two inputs, the number of stores (x_1) and the total area of stores (x_2) (unit: 1000 m^2) and one output, annual sales y_1 (unit: hundred million yen). The input and output data are reported in Table 1.

The last column of Table 1 shows the strongly and weakly congested units obtained by applying TS method.

Table 1. Chain stores data set.

DMU	x_1	x_2	y_1	Congestion status
1	2412	5480	41091	-
2	3163	6233	48367	Weak
3	3350	6798	56000	Weak
4	3371	7274	60940	-
5	3778	7992	69046	-
6	4020	8500	77347	-
7	5029	9246	85805	-
8	5164	9639	90433	-
9	5285	9981	95640	-
10	5618	10276	100257	-
11	5981	10521	105944	Weak
12	6217	10766	109857	Weak
13	6455	11144	116114	Weak
14	6674	11418	125404	Weak
15	6829	11717	131862	Weak
16	6995	11987	140817	-
17	7338	12463	150583	-
18	7946	13426	152943	Weak
19	8236	14147	155128	Weak
20	7722	15014	158714	Weak
21	7727	15022	161739	Weak
22	7822	16191	169786	-
24	7531	16969	167195	Strong
24	7201	17627	167187	-
25	7281	18364	165480	Strong
26	7053	19698	162847	Weak
27	6067	16176	154671	-

We use model (4) and Equation (5) to determine the strongly efficient units with respect to p_{convex} and the projection point of each DMU on the strongly efficient frontier of p_{convex}, respectively, then we use the projection point of the unit to determine the directional congestion of it in our proposed methods.

In the first scenario of our proposed methods, we determine the directional congestion of units along two input direction vectors $\vec{w_1} = (0.3, 0.7)$ and $\vec{w_2} = (0.9, 0.1)$ by applying Algorithm I and Algorithm II, respectively. The results are summarized in Table 2 and Table 3.

Table 2 reports the results of applying Algorithm I for the data set of chain stores.

Step 1. We solve model (7) to determine the maximum possible decrease in the components of the input vector of units along the input direction vector $\vec{w_1} = (0.3, 0.7)$. The second column of Table 2 shows the optimal value of this model.

If $\theta_o^* = 0$ then DMU_o is not directionally congested from the left along the input direction vector $\vec{w_1} = (0.3, 0.7)$. Otherwise, we go to Step 2.

Table 2. The directional congestion of units along $\vec{w_1} = (0.3, 0.7)$.

DMU	θ_o^*	$\vec{\delta_o}$	LHC	$\theta_o^{'*}$	$\vec{\delta_o'}$	RHC
1	0.000	-	No	0.000	-	No
2	0.000	-	No	0.000	-	No
3	0.088	-	No	0.000	-	No
4	0.204	-	No	0.574	-	No
5	0.217	-	No	0.788	-	No
6	0.240	-	No	0.845	-	No
7	0.061	-	No	1.474	-	No
8	0.101	-	No	1.352	-	No
9	0.132	-	No	1.251	-	No
10	0.089	-	No	1.073	-	No
11	0.027	-	No	0.905	-	No
12	0.005	-	No	0.795	-	No
13	0.010	-	No	0.678	-	No
14	0.000	-	No	0.584	-	No
15	0.011	-	No	0.512	-	No
16	0.013	-	No	0.442	-	No
17	0.000	-	No	0.312	-	No
18	0.000	-	No	0.102	-	No
19	0.000	-	No	0.000	-	No
20	0.284	-	No	0.076	1.000	Yes
21	0.284	-	No	0.075	1.000	Yes
22	0.388	-	No	0.000	-	No
23	0.502	1.000	Yes	0.025	1.000	Yes
24	0.594	-	No	0.063	1.000	Yes
25	0.628	1.000	Yes	0.013	1.000	Yes
26	0.724	-	No	0.000	-	No
27	0.655	-	No	0.087	1.000	Yes

Step 2. We solve system (8) to obtain the output direction vector $\vec{\delta_o} = (\delta_{1o}, ..., \delta_{so})$ along which the left-hand congestion occurs. The third column of Table 2 shows the output direction vectors for all units obtained by system (8). If system (8) has no solution for DMU_o then this unit is not directionally congested from the left along $\vec{w_1} = (0.3, 0.7)$. The fourth column of Table 2 shows the status of the left-hand directional congestion of units (LHC).

Step 3. We solve model (9) to determine the maximum possible increase in the components of the input vector of DMU_o along the input direction vector $\vec{w_1} = (0.3, 0.7)$. The fifth column of Table 2 shows the optimal value of this model.

If $\theta_o^{'*} = 0$ then DMU_o is not directionally congested from the right along the input direction vector $\vec{w_1} = (0.3, 0.7)$. Otherwise, we go to Step 4.

Step 4. We solve system (10) to obtain the output direction vector $\vec{\delta_o'} = (\delta_{1o}', ..., \delta_{so}')$ along which the right-hand congestion occurs. Column 6 of Table 2 shows the output direction vectors for all units obtained by system (10). If system (10) has no solution for DMU_o then this unit is not directionally congested from the right along $\vec{w_1} = (0.3, 0.7)$. Column 7 shows the status of the right-hand directional congestion of units (RHC).

Table 3. The directional congestion of units along $\vec{w_2} = (0.9, 0.1)$.

DMU	θ_o^*	$\vec{\delta_o}$	LHC	$\theta_o^{'*}$	$\vec{\delta_o'}$	RHC
1	0.000	-	No	0.000	-	No
2	0.191	-	No	0.000	-	No
3	0.182	-	No	0.000	-	No
4	0.134	-	No	0.196	1.000	Yes
5	0.174	-	No	0.202	1.000	Yes
6	0.186	-	No	0.223	1.000	Yes
7	0.329	-	No	0.045	1.000	Yes
8	0.322	-	No	0.076	1.000	Yes
9	0.317	-	No	0.102	1.000	Yes
10	0.348	-	No	0.065	1.000	Yes
11	0.382	-	No	0.018	1.000	Yes
12	0.397	-	No	0.003	1.000	Yes
13	0.403	-	No	0.007	1.000	Yes
14	0.413	-	No	0.000	-	No
15	0.413	-	No	0.007	1.000	Yes
16	0.417	-	No	0.007	1.000	Yes
17	0.427	-	No	0.000	-	No
18	0.438	-	No	0.000	-	No
19	0.431	-	No	0.000	-	No
20	0.340	-	No	0.047	1.000	Yes
21	0.340	-	No	0.046	1.000	Yes
22	0.294	-	No	0.000	-	No
23	0.220	-	No	0.017	1.000	Yes
24	0.140	-	No	0.045	1.000	Yes
25	0.111	1.000	Yes	0.009	1.000	Yes
26	0.000	-	No	0.000	-	No
27	0.033	-	No	0.302	1.000	Yes

Now, we determine the directional congested units along $\vec{w_2} = (0.9, 0.1)$ by applying Algorithm II. The results are summarized in Table 3. The second column of Table 3 reports the optimal value of model (7) with $\vec{w_2} = (0.9, 0.1)$. The third column shows the output direction vectors $\vec{\delta_o} = (\delta_{1o}, ..., \delta_{so})$ obtained by solving model (12). Column 4 shows the status of the left-hand directional congestion of units along $\vec{w_2} = (0.9, 0.1)$. The fifth column reports the optimal value of model (9) with $\vec{w_2} = (0.9, 0.1)$. Columns 6 shows the output direction vectors $\vec{\delta_o^{'}} = (\delta_{1o}^{'}, ..., \delta_{so}^{'})$ obtained by solving model (15). Finally, Column 7 shows the status of the right-hand directional congestion of units along $\vec{w_2} = (0.9, 0.1)$.

As we saw in Table 1, DMU_{23} and DMU_{25} are strongly congested by TS method. Also, in our proposed method, we obtain the output direction vector $\vec{\delta_o} > 0$, reported in Table 2 and Table 3, for $o \in \{23, 25\}$, so, according to Theorem 4, $DMU23$ and DMU_{25} are strongly congested units by our proposed methods. This shows the relationship between our proposed methods in the first scenario and the conventional methods for recognizing the congestion in the literature.

Table 4. The directional congestion of units in the second scenario.

DMU	$\vec{w_o}$	$\vec{\delta_o}$	LHC	$\vec{w_o^{'}}$	$\vec{\delta_o^{'}}$	RHC
1	-	-	No	-	-	No
2	(1.0, 0.0)	(1.0)	Yes	-	-	No
3	(1.0, 0.0)	(1.0)	Yes	-	(1.0)	No
4	-	-	No	(1.0, 0.0)	(1.0)	Yes
5	-	-	No	(0.0, 1.0)	(1.0)	Yes
6	-	-	No	(0.0, 1.0)	(1.0)	Yes
7	-	-	No	(0.0, 1.0)	(1.0)	Yes
8	-	-	No	(1.0, 0.0)	(1.0)	Yes
9	-	-	No	(1.0, 0.0)	(1.0)	Yes
10	-	-	No	(1.0, 0.0)	(1.0)	Yes
11	(1.0, 0.0)	(1.0)	Yes	(0.0, 1.0)	(1.0)	Yes
12	(1.0, 0.0)	(1.0)	Yes	(0.0, 1.0)	(1.0)	Yes
13	(1.0, 0.0)	(1.0)	Yes	(0.0, 1.0)	(1.0)	Yes
14	(1.0, 0.0)	(1.0)	Yes	(0.0, 1.0)	(1.0)	Yes
15	(1.0, 0.0)	(1.0)	Yes	(0.0, 1.0)	(1.0)	Yes
16	-	-	No	(0.0, 1.0)	(1.0)	Yes
17	-	-	No	(0.0, 1.0)	(1.0)	Yes
18	(1.0, 0.0)	(1.0)	Yes	(0.0, 1.0)	(1.0)	Yes
19	(1.0, 0.0)	(1.0)	Yes	-	-	No
20	(1.0, 0.0)	(1.0)	Yes	(0.0, 1.0)	(1.0)	Yes
21	(1.0, 0.0)	(1.0)	Yes	(0.0, 1.0)	(1.0)	Yes
22	-	-	No	-	-	No
23	(0.2, 0.8)	(1.0)	Yes	(0.0, 1.0)	(1.0)	Yes
24	-	-	No	(0.0, 1.0)	(1.0)	Yes
25	(0.0, 1.0)	(1.0)	Yes	(0.0, 1.0)	(1.0)	Yes
26	(0.0, 1.0)	(1.0)	Yes	-	-	No
27	-	-	No	(0.0, 1.0)	(1.0)	Yes

In the second scenario of our proposed method, none of the input and output directions are available. We apply algorithm III to recognize the directional congestion of the data set of chain stores. The results are summarized in Table 4.

Step 1. we solve system (17) and obtain the input direction vector $\vec{w_o}$ and the output direction vector $\vec{\delta_o}$, reported in columns 2 and 3 of Table 4, respectively. Column 4 shows the status of the left-hand directional congestion of units. As we saw in Table 1, the units 2, 3, 11, 12, 13, 14, 15, 18, 19, 20, 21, 26 are weakly congested and the units 23 and 25 are stronly congested by TS method. On the other hand, in our proposed method for the second scenario, we obtain the input direction vector $\vec{w_o} \geqq 0$ and the output direction vector $\vec{\delta_o} \geqq 0$ for all $o \in \{2, 3, 11, 12, 13, 14, 15, 18, 19, 20, 21, 26\}$, so, according to Theorem 3, the units 2, 3, 11, 12, 13, 14, 15, 18, 19, 20, 21, 26 are weakly congested by our proposed method. Also, we obtain the input direction vector $\vec{w_o} > 0$ and the output direction vector $\vec{\delta_o} > 0$ for $o \in \{23, 25\}$, so, according to Theorem 4, DMU_{23} and DMU_{25} are strongly congested units by our proposed method. This shows the relationship between our proposed method and the conventional methods for recognizing the congestion in the literature.

Step 2. we solve system (18) and obtain the input direction vector $\vec{w}'_o$ and the output direction vector $\vec{\delta}'_o$, reported in the fifth column of Table 4. Column 6 shows the status of the right-hand directional congestion of DMUs.

Example 2. In this example, the results of applying the proposed approach to the data set of basic research institutes in the Chinese Academy of Science (CAS) in 2010, reported in Yang [28], are presented. This dataset has 16 units with

Table 5. The data of units in Example 2.

DMU	x_1	x_2	y_1	y_2	y_3	y_4	Congestion status
1	252	117.945	436	133	184	31.558	No
2	37	29.431	243	127	43	15.3041	No
3	240	101.425	164	70	89	33.8365	weak
4	356	368.483	810	276	247	183.8434	No
5	310	195.862	200	55	111	12.9342	No
6	201	188.829	104	49	33	60.7366	No
7	157	131.301	113	49	45	72.5368	No
8	236	77.439	8	1	44	23.7015	Weak
9	805	396.905	371	118	89	216.9885	Strong
10	886	411.539	607	216	168	88.5561	Strong
11	623	221.428	314	49	89	45.3597	Strong
12	560	264.341	261	79	131	41.1156	Strong
13	1344	900.509	627	168	346	645.4150	No
14	508	344.312	971	518	335	205.4528	No
15	380	161.331	395	180	117	90.0373	Weak
16	132	83.972	229	138	62	32.6111	Weak

two inputs, the full-time equivalent of full-time research staff (x_1) and the amount of total income of each institute (x_2) to produce four outputs, the number of international papers indexed by Web of Science from Thompson Reuters (y_1), the number of highly-quality papers published in top research journals (y_2), the number of graduate student enrollment in 2009 (y_3) and the amount of external research funding from research contracts (y_4). The input and output data are reported in Table 5.

We use model (4) and Equation (5) to determine the strongly efficient units with respect to p_{convex} and the projection point of each DMU on the strongly efficient frontier of p_{convex}, respectively, then we use the projection point of DMU_o to determine the directional congestion of it in our proposed methods.

In the first scenario of our proposed methods, we determine the directional congestion of units along two input direction vectors $\vec{w_1} = (1.7, 0.3)$ and $\vec{w_2} = (1.1, 0.9)$ by applying Algorithm I and Algorithm II, respectively. The results are summarized in Tables 6 and 7, respectively.

Table 6. The directional congestion of units along $\vec{w_1} = (1.7, 0.3)$.

DMU	θ_o^*	$\vec{\delta_o}$	LHC	$\theta_o^{'*}$	$\vec{\delta_o^{'}}$	RHC
1	0.333	-	No	0.279	(0.4, 0.2, 0.4, 0.0)	Yes
2	0.000	-	No	0.000	-	No
3	0.357	(0.3, 0.6, 0.0, 0.1)	Yes	0.185	(0.1, 0.1, 0.5, 0.3)	Yes
4	0.000	-	No	0.976	(0.4, 0.4, 0.2, 0.0)	Yes
5	0.247	-	No	0.623	(0.3, 0.3, 0.3, 0.1)	Yes
6	0.049	-	No	1.431	(0.3, 0.3, 0.3, 0.1)	Yes
7	0.105	-	No	1.390	(0.3, 0.3, 0.2, 0.2)	Yes
8	0.406	(0.1, 0.7, 0.1, 0.1)	Yes	0.000	-	No
9	0.337	(0.2, 0.3, 0.4, 0.1)	Yes	0.051	(0.0, 0.8, 0.0, 0.2)	Yes
10	0.352	(0.1, 0.3, 0.2, 0.4)	Yes	0.000	-	No
11	0.407	(0.1, 0.6, 0.1, 0.2)	Yes	0.000	-	No
12	0.344	(0.2, 0.2, 0.4, 0.2)	Yes	0.145	(0.3, 0.3, 0.1, 0.3)	Yes
13	0.000	-	No	0.000	-	No
14	0.228	-	No	0.395	(0.2, 0.3, 0.2, 0.3)	Yes
15	0.365	(0.3, 0.5, 0.2, 0.0)	Yes	0.158	(0.2, 0.3, 0.2, 0.3)	Yes
16	0.218	-	No	0.776	(0.2, 0.3, 0.3, 0.2)	Yes

Table 6 reports the results of applying Algorithm I for the data set of 16 basic research institutes in CAS. The second column of Table 6 reports the optimal value of model (7) with $\vec{w_1} = (1.7, 0.3)$. The third column shows the output direction vectors $\vec{\delta_o} = (\delta_{1o}, ..., \delta_{so})$ obtained by solving system (8). Column 4 shows the status of the left-hand directional congestion of units along $\vec{w_1} = (1.7, 0.3)$. The fifth column reports the optimal value of model (9) with $\vec{w_1} = (1.7, 0.3)$. Column 6 shows the output direction vectors $\vec{\delta_o^{'}} = (\delta_{1o}^{'}, ..., \delta_{so}^{'})$ obtained by solving system (10). Finally, Column 7 shows the status of the right-hand directional congestion of units along $\vec{w_1} = (1.7, 0.3)$. As we saw in Table 6, the units 8, 9, 10, 11, 12 are strongly congested and the units 3, 15, 16 are weakly congested by TS method. In our proposed method, we obtain the output direction vector $\vec{\delta_o} > 0$ for all $o \in \{8, 9, 10, 11, 12\}$, so, according to Theorem 4, the units 8, 9, 10, 11, 12 are strongly congested by our proposed method. On the other hand, we obtain the output direction vector $\vec{\delta_o} \geqq 0$ for all $o \in \{3, 15, 16\}$, so, according to Theorem 3, the units 3, 15, 16 are weakly congested by our proposed method. This shows the relationship between our proposed method and the conventional methods for recognizing the congestion in the literature.

Table 7. The directional congestion of units along $\vec{w_2} = (1.1, 0.9)$.

DMU	θ_o^*	$\vec{\delta_o}$	LHC	$\theta_o^{'*}$	$\vec{\delta_o^{'}}$	RHC
1	0.743	-	No	1.748	(0.0, 0.2, 0.8, 0.0)	Yes
2	0.000	-	No	30.906	(0.4, 0.5, 0.1, 0.0)	Yes
3	0.759	-	No	1.576	(0.3, 0.3, 0.2, 0.2)	Yes
4	0.000	-	No	0.000	-	No
5	0.665	-	No	2.126	(0.3, 0.4, 0.3, 0.0)	Yes
6	0.229	-	No	1.890	(0.3, 0.3, 0.3, 0.1)	Yes
7	0.393	-	No	5.165	(0.3, 0.3, 0.3, 0.1)	Yes
8	0.000	-	No	0.000	-	No
9	0.769	-	No	0.122	(0.0, 0.0, 0.0, 1.0)	Yes
10	0.782	(0.1, 0.3, 0.2, 0.4)	Yes	0.000	-	No
11	0.814	(0.3, 0.5, 0.1, 0.1)	Yes	0.000	-	No
12	0.77	-	No	0.427	(0.3, 0.4, 0.2, 0.1)	Yes
13	0.681	-	No	0.000	-	No
14	0.654	-	No	1.731	(0.2, 0.4, 0.2, 0.2)	Yes
15	0.778	-	No	0.736	(0.3, 0.3, 0.1, 0.3)	Yes
16	0.590	-	No	6.010	(0.3, 0.3, 0.3, 0.1)	Yes

Now, we determine the directional congested units along $\vec{w_2} = (1.1, 0.9)$ by applying Algorithm II. The results are summarized in Table 7. The second column of Table 7 reports the optimal value of model (7) with $\vec{w_2} = (1.1, 0.9)$. The third column shows the output direction vectors $\vec{\delta_o} = (\delta_{1o}, ..., \delta_{so})$ obtained by solving model (12). Column 4 shows the status of the left-hand directional congestion of units along $\vec{w_2} = (1.1, 0.9)$. The fifth column reports the optimal value of model (9) with $\vec{w_2} = (1.1, 0.9)$. Columns 6 shows the output direction vectors $\vec{\delta_o^{'}} = (\delta_{1o}^{'}, ..., \delta_{so}^{'})$ obtained by solving model (15). Finally, Column 7 shows the status of the right-hand directional congestion of units along $\vec{w_2} = (1.1, 0.9)$. As we see in Table 7, we obtain the output direction vector $\vec{\delta_o} > 0$ for $o \in \{10, 11\}$, so, according to Theorem 4, DMU_{10} and DMU_{11} are strongly congested units by our proposed method and TS method.

In the second scenario of our proposed method, none of the input and output directions are available. We apply algorithm III to recognize the directional congestion of the data set of 16 basic institutes in CAE. The results are summarized in Table 8.

Table 8. The directional congestion of 16 units in the second scenario.

DMU	$\vec{w_o}$	$\vec{\delta_o}$	LHC	$\vec{w_o'}$	$\vec{\delta_o'}$	RHC
1	-	-	No	(0.4, 0.6)	(1.0, 0.0, 0.0, 0.0)	Yes
2	-	-	No	(0.6, 0.4)	(0.0, 1.0, 0.0, 0.0)	Yes
3	(0.5, 0.5)	(0.0, 0.3, 0.6, 0.1)	Yes	(0.4, 0.6)	(0.0, 0.0, 0.0, 1.0)	Yes
4	-	-	No	(0.7, 0.3)	(0.4, 0.6, 0.0, 0.0)	Yes
5	-	-	No	(0.5, 0.5)	(0.0, 1.0, 0.0, 0.0)	Yes
6	-	-	No	(0.6, 0.4)	(0.0, 1.0, 0.0, 0.0)	Yes
7	-	-	No	(0.6, 0.4)	(1.0, 0.0, 0.0, 0.0)	Yes
8	(0.6, 0.4)	(0.1, 0.5, 0.3, 0.1)	Yes	(0.3, 0.7)	-	No
9	(0.8, 0.2)	(0.2, 0.3, 0.4, 0.1)	Yes	(0.5, 0.5)	(0.0, 0.0, 1.0, 0.0)	Yes
10	(0.3, 0.7)	(0.1, 0.3, 0.2, 0.4)	Yes	(0.3, 0.7)	(0.0, 1.0, 0.0, 0.0)	Yes
11	(0.5, 0.5)	(0.1, 0.6, 0.1, 0.2)	Yes	(0.3, 0.7)	-	No
12	(0.3, 0.7)	(0.2, 0.4, 0.2, 0.2)	Yes	(0.4, 0.6)	(0.0, 1.0, 0.0, 0.0)	Yes
13	-	-	No	-	-	No
14	-	-	No	(0.5, 0.5)	(0.3, 0.7, 0.0, 0.0)	Yes
15	(1.0, 0.0)	(0.3, 0.4, 0.2, 0.1)	Yes	(0.4, 0.4)	(0.0, 1.0, 0.0, 0.0)	Yes
16	(1.0, 0.0)	(0.3, 0.4, 0.2, 0.1)	Yes	(0.4, 0.6)	(0.0, 1.0, 0.0, 0.0)	Yes

In the first step, we solve system (17) and obtain the input direction vector $\vec{w_o}$ and the output direction vector $\vec{\delta_o}$, reported in the second and third columns of Table 8. Column 4 shows the status of the left-hand directional congestion of units. It should be noted that, we obtain the input direction vector $\vec{w_o} > 0$ and the output direction vector $\vec{\delta_o} > 0$ for all $o \in \{8, 9, 10, 11, 12\}$, so, according to Theorem 4, the units 8, 9, 10, 11 and 12 are strongly congested by our proposed method in the second scenario and TS method.

In the next step, we solve system (18) and obtain the input direction vector $\vec{w_o'}$ and the output direction vector $\vec{\delta_o'}$, reported in columns 5 and 6. Column 7 shows the status of the right-hand directional congestion of DMUs. Therefore, the units 3, 9, 10, 12, 15 and 16 are directionally congested.

It should be noted that, Yang [28] only considers the situation that both the input and output direction vectors are predetermined. In the other word, if a unit has no directional congestion along these certain directions, then we cannot obtain any information about the existence of the congestion of this unit. He set the output direction vector as $\vec{\delta} = (1, 1, 1, 1)$ and determined the congestion of 16 basic institutes in CAE along different input direction vectors. For example, he recognized that DMU_1 has no congestion along the input direction vector $\vec{w_1} = (1.7, 0.3)$ and the output direction vector $\vec{\delta}$. Also, he showed that DMU_{15}

has no congestion along the input direction vector $\vec{w_2} = (1.1, 0.9)$ and the output direction vector $\vec{\delta}$. Hence, no information about the existence of congestion of DMU_1 and DMU_{15} can be obtained by his method. To tackle this issue, the first scenario of our proposed method considers the situation that only the input direction vector is available and then determines the output direction vector along which the congestion occurs.

As we see in Tables 6, our proposed method determines the output direction vector $\vec{\delta_1} = (0.4, 0.2, 0.4, 0.0)$ such that DMU_1 has the right-hand congestion along $\vec{w_1} = (1.7, 0.3)$ and $\vec{\delta_1}$. Also, as we see in Tables 7, our proposed method determines the output direction vector $\vec{\delta_{15}} = (0.3, 0.3, 0.1, 0.3)$ such that DMU_{15} has the right-hand congestion along $\vec{w_2} = (1.1, 0.9)$ and $\vec{\delta_{15}}$.

Yang [28] showed that DMU_1 has no congestion along the output direction vector $\vec{\delta} = (1, 1, 1, 1)$ and 9 different input direction vectors. But, we still cannot obtain any information about the congestion status of this unit. In the second scenario of our proposed methods, both the input and output directions are not specified and our proposed method determine the input and output direction vectors along which the congestion occurs. As we see in Table 8, our proposed method determines the input direction vector $\vec{w_1} = (0.4, 0.6)$ and the output direction vector $\vec{\delta_1} = (1.0, 0.0, 0.0, 0.0)$ along which DMU_1 has the right-hand congestion. So, our proposed methods are more powerful than the method of Yang [28].

5. Conclusion

The concepts of strong and weak congestion are two basic terms in the congestion literature. The definitions of strong and weak congestion have main drawbacks, for example, we do not know whether the congestion will occur or not if the DM increases or decreases the inputs and outputs dis proportionally. In this regard, Yang [28] introduced the concept of the directional congestion in DEA. He proposed methods to determine the directional congestion of units along the certain input and output directions. The main drawback of his methods is that if a unit has no congestion along the certain input and output directions, no information about the existence of the congestion along other directions can be obtained. This study considered the situation in which at least one input or output direction is not specified. Two different scenarios have been considered in this paper: (i) only the input direction vector is available. We presented two methods to recognize the directional congestion for this scenario. (ii) none of the input and output direction vectors are available. We proposed a system of inequalities to find the input and output direction vectors along which the congestion occurs in this scenario. This study addresses the relationship between our defined directional congestion and the classical definitions of strong and weak congestion. The validity of the proposed methods was demonstrated utilizing two numerical examples from literature.

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