Shortest Reconfiguration of Perfect Matchings via Alternating Cycles

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Abstract

Motivated by adjacency in perfect matching polytopes, we study the shortest reconfiguration problem of perfect matchings via alternating cycles. Namely, we want to find a shortest sequence of perfect matchings which transforms one given perfect matching to another given perfect matching such that the symmetric difference of each pair of consecutive perfect matchings is a single cycle. The problem is equivalent to the combinatorial shortest path problem in perfect matching polytopes. We prove that the problem is NP-hard even when a given graph is planar or bipartite, but it can be solved in polynomial time when the graph is outerplanar.

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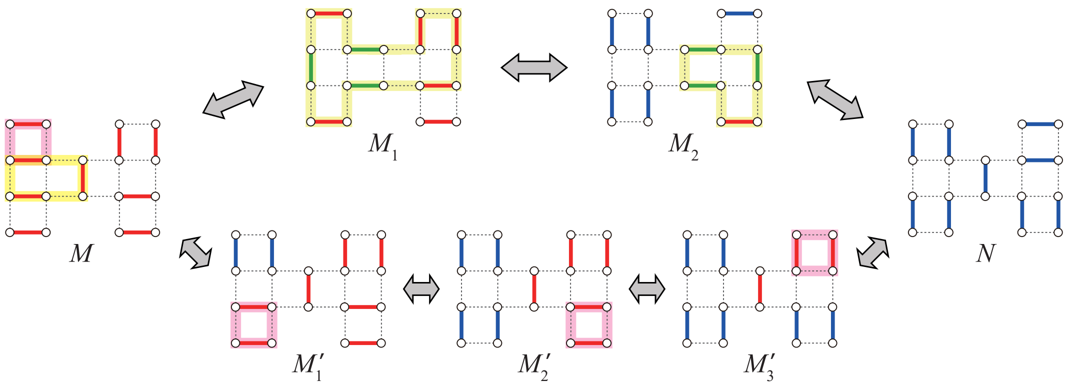


Figure 1 Two sequences of perfect matchings between M and N under the alternating cycle model. The sequence $\langle M, M_1, M_2, N\rangle$ following the yellow alternating cycles is shortest even though it touches the edge in $M \cap N$ twice. On the other hand, $\langle M, M'_1, M'_2, M'_3, N\rangle$ following the pink alternating cycles is not shortest although it touches only the edges in $M \triangle N$.

1 Introduction

Combinatorial reconfiguration is a fundamental research subject that sheds light on solution spaces of combinatorial (search) problems, and connects various concepts such as optimization, counting, enumeration, and sampling. In its general form, combinatorial reconfiguration is concerned with properties of the configuration space of a combinatorial problem. The configuration space of a combinatorial problem is often represented as a graph, but its size is usually exponential in the instance size. Thus, algorithmic problems on combinatorial reconfiguration are not trivial, and require novel tools for resolution. For recent surveys, see [11, 7].

Two basic questions have been encountered in the study of combinatorial reconfiguration. The first question asks the existence of a path between two given solutions in the configuration space, namely the *reachability* of the two solutions. The second question asks the shortest length of a path between two given solutions, if it exists. The second question is usually referred to as a *shortest reconfiguration problem*.

In this paper, we focus on reconfiguration problems of matchings, namely sets of independent edges. There are several ways of defining the configuration space for matchings, and some of them have already been studied in the literature [8, 9, 6, 3, 2]. We will explain them in Section 1.1.

We study yet another configuration space for matchings, which we call the *alternating path/cycle* model. The model is motivated by adjacency in matching polytopes, which we will see soon. In the model, we are given an undirected and unweighted graph G, and also an integer $k \geq 0$. The vertex set of the configuration space consists of the matchings in G of size at least k. Two matchings M and N in G are adjacent in the configuration space if and only if their symmetric difference $M \triangle N := (M \cup N) \setminus (M \cap N)$ is a single path or cycle. In particular, we are interested in the case where $k = |V(G)|/2$, namely the reconfiguration of *perfect matchings*. In that case, the model is simplified to the *alternating cycle* model since $M \triangle N$ cannot have a path. See Figure 1 as an example.

The reachability of two perfect matchings is trivial under the alternating cycle model: the answer is always yes. This is because the symmetric difference of two perfect matchings always consists of vertex-disjoint cycles. Therefore, we focus on the shortest perfect matching reconfiguration under the alternating cycle model.

1.1 Related Work[1]

Other Configuration Spaces for Matchings

As mentioned, reconfiguration problems of matchings have already been studied under different models [8, 9, 6, 3, 2]. These models chose more elementary changes as the adjacency on the configuration space. Then, the situation changes drastically under such models: even the reachability of two matchings is not guaranteed.

Matching reconfiguration was initiated by the work of Ito et al. [8]. They proposed the *token addition/removal* model of reconfiguration, in which we are also given an integer $k \geq 0$, and the vertex set of the configuration space consists of the matchings of size at least k.[2] Two matchings M and N are adjacent if and only if they differ in only one edge. Ito et al. [8] proved that the reachability of two given matchings can be checked in polynomial time.

Another model of reconfiguration is *token jumping*, introduced by Kamiński et al. [9]. In the token jumping model, we are also given an integer $k \geq 0$, and the vertex set of the configuration space consists of the matchings of size exactly k. Two matchings M and N are adjacent if and only if they differ in only two edges. Kamiński et al. [9, Theorem 1] proved that the token jumping model is equivalent to the token addition/removal model when two given matchings have the same size. Thus, using the result by Ito et al. [8], the reachability can be checked in polynomial time also under the token jumping model [9, Corollary 2].

On the other hand, the shortest matching reconfiguration is known to be hard. Gupta et al. [6] and Bousquet et al. [3] independently proved that the problem is NP-hard under the token jumping model. Then, the problem is also NP-hard under the token addition/removal model, because the shortest lengths are preserved under the two models [9, Theorem 1].

Recently, Bonamy et al. [2] studied the reachability of two perfect matchings under a model close to ours, namely the alternating cycle model *restricted to length four*. In the model, two perfect matchings M and N are adjacent if and only if their symmetric difference $M \triangle N$ is a cycle of length four. Then, the answer to the reachability is not always yes, and Bonamy et al. [2] proved that the reachability problem is PSPACE-complete under this restricted model.

Relation to Matching Polytopes

Our alternating cycle model (without any restriction of cycle length) for the perfect matching reconfiguration is natural when we see the connection with the simplex methods for linear optimization, or combinatorial shortest paths of the graphs of convex polytopes.

In the combinatorial shortest path problem of a convex polytope, we are given a convex polytope P, explicitly or implicitly, and two vertices v, w of P. Then, we want to find a shortest sequence $u_0, u_1, \ldots, u_t$ of vertices of P such that $u_0 = v, u_t = w$ and $\overline{u_i u_{i+1}}$ forms an edge of P for every $i = 0, 1, \ldots, t-1$. Often, we are only interested in the length of such a shortest sequence, and we are also interested in the maximum shortest path length among all pairs of vertices, which is known as the combinatorial diameter of the polytope P. The combinatorial diameter of a polytope has attracted much attention in the optimization community from the motivation of better understanding of simplex methods. Simplex methods for linear optimization start at a vertex of the feasible region, follow edges, and arrive at an optimal vertex. Therefore, the combinatorial diameter dictates the best-case

[1] Further related work can be found in the full version.
[2] Precisely, their model is defined in a slightly different way, but it is essentially the same as this definition.

behavior of such methods. The famous Hirsch conjecture states that every d-dimensional convex polytope with n facets has the combinatorial diameter at most $n - d$. This has been disproved by Santos [14], and the current best upper bound of $(n-d)^{\log_2 O(d/\log d)}$ for the combinatorial diameter was given by Sukegawa [15]. On the other hand, for the 0/1-polytopes (i.e., polytopes in which the coordinates of all vertices belong to $\{0,1\}$), the Hirsch conjecture holds [10].

The shortest perfect matching reconfiguration under the alternating cycle model can be seen as the combinatorial shortest path problem of a perfect matching polytope. The *perfect matching polytope* of a graph G is defined as follows. The polytope lives in $\mathbb{R}^{E(G)}$, namely each coordinate corresponds to an edge of G. Each vertex v of the polytope corresponds to a perfect matching M of G as $v_e = 1$ if $e \in M$ and $v_e = 0$ if $e \notin M$. The polytope is defined as the convex hull of those vertices. It is known that two vertices u, v of the perfect matching polytope form an edge if and only if the corresponding perfect matchings M, N have the property that $M \triangle N$ contains only one cycle [4]. This means that the graph of the perfect matching polytope is exactly the configuration space for perfect matchings under the alternating cycle model.

1.2 Our Contribution

To the best of the authors' knowledge, known results under different models do not have direct relations to our alternating cycle model, because their configuration spaces are different. In this paper, we thus investigate the polynomial-time solvability of the shortest perfect matching reconfiguration under the alternating cycle model. The results of our paper are two-fold.

1. The shortest perfect matching reconfiguration under the alternating cycle model can be solved in polynomial time if the input graph is outerplanar.
2. The shortest perfect matching reconfiguration under the alternating cycle model is NP-hard even when the input graph is planar or bipartite.

Since outerplanar graphs form a natural and fundamental subclass of planar graphs, our results exhibit a tractability border among planar graphs.

The hardness result for bipartite graphs implies that the computation of a combinatorial shortest path in a convex polytope is NP-hard even when an inequality description is explicitly given. This is because a polynomial-size inequality description of the perfect matching polytope can be explicitly written down from a given bipartite graph.

We point out that the hardness results have been independently obtained by Aichholzer et al. [1]. Indeed, they proved the hardness for planar bipartite graphs (i.e., an input graph is planar *and* bipartite).

Technical Key Points

Compared to recent algorithmic developments on reachability problems, only a few polynomial-time solvable cases are known for shortest reconfiguration problems. We now explain two technical key points, especially for algorithmic results on shortest reconfiguration problems.

The first point is the symmetric difference of two given solutions. Under several known models (not only for matchings) that employ elementary changes as the adjacency on the configuration space, the symmetric difference gives a (good) lower bound on the shortest reconfiguration. This is because any reconfiguration sequence (i.e., a path in the configuration space) between two given solutions must touch all elements in their symmetric difference at least once. For example, in Figure 1, the symmetric difference of two perfect matchings M and N consists of 16 edges and hence it gives the lower bound of $16/4 = 4$ under the alternating cycle model restricted to length 4 [2]. In such a case, if we can find a reconfiguration sequence touching only the elements in the symmetric difference (e.g., the sequence $\langle M, M'_1, M'_2, M'_3, N\rangle$ in Figure 1), then it is automatically the shortest under that model. However, this useful property does not hold under our alternating cycle model, because the length of an alternating cycle for reconfiguration is not fixed.

The second point is the characterization of *unhappy moves* that touch elements contained commonly in two given solutions. For example, the shortest reconfiguration sequence $\langle M, M_1, M_2, N\rangle$ in Figure 1 has an unhappy move, since it touches the edge in $M \cap N$ twice. In general, analyzing a shortest reconfiguration becomes much more difficult if such unhappy moves are required. A well-known example is the (generalized) 15-puzzle [13] in which the reachability can be determined in polynomial time, while the shortest reconfiguration is NP-hard. As illustrated in Figure 1, the shortest perfect matching reconfiguration requires unhappy moves even for outerplanar graphs, and hence we need to characterize the unhappy moves to develop a polynomial-time algorithm.

2 Problem Definition

In this paper, a graph always refers to an undirected graph that might have parallel edges and does not have loops. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. An edge subset $M \subseteq E$ is called a *matching* in G if no two edges in M share the end vertices. A matching M is *perfect* if $|M| = |V(G)|/2$.

A graph is *planar* if it can be drawn on the plane without edge crossing. Such a drawing is called a *plane* drawing of the planar graph. A *face* of a plane drawing is a maximal region of the plane that contains no point used in the drawing. There is a unique unbounded face, which is called the *outer face*. A planar graph is *outerplanar* if it has an *outerplane* drawing, i.e., a plane drawing in which all vertices are incident to the outer face.

For a matching M in a graph G, a cycle C in G is called M-*alternating* if edges in M and $E(G) \setminus M$ alternate in C. We identify a cycle with its edge set to simplify the notation. We say that two perfect matchings M and N are *reachable* (under the alternating cycle model) if there exists a sequence $\langle M_0, M_1, \ldots, M_t\rangle$ of perfect matchings in G such that

(i) $M_0 = M$ and $M_t = N$; and

(ii) $M_i = M_{i-1} \triangle C_i$ for some M_{i-1}-alternating cycle C_i for each $i = 1, \ldots, t$.

Such a sequence is called a *reconfiguration sequence* between M and N, and its *length* is defined as t.

For two perfect matchings M and N, the subgraph $M \triangle N$ consists of disjoint M-alternating cycles $C_1, \ldots, C_t$. Thus it is clear that M and N are always reachable for any two perfect matchings M and N by setting $M_i = M_{i-1} \triangle C_i$ for $i = 1, \ldots, t$. In this paper, we are interested in finding a *shortest* reconfiguration sequence of perfect matchings. That is, the problem is defined as follows:

Shortest Perfect Matching Reconfiguration
Input: A graph G and two perfect matchings M and N in G
Find: A shortest reconfiguration sequence between M and N.

We denote by a tuple $I = (G, M, N)$ an instance of Shortest Perfect Matching Reconfiguration. Also, we denote by $\mathrm{OPT}(I)$ the length of a shortest reconfiguration sequence of an instance I. We note that it may happen that $\mathrm{OPT}(I)$ is much shorter than the number of disjoint M-alternating cycles in $M \triangle N$ (see Figure 1).

3 Polynomial-Time Algorithm for Outerplanar Graphs

In this section, we prove that there exists a polynomial-time algorithm for SHORTEST PERFECT MATCHING RECONFIGURATION on an outerplanar graph, as follows.

▶ **Theorem 1.** SHORTEST PERFECT MATCHING RECONFIGURATION *on outerplanar graphs G can be solved in $O(|V(G)|^5)$ time.*

We give such an algorithm in this section. Let $I = (G, M, N)$ be an instance of the problem such that $G = (V, E)$ is an outerplanar graph. We first observe that it suffices to consider the case when G is 2-connected.

▶ **Lemma 2** ($*$[3]). *Let $I = (G, M, N)$ be an instance of* SHORTEST PERFECT MATCHING RECONFIGURATION*, and $G_1, \dots, G_p$ be the 2-connected components of G. Furthermore, for every $i = 1, \dots, p$, let $I_i = (G_i, M \cap E(G_i), N \cap E(G_i))$ be an instance of* SHORTEST PERFECT MATCHING RECONFIGURATION*. Then, $\mathrm{OPT}(I) = \sum_{i=1}^{p} \mathrm{OPT}(I_i)$.*

Since the 2-connected components of a graph can be found in linear time, the reduction to 2-connected outerplanar graphs can be done in linear time, too.

We fix an outerplane drawing of a given 2-connected outerplanar graph G, and identify G with the drawing for the sake of convenience. We denote by C_{out} the outer face boundary. Then C_{out} is a simple cycle since G is 2-connected. We denote the set of the *inner edges of* G by $E_{\text{in}} = E \setminus C_{\text{out}}$. In other words, E_{in} is the set of chords of C_{out}.

3.1 Technical Highlight

As mentioned in Introduction, there are two technical key points to develop a polynomial-time algorithm for SHORTEST PERFECT MATCHING RECONFIGURATION: a lower bound on the length of a shortest reconfiguration sequence, and the characterization of unhappy moves. We here explain our ideas roughly, and will give detailed descriptions in the next subsections.

Since G is planar, we can define its "dual-like" graph G^*. Then, G^* forms a tree since G is outerplanar and 2-connected. (The definition of G^* will be given in Section 3.2, and an example is given in Figure 2.) We make a correspondence between an edge in G^* and a set of edges in G. Then, we will define the length $\ell(e^*)$ of each edge e^* in G^* so that it represents the "gap" between M and N when we are restricted to the edges in the corresponding set of e^*. It is important to notice that any cycle C in G corresponds to a subtree of G^*, and vice versa. Indeed, we focus on a cut C^* of G^* clipping the subtree from G^*, that is, the set of edges in G^* leaving the subtree. If we apply an M-alternating cycle C to a perfect matching M of G, then it changes lengths $\ell(e^*)$ of the edges e^* in the corresponding cut C^*.

For our algorithm, we need a (good) lower bound for the length of a shortest reconfiguration sequence between two given perfect matchings M and N. Recall that $|M \triangle N|$ does not give a good lower bound under the alternating cycle model. This is because we can take a cycle of an arbitrary (non-fixed) length, and hence $|M \triangle N|$ can decrease drastically by only a single alternating cycle. Furthermore, no matter how we define the length $\ell(e^*)$ of each edge e^* in G^*, the total length of *all* edges in G^* does not give a good lower bound. This is because a cycle C of non-fixed length in G may correspond to a cut C^* having many edges in G^*, and hence it can change the total length drastically. Our key idea is to focus on the total length of each *path* in G^*, that is, we take the *diameter* of G^* (with respect to length ℓ)

[3] The symbol ($*$) means that the proof is postponed to the full version.

as a lower bound. Then, because G^* is a tree, any path in G^* can contain at most two edges from the corresponding cut C^*. Therefore, regardless of the cycle length, the diameter of G^* can be changed by only these two edges. By carefully setting the length $\ell(e^*)$ as in (1), we will prove that the diameter of G^* is not only a lower bound, but indeed gives the shortest length under the assumption that $E_{\text{in}} \cap M \cap N$ is empty. Therefore, the real difficulty arises when $E_{\text{in}} \cap M \cap N$ is not empty.

In the latter case, we will characterize the unhappy moves. Assume that we know the set $F \subseteq E_{\text{in}} \cap M \cap N$ of chords that are *not* touched in a shortest reconfiguration sequence between M and N; in other words, *all* chords in $(E_{\text{in}} \cap M \cap N) \setminus F$ must be touched for unhappy moves in that sequence. Then, we subdivide a given outerplanar graph G into subgraphs $G_1, \ldots, G_{|F|+1}$ along the chords in F. Notice that each edge in F appears on the outer face boundaries in two of these subgraphs. Furthermore, each chord e in these subgraphs satisfies $e \in (E_{\text{in}} \cap M \cap N) \setminus F$ if $e \in M \cap N$. Therefore, *all* chords in these subgraphs are touched for unhappy moves as long as they are in $M \cap N$. Under this assumption, we will prove that the diameter of G_i^* gives the shortest length of a reconfiguration sequence between $M \cap E(G_i)$ and $N \cap E(G_i)$. Thus, we can solve the problem in polynomial time if we know F which yields a shortest reconfiguration sequence between M and N. Finally, to find such a set F of chords, we construct a polynomial-time algorithm which employs a dynamic programming method along the tree G^*.

3.2 Preliminaries: Constructing a Dual Graph

Let $I = (G, M, N)$ be an instance of Shortest Perfect Matching Reconfiguration such that G is a 2-connected outerplanar graph. Since G is planar, we can define the *dual* of G. In fact, we here construct a graph G^* obtained from the dual by applying a slight modification as follows. The construction is illustrated in Figure 2. Let V^* be the set of faces (without the outer face) of G. For a face $v^* \in V^*$, let $E_{v^*} \subseteq E(G)$ be the set of edges around v^*. We denote the set of faces touching the outer face by U^*, i.e., $U^* = \{v^* \in V^* \mid E_{v^*} \cap C_{\text{out}} \neq \emptyset\}$. We make a copy of U^*, denoted by $\tilde{U}^*$. We set the vertex set of G^* to be $V^* \cup \tilde{U}^*$. For v^*, w^* in V^*, an edge v^*w^* in G^* exists if and only if the faces v^* and w^* share an edge in E_{in}, i.e., $|E_{v^*} \cap E_{w^*}| = 1$. Also G^* has an edge between u^* and $\tilde{u}^*$ for every $u^* \in U^*$, where $\tilde{u}^* \in \tilde{U}^*$ is the copy of u^*. Thus the edge set of G^* is given by

$$E(G^*) = \{v^*w^* \mid v^*, w^* \in V^*,\ |E_{v^*} \cap E_{w^*}| = 1\} \cup \{u^*\tilde{u}^* \mid u^* \in U^*\}.$$

The first part is denoted by E_{in}^*, and the second part is denoted by $\tilde{E}^*$. We observe that G^* is a tree, since G is 2-connected and outerplanar. A face of G that touches only one face (other than the outer face) is called a *leaf in* $G^* - \tilde{U}^*$. We note that there is a one-to-one correspondence between edges in E_{in} of G and E_{in}^* of G^*. For an edge subset $F \subseteq E_{\text{in}}$, F^* denotes the corresponding edge subset in G^*, that is, $F^* = \{e^* \in E_{\text{in}}^* \mid e \in F\}$. Conversely, for an edge subset $F^* \subseteq E(G^*)$, F denotes the corresponding edge subset in E_{in}, that is, $F = \{e \in E_{\text{in}} \mid e^* \in F^* \cap E_{\text{in}}^*\}$. We extend this correspondence to $\tilde{E}^*$, that is, $u^*\tilde{u}^* \in \tilde{E}^*$ corresponds to the edge set $E_{u^*} \cap C_{\text{out}}$ for $u^* \in U^*$, and vice versa.

It follows from the duality that there is a relationship between a cut in G^* and a cycle in G. Suppose that we are given a cycle C $(\neq C_{\text{out}})$ in G. Then, since G is outerplanar, the cycle C surrounds a set X^* of faces such that X^* does not have the outer face. The set X^* induces a connected graph (subtree) in G^*, and the set of edges leaving from X^* yields a cut $C^* = \{e^* = v^*w^* \mid v^* \in X^*, w^* \in V(G^*) \setminus X^*\}$. Conversely, let $X^* \subseteq V^*$ be a vertex subset of G^* such that the subgraph induced by X^* is connected. Then the set of edges leaving from X^* yields a cut C^* in G^*, which corresponds to a cycle in G.

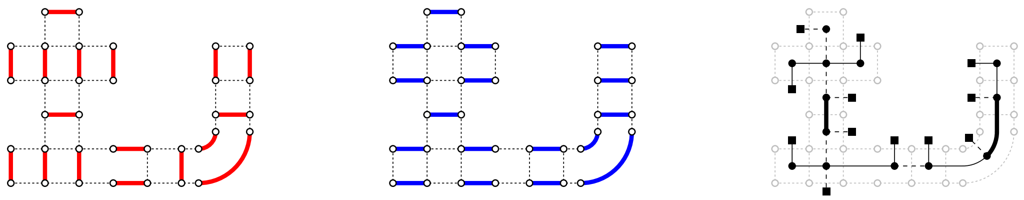


Figure 2 The construction of G^* and the length function ℓ. In (c), the edge lengths are depicted by different styles: thick solid lines represent edges of length two, thin solid lines represent edges of length one, and dotted lines represent edges of length zero.

We classify faces in U^* into two groups. For a face u^* in U^*, the edge set $E_{u^*} \cap C_{\mathrm{out}}$ forms a family $\mathcal{P}_{u^*}$ of disjoint paths. Since M and N are perfect matchings, each path P in $\mathcal{P}_{u^*}$ is both M-alternating and N-alternating. In addition, P satisfies either

(i) $E(P) \subseteq M \triangle N$, or

(ii) $(M \triangle N) \cap E(P) = \emptyset$.

Furthermore, we observe that either (i) holds for *every* path P in $\mathcal{P}_{u^*}$, or (ii) holds for *every* path P in $\mathcal{P}_{u^*}$. Indeed, since $M \triangle N$ consists of disjoint cycles, if some path P in $\mathcal{P}_{u^*}$ satisfies (i), then P is included in a cycle C in $M \triangle N$ that separates u^* from the outer face. Since the other paths in $\mathcal{P}_{u^*}$ touch the outer face, they are on C. Thus every path satisfies (i), which shows the observation. We divide U^* into two groups U_1^* and U_2^* where each face in U_1^* satisfies (i) for every path, while each face in U_2^* satisfies (ii) for every path.

For an edge e^* in $E(G^*)$, we define the length $\ell(e^*)$ to be

$$\ell(e^*) = \begin{cases} |M \cap \{e\}| + |N \cap \{e\}| & \text{if } e^* \in E^*_{\mathrm{in}}; \\ 1 & \text{if } e^* = u^*\tilde{u}^* \in \tilde{E}^* \text{ such that } u^* \in U_1^*; \\ 0 & \text{if } e^* = u^*\tilde{u}^* \in \tilde{E}^* \text{ such that } u^* \in U_2^*. \end{cases} \tag{1}$$

See Figure 2 for an example. Let $\ell(u^*, v^*)$ be the length of the (unique) path from u^* to v^* in G^*. We define the *gap* between M and N in the graph G as the diameter of G^*, that is, we define $\mathrm{gap}(I) = \max\{\ell(u^*, v^*) \mid u^*, v^* \in V(G^*)\}$. This value is simply denoted by $\mathrm{gap}(M, N)$ if G is clear from the context.

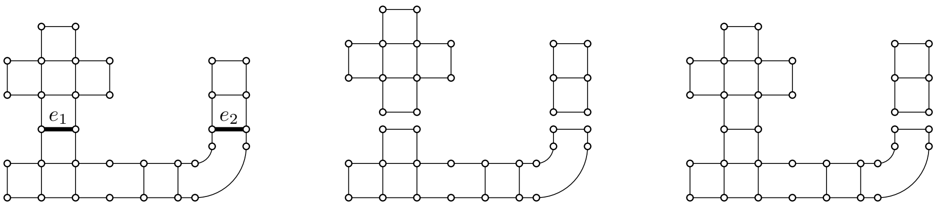
3.3 Characterization for the Disjoint Case

Let $I = (G, M, N)$ be an instance of Shortest Perfect Matching Reconfiguration such that G is a 2-connected outerplanar graph. In this subsection, we show that if $E_{\mathrm{in}} \cap M \cap N$ is empty, we can characterize the optimal value with $\mathrm{gap}(I)$, which leads to a simple polynomial-time algorithm for this case. We note that if $E_{\mathrm{in}} \cap M \cap N$ is empty, then no edge in E_{in} belongs to both M and N, and hence $\ell(e^*)$ can only take 0 or 1.

▶ **Lemma 3** ($*$). *It holds that* $\mathrm{gap}(M, N)$ *is even.*

A main theorem of this subsection is to give a characterization of the optimal value with $\mathrm{gap}(M, N)$.

▶ **Theorem 4.** *Let* $I = (G, M, N)$ *be an instance of* Shortest Perfect Matching Reconfiguration *such that* G *is a 2-connected outerplanar graph. If* $E_{\mathrm{in}} \cap M \cap N$ *is empty, then it holds that* $\mathrm{OPT}(I) = \mathrm{gap}(M, N)/2$.



(a) G and $E'_{\text{in}} = \{e_1, e_2\}$. (b) G_1, G_2, G_3 when $F = E'_{\text{in}}$. (c) G_1, G_2 when $F = \{e_2\}$.

Figure 3 Decomposition of the outerplanar graph in Figure 2. The edges in E'_{in} are shown with bold lines.

Proof. To show the theorem, we first prove the following claim.

▷ Claim 5 ($*$). For any M-alternating cycle C, it holds that $\text{gap}(M, N) \leq \text{gap}(M \triangle C, N) + 2$.

Consider a shortest reconfiguration sequence $\langle M_0, M_1, \ldots, M_t \rangle$ from $M_0 = M$ to $M_t = N$. Then, $t = \text{OPT}(I)$. For each $i = 1, \ldots, t$, it then holds that $\text{gap}(M_{i-1}, N) \leq \text{gap}(M_i, N) + 2$. By repeatedly applying the above inequalities, we obtain

$$\text{gap}(M, N) = \text{gap}(M_0, N) \leq \text{gap}(M_t, N) + 2t = 2t = 2\text{OPT}(I)$$

since $\text{gap}(M_t, N) = 0$. Hence it holds that $\text{OPT}(I) \geq \text{gap}(M, N)/2$.

It remains to show that $\text{OPT}(I) \leq \text{gap}(M, N)/2$. We prove the following claim.

▷ Claim 6 ($*$). There exists an M-alternating cycle C such that $\text{gap}(M, N) = \text{gap}(M \triangle C, N) + 2$.

For a perfect matching M_{i-1} in G, it follows from Claim 6 that there exists an M_{i-1}-alternating cycle C_i such that $\text{gap}(M_{i-1}, N) = \text{gap}(M_{i-1} \triangle C_i, N) + 2$. Define $M_i = M_{i-1} \triangle C_i$, and repeat finding an alternating cycle satisfying the above equation. The repetition ends when $\text{gap}(M_i, N) = 0$, which means that $M_i = N$. The number of repetitions is equal to $\text{gap}(M, N)/2$, and therefore, we have $\text{OPT}(I) \leq \text{gap}(M, N)/2$. Thus the proof is complete. ◀

3.4 General Case

Let $I = (G, M, N)$ be an instance of Shortest Perfect Matching Reconfiguration such that G is a 2-connected outerplanar graph. Define $E'_{\text{in}} = E_{\text{in}} \cap M \cap N$. In this subsection, we deal with the general case, that is, E'_{in} is not necessarily empty. Then, there is a case when changing an edge in E'_{in} reduces the number of reconfiguration steps as in Figure 1. We call such a move an *unhappy move*. The key idea of our algorithm is to detect a set of edges necessary for unhappy moves.

Since G is outerplanar and 2-connected, any $F \subseteq E'_{\text{in}}$ divides the inner region of C_{out} into $|F| + 1$ parts $R_1, \ldots, R_{|F|+1}$. For each $i = 1, \ldots, |F| + 1$, let G_i be the subgraph of G consisting of all the vertices and the edges in R_i and its boundary. Thus, each edge $e \in F$ appears on the outer face boundaries in two of these subgraphs. See Figure 3. Let $\mathcal{G}_F = \{G_1, \ldots, G_{|F|+1}\}$. Note that each graph in $\mathcal{G}_F$ is outerplanar and 2-connected. For each $H \in \mathcal{G}_F$, let $I_H = (H, M \cap E(H), N \cap E(H))$. We now show the following theorem.

▶ **Theorem 7.** $\text{OPT}(I) = \dfrac{1}{2} \min\limits_{F \subseteq E'_{\text{in}}} \sum\limits_{H \in \mathcal{G}_F} \text{gap}(I_H)$.

Proof. Let $\langle M_0, M_1, \ldots, M_t\rangle$ be a shortest reconfiguration sequence from $M_0 = M$ to $M_t = N$. We denote by C_i the M_{i-1}-alternating cycle with $M_i = M_{i-1} \triangle C_i$. Define $F_{\mathrm{opt}} = \{e \in E'_{\mathrm{in}} \mid e \notin C_i, \forall i\}$, which is the set of edges in E'_{in} that are not touched in the shortest reconfiguration sequence. Then C_i is contained in some $H \in \mathcal{G}_{F_{\mathrm{opt}}}$, and can be used to obtain a reconfiguration sequence from $M \cap E(H)$ to $N \cap E(H)$ in H. Therefore, we have

$$\mathrm{OPT}(I) = \sum_{H \in \mathcal{G}_{F_{\mathrm{opt}}}} \mathrm{OPT}(I_H). \tag{2}$$

We can also see that

$$\mathrm{OPT}(I) \le \sum_{H \in \mathcal{G}_F} \mathrm{OPT}(I_H) \tag{3}$$

for any $F \subseteq E'_{\mathrm{in}}$.

To evaluate $\mathrm{OPT}(I_H)$ for $H \in \mathcal{G}_F$, we slightly modify the instance I_H by replacing every inner edge of H contained in $M \cap N$ by two parallel edges each in M and N, respectively. The obtained graph is denoted by H', and the corresponding instance is denoted by $I_{H'}$. Since a reconfiguration sequence for $I_{H'}$ can be converted to one for I_H, it holds that $\mathrm{OPT}(I_H) \le \mathrm{OPT}(I_{H'})$, and hence

$$\mathrm{OPT}(I) \le \sum_{H \in \mathcal{G}_F} \mathrm{OPT}(I_H) \le \sum_{H \in \mathcal{G}_F} \mathrm{OPT}(I_{H'}) \tag{4}$$

holds for any $F \subseteq E'_{\mathrm{in}}$ by (3). Moreover, by the definition of F_{opt}, there exists an index i such that $e \in C_i$ for any $e \in E'_{\mathrm{in}} \setminus F_{\mathrm{opt}}$. Therefore, for $H \in \mathcal{G}_{F_{\mathrm{opt}}}$, the shortest reconfiguration sequence for I_H can be converted to a reconfiguration sequence for $I_{H'}$. Thus, $\mathrm{OPT}(I_H) \ge \mathrm{OPT}(I_{H'})$ holds for $H \in \mathcal{G}_{F_{\mathrm{opt}}}$, and hence

$$\mathrm{OPT}(I) = \sum_{H \in \mathcal{G}_{F_{\mathrm{opt}}}} \mathrm{OPT}(I_H) \ge \sum_{H \in \mathcal{G}_{F_{\mathrm{opt}}}} \mathrm{OPT}(I_{H'}) \tag{5}$$

by (2). By (4) and (5), we obtain

$$\mathrm{OPT}(I) = \min_{F \subseteq E'_{\mathrm{in}}} \sum_{H \in \mathcal{G}_F} \mathrm{OPT}(I_{H'}), \tag{6}$$

and F_{opt} is a minimizer of the right-hand side.

By (6) and Theorem 4, we obtain

$$\mathrm{OPT}(I) = \frac{1}{2} \min_{F \subseteq E'_{\mathrm{in}}} \sum_{H \in \mathcal{G}_F} \mathrm{gap}(I_{H'}), \tag{7}$$

because each $I_{H'}$ satisfies the condition in Theorem 4. Since $(H')^*$ is obtained from H^* by subdividing some edges of length two into two edges of length one, the diameter of $(H')^*$ is equal to that of H^*, that is, $\mathrm{gap}(I_{H'}) = \mathrm{gap}(I_H)$. Therefore, we obtain the theorem by (7). ◀

As an example, we apply this theorem to the instance in Figure 2. See Figure 3(c). If F consists of only the right thick edge in Figure 2(c), then $\mathcal{G}_F$ consists two graphs G_1 and G_2 such that $\mathrm{gap}(I_{G_1}) = 6$ and $\mathrm{gap}(I_{G_2}) = 2$. Since we can check that such F attains the minimum in the right-hand side of Theorem 7, we obtain $\mathrm{OPT}(I) = 4$ by Theorem 7.

In order to compute the value in Theorem 7 efficiently, we reduce the problem to Min-Sum Diameter Decomposition, whose definition will be given later.

For $F \subseteq E'_{\mathrm{in}}$, let F^* be the edge subset of E^*_{in} corresponding to F, and let $\mathcal{G}_F = \{G_1, \dots, G_{|F|+1}\}$. Then, $G^* - F^*$ consists of $|F|+1$ components $T_1, T_2, \dots, T_{|F|+1}$ such that T_i coincides with G^*_i (except for the difference of edges of length zero) for $i = 1, \dots, |F|+1$. In particular, for each i, we have $\mathrm{gap}(I_{G_i}) = \max\{\ell(u^*, v^*) \mid u^*, v^* \in V(T_i)\}$, where ℓ is the length function on $E(G^*)$ defined by the instance $I = (G, M, N)$. We call $\max\{\ell(u^*, v^*) \mid u^*, v^* \in V(T_i)\}$ the *diameter* of T_i, which is denoted by $\mathrm{diam}_\ell(T_i)$. Then, Theorem 7 shows that

$$\mathrm{OPT}(I) = \frac{1}{2} \min_{F \subseteq E'_{\mathrm{in}}} \sum_{i=1}^{|F|+1} \mathrm{diam}_\ell(T_i). \tag{8}$$

Therefore, we can compute $\mathrm{OPT}(I)$ by solving the following problem in which $T = G^*$ and $E_0 = (E'_{\mathrm{in}})^*$.

Min-Sum Diameter Decomposition
Input: A tree T, an edge subset $E_0 \subseteq E(T)$, and a length function $\ell : E(T) \to \mathbb{Z}_{\geq 0}$
Find: An edge set $F \subseteq E_0$ that minimizes $\sum_{T'} \mathrm{diam}_\ell(T')$, where the sum is taken over all the components T' of $T - F$.

In the subsequent subsection, we show that Min-Sum Diameter Decomposition can be solved in time polynomial in $|V(T)|$ and $L := \sum_{e \in E(T)} \ell(e)$.

▶ **Theorem 8.** Min-Sum Diameter Decomposition *can be solved in* $O(|V(T)|L^4)$ *time, where* $L := \sum_{e \in E(T)} \ell(e)$.

Since (8) shows that Shortest Perfect Matching Reconfiguration on outerplanar graphs is reduced to Min-Sum Diameter Decomposition in which $L = O(|V(T)|)$, we obtain Theorem 1.

3.5 Algorithm for Min-Sum Diameter Decomposition

The remaining task is to show Theorem 8, that is, to give an algorithm for Min-Sum Diameter Decomposition that runs in $O(|V(T)|L^4)$ time. For this purpose, we adopt a dynamic programming approach.

We choose an arbitrary vertex r of a given tree T, and regard T as a rooted tree with the root r. For each vertex v of T, we denote by T_v the subtree of T which is rooted at v and is induced by all descendants of v in T. (See Figure 4(a).) Thus, $T = T_r$ for the root r. Let $w_1, w_2, \dots, w_q$ be the children of v, ordered arbitrarily. For each $j \in \{1, 2, \dots, q\}$, we denote by T^j_v the subtree of T induced by $\{v\} \cup V(T_{w_1}) \cup V(T_{w_2}) \cup \dots \cup V(T_{w_j})$. For example, in Figure 4(b), the subtree T^j_v is surrounded by a thick dotted rectangle. For notational convenience, we denote by T^0_v the tree consisting of a single vertex v. Then, $T_v = T^0_v$ for each leaf v of T. Our algorithm computes and extends partial solutions for subtrees T^j_v from the leaves to the root r of T by keeping the information required for computing (the sum of) diameters of a partial solution.

We now define partial solutions for subtrees. For a subtree T^j_v and an edge subset $F' \subseteq E_0 \cap E(T^j_v)$, the *frontier* for F' is the component (subtree) in $T^j_v - F'$ that contains the root v of T^j_v. We sometimes call it the *v-frontier* for F' to emphasize the root v. For three integers $x, y, z \in \{0, 1, \dots, L\}$, the edge subset F' is called an *(x, y, z)-separator* of T^j_v if the following three conditions hold. (See also Figure 4(c).)

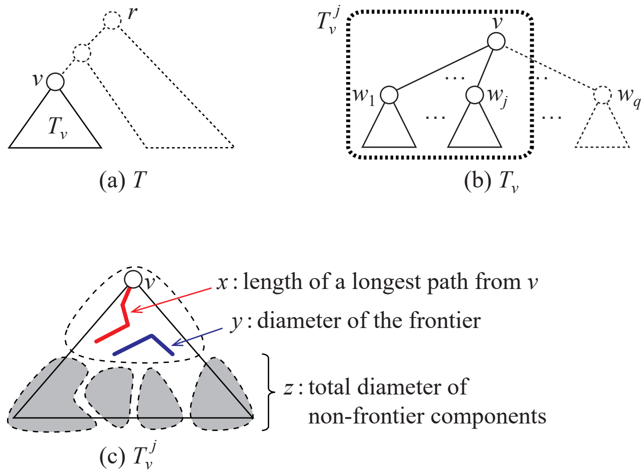


Figure 4 (a) Subtree T_v in the whole tree T, (b) subtree T_v^j in T_v, and (c) an (x, y, z)-separator of T_v^j.

- $x = \max\{\ell(v, u) \mid u \in V(T_{F'})\}$, where $T_{F'}$ is the v-frontier for F'. That is, the longest path from v to a vertex in $T_{F'}$ is of length x.
- $y = \mathrm{diam}_\ell(T_{F'})$, that is, y denotes the diameter of the v-frontier $T_{F'}$ for F'.
- $z = \sum_{T'} \mathrm{diam}_\ell(T')$, where the sum is taken over all the components T' of $(T - F') \setminus T_{F'}$.

Note that $x \le y$ always holds for an (x, y, z)-separator of T_v^j. We then define the following function: for a subtree T_v^j and two integers $x, y \in \{0, 1, \dots, L\}$, we let

$$f(T_v^j; x, y) = \min\left\{z \mid T_v^j \text{ has an } (x, y, z)\text{-separator}\right\}.$$

Note that $f(T_v^j; x, y)$ is defined as $+\infty$ if T_v^j does not have an (x, y, z)-separator for any $z \in \{0, 1, \dots, L\}$. Then, the optimal objective value to Min-Sum Diameter Decomposition can be computed as $\min\{y + f(T; x, y) \mid x, y \in \{0, 1, \dots, L\}\}$.

For a given tree T, our algorithm computes $f(T_v^j; x, y)$ for all possible triplets (T_v^j, x, y) from the leaves to the root r of T. The algorithm runs in $O(|V(T)|L^4)$ time in total. (The details are explained in the full version.) Note that we can easily modify the algorithm so that we obtain not only the optimal value but also an optimal solution. This completes the proof of Theorem 8.

We note here that the algorithm can be modified so that the running time is bounded by a polynomial in $|V(T)|$ by replacing the domain $\{0, 1, \dots, L\}$ of x and y with $D := \{\ell(u, v) \mid u, v \in V(T)\}$. This modification is valid, because $f(T_v^j; x, y) = +\infty$ unless $x, y \in D$. Since $|D| = O(|V(T)|^2)$, the modified algorithm runs in $O(|V(T)||D|^4) = O(|V(T)|^9)$ time. Note that, although this bound is polynomial only in $|V(T)|$, it is worse than $O(|V(T)|L^4)$ when $L = O(|V(T)|)$.

4 NP-Hardness for Planar Graphs and Bipartite Graphs

In this section, we prove that Shortest Perfect Matching Reconfiguration is NP-hard even when the input graph is planar or bipartite.

▶ **Theorem 9.** Shortest Perfect Matching Reconfiguration *is* NP*-hard even for planar graphs of maximum degree three.*

We reduce the Hamiltonian Cycle problem, which is known to be NP-complete even when a given graph is 3-regular and planar [5].

Hamiltonian Cycle
Input: A 3-regular planar graph $H = (V, E)$
Question: Decide whether H has a Hamiltonian cycle, i.e., a cycle that goes through all the vertices exactly once.

Proof. Let H be a 3-regular planar graph, which is an instance of Hamiltonian Cycle. For each vertex $v \in V(H)$, we define a 8-vertex graph D_v (see also the top right in Figure 5):

$$V(D_v) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\},$$
$$E(D_v) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_4v_5, v_5v_7, v_3v_6, v_6v_8\}.$$

We construct an instance $I = (G, M, N)$ of our problem as follows. (See Figure 5 as an example.) We subdivide each edge $e = uv$ in H twice, and the obtained vertices are denoted by u_e and v_e, where u_e is closer to u. Then, for each vertex $v \in V(H)$, we replace v with the graph D_v, and connect v_7 to $v_{e_v^{(1)}}$ and $v_{e_v^{(2)}}$, v_8 to $v_{e_v^{(2)}}$ and $v_{e_v^{(3)}}$, where $e_v^{(1)}$, $e_v^{(2)}$, $e_v^{(3)}$ are edges incident to v and the order follows the plane drawing of H. Let $E_v = \{v_7v_{e_v^{(1)}}, v_7v_{e_v^{(2)}}, v_8v_{e_v^{(2)}}, v_8v_{e_v^{(3)}}\}$. The resulting graph is denoted by G, i.e., G is defined as follows:

$$V(G) = \bigcup_{v\in V(H)} V(D_v) \cup \bigcup_{e=uv\in E(H)} \{u_e, v_e\},$$
$$E(G) = \left(\bigcup_{v\in V(H)} E(D_v) \cup E_v\right) \cup \{u_ev_e \mid e \in E(H)\}.$$

It follows that G is a planar graph of maximum degree three. Furthermore, we define initial and target perfect matchings M and N in G, respectively, to be

$$M = \{v_1v_2, v_3v_4, v_5v_7, v_6v_8 \mid v \in V(H)\} \cup \{u_ev_e \mid e \in E(H)\},$$
$$N = \{v_1v_4, v_2v_3, v_5v_7, v_6v_8 \mid v \in V(H)\} \cup \{u_ev_e \mid e \in E(H)\}.$$

This completes the construction of our corresponding instance $I = (G, M, N)$. The construction can be done in polynomial time.

We then give the following claim.

▷ Claim 10 ($*$). H has a Hamiltonian cycle if and only if $\mathrm{OPT}(I) = 2$.

This completes the proof of Theorem 9. ◀

The hardness for bipartite graphs of maximum degree three can be obtained with a similar proof; the reduction uses the Directed Hamiltonian Cycle problem which is NP-complete even when input directed graphs have the maximum in-degree two and the maximum out-degree two [12]. The details are deferred to the full version.

▶ **Theorem 11** ($*$). Shortest Perfect Matching Reconfiguration *is* NP*-hard even for bipartite graphs of maximum degree three.*

The proofs actually show that Shortest Perfect Matching Reconfiguration is NP-hard to approximate within a factor of less than 3/2.

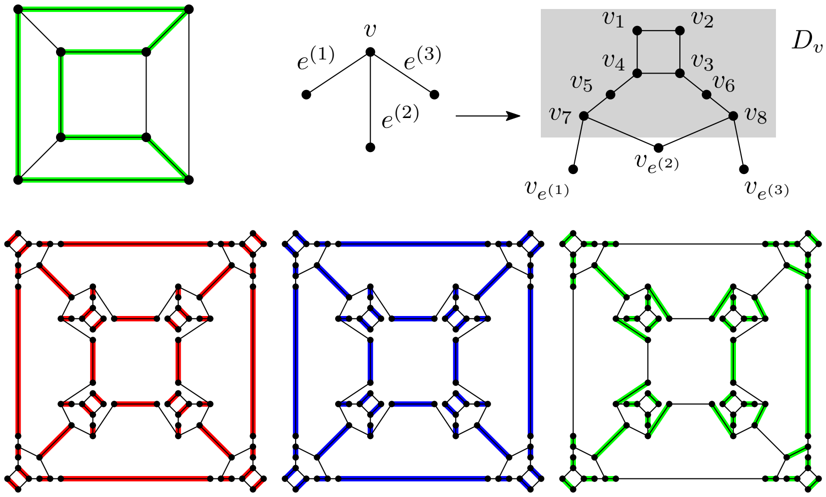


Figure 5 Reduction for planar graphs of maximum degree three. Top left: a yes instance H of Hamiltonian Cycle with a green Hamiltonian cycle. Top right: the constructed fragment D_v. Bottom left: The initial perfect matching M (red). Bottom middle: The target perfect matching N (blue). Bottom right: The perfect matching obtained as $M \triangle C$, where C corresponds to the Hamiltonian cycle of H.

5 Conclusion

In this paper, we studied the shortest reconfiguration problem of perfect matchings under the alternating cycle model, which is equivalent to the combinatorial shortest path problem on perfect matching polytopes. We prove that the problem can be solved in polynomial time for outerplanar graphs, but it is NP-hard, and even APX-hard for planar graphs and bipartite graphs.

Several questions remain unsolved. For polynomial-time solvability, our algorithm runs only for outerplanar graphs, and it looks difficult to extend the algorithm to other graph classes. A next step would be to try k-outerplanar graphs for fixed $k \geq 2$.

One way to tackle NP-hard cases is approximation. We only know the NP-hardness of approximating within a factor of less than 3/2. We believe the existence of a polynomial-time constant-factor approximation. Note that we do not obtain a constant-factor approximation by flipping alternating cycles in the symmetric difference of two given perfect matchings one by one.

This paper was mainly concerned with reconfiguration of perfect matchings. Alternatively, we may consider reconfiguration of maximum matchings, or maximum-weight matchings. In those cases, we need to adopt the alternating path/cycle model. Then, the question is related to the combinatorial shortest path problem on faces of matching polytopes. Note that the perfect matching polytope is also a face of the matching polytope. Therefore, the study on maximum-weight matchings will be a generalization of this paper.

To the best of the authors' knowledge, the combinatorial shortest path problem of 0/1-polytopes has not been well investigated while the adjacency in 0/1-polytopes has been extensively studied in the literature. This paper opens up a new perspective for the study of combinatorial and computational aspects of polytopes, and connects them with the study of combinatorial reconfiguration.

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