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Abstract :

A feedback loop to a 3D chaotic system with only six-terms on the right-hand of the equations and only two nonlinearities is applied to intentionally build a minimalist novel 4D chaotic system. Simulations depict the coexistence of strange attractors not by the modification of an unique or several parameters but surprisingly by a slight modification of the intial conditions. It is acknowledged that a strange attractor is locally unstable but globally stable. Our experiementation displays that strange attractors could be unstable at all scales.

I. Introduction

Morphology of the strange attractors as mathematical "objects" could have relevance to report the whole envelop of amplitudes and frequencies reached by the chaotic oscillations in the phase space.
One could well argue that their shapes are just the condensed summary of long listings of time series. However, the morphological plasticity have a significance introduced by the sensitive dependence on parameters (SDP) of the given attractor and helps the modeler to configurate adaquately its system. Such pertinence arises more intensely when the chaotic behaviors become captured by the sensitive dependance on initial conditions (SDIC) as displayed in a previous 3D chaotic system (Bouali, 2013). Its properties are in full contradiction to the findings of the Butterfly Effect (Lorenz, 1993), indicating that small gaps between the initial conditions lead the trajectories to different dynamics, but converge asymptotically toward an unique attractor. Locally unstable but globally stable, a strange attractor..
The primary objective of this paper is to experiment the findings of an interactive perturbation of this system of 2013 violating the SDIC scopes.
To this end, a feedback loop altering the structural stability of the model and extending the model to the fourth dimension is applied. We expect the emergence of a morphogenesis at the sense of Thom (2018 for the last edition).

The experimentation represents a preliminary step to the topological study of strange attractors (Gilmore and Lefranc, 2011; Letellier and Gilmore, 2013).
The secondary objective is represented by the self-selected constraint to develop a 4D version with the minimum number of terms in the right-hand of the equations.
In light of the coexisting attractors computed from such unique 4D dynamical system only with SDIC, we indicate its suitability as a chaotic generator for secure communication schemes.
In section III, the morphological variety of the phase portraits are described as well as the feature of the (planar) rotated copy for any phase portrait found. Section IV presents remarks on the origin of the attractor plasticity and the high suitability of the 4D model as a chaotic generator for cryptology.

II. A Minimalist 4D chaotic system

To achieve our goal to intentionally construct a 4D chaotic model with the minimum extent possible set of terms on the right-hand side, we propose the following system:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{\alpha\, x\, (1 - y) - \beta z} \\ \dot{\boldsymbol{y}} = \boldsymbol{-(1 - x^2)\, y} \\ \dot{\boldsymbol{z}} = \boldsymbol{-\mu\, v} \\ \dot{\boldsymbol{v}} = \boldsymbol{-\rho\, z} \end{cases}$$

where the overdot denotes a time derivative, x, y, z, and v, the four state variables of the model, and α, β, μ, ρ assumed to be positive parameters.
Such seven-term system embeding only two nonlinearity, a quadratic term, and one cubic item, respectively, xy, and x^2y, expresses the minimalist extension of a 3D system having an exclusive silhouette (Bouali, idem, 2013) with a fourth state variable.
Requirement to investigate the patterns of its phase portrait lead us to retain the set of parameters P (α, β, μ, ρ) = (3, 0.1, 1, 0.2, 0.01).

II.1. Basic mathematical properties

The equilibrium coordinates could be found by setting: $\dot{\boldsymbol{x}} = \dot{\boldsymbol{y}} = \dot{\boldsymbol{z}} = \dot{\boldsymbol{v}} = \boldsymbol{0}$.
Thus, the solutions of the following system :

$$\begin{cases} \boldsymbol{0} = \boldsymbol{\alpha\, x\, (1 - y) - \beta z} \\ \boldsymbol{0} = \boldsymbol{-(1 - x^2)\, y} \\ \boldsymbol{0} = \boldsymbol{-\mu\, v} \\ \boldsymbol{0} = \boldsymbol{-\rho\, z} \end{cases}$$

give the coordinates of equilibria.
In the first equation, and substituting z = 0 from the last relation, we obtain x = 0 or y = 1. Firstly, the second equation gives for x= 0, y= 0, and secondly for y =1, x = ±1.
The elementary attributes of these three equilibria given by the corresponding eigenvalues λ_i are found by solving the characteristic equation $|J - \lambda I| = 0$ where J, the Jacobian of the model, and I, the unit matrix. This kind of calculation is easy to achieve since the non-zero

terms of the Jacobian are the least numerous (7 out of 16). In Table 1, coordinates, eigenvalues and features of the stability related to the equilibria are reported.

Table 1. Index and Stability of the Equilibria

Coordinates of the equilibia	Corresponding characteristic equation, $\|J - \lambda I\| = 0$, and eigenvalues	Index and stability
E_0 (x_0, y_0, z_0, v_0) = (0, 0, 0, 0)	$(\lambda^2 + 0.002)(1+\lambda)(3-\lambda) = 0$ $\lambda_1 = \frac{-i}{10\sqrt{5}}$ $\lambda_2 = \frac{i}{10\sqrt{5}}$ $\lambda_3 = -1$ $\lambda_4 = 3$	Index-1 Unstable: Spiral saddle point
E_1 (x_1, y_1, z_1, v_1) = (1, 1, 0, 0) **&** **E_2 (x_2, y_2, z_2, v_2) = (-1, 1, 0, 0)**	$(\lambda^2 + 0.002)(6+\lambda^2) = 0$ $\lambda_1 = \frac{-i}{10\sqrt{5}}$ $\lambda_2 = \frac{i}{10\sqrt{5}}$ $\lambda_3 = -i\sqrt{6}$ $\lambda_4 = i\sqrt{6}$	Index-0 Neutrally stable: centers

(1) Index reports the number of eigenvalues with real parts Re(λ) > 0. From 1 to 4, it indicates the degree of instability. Index-0: null or negative real parts of all eigenvalues of the equilibrium characterize its stability.

II.2. A nonuniformly dissipativity

To qualify the dissipative nature of the 4D dynamical system, the divergence of the whole vector field could be derived from the following formulation:

$$\boldsymbol{div.(Volume)} = \frac{\boldsymbol{\partial Volume}/\boldsymbol{\partial t}}{\boldsymbol{Volume}} = \boldsymbol{Tr(J)} = \frac{\boldsymbol{\partial \dot{x}}}{\boldsymbol{\partial x}} + \frac{\boldsymbol{\partial \dot{y}}}{\boldsymbol{\partial y}} + \frac{\boldsymbol{\partial \dot{z}}}{\boldsymbol{\partial z}} + \frac{\boldsymbol{\partial \dot{v}}}{\boldsymbol{\partial v}}$$

To this end, Tr (J), the sum of the diagonal terms of the Jacobian should be negative to attest the dissipativity of the flow:

$$\boldsymbol{Tr(J)} = \boldsymbol{\alpha\,(1 - y) - (1 - x^2)} < 0$$

where α, positive value.

Dissipativity and volume contraction of the flow are accurately identified when these state variables of the flow, x, and y (and not including either z or v) fulfil the required condition:

$$\boldsymbol{\alpha\,(1 - y) < (1 - x^2)}$$

The system arises nonuniformly dissipative. However, in the precise case of the above condition is met, orbits are ultimately limited in a specific fractal-dimensional subspace of zero volume.

The set of (x, y) that verifies this inequality marks a domain bounded by an separatrix where the flow of the dynamics is dissipative. Besides, the trajectories are not short-term transient dynamics and report the zones of chaotic behavior -and not that regular- within the phase space (Joglekar and al., 2014; Lai and Tél, 2011; Motter et al., 2005; Kantz and Grassberger, 1985).
On the other hand, focusing the 4D model with its P parameters, it is pertinent to compute the Lyapunov exponents spectrum, the most useful measure to estimate the chaos. The system displays a chaotic nature since for β, varying from 0.01 to 0.21, the dominant Lyapunov exponent reaches a positive value (Fig. 1).

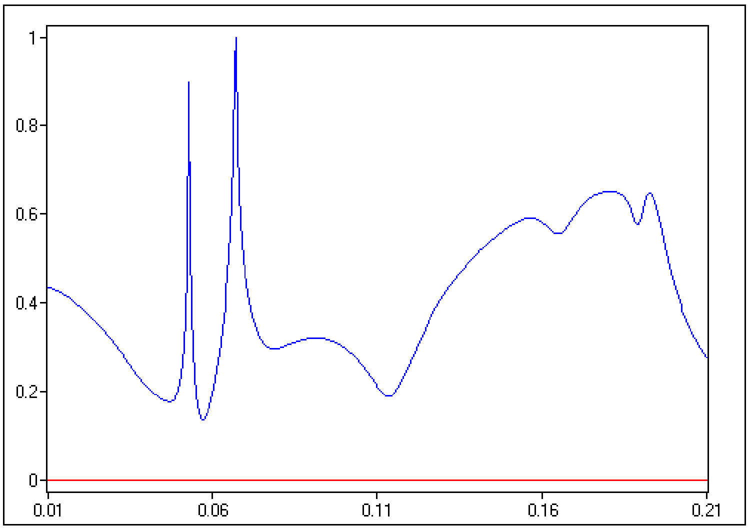

Fig. 1. Lyapunov exponent spectrum for the 4D chaotic system keeping unchanged the specification of parameters P, except β which varies in the range [0.01, 0.21]

III. A Collection of Attractors with Distinct Morphologies

We simulate the 4D system keeping unchanged the P parameters and all the graphical representations of the trajectories will be projected within the (x, y, z) space. This option will assist the analysis of morphological forms and their basins of attraction.

III.1. A non-standard SDIC

Surprisingly, the representation of these normalized 3D projections of the space phases depicts a noteworthy morphological variety of the portraits even if the system and the values of its parameters are kept unchanged. A non-standard butterfly effect is highlighted.
The first two projections (fig. 2a, and 2b) display asymmetrical portraits and identifiable saddle-focus homoclinic bifurcations leading to a very complex dynamics. Furthermore, the third projection (fig. 2c) depicts embedded rolls without, here also, any axis of symmetry.
Figure 2d shows a lemon-like shape of attractor whose initial conditions seem to belong to a zone of regular motion within the basin of attraction.

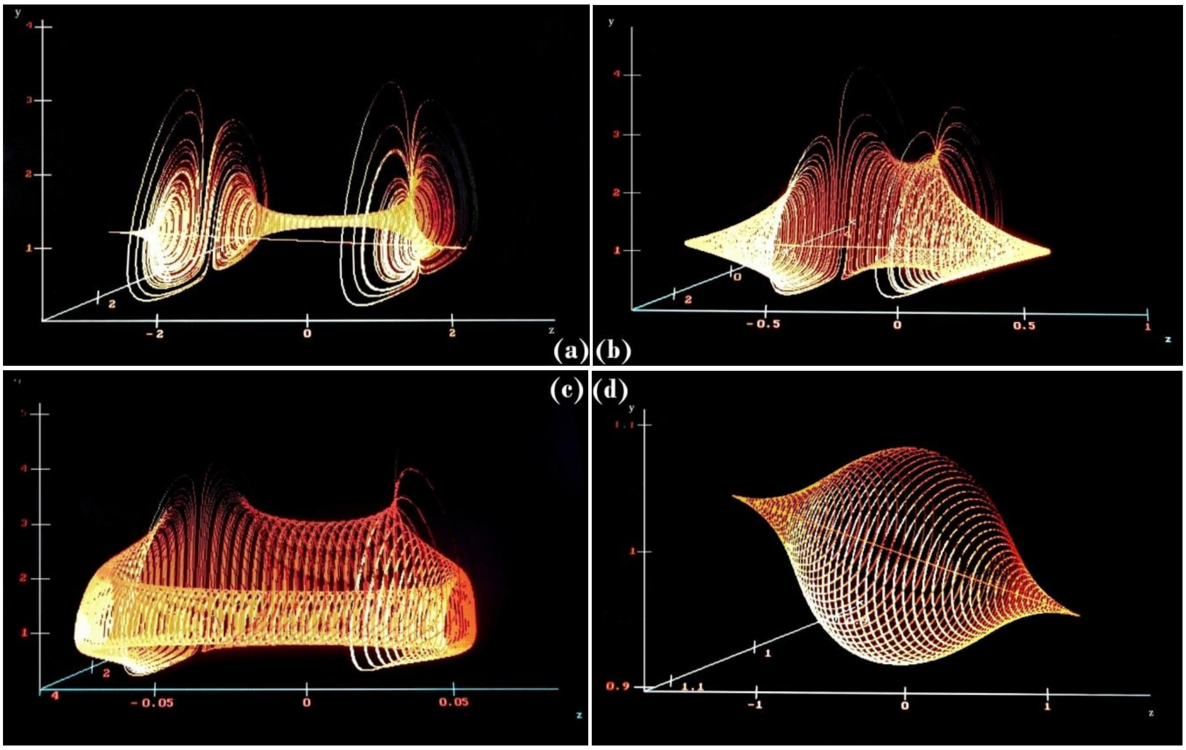

Fig. 2. Simulation of the 4D chaotic system with the (unchanged) set of parameters P. The 4D trajectories projected on the space (x, y, z) do not converge to an unique attractor whereas only the initial conditions have been modified. The Initial Conditions (a) IC_a=(x_a, y_a, z_a, v_a)= (2, 2, 2, 2), (b) IC_b=(x_b, y_b, z_b, v_b)= (0.5, 0.5, 0.5, 0.5), (c) IC_c=(x_c, y_c, z_c, v_c)= (0.05, 0.05, 0.05, 0.05), and (d) IC_d=(x_d, y_d, z_d, v_d)= (1, 1, 1, 1).

III.2. A rotated copy by π radians for any attractor

On the other hands, simulations have showed that another type of SDIC operates, albeit with a null impact on morphology. Indeed, for a small variation of the initial conditions, an identical portrait of the given strange attractor could be created, however with a planar half-turn. This is in the same line of the Chua model (Chua and al., 1986; Chua, 1992), or the Bouali model (Bouali, 1999) since under a peculiar specification of parameters, each system creates two strange attractors obviously with a weak difference of initial conditions. Although located in two sub-basin of attraction, the attractors are in reversed position. Nevertheless, this duplication is only the effect of the least spectacular SDIC

The present 4D system creates a pair of portraits for any particular morphology detected in the entire basin of attraction (Fig. 3). To note that figure 3a is similar of that of figure 2a. Another example of a pair of portraits is given in Figure 4. Here, the figure 4b is similar of that of figure 2b.

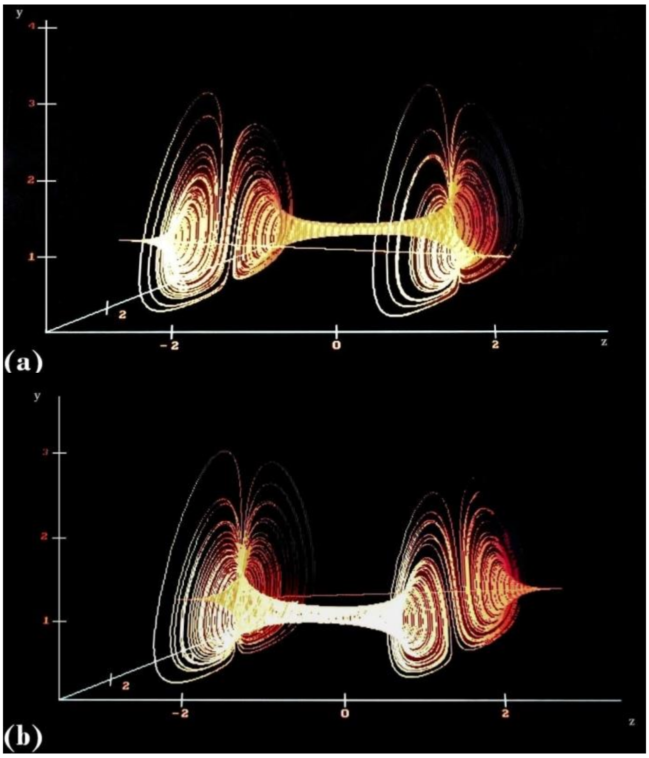

Fig. 3. A pair of 4D trajectories projected on the space (x, y, z) simulated with the (unchanged) set of parameters P and (a) IC_a=(x_a, y_a, z_a, v_a)= (2, 2, 2, 2), and (b) IC_b=(x_b, y_b, z_b, v_b)= (2, 2, -2, -2).

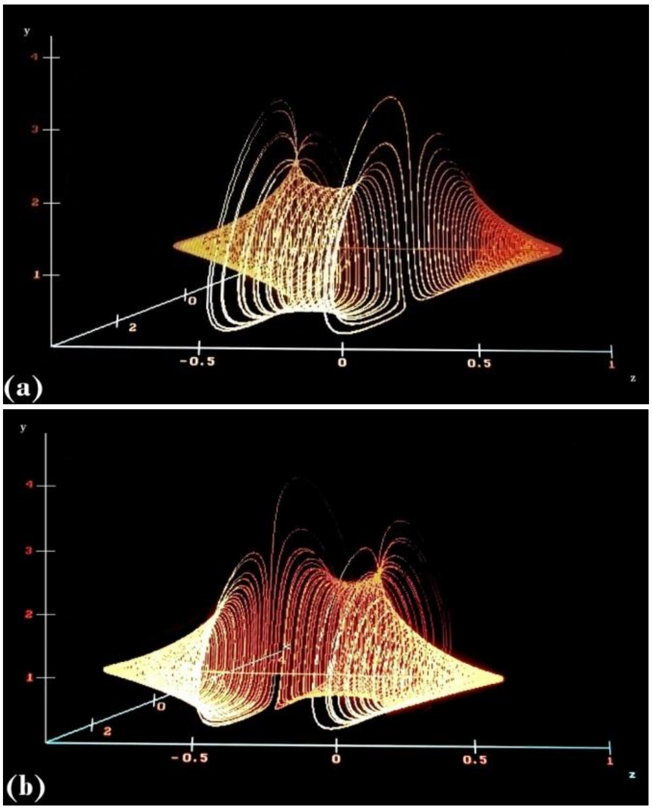

Fig.4. Another pair of portraits on the space (x, y, z) simulated with the (unchanged) set of parameters P and (a) IC_a=(x_a, y_a, z_a, v_a)= (-0.5, 0.5, -0.5, -0.5), and (b) IC_b=(x_b, y_b, z_b, v_b)= (0.5, 0.5, 0.5, 0.5).

In fact, the powerful SDIC able to drive trajectories toward distinct strange attractors stemms from the original 3D system (Bouali, idem, 2013). Embedded attractors phenomenon had been also reported. However, no identifiable morphogenesis could be noticed despite the light dissimilarity of rolls and wings of the attractors in the three dimensional space. Comparatively, therefore, it is the feedback of the fourth state variable by means of a supplementary dimension that triggered a complex mutation of the attractor morphology.

IV. Concluding Remarks

The Butterfly Effect as the most explicit feature of a strange attractor constitutes a blazing legacy of the wide scientific contribution Edward Norton Lorenz (1963). The SDIC reports that a small gap between two initial conditions in its idealized meteorological system deflects their paths in a way that no prediction of their drift could be done. Indeed, their gap grows at exponentially distance. Such singular phenomenon states that a strange attractor is locally unstable. However, it is globally stable since the trajectories converge to the same attractor.
Our 4D system moves beyond this result.
The main outcome of our paper is precisely that a strange attractor is not only locally unstable but also it could be globally unstable. Such novel SDIC allows a marked volatility of the chaotic dynamics at two levels. On the one hand, the bifurcation to a different strange attractor by a small variation of the initial conditions. On the other hand, the jump toward a rotated copy of any strange attractor by a slight change of the previous initial conditions even if this result had been observed elsewhese.
Such model could be used to preserve cryptography against hacker attacks of the communication systems based on the synchronization of chaotic oscillators (Frasca and al., 2018; Huang, 2004; Kocarev and Parlitz, 1995; Pecora and Carroll, 1990). Even if the oscillator parameters usually used as secret keys are recovered, the chaotic system remains unknow due to its drastic SDIC associating both volatility of morphology or rotation.
Indeed, capture of both parameters and initial conditions appears costly in lenght time of computation, and highly uncertain in case of jump from a chaotic generator to an other. In this case, the predictability horizon stands far beyond the computation time and the physical latences of the (hacking) circuit receiver.

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