C^1 POSITIVITY CONSTRAINED INTERPOLATION BY WEIGHTED RATIONAL CUBIC TRIANGLES

by

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ABSTRAK

Pembinaan interpolan bivariat kepositifan C^1 kepada data berselerak dengan menggunakan tampalan segi tiga Bézier kubik nisbah dipertimbangkan. Permukaan interpolasi dibina cebis demi cebis sebagai gabungan cembung tiga segi tiga Bézier kubik nisbah. Syarat cukup keselanjaran C^1 sepanjang sempadan sepunya dua segi tiga Bézier kubik nisbah yang bersebelahan dipaparkan. Syarat cukup untuk kepositifan segi tiga Bézier kubik nisbah diterbitkan. Nilai awal ordinat Bézier ditentukan dengan data dan kecerunan anggaran pada tapak data manakala pemberat diberi nilai satu. Ordinat Bézier sempadan dan pemberat diubahsuai jika perlu supaya tampalan yang dihasilkan memenuhi syarat keselanjaran C^1 dan syarat kepositifan. Skema untuk membina interpolan C^1 mengekalkan kepositifan adalah setempat. Beberapa contoh berangka digambarkan.

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ABSTRACT

The construction of positivity bivariate C^1 interpolants to scattered data using rational cubic Bézier triangular patches is considered. The interpolating surface is formed piecewise as a convex combination of three rational cubic Bézier triangles. Sufficient C^1 continuity conditions along the common boundary of two adjacent rational cubic Bézier triangles are shown. The sufficient positivity conditions on rational cubic Bézier triangle are derived. The initial values of the Bézier ordinates are determined by the data and the estimated gradient at the data sites while the weights are given the value one. The boundary Bézier ordinates and the weights are modified if necessary so that the resulting patch satisfies the C^1 continuity and positivity conditions. The scheme for constructing C^1 non-negativity preserving interpolant is local. Several numerical examples are illustrated.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Computer-aided geometry design (CAGD) is a young field dealing with mathematical description of 2D and 3D shapes used in computer graphics, analysis, manufacturing, etc. While this field may be mathematical in nature, it stretches across several disciplines such as computer science and engineering fields. Today, CAGD is widely used for computer description of parts such as car bodies or phone bodies and they are also used in representing scientific data such as terrain models or fossil bones. The first work in this field began in the mid-1960s. In 1974, R.E. Barnhill and R.F. Riesenfeld coined the term “CAGD” in connection with a conference held at the University of Utah. This event was considered as the founding event of the field.

There are several ways to represent 3D shapes in CAGD, some common concepts including but not limited to free-form deformation (FFD), tensor-product surface, Bézier surface, NURBS and rational Bézier surface. Rational Bézier surface is chosen as the interpolation scheme in this thesis due to its popularity. It is easy to use, and it is also not as complex compared to NURBS, although NURBS is considered as a generalization of rational Bézier surface.

Bézier surface is a numeric function that is piecewise defined by polynomial functions used in computer graphics. It is named after Pierre Bézier, a French engineer who invented it in 1962, while he was working in Renault automotive company. He used it to design automobile bodies. The shape of the Bézier surface is controlled by a finite number of control points. Piecewise Bézier surfaces are commonly used to represent any arbitrary 3D shape with certain level of continuity property. Bézier surface is popular due to it is easy to compute and easy to use. It can be stitched together to represent any 3D shapes one can imagine and have great continuity properties. A more controllable and versatile surface can be achieved if a scalar weight is assigned to each control point in Bézier surface. This leads to a rational Bézier surface. If all the values of weights assigned to the control points are 1, the rational Bézier surface can be reduced to a standard polynomial Bézier surface.

Sometimes the data collected from experiments are positive data, but the interpolating surfaces generated to fit the data might not always be positive. A surface that defined on a region is said to be positive if the entire surface lies above or just touching with the Cartesian *xy*-plane, that is the coordinate z of the surface is greater than or equal to 0 for all the points in the domain region of the surface. In these cases, the positivity preservation of an interpolant is essential. For examples, disaster related data such as flood, wind storm and slides, pollution data such as gas emission, chemical experiments such as stability of radioactive substance, forecast data such as rainfall amounts, population statistics and probability distribution should all be presented in positive.

On the other hand, smoothness is also an important property to have for a surface in order to produce visually pleasing graphical results or to achieve specified analytical continuity. Smoothness of a surface in mathematical analysis is a property measured by number of derivatives it has which are continuous. It circumvents certain nasty behaviour, infinities and paradoxes. The smoothness discussed in this thesis is C^1 continuity that is positional and tangential continuity.

1.2 Literature Review

Visualization of 3D data into surfaces and contour maps is important not only in CAGD, but also in scientific research. An interpolant is visualized through the data and is used in surface drawing. In modern day, visualization of 3D data alone is not enough. Since 1980's positivity preservation problem (also known as non-negativity preservation) has been studied and carried out by number of researchers, such as (Dougherty *et al.*, 1989), (Fisher *et al.*, 1991), (Goodman *et al*., 1991), (Ong & Unsworth, 1992), (Brodlie *et al*., 1995), (Ong & Wong, 1996), (Chan & Ong, 2001), (Piah, *et al*., 2005), (Hussain & Hussain, 2009), (Schumaker & Speleers, 2010) and (Lai & Meile, 2015).

(Brodlie *et al*., 1995) considered the problem of generating interpolants subject to linear constraints as lower and upper bounds. Sufficient positivity conditions on first partial derivatives and second mixed partial derivatives were derived from the result of (Schmidt & Heß, 1988). These derivatives were estimated and projected onto the valid intervals defined by sufficient positivity conditions.

(Ong & Wong, 1996) described a local C^1 scattered data interpolation scheme subject to constant lower and upper bounds. The side vertex method for interpolation in triangles was used with rational cubics for univariate interpolation along the line segments joining a vertex to the opposite edge of the triangle. By using the results in (Goodman & Said, 1991), the weights of the rational cubics are adjusted to ensure the resulting interpolant is positive.

In (Chan & Ong, 2001) and (Piah *et al*., 2005), a local C^1 range restricted scattered data interpolation scheme was presented. The interpolating surface is obtained piecewise as the convex combination of three cubic Bézier triangular patches. Positivity was preserved by adjusting the first partial derivatives at data sites and determining the lower bound on the Bézier ordinates. Piah *et al*., (2005) proposed a more relaxed lower bound for the Bézier ordinates.

In (Hussain & Hussain, 2009), a local C^1 positivity preserving scheme was developed using Bernstein-Bézier rational cubic function. The sufficient positivity conditions are derived by imposing lower bound on the inner and boundary Bézier ordinates. The derivation of lower bound on Bézier ordinates was motivated by the ideas in (Piah *et al*., (2005). However, the weights acted as free parameters which used to refine the shape of the interpolant.

(Schumaker & Speleers, 2010) constructed a positive bivariate interpolant to scattered data using splitting method with C^1 Bernstein Bézier spline. Positivity of the spline is insured by adjusting gradients at the data points when the data set is positive. The gradients are adjusted such that the Bézier ordinates satisfy the lower bound of positivity. The proposed method is local.

Last but not least, (Lai & Meile, 2015) proposed a constrained minimal energy method to find a C^1 smooth interpolation of positive data values over scattered location using bivariate splines. Uniqueness and existence of the minimizer were established under mild assumptions on the data locations and triangulations. Then, the classic projected gradient algorithm was used to find the minimizer using a simplified positive constraint.

1.3 Motivation

In the recent 20 years, there are an increasing number of studies in shape preserving interpolation and approximation. Despite of that, there were not many studies especially in shape preserving surface interpolation using rational Bézier as shape preserving interpolant and utilized the benefit of weight parameters in rational

Bézier. This has created a motivation to study further on how to manipulate weight parameters in rational Bézier to control the shape of the resulting surface, and how they can be used to preserve surface positivity.

1.4 Problem Statement

Consider a matter where a set of positive 3D data needs to be presented graphically. A surface that interpolates the given data and preserves the positivity inherent in the data is pursued. Moreover the resulting interpolant should possess positional and tangential continuities. Rational Bézier spline is utilized to construct the desired surface. A new challenge in this study is to achieve surface positivity by using rational Bézier as interpolant and adjusting weight parameters in the rational Bézier. Minimum point of rational Bézier patch is computed and analysed such that it will be positive which implies the whole surface is positive.

1.5 Thesis Objective

The main objective in this thesis is to develop an alternative method to achieve C^1 positive surface using rational Bézier surface. Differ from other researches, this research deals on manipulating weight parameters in rational Bézier patches to obtain positive patches. The patches are constructed piecewise to form a C^1 continuous surface. A set of new sufficient conditions of positivity preservation for rational Bézier patch should be derived, and imposed onto the weight parameters to produce positive surface.

1.6 Methodology

In this thesis, the construction of C^1 positivity preserving interpolants to scattered data using rational cubic Bézier triangular patches is considered. Rational cubic Bézier is used because it is the lowest patch to be able to produce C^1 positive interpolant. The data are triangulated such that the data points are used as the vertices of triangulation. The resulting interpolant to the data is a piecewise convex combination of three rational cubic Bézier triangular patches. The initial values of the Bézier ordinates of a rational cubic Bézier triangular patch are determined by the given data and the gradients specified at the data sites, while all the weights of the rational Bézier patch are initially set to have the value 1. The boundary Bézier ordinates of rational Bézier patch are constrained by a lower bound of positivity conditions. They are modified if necessary by adjusting the gradients at the data sites so that the boundary curves of the patch are positive. Instead of imposing lower bound to inner Bézier ordinate, the weights of rational Bézier patch are utilized to obtain positivity of the patch. The weights at three corners of the patch are increased if necessary such that the interior patch is positive. The scheme for constructing the positivity preserving interpolant is local. The interpolant to be constructed has a smoothness of C^1 continuity.

1.7 Thesis Outline

The layout of this thesis is arranged in the following sequence. Chapter 1 provides a brief introduction to CAGD, a comprehensive literature review, the problem statements, objectives and methodology of this study. Chapter 2 gives an introduction to rational cubic Bézier triangular patch and some useful intrinsic properties of the rational patch such as endpoint interpolation, convex hull property as well as effect of weights towards the patch. The directional derivative equations of rational cubic Bézier triangular patches are also provided which will be used in the surface continuity.

C^1 continuity conditions for two adjacent rational cubic Bézier triangular patches are presented in Chapter 3. The effect of corner weights towards C^1 continuity is also discussed in this chapter. In Chapter 4, sufficient positivity conditions proposed in (Piah *et al*., 2005) for cubic Bézier patch are stated. The core of this chapter discusses on the formulation of conditions for rational cubic Bézier triangular patch to be positive. The work of derivation is inspired by the one shown in (Pial *et al*., 2005). A way to determine minimum point of rational cubic surface is proposed in which up to 10 different cases are studied. A set of sufficient positivity conditions for a rational cubic Bézier patch to be positive is derived.

In Chapter 5, a local scheme for C^1 positivity preserving interpolation to scattered data is presented. Given positive data, the domain is triangulated by using Delaunay triangulation method (Fang & Piegl, 1992). The first order partial derivatives with respect to x and y at each data site are estimated using the method introduced in (Goodman *et al*., 1994). The estimated derivatives are used to determine the initial values of boundary Bézier ordinates, while the inner Bézier ordinate is determined according to C^1 continuity conditions. If necessary, the boundary ordinates are modified by changing the estimated derivatives at the data sites. The weight parameters are adjusted in such that the positivity conditions derived in Chapter 4 are fulfilled. Lastly in Chapter 6, a few numerical examples are visualized with the proposed C^1 positivity preserving interpolation scheme. Conclusion and future works are included as well.

CHAPTER 2

WEIGHTED RATIONAL BÉZIER TRIANGULAR PATCH

Rational Bézier triangular patch is a mathematical model that has been widely used in Computer Aided Design. Compared to regular polynomial Bézier patch, rational Bézier patch has an additional feature – adjustable weights which provide better modelling to arbitrary shapes. Its interpolation behaviour is that the patch generated touches the vertex points and stretches toward other points, while the weight can further attracts or repels the patch depending on the location of the weight. Rational cubic Bézier triangular patch is chosen for this research as it gives enough degree of freedom to make necessary adjustments to the shape of the patch in conjunction with smoothness requirements. This chapter gives the definition of rational cubic Bézier triangular patch and its properties.

2.1 Weighted Rational Cubic Bézier Triangular Patch

Consider a triangle T with vertices $V_i = (x_i,\ y_i)$, for $i = 1$, 2, 3. The barycentric coordinates (u, v, w) of any point V on T can be written as

$$V = uV_1 + vV_2 + wV_3 ,$$

with $u + v + w = 1$ and $u,\ v,\ w \geq 0$. As in (Farin, 1996), a rational cubic Bézier triangular patch R on T can be defined as

$$R(u,v,w) = \frac{\sum\limits_{\substack{i+j+k=3 \\ i,j,k\geq 0}} b_{i,j,k}\,\omega_{i,j,k}\,B^3_{i,j,k}(u,v,w)}{\sum\limits_{\substack{i+j+k=3 \\ i,j,k\geq 0}} \omega_{i,j,k}\,B^3_{i,j,k}(u,v,w)} \tag{2.1}$$

where $B_{i,j,k}^3$ are bivariate Bernstein polynomials of degree 3 as

$$B_{i,j,k}^3(u,v,w) = \frac{3!}{i!\,j!\,k!} u^i v^j w^k ,$$

$b_{i,j,k} \in \mathbb{R}$ are the Bézier ordinates of patch R and $\omega_{i,j,k} \in \mathbb{R}$ are the weights, with $i+j+k=3$ and $i,\ j,\ k \geq 0$. By $V_i = (x_i,\ y_i)$, for $i=1$, 2, 3, the patch R can be represented in parametric form with Bézier points

$$\boldsymbol{b}_{i,j,k} = \left(\frac{i}{3}x_1 + \frac{j}{3}x_2 + \frac{k}{3}x_3,\quad \frac{i}{3}y_1 + \frac{j}{3}y_2 + \frac{k}{3}y_3,\quad b_{i,j,k}\right), \tag{2.2}$$

refer to Figure 2.1, which form a control net in space. Note that when all the weights $\omega_{i,j,k}$ are the same, a Bézier polynomial patch can be obtained

$$R(u,v,w) = \sum_{\substack{i+j+k=3 \\ i,j,k\geq 0}} b_{i,j,k} B_{i,j,k}^3(u,v,w) . \tag{2.3}$$

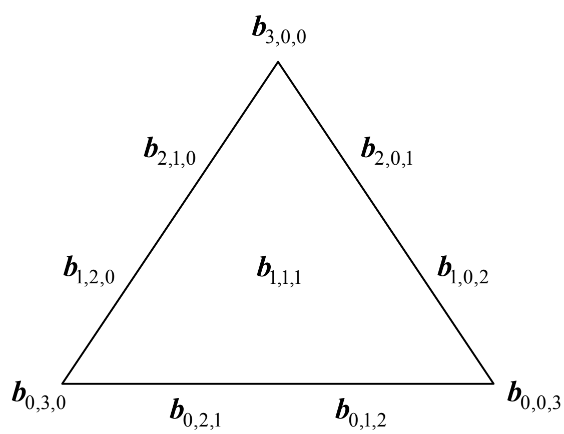

Figure 2.1: Bézier points of rational cubic Bézier triangular patch

2.2 Properties of Rational Cubic Bézier Triangular Patch

This section expresses a few useful properties of rational cubic Bézier patch. They are end-point interpolation property, convex hull property and effect of the weight parameters.

2.2.1 End-point Interpolation

For a rational cubic Bézier triangular patch, the three corners of the control net, i.e. $\boldsymbol{b}_{3,0,0}$, $\boldsymbol{b}_{0,3,0}$, $\boldsymbol{b}_{0,0,3}$ lie on the Bézier patch, as shown in Figure 2.2. This leads to

$$R(V_1) = b_{3,0,0},$$

$$R(V_2) = b_{0,3,0},$$

$$R(V_3) = b_{0,0,3}.$$

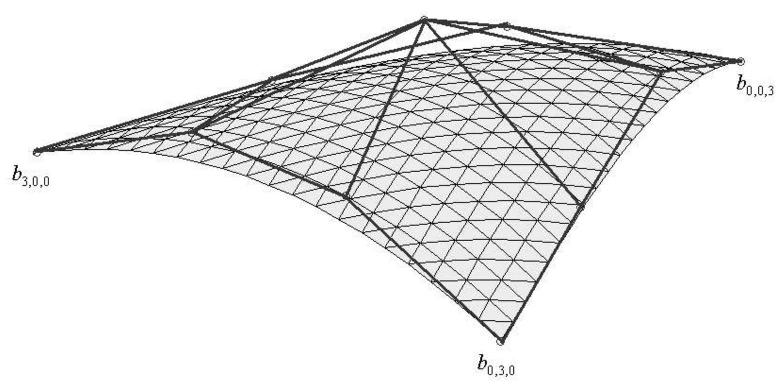

Figure 2.2: Rational cubic Bézier patch and its control net

We should note that the boundary curves of the patch are rational cubic Bézier curves which are determined by the boundary Bézier ordinates and the corresponding boundary weights of the patch. For example, along $u = 0$, a rational curve is

$$R(0,v,w) = \frac{\sum_{j+k=3} b_{0,j,k}\omega_{0,j,k}B^3_{0,j,k}(0,v,w)}{\sum_{j+k=3} \omega_{0,j,k}B^3_{0,j,k}(0,v,w)}$$

$$= \frac{\sum_{k=0}^{3} b_{0,3-k,k}\omega_{0,3-k,k}B^3_k(w)}{\sum_{k=0}^{3} \omega_{0,3-k,k}B^3_k(w)} \tag{2.4}$$

where

$$B^3_k(w) = \frac{3!}{(3-k)!k!}(1-w)^{3-k}w^k,$$

for $k = 0$, 1, 2, 3, and $0 \le w \le 1$.

2.2.2 Convex Hull Property

Suppose that all the weights $\omega_{i,j,k} > 0$, the rational Bézier triangular patch lies completely inside the convex hull of its control net. It is shown as the shaded region in Figure 2.3. Thus, if all the Bézier ordinates are non-negative then so is the patch.

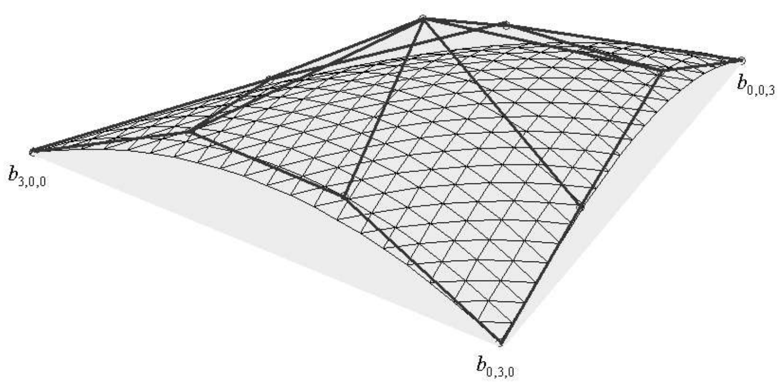

Figure 2.3: Convex hull property

2.2.3 Effects of Weight towards Patch

One of the advantages of using rational Bézier triangular patch is the weights $\omega_{i,j,k}$ can be manipulated as additional design parameters. By easy calculation, the parametric form of rational cubic Bézier patch, denoted as $\boldsymbol{R}$, has

$$\lim_{\omega_{r,s,t}\to\infty} \boldsymbol{R}(u,v,w) = \lim_{\omega_{r,s,t}\to\infty} \frac{\sum\limits_{\substack{i+j+k=3,\\ i,j,k\geq 0}} \omega_{i,j,k}\boldsymbol{b}_{i,j,k}B^{3}_{i,j,k}(u,v,w)}{\sum\limits_{\substack{i+j+k=3,\\ i,j,k\geq 0}} \omega_{i,j,k}B^{3}_{i,j,k}(u,v,w)}$$

$$= \lim_{\omega_{r,s,t}\to\infty} \frac{\boldsymbol{b}_{r,s,t}B^{3}_{r,s,t}(u,v,w)}{B^{3}_{r,s,t}(u,v,w)}$$

$$= \boldsymbol{b}_{r,s,t}\,.$$

For example, when $\omega_{3,0,0}$ tends to infinity, it results in the parametric patch $\boldsymbol{R}$ tends to the Bézier point $\boldsymbol{b}_{3,0,0}$. Figure 2.4 shows the effect of weight $\omega_{3,0,0}$ towards the patch *R*. The horizontal line indicates a plane. Note that the patch generated with different positive weight values will always be inside the convex hull of control points.

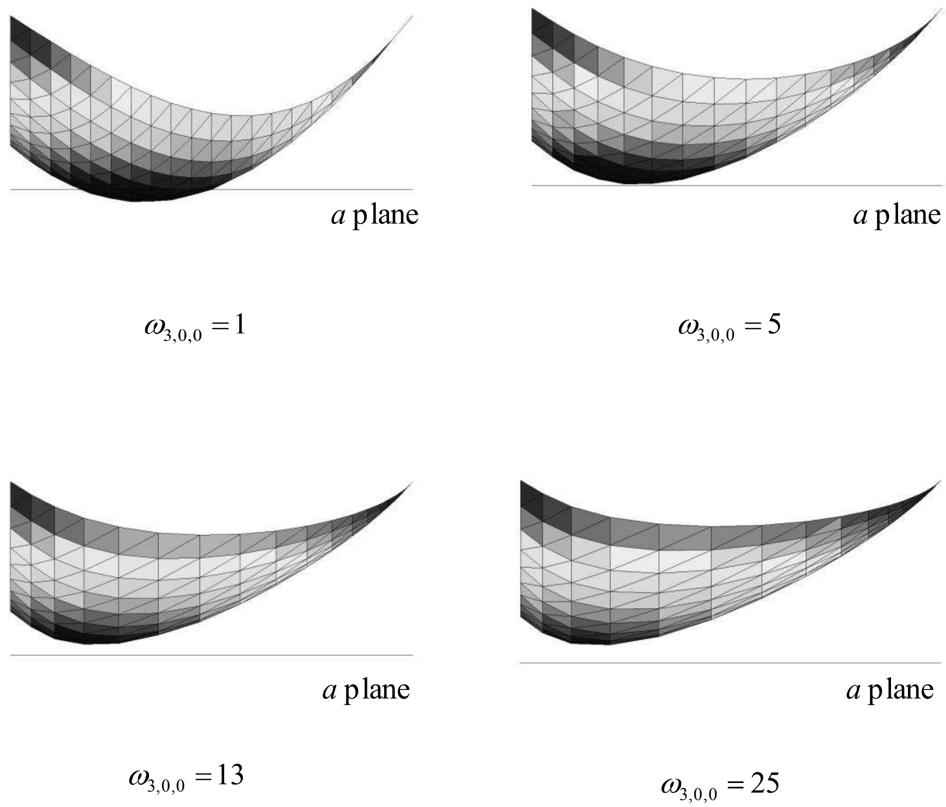

Figure 2.4: Patch *R* with different weight values

To be more precise, let $\boldsymbol{R}(u,v,w)$ be interpreted as a point on the rational patch defined with the weights $\omega_{i,j,k}$, $i+j+k=3$ and $i,\ j,\ k \geq 0$. Suppose one of the weights, denoted as $\omega_{r,s,t}$, is changed to $\hat{\omega}_{r,s,t} = \omega_{r,s,t} + \Delta\omega$, where $\Delta\omega \in \mathbb{R}$, while the others remain unchanged. Let $\hat{\boldsymbol{R}}(u,v,w)$ be the point on the new rational patch produced using $\hat{\omega}_{r,s,t}$. For simplicity, let

$$\boldsymbol{R}(u,v,w) = \frac{\boldsymbol{S}(u,v,w)}{W(u,v,w)},$$

$$\hat{\boldsymbol{R}}(u,v,w) = \frac{\hat{\boldsymbol{S}}(u,v,w)}{\hat{W}(u,v,w)}.$$

Then

$$\hat{\boldsymbol{R}}(u,v,w)-\boldsymbol{R}(u,v,w)=\frac{\hat{\boldsymbol{S}}(u,v,w)}{\hat{W}(u,v,w)}-\frac{\boldsymbol{S}(u,v,w)}{W(u,v,w)}$$

$$=\frac{\boldsymbol{S}(u,v,w)+\boldsymbol{b}_{r,s,t}\Delta\omega B^3_{r,s,t}(u,v,w)}{\hat{W}(u,v,w)}-\frac{\boldsymbol{S}(u,v,w)}{W(u,v,w)}\frac{W(u,v,w)+\Delta\omega B^3_{r,s,t}(u,v,w)}{\hat{W}(u,v,w)}$$

$$=\frac{\Delta\omega B^3_{r,s,t}(u,v,w)}{\hat{W}(u,v,w)}(\boldsymbol{b}_{r,s,t}-\boldsymbol{R}(u,v,w))\,.$$

It is obvious that the patch $\boldsymbol{R}$ moves toward the point $\boldsymbol{b}_{r,s,t}$ when we increase the value of $\omega_{r,s,t}$. Conversely, the patch moves further away from $\boldsymbol{b}_{r,s,t}$ if $\omega_{r,s,t}$ is decreased.

2.3 Directional Derivative of Rational Bézier Patch

Here, the directional derivative of parametric patch $\boldsymbol{R}$ is considered. The detailed formulation can be obtained from (Farin, 1996). Let

$$\boldsymbol{R}(u,v,w)=\frac{\boldsymbol{S}(u,v,w)}{W(u,v,w)}$$

where $\boldsymbol{S}$ is a vector-valued function with Bézier points given in (2.2). Let $D=(d,e,f)$ be a direction defined on the triangle T with barycentric coordinates $d+e+f=0$. The directional derivatives for $\boldsymbol{S}(u,v,w)$ and $W(u,v,w)$ with respect to direction D are

$$\begin{aligned}\frac{\partial \boldsymbol{S}}{\partial D}(u,v,w) &= d\frac{\partial}{\partial u}\boldsymbol{S}(u,v,w)+e\frac{\partial}{\partial v}\boldsymbol{S}(u,v,w)+f\frac{\partial}{\partial w}\boldsymbol{S}(u,v,w)\\ &=3\sum_{\substack{i,j,k\ge 0\\ i+j+k=2}}(d\omega_{i+1,j,k}\boldsymbol{b}_{i+1,j,k}+e\omega_{i,j+1,k}\boldsymbol{b}_{i,j+1,k}+f\omega_{i,j,k+1}\boldsymbol{b}_{i,j,k+1})B^2_{i,j,k}(u,v,w),\end{aligned}$$

$$\frac{\partial W}{\partial D}(u,v,w) = d\frac{\partial}{\partial u}W(u,v,w)+e\frac{\partial}{\partial v}W(u,v,w)+f\frac{\partial}{\partial w}W(u,v,w)$$

$$= 3 \sum_{\substack{i,j,k \geq 0 \\ i+j+k=2}} (d\omega_{i+1,j,k} + e\omega_{i,j+1,k} + f\omega_{i,j,k+1}) B^2_{i,j,k}(u,v,w) . \tag{2.5}$$

Hence, the directional derivative of $\boldsymbol{R}$ can easily be defined by the quotient rule of differentiation as

$$\frac{\partial \boldsymbol{R}}{\partial D}(u,v,w) = \frac{\frac{\partial \boldsymbol{R}}{\partial D}(u,v,w) W(u,v,w) - \boldsymbol{R}(u,v,w) \frac{\partial W}{\partial D}(u,v,w)}{W(u,v,w)^2} .$$

Along the edge of T, such as $u = 0$, the derivatives in (2.5) can be simplified to as

$$\begin{aligned} \frac{\partial \boldsymbol{S}}{\partial D}(0,v,w) = {} & 3(d\omega_{1,2,0}\boldsymbol{b}_{1,2,0} + e\omega_{0,3,0}\boldsymbol{b}_{0,3,0} + f\omega_{0,2,1}\boldsymbol{b}_{0,2,1}) B^2_{0,2,0}(0,v,w) \\ & + 3(d\omega_{1,1,1}\boldsymbol{b}_{1,1,1} + e\omega_{0,2,1}\boldsymbol{b}_{0,2,1} + f\omega_{0,1,2}\boldsymbol{b}_{0,1,2}) B^2_{0,1,1}(0,v,w) \\ & + 3(d\omega_{1,0,2}\boldsymbol{b}_{1,0,2} + e\omega_{0,1,2}\boldsymbol{b}_{0,1,2} + f\omega_{0,0,3}\boldsymbol{b}_{0,0,3}) B^2_{0,0,2}(0,v,w) , \end{aligned}$$

$$\begin{aligned} \frac{\partial W}{\partial D}(0,v,w) = {} & 3(d\omega_{1,2,0} + e\omega_{0,3,0} + f\omega_{0,2,1}) B^2_{0,2,0}(0,v,w) \\ & + 3(d\omega_{1,1,1} + e\omega_{0,2,1} + f\omega_{0,1,2}) B^2_{0,1,1}(0,v,w) \\ & + 3(d\omega_{1,0,2} + e\omega_{0,1,2} + f\omega_{0,0,3}) B^2_{0,0,2}(0,v,w) . \end{aligned}$$

Note that when all the corresponding weights are the same, i.e.

$$\omega_{1,2,0} = \omega_{1,1,1} = \omega_{1,0,2} = \omega_{0,3,0} = \omega_{0,2,1} = \omega_{0,1,2} = \omega_{0,0,3} \quad (\text{assume} = 1)$$

then

$$\begin{aligned} \frac{\partial \boldsymbol{R}}{\partial D}(0,v,w) = {} & 3(d\boldsymbol{b}_{1,2,0} + e\boldsymbol{b}_{0,3,0} + f\boldsymbol{b}_{0,2,1}) B^2_{0,2,0}(0,v,w) \\ & + 3(d\boldsymbol{b}_{1,1,1} + e\boldsymbol{b}_{0,2,1} + f\boldsymbol{b}_{0,1,2}) B^2_{0,1,1}(0,v,w) \\ & + 3(d\boldsymbol{b}_{1,0,2} + e\boldsymbol{b}_{0,1,2} + f\boldsymbol{b}_{0,0,3}) B^2_{0,0,2}(0,v,w) , \end{aligned} \tag{2.6}$$

which is exactly identical to the directional derivative of Bézier polynomial patch of (2.3) along the edge $u = 0$. Similar argument holds for the other two edges, $v = 0$ and $w = 0$.

CHAPTER 3

C^1 CONTINUITY CONDITIONS

This chapter emphasizes on how to obtain a smooth joint between two adjacent rational cubic Bézier patches. Let the rational patches be defined by (2.1) on the triangles $V_1V_2V_3$ and $W_1W_2W_3$ respectively, where $V_2 = W_2$ and $V_3 = W_3$. Let $b_{i,j,k}$ and $c_{i,j,k}$ denote the corresponding Bézier ordinates, see Figure 3.1. Suppose the weights towards $b_{1,0,2}$, $b_{1,1,1}$, $b_{1,2,0}$, $b_{0,0,3}$, $b_{0,1,2}$, $b_{0,2,1}$, $b_{0,3,0}$, and $c_{0,0,3}$, $c_{0,1,2}$, $c_{0,2,1}$, $c_{0,3,0}$ $c_{1,0,2}$, $c_{1,1,1}$, $c_{1,2,0}$ are set to be 1, then as in (2.6) the derivatives of the rational patches along the common edge are equal to the derivatives of Bézier polynomial patches which defined with the Bézier ordinates $b_{i,j,k}$ and $c_{i,j,k}$ respectively.

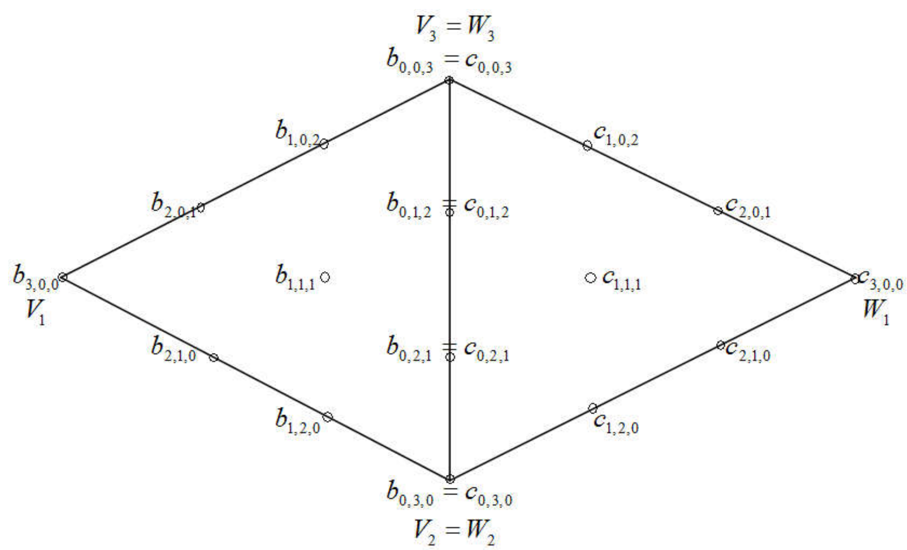

Figure 3.1: A pair of cubic Bézier triangles

3.1 C^1 Conditions Between Two Adjacent Bézier Triangles

In 2003, Farin described the sufficient and necessary conditions for two adjacent Bézier triangles to be joined C^1 continuously, which is positional and tangential continuity. Suppose the Bézier patches join at a common boundary along the edge V_2V_3 by $b_{0,0,3} = c_{0,0,3}$, $b_{0,1,2} = c_{0,1,2}$, $b_{0,2,1} = c_{0,2,1}$ and $b_{0,3,0} = c_{0,3,0}$. The conditions

$$c_{1,0,2} = \alpha b_{1,0,2} + \beta b_{0,1,2} + \gamma b_{0,0,3}, \tag{3.1}$$

$$c_{1,1,1} = \alpha b_{1,1,1} + \beta b_{0,2,1} + \gamma b_{0,1,2}, \tag{3.2}$$

$$c_{1,2,0} = \alpha b_{1,2,0} + \beta b_{0,3,0} + \gamma b_{0,2,1} \tag{3.3}$$

where $W_1 = \alpha V_1 + \beta V_2 + \gamma V_3$, ensure the C^1 continuity along the common boundary of adjacent Bézier patches. Conditions (3.1) and (3.3) are fulfilled if the related Bézier ordinates are determined by using the partial derivatives of surface specified at vertices V_2 and V_3 respectively. To satisfy condition (3.2), the ideas originated in (Goodman & Said, 1991) were considered, in which the normal derivative of the patch is required to vary linearly along the boundary.

Consider the triangle T by $V_1V_2V_3$. Let E_i denote the opposite side of vertex V_i on triangle T, e_i be the direction along the edge E_i and n_i be the inward normal direction to the edge E_i, where $i = 1$, 2, 3, see Figure 3.2 below.

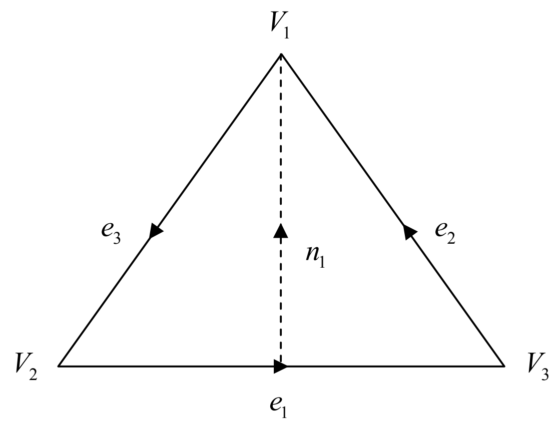

Figure 3.2: Notations on a triangle

By simple calculation,

$$n_1 = -e_3 + h_1 e_1$$

where $h_1 = \dfrac{e_3 \bullet e_1}{e_1 \bullet e_1}$. The symbol "$\bullet$" indicates the dot product of two vectors. Then, vector n_1 can be written in the barycentric form as

$$n_1 = (1,\ -h_1 - 1,\ h_1)\,. \tag{3.4}$$

Similarly, the normal directions to the edges E_2 and E_3, denoted as n_2 and n_3 respectively, can be defined in the barycentric form as

$$n_2 = (h_2,\ 1,\ -h_2 - 1)\,,$$

$$n_3 = (-h_3 - 1,\ h_3,\ 1)$$

where $h_2 = \dfrac{e_2 \bullet e_1}{e_2 \bullet e_2}$ and $h_3 = \dfrac{e_3 \bullet e_2}{e_3 \bullet e_3}$. Using (3.4), the normal derivative of the rational Bézier patch *R* on edge E_1 is

$$\begin{aligned}\frac{\partial R}{\partial n_1}(0,v,w) &= \frac{\partial R}{\partial u}(0,v,w) - (h_1+1)\frac{\partial R}{\partial v}(0,v,w) + h_1\frac{\partial R}{\partial w}(0,v,w)\\ &= 3A_0 + 6(A_1 - A_0)w + 3(A_2 - 2A_1 + A_0)w^2\end{aligned}$$

where

$$A_0 = b_{1,2,0} - (1+h_1)b_{0,3,0} + h_1 b_{0,2,1},$$

$$A_1 = b_{1,1,1} - (1+h_1)b_{0,2,1} + h_1 b_{0,1,2},$$

$$A_2 = b_{1,0,2} - (1+h_1)b_{0,1,2} + h_1 b_{0,0,3},$$

and $v = 1 - w$. Since the normal derivative of *R* is required to vary linearly along E_1, the coefficient of w^2 is set to be zero, i.e.

$$(b_{1,2,0} - b_{0,3,0} - h_1 b_{0,3,0} + h_1 b_{0,2,1}) - 2(b_{1,1,1} - b_{0,2,1} - h_1 b_{0,2,1} + h_1 b_{0,1,2})$$
$$+ (b_{1,0,2} - b_{0,1,2} - h_1 b_{0,1,2} + h_1 b_{0,0,3}) = 0.$$

Solving the equation for the ordinate $b_{1,1,1}$ and denoting the value by

$$b^1_{1,1,1} = \frac{1}{2}[(b_{1,2,0} - b_{0,3,0} + b_{1,0,2} - b_{0,1,2} + 2b_{0,2,1})$$
$$+ h_1(-b_{0,3,0} + b_{0,2,1} - b_{0,1,2} + b_{0,0,3} + 2b_{0,2,1} - 2b_{0,1,2})]. \quad (3.5)$$

The ordinates on the right side of Equation (3.5) are known since they are completely determined by the partial derivatives of surface specified at vertices. Using similar way along edges E_2 and E_3, $b^2_{1,1,1}$ and $b^3_{1,1,1}$ are obtained as

$$b^2_{1,1,1} = \frac{1}{2}[(b_{012} - b_{003} + b_{210} - b_{201} + 2b_{102})$$
$$+ h_2(-b_{0,0,3} + b_{1,0,2} - b_{2,0,1} + b_{3,0,0} + 2b_{1,0,2} - 2b_{2,0,1})], \quad (3.6)$$

$$b^3_{1,1,1} = \frac{1}{2}[(b_{2,0,1} - b_{3,0,0} + b_{0,2,1} - b_{1,2,0} + 2b_{2,1,0})$$
$$+ h_3(-b_{3,0,0} + b_{2,1,0} - b_{1,2,0} + b_{0,3,0} + 2b_{2,1,0} - 2b_{1,2,0})]. \quad (3.7)$$

3.2 Convex Combination Patch

Note that the inner Bézier ordinate $b_{1,1,1}$ needs to fulfil equations (3.5), (3.6) and (3.7) in order for the patch R to have C^1 continuity across all three boundaries. This over determined problem can be solved by defining a convex combination patch of three rational cubic Bézier triangular patches. The three rational Bézier patches, denoted as R_1, R_2 and R_3, are defined to have different inner Bézier ordinates $b^i_{1,1,1}$, $i=1$, 2, 3, respectively, and have the same boundary Bézier ordinates. They also have different weight values at three vertices while the rest of the weights equal to 1,

$$R_1(u,v,w) = \frac{\begin{array}{l}\omega^1_{3,0,0}b_{3,0,0}B^3_{3,0,0} + b_{0,3,0}B^3_{0,3,0} + b_{0,0,3}B^3_{0,0,3} + b_{2,1,0}B^3_{2,1,0} + b_{1,2,0}B^3_{1,2,0} \\ +b_{2,0,1}B^3_{2,0,1} + b_{1,0,2}B^3_{1,0,2} + b_{0,2,1}B^3_{0,2,1} + b_{0,1,2}B^3_{0,1,2} + b^1_{1,1,1}B^3_{1,1,1}\end{array}}{\begin{array}{l}\omega^1_{3,0,0}B^3_{3,0,0} + B^3_{0,3,0} + B^3_{0,0,3} + B^3_{2,1,0} + B^3_{1,2,0} + B^3_{2,0,1} + B^3_{1,0,2} + B^3_{0,2,1} \\ +B^3_{0,1,2} + B^3_{1,1,1}\end{array}}, \tag{3.8}$$

$$R_2(u,v,w) = \frac{\begin{array}{l}b_{3,0,0}B^3_{3,0,0} + \omega^2_{0,3,0}b_{0,3,0}B^3_{0,3,0} + b_{0,0,3}B^3_{0,0,3} + b_{2,1,0}B^3_{2,1,0} + b_{1,2,0}B^3_{1,2,0} \\ +b_{2,0,1}B^3_{2,0,1} + b_{1,0,2}B^3_{1,0,2} + b_{0,2,1}B^3_{0,2,1} + b_{0,1,2}B^3_{0,1,2} + b^2_{1,1,1}B^3_{1,1,1}\end{array}}{\begin{array}{l}B^3_{3,0,0} + \omega^2_{0,3,0}B^3_{0,3,0} + B^3_{0,0,3} + B^3_{2,1,0} + B^3_{1,2,0} + B^3_{2,0,1} + B^3_{1,0,2} + B^3_{0,2,1} \\ +B^3_{0,1,2} + B^3_{1,1,1}\end{array}}, \tag{3.9}$$

$$R_3(u,v,w) = \frac{\begin{array}{l}b_{3,0,0}B^3_{3,0,0} + b_{0,3,0}B^3_{0,3,0} + \omega^3_{0,0,3}b_{0,0,3}B^3_{0,0,3} + b_{2,1,0}B^3_{2,1,0} + b_{1,2,0}B^3_{1,2,0} \\ +b_{2,0,1}B^3_{2,0,1} + b_{1,0,2}B^3_{1,0,2} + b_{0,2,1}B^3_{0,2,1} + b_{0,1,2}B^3_{0,1,2} + b^3_{1,1,1}B^3_{1,1,1}\end{array}}{\begin{array}{l}B^3_{3,0,0} + B^3_{0,3,0} + \omega^3_{0,0,3}B^3_{0,0,3} + B^3_{2,1,0} + B^3_{1,2,0} + B^3_{2,0,1} + B^3_{1,0,2} + B^3_{0,2,1} \\ +B^3_{0,1,2} + B^3_{1,1,1}\end{array}}. \tag{3.10}$$

The convex combination of R_1, R_2 and R_3 is defined as

$$R(u,v,w) = c_1(u,v,w)R_1(u,v,w) + c_2(u,v,w)R_2(u,v,w) + c_3(u,v,w)R_3(u,v,w)$$

where the combination coefficients are

$$c_1(u,v,w) = \frac{v^2w^2}{u^2v^2 + u^2w^2 + v^2w^2},$$

$$c_2(u,v,w) = \frac{u^2w^2}{u^2v^2 + u^2w^2 + v^2w^2},$$

$$c_3(u,v,w) = \frac{u^2v^2}{u^2v^2 + u^2w^2 + v^2w^2}. \tag{3.11}$$

Note that these blending functions c_1, c_2 and c_3 have some useful properties. We have

(i) $c_i(E_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

(ii) $c_1 + c_2 + c_3 = 1$

(iii) $\frac{\partial c_i}{\partial D}(E_j) = 0$, for $\forall\, i,\ j = 1, 2, 3$ (3.12)

where $\frac{\partial}{\partial D}$ indicates the differentiation with respect to a direction D. When R_1, R_2 and R_3 are respectively C^1 continuous along the edges E_1, E_2 and E_3, the properties in (3.12) will enable the convex combination patch R to be C^1 continuous along all the three edges.

3.3 Effect of Corner Weights to C^1 Continuity

This sections discusses the effect of the weights $\omega^1_{3,0,0}$, $\omega^2_{0,3,0}$, $\omega^3_{0,0,3}$ at the three corners onto C^1 continuity across the boundaries of rational Bézier patch $R_i(u,v,w)$, $i=1$, 2, 3. Let us consider the patch R_1. Figure 3.3 shows the corresponding corner weights.

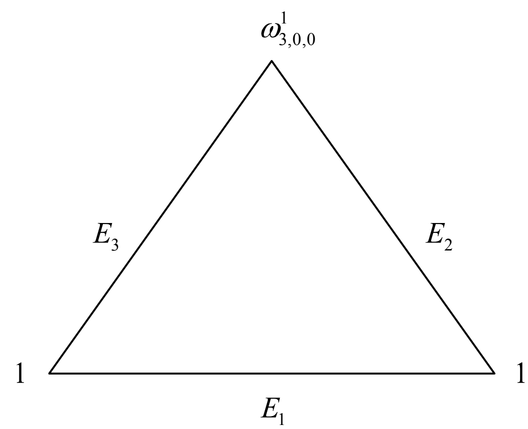

Figure 3.3: Three corner weights of R_1

By referring to equations (2.4) and (2.6), notice that $\omega^1_{3,0,0}$ does not exist in these equations when $u=0$. This concludes that any modification on the weight $\omega^1_{3,0,0}$ will not affect the boundary curve along the opposite edge E_1 and also the C^1 continuity across the edge. The same argument holds for the patches R_2 and R_3. However, the modification of $\omega^1_{3,0,0}$ will change the boundary curves of R_1 along the edges E_2 and E_3. But, this alteration of boundaries will not appear in the resulting convex combination patch R in (3.11) due to the properties of blending functions in (3.12).

CHAPTER 4

POSITIVITY CONDITIONS

The conditions for a rational cubic Bézier triangular patch to be positive are derived in this chapter. The formulation of these positivity conditions is motivated by the analysis shown in (Piah *et al*., 2005). The conditions are basically created based on the requirement that the minimum point of rational cubic Bézier patch must be positive. A positive patch means any point on the patch is greater than or equals to zero, that is the patch lies above the *xy*-plane. Section 4.1 will describe the sufficient positivity conditions for a cubic Bézier triangular patch which is the result from (Piah *et al.,* 2005). A lower bound is employed to restrict the Bézier ordinates such that the patch preserves positivity. In Section 4.2 the positivity conditions for rational cubic Bézier triangular patch will be introduced. A lower bound is imposed to the boundary Bézier ordinates and the weights of the rational Bézier patch are manipulated to obtain positive surface. Note that all the Bézier ordinates $b_{i,j,k}$ in this chapter are scalar numbers.

4.1 Sufficient Positivity Conditions for Cubic Bézier Triangular Patch

In 2001, Chan and Ong derived sufficient conditions for a cubic Bézier triangular patch to be positive where a lower bound was imposed on the Bézier ordinates of the Bézier patch. In 2005, Piah *et al*. proposed a more relaxed lower bound onto the Bézier ordinates. The derivation of this lower bound is shown as follows. Consider a cubic Bézier patch

$$P(u,v,w)=\sum_{\substack{i+j+k=3\\ i,j,k\geq 0}} b_{i,j,k}B^3_{i,j,k}(u,v,w)$$

$$=u^3b_{3,0,0}+v^3b_{0,3,0}+w^3b_{0,0,3}+3u^2vb_{2,1,0}+3uv^2b_{1,2,0}$$

$$+3u^2wb_{2,0,1}+3uw^2b_{1,0,2}+3v^2wb_{0,2,1}+3vw^2b_{0,1,2}+6uvwb_{1,1,1} \qquad (4.1)$$

where $b_{i,j,k}\in\mathbb{R}$, $u,v,w\geq 0$ and $u+v+w=1$. Note that this cubic patch can be obtained from rational cubic in (2.1) by assuming that all the weights are equal to 1, i.e. $\omega_{i,j,k}=1$, for $i+j+k=3$.

Given that all three Bézier ordinates at vertices are positive, i.e. $b_{3,0,0}$, $b_{0,3,0}$, $b_{0,0,3}>0$. A lower bound is determined for the remaining Bézier ordinates such that the patch $P(u,v,w)\geq 0$, if all the remaining Bézier ordinates have values greater than or equal to that lower bound. Let $b_{3,0,0}=A$, $b_{0,3,0}=B$, $b_{0,0,3}=C$ where $A,\ B,\ C>0$, and let the other Bézier ordinates $b_{2,1,0}$, $b_{1,2,0}$, $b_{1,0,2}$, $b_{2,0,1}$, $b_{0,1,2}$, $b_{0,2,1}$, $b_{1,1,1}=-m$, with $m>0$. Equation (4.1) can be rewritten as

$$P(u,v,w)=Au^3+Bv^3+Cw^3-3m(u^2v+u^2w+uv^2+uw^2+v^2w+vw^2+2uvw)$$

$$=Au^3+Bv^3+Cw^3-m(1-u^3-v^3-w^3)$$

$$=(A+m)u^3+(B+m)v^3+(C+m)w^3-m\,. \qquad (4.2)$$

From (4.2), it is obvious that when $m\leq 0$, $P(u,v,w)>0$ and $P(u,v,w)$ decreases when m increases. Now we shall find the value of m when the minimum of $P(u,v,w)$ is 0. The partial derivatives of P are