Consistent Digital Curved Rays and Pseudoline Arrangements

Jinhee Chun
GSIS Tohoku University, Sendai, Miyagi, Japan
jinhee@dais.is.tohoku.ac.jp

Kenya Kikuchi
GSIS Tohoku University, Sendai, Miyagi, Japan
kenya@dais.is.tohoku.ac.jp

Takeshi Tokuyama
Kwansei-Gakuin University, Sanda, Hyōgo, Japan
tokuyama@kwansei.ac.jp

Abstract

Representing a family of geometric objects in the digital world where each object is represented by a set of pixels is a basic problem in graphics and computational geometry. One important criterion is the consistency, where the intersection pattern of the objects should be consistent with axioms of the Euclidean geometry, e.g., the intersection of two lines should be a single connected component. Previously, the set of linear rays and segments has been considered. In this paper, we extended this theory to families of curved rays going through the origin. We further consider some psudoline arrangements obtained as unions of such families of rays.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases Computational Geometry, Digital Geometry, Spanning Tree, Graph Drawing

Digital Object Identifier 10.4230/LIPIcs.ESA.2019.32

Funding This work is partially supported by MEXT JSPS Kakenhi 17K19954, 17K00002 and 18H05291.

1 Introduction

The representation of geometric objects in the pixel world does not always satisfy geometric properties such as Euclidean axioms. Figure 1 shows that a naive definition of digital lines may cause inconsistency. In Figure 1, the intersection of a pair of digital lines is divided into three connected components (in the 4-neighbor topology), while it is desired that the intersection should be connected to imitate the Euclidean axiom that two non-parallel lines intersect at a point. Thus, it is important to seek for a digital representation of a family of geometric objects such that they satisfy a digital version of geometric axioms.

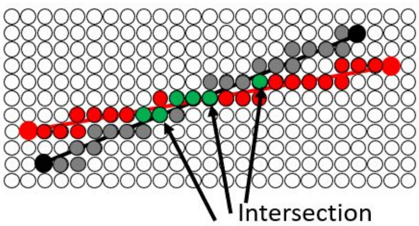

Figure 1 Inconsistency of intersection (green pixels) of two digital line segments.

Geometric consistency is important in implementation of algorithms of computational geometry. In geometric computation, we often experience that finite-precision computation is suffered by geometric inconsistency. For example, the divide and conquer algorithm to construct a Voronoi diagram given in the textbook of Preparata and Shamos [6] is known to be difficult to implement. The algorithm needs to compute the intersection point of two (possibly nearly parallel) lines, and then later decides whether the intersection point is above or below another line. Therefore, we may need to compute the intersection point precisely beyond the precision of the system to avoid inconsistency causing a wrong decision. It is a difficult task to avoid such geometric inconsistency. After the seminal paper of Greene and Yao [7], many approaches to overcome the geometric inconsistency in finite precision computation have been proposed. Snap rounding [10, 8, 9] is one of the approaches, which systematically replaces line segments with piecewise linear segments. Another approach implemented in several softwares is the dynamic control of the precision [15, 13].

The pixel-based consistent representation of digital objects would lead to an additional methodology for consistent geometric computation. In general, it is a difficult task to convert families of geometric objects into families of digital objects without geometric inconsistency. However, we have hope if we restrict the task on some fundamental curves to represent basic geometric objects.

In this paper, we propose the *consistent digital curved rays* generalizing consistent digital rays for straight lines [5, 14]. We also show constructions of digital rays that consistently approximate some pseudoline arrangements in the first quadrant.

We consider the triangular region Δ defined by $\{(x, y) : x \geq 0, y \geq 0, x + y \leq n\}$ in the plane, and the integer grid $G = \{(i, j) : i, j \in \{0, 1, \ldots, n\}, i + j \leq n\}$ in the region. We can also handle a square region, but use Δ to make the description easier.

Each element of G is called a *pixel* (usually, a pixel is a square, but we represent it by its lower-left-corner grid point in this paper). A pixel is called a *boundary pixel* if it lies on $x + y = n$. We consider an undirected graph structure under the four-neighbor topology such that $(i, j) \in G$ is connected to $(k, \ell) \in G$ if $(k, \ell) \in \{(i-1, j), (i, j-1), (i+1, j), (i, j+1)\}$.

A *digital ray* $S(p)$ is a path in G from the origin o to p, where $S(o) = \{o\}$ is a zero-length path. Let us consider a family $\{S(p) : p \in G\}$ of digital rays, where a digital ray is uniquely assigned to each $p \in G$. The family is called *consistent* if the following three conditions hold:

1. If $q \in S(p)$, then $S(q) \subseteq S(p)$.
2. For each $S(p)$, there is a (not necessarily unique) boundary pixel r such that $S(p) \subseteq S(r)$.
3. Each $S(p)$ is a shortest path from o to p in G.

The consistency implies that the union of paths $S(p)$ form a spanning tree T of G such that all leaves are boundary pixels, and accordingly the intersection of two digital rays consists of single connected component. See the pictures (a) and (b) of Figure 2for the illustration. The tree T and also the family of digital rays are called CDR (*Consistent Digital Rays*).

Previously, the theory has been considered only for digital straightness[11]. Lubby [14] first gave a construction of CDR where each $S(p)$ simulates a linear ray within Hausdorff distance $O(\log n)$, and showed that the bound is asymptotically tight. Here, the Hausdorff distance between objects P and Q is $\max\{\max_{p \in P} \min_{q \in Q} d(p, q), \max_{q \in Q} \min_{p \in P} d(p, q)\}$, where $d(p, q)$ is the Euclidean distance between p and q. The construction was re-discovered by Chun et al.[5] to give further investigation, and Christ et al.[4] gave a construction of consistent digital line segments where the lines need not go through the origin. There are several works on different characterizations and variations [1, 2, 3].

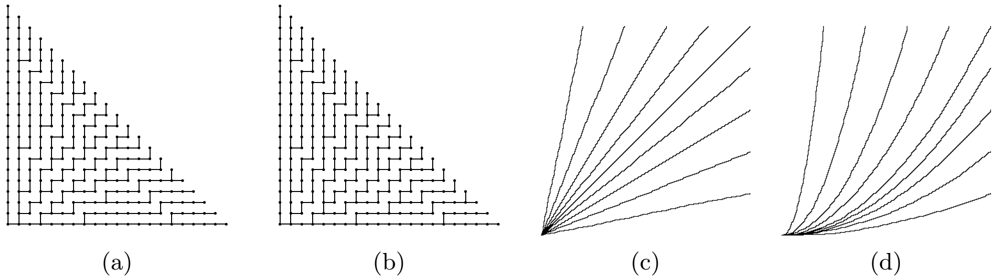

Figure 2 CDR for linear rays and parabolic rays in the triangular region of a 20×20 grid, and sampled linear and parabola digital rays in a 300×300 square grid.

We will extend the theory to families of curves with the same topology as linear rays. In Figure 2, the combinatorial difference between two CDRs can be observed. The difference leads to the visual difference of digital rays illustrated in Figure 2, where it can be seen that the digital rays in (b) approximate parabolas as shown in (d) extended to a sufficiently large grid, while (a) approximates linear rays as shown in (c).

A family $\mathcal{F}$ of nondecreasing curves in Δ is called a *ray family* if each curve goes through the origin o, and for each point $(x, y) \in \Delta \setminus \{o\}$ there exists a unique curve of $\mathcal{F}$ going through it. We call an element of $\mathcal{F}$ a *ray*. Accordingly, each pair of rays intersect each other only at the origin. A typical example is the family of parabolas $y = ax^2$ for $a \geq 0$.

We give a construction method of CDR $T_{\mathcal{F}}$ in G such that the (unique) ray of $\mathcal{F}$ connecting o and a pixel p is approximated by the path $S(p)$ of $T_{\mathcal{F}}$ well. In order to theoretically gurantee the goodness of the approximation, we give an $O(\sqrt{n \log n})$ bound of the Hausdorff distance for several ray families, where the unit is given by the pixel size. Although the theoretical bound is much worse than the known $\Theta(\log n)$ optimal bound for the linear ray [5, 14], it is the first nontrivial result for curved rays as far as the authors know.

Then, we investigate the structure of unions of CDRs. Our results include a new interpretation of CDS, and generalize it to a digitized pseudoline arrangement (i.e. set of paths interesecting at monst once to each other) given as union of translated copies of a ray family. Moreover, we deal with digitization of the arrangement given as a union of families of constant degree homogeneous polynomial curves to show that they can be consistently discretized to form a pseudoline arrangement in a subregion of Δ excluding a constant number of rows and columns, and a constant-area triangle.

We have implemented our construction algorithm of CDR for several families of rays, and our experimental result shows that the Hausdorff distance is only 12 for $n = 2^{14}$ for the parabola rays.

2 Consistent digital rays and their properties

The set of pixels of G on the diagonal $x + y = k$ for $k = 0, 1, \ldots, n$ is called the level set $L(k)$. We implicitly give a direction of edges from lower towards higher levels, and call an edge of G between nodes $u \in L(k-1)$ and $v \in L(k)$ an incoming edge to (resp. outgoing edge from) v (resp. u).

Consider a CDR T. Any node of T has exactly one incoming edge, and at most two outgoing edges of T. The following observation was given by Chun et al.[5] (see Figure 3 for its illustration).

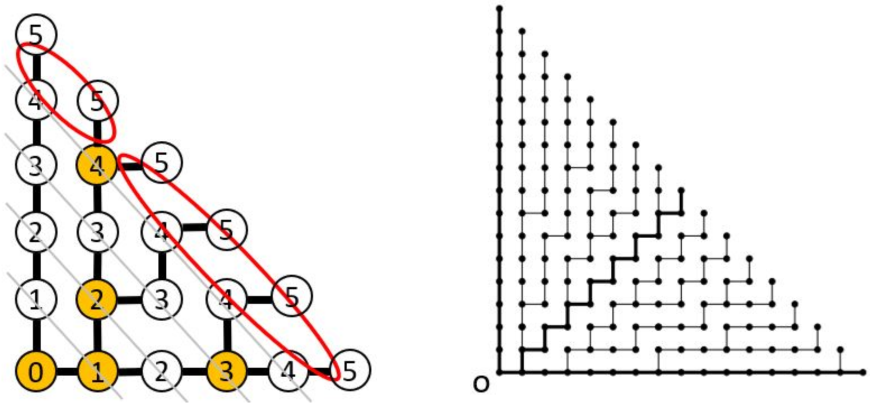

Figure 3 The branching nodes (colored yellow) and partition of incoming edges to the 5th level (left picture) of the CDR (right picture) of linear rays.

▶ **Lemma 1.** *In the level set $L(k)$ for $k \geq 1$, there exists a real value $0 < x(k) < k$ such that the incoming edge of T to each node whose x-value is smaller than (resp. larger than or equal to) $x(k)$ is vertical (resp. horizontal). Accordingly, there exists a unique branching node of T in $L(k-1)$ (colored yellow in Figure 3).*

Thus, a CDR is completely characterized by the integer sequence $\lceil x(1) \rceil, \lceil x(2) \rceil, \ldots, \lceil x(n) \rceil$, where $1 \leq x(i) \leq i$. We call $x(k)$ the *separating position* on $L(k)$. The following lemma is easy to verify.

▶ **Lemma 2.** *A (unique) CDR exists for each of $(n-1)!$ possible sequences as above.*

Our task is to find a CDR among those candidates to approximate a given family of rays as good as possible.

2.1 CDR for linear rays revisited

The CDR of linear rays can be obtained by selecting $x(k)$ as uniformly as possible from $[1, k]$.

Let us consider the binary representation $k = \sum_{i=0}^{\infty} a(i)2^i$ of a natural number k. The van der Corput sequence (see [12]) is the sequence that is defined by a function $V(k) = \sum_{i=1}^{\infty} a(i)2^{-i}$ from natural numbers to $[0, 1]$. We remove $V(0) = 0$ from our consideration so that the range becomes $(0, 1]$. For example, for $6 = 2 + 4 = 110_2$, $V(6) = 0.11_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. Here, a sequence of digits with subscript 2 means 2-adic representation of numbers.

The van der Corput sequence is known to be a low discrepancy sequence: There is a nonnegative constant c such that for each n and a range $[a, b]$ in $(0, 1]$, the number of $k \leq n$ satisfying $V(k) \in [a, b]$ differs from $(b-a)n$ at most $c \log n$. In particular, for each $m < n$, the set $\{V(i) : m \leq i \neq n\}$ gives an almost uniform distribution on $[0, 1]$.

We can set $x(k) = kV(k)$ to obtain a CDR. This CDR is exactly same as the one given by Chun et al.[5], and it has been shown that it approximates the linear rays emanating from the origin with the optimal $\Theta(\log n)$ distance bound. In order to generalize to the curved rays, we give the following interpretation.

Consider a line $y = ax$ intersecting $x + y = k$ at $q = (x_0, k - x_0)$. By definition, its slope is a, which is $\frac{k-x_0}{x_0}$. Naturally, we need to approximate the line segment of slope $\frac{k-x_0}{x_0}$ with a grid path in a neighborhood of q in order to globally approximate a line by the path. Ideally the ratio of vertical edges to the horizontal edges in the path should be $\frac{k-x_0}{x_0}$ in the neighborhood. If we set $x_0 = kt_0$, the ratio is $\frac{1-t_0}{t_0}$.

By the definition of the separating position $x(k)$, the edge incoming to q is vertical if and only if q lies on the left of $x(k)$. Let $x(k) = kt(k)$ and we take $t(k) = \theta$ for a uniformly random variable θ on $(0, 1]$. Then, q is on the left of $x(k)$ if and only if $t_0 < \theta$, and its probability is $1 - t_0$. Thus, the incoming edge becomes horizontal and vertical with probabilities t_0 and $1 - t_0$, respectively. Hence, the ratio between them is $\frac{1-t_0}{t_0}$ as desired.

The construction can be derandomized by replacing θ by $V(k)$ for each k. This derandomization also improves the Hausdorff distance bound.

We would like to extend this argument for other families of curves.

3 CDR for families of curves

Let us give a construction method of CDR applicable to several families of curves. We start with a family of parabolas as a typical example for improving the readability, and then discuss more general cases for which we will prove an upper bound for the Hausdorff distance between rays and digital rays.

3.1 CDR for a family of parabolas

3.1.1 Construction of CDR

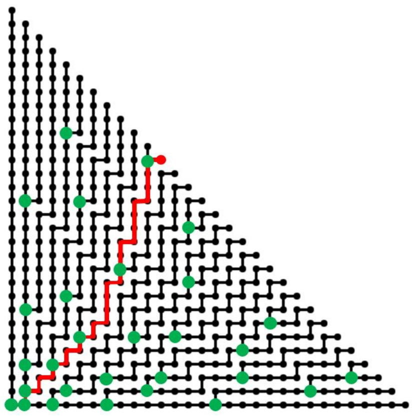

Figure 4 CDR T_{para}. Green nodes are branching nodes. Red path gives a digital parabolic ray.

Let us consider the family $y = ax^2$ $(a \geq 0)$ of parabolas that have the origin o as their apex. We include the y-axis $x = 0$ in the family (this convention is applied to all other cases).

Consider a parabola $C : y = ax^2$ intersecting the level $x + y = k$ at $q = (x_0, k - x_0)$. The slope of tangent at q is $2ax_0$, which is $\frac{2y_0}{x_0} = \frac{2(k-x_0)}{x_0} = \frac{2(1-t_0)}{t_0}$ if we set $x_0 = kt_0$.

Analogously to the linear case, if we would like to have a digital ray nicely approximate C, the curve C in a neighborhood of q should be approximated by a path that contains the horizontal and vertical edges with the probabilities $\frac{t_0}{2-t_0}$ and $\frac{2(1-t_0)}{2-t_0}$, respectively.

Thus, we should select the separating position $x(k) = kt(k)$ to be located on the left of q with probability $\frac{t_0}{2-t_0}$. We consider a monotonically increasing function F_k in the range $[0, 1]$ and set $t(k) = F_k(\theta)$ for the uniformly random variable θ on $(0, 1]$. The probability that $x_0 = kt_0 < x(k)$ is the probability that $F_k^{-1}(t_0) < \theta$ from the monotonicity of F_k. Because of uniformity, this probability equals to $F_k^{-1}(t_0)$.

Thus, we should set $F_k^{-1}(t) = \frac{t}{2-t}$ to meet our requirement, and $F_k(\theta) = \frac{2\theta}{\theta+1}$. This is indeed monotonically increasing as we desired[1]. We then derandomize the process by replacing θ with $V(k)$, and we set $t(k) = \frac{2V(k)}{V(k)+1}$ and hence $x(k) = \frac{2kV(k)}{V(k)+1}$ to construct a CDR T_{para} illustrated in Figure 4 deterministically.

The bound for the Hausdorff distance between a parabola ray and its corresponding digital ray in T_{para} is given in the following theorem, although its proof will be given later for a more general form.

▶ **Theorem 3.** *For each node $p \in G$, the Hausdorff distance between the parabola ray going through p and the path $S(p)$ from p towards the origin in the CDR T_{para} is $O(\sqrt{n \log n})$.*

3.2 Homogeneous polynomials

Let us consider the family $\mathcal{F}_j$ of curves defined by $y = f_a(x) = ax^j$ for $a \geq 0$. Here, the slope of tangent of a curve at (x, y) is $f'_a(x) = jax^{j-1}$, which equals jy/x. Thus, analogously to the parabola case, we have $F_k^{-1}(t) = \frac{t}{j-(j-1)t}$ and $F_k(\theta) = \frac{j\theta}{1+(j-1)\theta}$. Applying derandomization to replace θ by $V(k)$, we set $x(k) = \frac{jkV(k)}{(j-1)V(k)+1}$ for $k = 1, 2, \ldots, n$ to define a CDR $T_{\mathcal{F}_j}$. The following theorem is analogously obtained to the parabola case.

▶ **Theorem 4.** *For each node $p \in G$, the Hausdorff distance between the ray in $\mathcal{F}_j$ going through p and the path $S(p)$ from p towards the origin in the CDR $T_{\mathcal{F}_j}$ is $O(\sqrt{n \log n})$.*

3.3 Handling general ray families

3.3.1 Framework for a diffused ray family

Recall that a family $\mathcal{F}$ of nondecreasing curves in Δ is called *ray family* if each curve (called ray) goes through the origin o, and for each point $(x, y) \in \Delta \setminus \{o\}$ there exists a unique curve of $\mathcal{F}$ going through it. We call a ray family *smooth* if every curve is differentiable. Let us consider the slope $\tau(t, z)$ at $p = (tz, z - tz)$ $(0 < t \leq 1)$ of the unique curve of $\mathcal{F}$ going through p. We assume that we can compute $\tau(t, z)$ for a given p efficiently, and its computation time will be regarded as the unit of the time complexity.

▶ **Definition 5.** *A smooth ray family $\mathcal{F}$ is called diffused (resp. weakly diffused) if $\tau(t, z)$ is continuous and decreasing (resp. nonincreasing) in t for each fixed $z > 0$.*

Intuitively, the diffusedness means that the rays always expand: The distance between two curves along the off-diagonal $x + y = k$ is increasing in k, since the right curve has a smaller slope than the left one. It can be considered as a continuous counterpart of the property of CDR given in Lemma 1 that vertical edges are incoming to the left of $x(k) = kt(k)$ while horizontal edges incoming to the right of it in each level $L(k)$. The families of parabolas and homogeneous polynomials are diffused.

Now, we consider construction of a CDR for a diffused family $\mathcal{F}$. We want to control so that the probability that the edge incoming to a pixel $q = (tk, k - tk)$ in $L(k)$ is horizontal with probability $g_k(t) = \frac{1}{1+\tau(t,k)}$, so that the ratio of probabilities to have a vertical edge against a horizontal edge becomes $\tau(t, k)$ for each of $t = j/k$ $(j = 1, 2, \ldots, k)$.

We would like to find a monotonically increasing function F_k such that $t(k) = F_k(\theta)$ for a uniformly random variable θ on $(0, 1]$ so that the incoming edge to $(tk, k - tk)$ becomes horizontal with probability $g_k(t)$.

[1] The function F_k is independent of k, but it is not always true for the more general cases.

Since the family is diffused, $\tau(t, k)$ is decreasing in t, and hence g_k is increasing. Therefore, it has the inverse function g_k^{-1} that is also increasing.

We set $F_k = g_k^{-1}$ to attain our requirement. Indeed, the condition that $x(k)$ lies on the left of q is that $t(k) < t$, which means $g_k(t(k)) \leq g_k(t)$ because of the monotonicity of g_k. Since $g_k(t(k)) = g_k(F_k(\theta)) = \theta$, this happens if the value of θ is smaller than $g_k(t)$, and hence the probability is $g_k(t)$ as we desire.

By evaluating $F_k(\theta)$ at a given θ, we have $t(k)$ for $k = 1, 2, \ldots, n$, and hence obtain a CDR for $\mathcal{F}$. Approximate evaluation is sufficient for our purpose, since only the value $\lceil x(k) \rceil = \lceil kt(k) \rceil$ is necessary for the construction of CDR. Since $\tau(t, z)$ can be computed in the unit time, we can compute $g_k(t)$ Since $g_k(t)$ is an increasing function, the value $\lceil kt(k) \rceil$ for $t(k) = t_\theta = F_k(\theta)$ can be computed by binary searching over $t \in \{1/k, 2/k, \ldots k-1/k\}$ to find the value t_0 such that $g(t_0 - 1/k) < \theta \leq g(t_0)$.

We then derandomize the process replacing θ by $V(k)$ for each k.

3.3.2 Upper bound of the Hausdorff distance

We give the analysis for the Hausdorff distance between a curved ray and its digitized ray. We consider the derandomized version here, and the analysis for the randomized version is given later.

For a differentiable curve $C \in \mathcal{F}$, consider the intersection point $p_C(z) = (x_C(z), z - x_C(z))$ with the line $x + y = z$ for $0 < z \leq n$. Let $s_C(z)$ be the slope of C at $p_C(z)$. Our analysis depends on the property of the function $s_C(z)$.

▶ **Definition 6.** *Given a function $y = f(x)$ defined on an interval I, if I can be decomposed into a minimum number of consecutive subintervals such that $f(x)$ is monotone (either nonincreasing or nondecreasing) on each subinterval, the number of subintervals is called the wave number of f. It is infinity if there is no such decomposition into a finite number of subintervals.*

The wave number of $s_C(z)$ is intuitively the length of the alternating sequence of consecutive convex segments and concave segments of C.

▶ **Definition 7.** *The wave number of $\mathcal{F}$ is the supremum of the wave numbers of $s_C(z)$ over all $C \in \mathcal{F}$ on the interval $(0, n]$ of z.*

▶ **Theorem 8.** *If $\mathcal{F}$ is a diffused family of rays with the wave number w, the Hausdorff distance between the ray C going through p in $\mathcal{F}$ and the path $P = S(p)$ from p towards the origin in $T_{\mathcal{F}}^{det}$ is bounded by $O(\sqrt{wn \log n})$ for any node $p \in G$.*

For the families of parabolas and homogeneous polynomials, we can verify that the wave number is 1, and thus we have Theorems 3 and 4 as corollaries.

In order to prove Theorem 8, we prepare two lemmas. The first one (Lemma 9) is well-known (see e.g. [12]). The area of a planar region X is denoted by $A(X)$.

▶ **Lemma 9.** *Consider the set of points $S = \{(k, V(k)) : k = 0, 1, 2, \ldots, n\}$ in the region $X = [0, n] \times [0, 1]$. Then, for any axis parallel rectangle R in X, the difference (called discrepancy) between the number of points in $S \cap R$ and the area of $A(R)$ is $O(\log n)$.*

The following Lemma 10 gives a discrepancy bound of S with respect to a region below a curve of a function.

▶ **Lemma 10.** *Consider the set of points $S = \{(k, V(k)) : k = 0, 1, 2, \ldots, n\}$. Let $f(x)$ be a continuous function from $[0, n]$ to $[0, 1]$ with a wave number w, and let $Q_I(f) = \{(x, y) : 0 \leq y \leq f(x), x \in I\}$ for any given interval $I \subset [0, 1]$. Then, the discrepancy $||S \cap Q_I(f)| - A(Q_I(f))|$ is bounded by $c\sqrt{wn \log n}$ for a suitable constant c.*

Proof. Since we can decompose the interval I into w subintervals such that f is monotone on each of them, it suffices to consider the case $w = 1$. Indeed, if subintervals have lengths $n_1, n_2, \ldots, n_w$ and has discrepancies $c\sqrt{n_i \log n_i}$ for $i = 1, 2, \ldots, w$, the sum $\sum_{i=1}^{w} c\sqrt{n_i \log n_i}$ is bounded by $c\sqrt{wn \log n}$, where the minimum it attained if $n_i = n/w$ for every i. If $w = 1$, f is either nonincreasing or nondecreasing, and without loss of generality, we assume f is nondecreasing.

We divide $Q_I(f)$ into its intersections with consecutive vertical strips of width $\sqrt{n \log n}$ (possibly the last one is skinnier). Let A_i for $i = 1, 2, \ldots, M = \lceil \sqrt{n/\log n} \rceil$ be the strips.

Suppose s_i and t_i are x-values of the leftmost and rightmost boundary of A_i, respectively. Now, within the strip A_i, $Q_I(f)$ is contained in a rectangle R_i whose height is $f(t_i)$, and contains another rectangle R'_i whose height is $f(s_i)$. Since $0 \leq f(x) \leq 1$ and f is nondecreasing we can easily see that the difference of areas of $\cup_{i=1}^{M} R_i$ and $\cup_{i=1}^{M} R'_i$ is at most the area of a rectangle of height 1 and width $\sqrt{n \log n}$. For a union of M rectangles, we can apply Lemma 9, and the number of points of S in $\cup_{i=1}^{M} R_i$ is at most $A(Q_I(f)) + \sqrt{n \log n} + O(M \log n)$, and that in $\cup_{i=1}^{M} R'_i$ is at least $A(Q_I(f)) - \sqrt{n \log n} - O(M \log n)$. Since $M < \sqrt{n/\log n} + 1$, we have the lemma. ◀

We remark that for the discrepancy in Lemma 10, an $\Omega(\sqrt{n})$ lower bound is known even if f is a linear function (see [12]).

Now let us give a proof for Theorem 8.

The basic idea is that if P goes too far from C on a level, then it cannot come back to the same destination point p without violating the discrepancy condition given in Lemma 10.

Without loss of generality, we can assume that p is a boundary element located on $L(n)$. For each diagonal $x + y = k$, the intersection of C (resp. P) with it is denoted by $q_C(k) = (x_C(k), y_C(k))$ and $q_P(k) = (x_P(k), y_P(k))$, respectively. Then, the Hausdorff distance is bounded by $\sqrt{2} \max_{1 \leq k \leq n} |x_C(k) - x_P(k)|$, and it suffices to show that there is a constant c' such that $|x_C(k) - x_P(k)| \leq c'\sqrt{wn \log n}$. We take $c' > c$, where c is the constant given in Lemma 10.

Assume on the contrary there exists an index s such that $|x_C(s) - x_P(s)| > c' w \sqrt{n \log n}$. Without loss of generality, we can assume that $x_C(s) > x_P(s)$, since the other case can be handled analogously.

There exists an index m such that $x_C(i) - x_P(i) > 0$ for $s \leq i < m$ and $x_C(m) - x_P(m) \leq 0$ because $x_P(n) = x_C(n)$ (both P and C need to go through p). In other words, the path P lies on the left of C in $L(k)$ for $s \leq k < m$ and first comes back to the (almost) same position on $L(m)$. Let I be the interval $(s, m]$. Thus, we have

$$x_P(m) - x_P(s) > x_C(m) - x_C(s) + c'\sqrt{wn \log n}. \tag{*}$$

In the derandomized construction, $V(k)$ is used (instead of θ) to determine $t(k)$. In our construction method, P has a horizontal incoming edge at $L(k)$ if and only if $g_k(t_P(k)) \geq V(k)$, where $t_P(k) = \frac{x_P(k)}{k}$. By the monotonicity of g_k, $g_k(t_C(k)) \geq g_k(t_P(k))$ if $k \in I$, and this implies $g_k(t_C(k)) \geq V(k)$.

The integer-valued function $x_C(k)$ is extended to a continuous function $x_C(z)$ that gives the x-value of the intersection point of $x + y = z$ and the ray C for a real value $z \in (0, n]$. Moreover, the function $g_k(t) = \frac{1}{1+\tau(t,k)}$ can be extended to $g(t, z) = \frac{1}{1+\tau(t,z)}$.

We define $\varphi_C(z) = g(t_C(z), z) = \frac{1}{1+\tau(t_C(z),z)}$. Recall that $\tau(t_C(z_0), z_0) = dy/dx|_{z=z_0}$ is the slope of C at $z = z_0$, and hence

$$\varphi_C(z_0) = \frac{1}{1 + \frac{dy}{dx}|_{z=z_0}} = \frac{dx}{dx+dy}|_{z=z_0} = \frac{dx}{dz}|_{z=z_0}.$$

Thus, $\varphi_C(z)$ is the ratio of the increase of $x_C(z)$ to the increase of z in the infinitesimal neighbor of z_0. The wave number of $\varphi_C(z)$ is the same as that of $\tau(t_C(z), z)$, since $\varphi_C(z)$ is increasing in an interval I if and only if $\tau(t_C(z), z)$ is decreasing. We can observe that $\tau(t_C(z), z) = s_C(z)$ by definition, and hence the wave number of $\varphi_C(z)$ is the same as that of $s_C(z)$, and bounded by w. Also, the range of $\varphi_C(z)$ is in $(0, 1]$.

If the incoming edge of P is horizontal, $\varphi_C(k) = g_k(t_C(k)) \geq V(k)$ as shown above, and this condition is equivalent to $(k, V(k)) \in Q_I(\varphi_C)$, since $Q_I(\varphi_C) = \{(z, x) : 0 \leq x \leq \varphi_C(z), s < z \leq m\}$. Let S be the set of points $(k, V(k))$ for $k = s, s+1, \ldots, m-1$. The difference of the x-values of P at $z = s$ and $z = m$ is the number of horizontal edges in the interval, which is hence bounded by $|Q_I(\varphi_C) \cap S|$.

On the other hand, $A(Q_I(\varphi_C)$ equals the difference of x-value of C at s and m, since

$$A(Q_I(\varphi_C)) \quad = \quad \int_{s<z<m} \varphi_C(z)dz = \int_{s<z<m} \frac{dx_C(z)}{dz} dz = x_C(m) - x_C(s).$$

Since the wave number of φ_C is bounded by w and its range is in $(0, 1]$, Lemma 10 says that $|Q_I(\varphi_C) \cap S| - A(Q_I(\varphi_C)) < c\sqrt{wn \log n}$. Thus, we have

$$x_P(m) - x_P(s) \leq |Q_I(\varphi_C) \cap S| \leq A(Q_I(\varphi_C)) + c\sqrt{wn \log n} = x_C(m) - x_C(s) + c\sqrt{wn \log n}.$$

Therefore, $x_P(m) - x_P(s) \leq x_C(m) - x_C(s) + c\sqrt{wn \log n}$. Compared with $(*)$, we have $c > c'$, and obtain a contradiction.

3.3.3 Analysis for the randomized version

We would like to mention the quality of the randomized construction of a CDR.

▶ **Definition 11.** *Consider a continuous function f defined on an interval $I = (k, m]$ with the range $[0, 1]$, where $0 < k < m < n$ are positive integers. let $\bar{f}$ be the linear interpolation using the values of f on integer abascissae, which is the piecewise linear curve connecting $(k, f(k)), (k+1, f(k+1)), \ldots, (m, f(m))$ by linear segments of width 1. The* discretization error *of f on I is $|A(Q_I(f)) - A(Q_I(\bar{f}))|$.*

The following is easy to see.

▶ **Lemma 12.** *If the wave number of f is bounded by w, the discretization error of f is at most w.*

Now, given a function f from $[0, n]$ to $[0, 1]$, consider a $\{0, 1\}$-valued random variable $X_f(i)$ for each $i = 1, 2, \ldots, n$ such that it becomes 1 if and only if a uniformly randomly number (chosen independently for each i) in $[0, 1]$ becomes less than or equals to $f(i)$. Let $X_f = \sum_{i=1}^n X_f(i)$. Then the expected value $E(X) = \sum_{i=1}^n E(X_i)$ equals $\sum_{i=1}^n f(i) = A(Q_I(\bar{f})) + \frac{f(m)-f(k)}{2}$. Note that $A(Q_I(\bar{f})) \leq n$, thus $E(X) \leq n + 1/2$.

We can apply Chernoff's inequality, and obtain a constant $c(r)$ such that $|X - Q(\bar{f})| \geq c(r)\sqrt{n} \log n$ with a probability $1 - n^{-r-3}$ for any given constant r. Now we are ready to analyze the randomized construction of CDR.

▶ **Theorem 13.** *If $\mathcal{F}$ is a diffused family of rays with the wave number w, the largest Hausdorff distance between a ray and the corresponding digital ray in $T_{\mathcal{F}}^{rand}$ is $O(\sqrt{n}\log n + w)$ with probability $1 - n^{-r}$ for any fixed $r > 0$.*

Proof. Analogously to the deterministic version, For any path p in the CDR corresponding a curve $C = C(p) \in F$, the Hausdorff distance from p to C is bounded by the maximum difference of $A(Q_I(\varphi(C)))$ and $X_{Q_I(\varphi(C)}$ over all I. Since there are $O(n)$ paths from the root to leaves, and there are $O(n^2)$ intervals $[k, m]$, there are $O(n^3)$ choices. Thus, with probability $1-n^{-r}$, $|A(Q_I(\varphi(\bar{C}(p)))) - X(Q_I(\varphi(C(p)))| \leq c(r)\sqrt{n}\log n$ for all p and I. Thus, we have the theorem. ◀

The above upper bound is worse than the deterministic version by a $\sqrt{\log n}$ factor if w is a small constant, while it is theoretically better if $w > \log n$.

3.4 Family of curves linear in a parameter

Let us consider a nondecreasing differentiable function $y = f(x)$ for $x \in [0, n]$ such that $f(0) = 0$ and $f(x) > 0$ for $x > 0$. We define the family $\mathcal{F} = \{C_a : a \geq 0\}$ of curves, where C_a is defined by $y = af(x)$. It is clear that this gives a ray family.

If C_a goes through (x_0, y_0), then $a = \frac{y_0}{f(x_0)}$. The slope of the curve C_a at (x_0, y_0) is $af'(x_0)$, which is (eliminating a) $\frac{f'(x_0)y_0}{f(x_0)}$. We consider the slope $\tau(x,k) = \frac{(k-x)f'(x)}{f(x)}$ along the diagonal $x + y = k$ for each k.

If $\mathcal{F}$ is diffused, the framework in the previous subsection works. Although the explicit form of F_k might not be obtained, we can apply binary search to compute $F_k(z)$ for a given z utilizing the monotonicity. Thus, we can compute $x(k) = kF_k(V(k))$ within the pixel precision in $O(\log n)$ time.

Diffusedness and the wave number depend on f. The following lemma is easy to observe.

▶ **Lemma 14.** *If f is a strictly increasing and concave function, the family $\mathcal{F}$ is diffused, and its wave number is 1.*

Note that the family of rays $\{y = f(a^{-1}x) : a > 0\}$ for a convex function f can be also handled, since this family is $\{x = af^{-1}(y)\}$ and $f^{-1}(y)$ is a concave function, e.g., the families of parabolas and homogeneous polynomials could be regarded in this form. Let us see some typical examples.

▶ **Example 15.** Let $\mathcal{F}_{sig}$ be the family of curves $y = a\sigma(x), 0 \leq a$, where $\sigma(x) = \frac{1}{1+e^{-x}} - \frac{1}{2}$ is the shifted sigmoid function. The curve $y = \sigma(x)$ is strictly increasing and concave; hence, the family is diffused with the wave number 1, and we have the $O(\sqrt{n \log n})$ bound.

Here, $\tau(x,k) = \frac{(k-x)e^{-x}}{(1+e^{-x})^2\sigma(x)}$. The function $g_k = F_k^{-1}$ can be analytically given, but it is a complicated function such that it is difficult to find an explicit formula for F_k. Thus, we apply the binary searching method to find a value of $F_k(V(k))$ in our experiment.

▶ **Example 16.** Consider the family of curves $y = a\log(x+1)$, then similarly we have a CDR with the $O(\sqrt{n \log n})$ distance bound.

▶ **Example 17.** The sine curve $y = \sin(x)$ is not monotone. Therefore, we define $\tilde{\sin}(x)$ by $\tilde{\sin}(x) = 0$ for $x < 0$, $\tilde{\sin}(x) = \sin x$ for $0 \leq x \leq \pi/2$ and $\tilde{\sin}(x) = 1$ for $x > \pi/2$. The curve $y = \tilde{\sin}(x)$ is monotonically nondecreasing and differentiable, and we will apply our CDR construction for the family of curves $y = a\ \tilde{\sin}(x)$ for $a \geq 0$.

Here, the family is weakly-diffused but not diffused, since there are many parallel horizontal lines intersecting each level. However, it is clear that in the region $x > \pi/2$ where the rays becomes horizontal, we can set all edges horizontal. Thus, we can still apply our method to have the $O(\sqrt{n \log n})$ bound.

The obtained CDRs are illustrated in Figure 6 in the section to give experimental results.

4 Union of CDRs with consistency

A CDR is characterized by the sequence $\mathbf{m} : m(1), m(2), \dots m(n)$ where $1 \leq m(i) = \lceil x(k) \rceil \leq k$, and we denote the CDR by $T(\mathbf{m})$. We denote $\mathbf{m} \succeq \mathbf{m'}$ if $m(i) \geq m(i)'$ for all $1 \leq i \leq n$. $\succeq$ is a partial ordering. We write $\mathbf{m} \succ \mathbf{m'}$ if $\mathbf{m} \succeq \mathbf{m'}$ and $\mathbf{m} \neq \mathbf{m'}$.

Consider $\mathcal{P}(T(\mathbf{m})) \cup \mathcal{P}(T(\mathbf{m'}))$, where $\mathcal{P}(T)$ means the set of paths from the root towards leaf vertices in T.

Let $x_P(k)$ be the x-value of the pixel of a path P (from the root to a leaf) in a CDR on the level $L(k)$.

▶ **Definition 18.** *We say a path P_1 is* steeper *than another path P_2 in a different CDR if there is an index $0 \leq k \leq n$ such that $x_{P_1}(i) \leq x_{P_2}(i)$ for $i \leq k$ and $x_{P_1}(i) > x_{P_2}(i)$ for $i > k$. We say the level $L(k)$ the break level of P_1 and P_2.*

The above definition implies that P_1 lies below or on P_2 up to the break level, and it lies strictly above P_2 after it. We allow $k = n$, which means P_1 never goes above P_2. We say the pair of paths have a *singular separation* on a level $L(i)$ if $x_{P_1}(i) = x_{P_2}(i)$ and $x_{P_1}(i+1) > x_{P_2}(i+1)$. Thus, the paths cross each other at most once, although they may touch and singularly separate several times before the break level. We say P_1 and P_2 *semi-consistently intersect* if one is steeper than the other. Moreover, if there is no singular separation, we say they *consistently intersect* each other.

▶ **Theorem 19.** *If $\mathbf{m} \succ \mathbf{m'}$, any path $P_1 \in \mathcal{P}(T(\mathbf{m}))$ is steeper than any path $P_2 \in \mathcal{P}(T(\mathbf{m'}))$. Moreover, if a singular separation happens on a level $L(i)$, $m(i+1) = m'(i+1) = x_{P_1}(i)+1 = x_{P_2}(i) + 1$.*

Proof. Since $\mathbf{m} \succ \mathbf{m'}$, if a vertical edge comes in $p \in L(k)$ in $T_{\mathbf{m'}}$, a vertical edge comes in p in $T_{\mathbf{m}}$, too. This further implies that if such p is located on the right of another pixel q on $L(k)$, every incoming edge to q must be also vertical in both trees.

Therefore, if P_2 lies strictly on the right of P_1 on a level $L(k)$, whenever P_2 selects a vertical incoming edge in $L(k+1)$, P_1 also must select a vertical edge. Thus, inductively the horizontal distance never decreases after the break level, and hence P_1 never meets P_2 again.

Next, we consider what happens at a singular separation. Then, P_1 and P_2 goes through a same point $p = (x, k-x)$ in a level $L(k)$, and P_1 selects a horizontal and P_2 selects a vertical edge towards $L(k+1)$. Then, the vertices of P_1 and P_2 are at positions $q = (x+1, k-x)$ and $q' = (x, k+1-x)$, respectively. Since the incoming edge of $T_{\mathbf{m}}$ to q is horizontal and that of $T_{\mathbf{m'}}$ to q' is vertical, $m(i+1) \leq x+1$ and $m'(i+1) > x$. Since $m(i+1) \geq m'(i+1)$, this happens only if $m(i+1) = m'(i+1) = x+1$. ◀

We note that the condition $m(i+1) = m'(i+1) = x_{P_1}+1 = x_{P_2}+1$ for a singular level $L(i)$ means that both $T_{\mathbf{m}}$ and $T_{\mathbf{m'}}$ have the branching node $p = (m(i+1)-1, i-m(i+1)+1)$ on the level $L(i)$ simultaneously. Both P_1 and P_2 goes through p, and P_1 selects the horizontal while P_2 selects the vertical branch. We say a node p a *singular point* if it is a shared branching node of $T_{\mathbf{m}}$ a $T_{\mathbf{m'}}$, and hence a singular separation only occurs at a singular point.

A region $R \subseteq \Delta$ is called a *slanted-quadrant* if it is defined as $\{(x, y) \in \Delta \mid x \geq a, y \geq b, x+y \geq c\}$ for nonnegative numbers a, b, and c. We say a set of digital rays semi-consistently (resp. consistently) approximates a family of curves intersecting at most once to each other in a slanted-quadrant R if each pair of digital rays semi-consistently (resp. consistently) intersect each other if they are restricted to $G \cap R$.

Suppose that families $\mathcal{F}$ and $\mathcal{F}'$ has CDRs $T(\mathbf{m})$ and $T(\mathbf{m}')$ for $\mathbf{m} \succ \mathbf{m}'$, respectively. Assume that $\mathcal{F} \cup \mathcal{F}'$ forms a pseudoline arrangement in a slanted quadrant R. Theorem 19 assures that $\mathcal{P}(T(\mathbf{m})) \cup \mathcal{P}(T(\mathbf{m}'))$ consistently approximates $\mathcal{F} \cup \mathcal{F}'$ in R if there is no singular point in R.

4.1 Union of translated copies of a CDR

For the sequence $\mathbf{m}$ and a nonnegative integer s, we define a new sequence $\mathbf{m^s}$ by $m^s(k) = \min(m(k)+s, k)$. Similarly, for a negative integer s, we define $\mathbf{m^s}$ by $m^s(k) = \max(m(k)+s, 1)$. The following lemma is obvious.

▶ **Lemma 20.** *If $s \geq 1$, $\mathbf{m}^s \succ \mathbf{m}$. If $s \leq -1$, $\mathbf{m} \succ \mathbf{m}^s$.*

▶ **Theorem 21.** *Suppose that $\mathcal{F}$ and $\mathcal{F}^s$ are ray families digitized by $T(\mathbf{m})$ and $T(\mathbf{m}^s)$, respectively. Assume that $\mathcal{F} \cup \mathcal{F}^s$ forms a pseudoline arrangement in $\Delta_1 : \{(x, y) \in \Delta \mid x \geq 1, y \geq 1\}$. Then $\mathcal{P}(T(\mathbf{m})) \cup \mathcal{P}(T(\mathbf{m}^s))$ consistently approximate $\mathcal{F} \cup \mathcal{F}^s$ in Δ_1.*

Proof. Semi-consistency is clear from Theorem 19 and Lemma 20. Consider the location of a singular point $p = (m(i+1) - 1, i - m(i+1) + 1)$ for a pair of paths. However, Theorem 19 says that $m(i+1) = m^s(i+1)$. This only happens either $m(i+1) = m^s(i+1) = 1$ or $m(i+1) = m^s(i+1) = i+1$, and hence $p = (0, i)$ or $p = (i, 0)$. Thus the singular points only locate on the coordinate axises. Thus, we have the theorem. ◀

For a CDR $T = T(\mathbf{m})$, we define $\mathcal{U}^K(T) = \cup_{-K \leq i \leq K} \mathcal{P}(T(\mathbf{m}^s))$. It follows from Theorem 21 that $\mathcal{U}^K(T)$ consistently approximates a pseudoline arrangement represented as a union of associated ray families.

▶ **Example 22** (Consistent digital line arrangement)**.** Let us consider the family $\mathcal{F}_1$ of linear rays. Define $\mathcal{F}_1^s$ for $s \geq 0$ (resp. $s \leq -1$) to be the set of rays starting with horizontal (resp. vertical) rays, and continue to linear rays with positive slopes emanating from $(s, -s)$. Let T be the CDR for $\mathcal{F}_1$ constructed in Section 2.1. Then $\mathcal{U}^K(T)$ consistently digitize $\cup_{-K \leq s \leq K} \mathcal{F}_1^s$ in Δ_1 with the $O(\log n)$ distance bound.

Indeed, for $s > 0$, the structure of $T(\mathbf{m}^s)$ in $\Delta \cap \{(x, y) : x \geq s\}$ is same as the tree obtained by connecting the forest of $T \cap \{(x, y) : y \geq s\}$ by a horizontal path. The case $s < 0$ is similar. Thus, the discrepancy bound remains as same as the one for T.

Note that although we only consider the lines with positive slopes, we can easily mix it with those with negative slopes (obtained by a mirror construction) without losing consistency. The above example shows that we can consistently digitize the line segments in the first quadrant. However, it is weaker than [4] since we only deal with segments on the lines going through $(s, -s)$ for integers s, and we need a finer precision to represent short segments.

▶ **Example 23** (Consistent digital pseudoline arrangement of shifted parabola rays)**.** Let us consider the family $\mathcal{F}_2$ consisting of curves defined by $y = ax^2$ $(a > 0)$. Define $\mathcal{F}_2^s$ to be the set of the right halves of parabolas with the apex $(s, -s)$. Then $\mathcal{U}^K(T_{para})$ consistently digitize $\cup_{-K \leq s \leq K} \mathcal{F}_2^s$ in Δ_1 with the $O(\sqrt{n \log n})$ distance bound.

4.2 Union of homogeneous polynomial families

In this section, we assume that the CDRs are constructed deterministically. Let us consider $\mathcal{F}_{i,j} = \mathcal{F}_i \cup \mathcal{F}_j$ for $1 \le i \le j$, where $\mathcal{F}_i$ is the family of homogeneous polynomial curves of degree i. Let $\mathbf{m}^{(i)}$ be the sequence (do not confuse this with $\mathbf{m}^s$ given above) corresponding to $T_{\mathcal{F}_i}$ deterministically constructed. Naturally, each pair of curves $f \in \mathcal{F}_i$ and $g \in \mathcal{F}_j$ intersect once in the first quadrant other than the origin, and thus behaves as a pseudoline arrangement in the region $x > 0, y > 0$. We consider the union $\mathcal{T}_{i,j} = \mathcal{P}(T_{\mathcal{F}_i}) \cup \mathcal{P}(T_{\mathcal{F}_j})$ to approximate curves in $\mathcal{F}_{i,j}$.

▶ **Lemma 24.** $\mathbf{m}^{(j)} \succeq \mathbf{m}^{(i)}$ *for* $1 \le i \le j$.

Proof. Recall that $x(k) = \frac{jkV(k)}{(j-1)V(k)+1}$ in the construction of $T_{\mathcal{F}_j}$. Thus, $m^{(j)}(k) = \lceil \frac{jkV(k)}{(j-1)V(k)+1} \rceil$. Since $\frac{ikV(k)}{(i-1)V(k)+1} \le \frac{jkV(k)}{(j-1)V(k)+1}$ if $i < j$, we have the lemma. ◀

▶ **Theorem 25.** *$\mathcal{T}_{1,2}$ consistently approximates $\mathcal{F}_{1,2}$ in the region $\{(x,y) \in \Delta \mid x \ge 3, y \ge 3\}$. For $i \ge 2$, $\mathcal{T}_{i,i+1}$ consistently approximates $\mathcal{F}_{i,i+1}$ in the region $R(i) = \{(x,y) \in \Delta \mid x + y \ge 4(i+1)(i+2), x \ge 4i, y \ge 4(i+1)\}$.*

Proof. $\mathcal{T}_{i,i+1}$ semi-consistently approximates $\mathcal{F}_{i,i+1}$ in Δ, although the existence of multiple singular points prevents the consistency.

Thus, we study location of singular points to find a subregion R to attain the consistency.

Consider a singular point p in a level $L(k)$. Recall that $p = (m^{(i)}(k+1), k - m^{(i+1)}(k+1))$ and $m^{(i)}(k+1) = m^{(i+1)}(k+1)$ at a singular point p in a level $L(k)$. Since $m^{(i)}(k+1) = m^{(i+1)}(k+1)$, we have

$$\frac{(i+1)(k+1)V(k+1)}{iV(k+1)+1} - \frac{i(k+1)V(k+1)}{(i-1)V(k+1)+1} < 1.$$

For the case $i = 1$, suppose that a singular point appears on $L(k)$, and let $v = V(k+1)$, and $K = k+1$. Then we have $\frac{2Kv}{v+1} - Kv < 1$, which means $Kv(1-v) < v+1$. We assume that $K \ge 6$ and $\frac{3}{K} \le v \le 1 - \frac{3}{K}$. Then, $Kv(1-v) \ge 3(1-\frac{3}{K}) = 3 - \frac{9}{K}$ and hence $3 - \frac{9}{K} < 1 + 1 - \frac{3}{K}$ and hence $K < 6$, and we have contradiction.

Thus, $k+1 < 6$ or $(k+1)v < 3$ or $(k+1)v > k+1-3 = k-2$. The position of the singular point p is $\lceil (k+1)v \rceil, k - \lceil (k+1)v \rceil$, and hence it is located either in the region $x + y \le 4$, $x \le 2$ or $y \le 2$. Thus, we have the theorem for $\mathcal{T}_{1,2}$.

For $i \ge 2$, $p = (m^{(i)}(k+1), k - m^{(i)}(k+1))$ and $m^{(i)}(k+1) = m^{(i+1)}(k+1)$ at the singular point p. For simplifying the formulas, we set $K = k+1$, $v = V(k+1)$, and $c = 2(i+1)(i+2)$. We assume $K > 2c$ since otherwise the singular point is outside R.

Now, we have

$$\frac{(i+1)Kv}{iv+1} - \frac{iKv}{(i-1)v+1} < 1.$$

This is transformed to

$$Kv(1-v) < (iv+1)((i+1)v+1). \qquad (*)$$

We will first show that it gives a contradiction if $\frac{4}{K} \le v \le 1 - \frac{c}{K}$.

For the case where $1/2 \le v \le 1 - \frac{c}{K}$, $v(1-v)$ take its minimum at $v = 1 - \frac{c}{K}$, and the right hand side is at most $(i+1)(i+2)$. Thus we have

$$K\frac{c}{K}(1 - \frac{c}{K}) = c(1 - \frac{c}{K}) < (i+1)(i+2)$$

Since $K > 2c$, $c(1-1/2) < (i+1)(i+2)$ and hence $c < 2(i+1)(i+2)$ and it contradicts the definition of c.

If $\frac{1}{i+1} \leq v < \frac{1}{2}$, substituting $K > 4(i+1)(i+2)$, $(*)$ implies that

$$4(i+1)(i+2)v(1-v) < (iv+1)((i+1)v+1).$$

Cleaning up the formula, we have

$$-(5i^2+13i+8)v^2 + (4i^2+10i+7)v - 1 < 0.$$

Since $v < 1/2$, we replace v^2 by $v/2$, we have

$$-(5i^2+13i+8)\frac{v}{2} + (4i^2+10i+7)v - 1 < 0.$$

And hence $(3i^2+7i+6)v < 2$. This does not happen if $v \geq \frac{1}{i+1}$.

If $\frac{4}{K} \leq v \leq \frac{1}{i+1}$, the left hand of $(*)$ takes minimum at $v = \frac{4}{K}$, and the right hand takes maximum at $v = \frac{1}{i+1}$, and we have

$$4(1-\frac{4}{K}) < 2\frac{2i+1}{i+1}.$$

Since $K > 4(i+1)(i+2)$,

$$1 - \frac{1}{(i+1)(i+2)} < 1 - \frac{1}{2(i+1)}.$$

This does not happen since $i+2 > 2$.

Thus, we have either $v < \frac{4}{K}$ or $v > 1 - \frac{c}{K}$.

In the former case that $v < \frac{4}{K}$. The x-value of the singular point is $\lceil \frac{iKv}{(i-1)v+1} \rceil$. $\frac{iKv}{(i-1)v+1}$ is monotonically increasing in v (if $v > 0$). Thus, we have it is less than $\frac{4i}{1+4(i-1)/K}$, which takes maximum at $K = \infty$. Thus, the x-value of the singular point is less than $4i$.

In the latter case, consider the y- value $k - \lceil \frac{(i+1)Kv}{iv+1} \rceil$ of the singular point, which is at most $K - \frac{(i+1)Kv}{iv+1} = \frac{K-Kv}{iv+1}$. It takes the maximum $\frac{c}{i(1-\frac{c}{K})+1}$ at $v = 1 - \frac{c}{K}$, and since $K > 2c$ it is less than $\frac{2c}{i+2} = 4(i+1)$. ◀

The following is a straightforward corollary.

▶ **Corollary 26.** *$\mathcal{T}_{i,j}$ consistently approximates $\mathcal{F}_{i,j}$ for any $j > i$ in the region $R(i)$. Accordingly, $\mathcal{T}_{\leq d} = \cup_{1\leq i\leq d}\mathcal{P}(T_i)$ consistently approximates $\mathcal{H}_d = \cup_{1\leq i\leq d}\mathcal{F}_i$ in the region $R(d-1)$.*

5 Experimental results

We have implemented our method and constructed CDR for the constant-multiplied curves. Figure 5 and Figure 6 illustrate CDRs for polynomial curves, sine, sigmoid, and logarithmic rays.

Figure 7 shows the selected paths approximating the curves towards equally-spaced sampled points on the boundary of square regions.

For each grid width $n = 2^m$ up to $n = 2^{14}$, the worst-case Hausdorff distance between parabolas and digital rays in T_{para} is given in Figure 8, where it is about 12 for $n = 2^{14}$. The dependency of worst-case Hausdorff distance on n have the similar behavior for each of other types of curve.

The error for $n = 2^{14}$ is about 11.2, 13.4, 15.0 for sine, sigmoid and logarithmic curves, respectively. Note that the values are real numbers since we consider Hausdorff distance based on Euclidean distance.

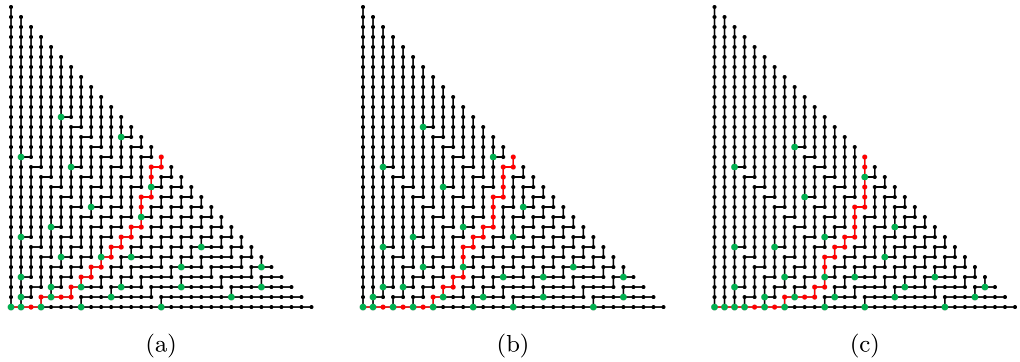

Figure 5 CDRs for (a) $y = ax^2$, (b) $y = ax^3$, and (c) $y = ax^4$. Green nodes are branching nodes. Red paths are the digital curves towards $p = (15, 15)$.

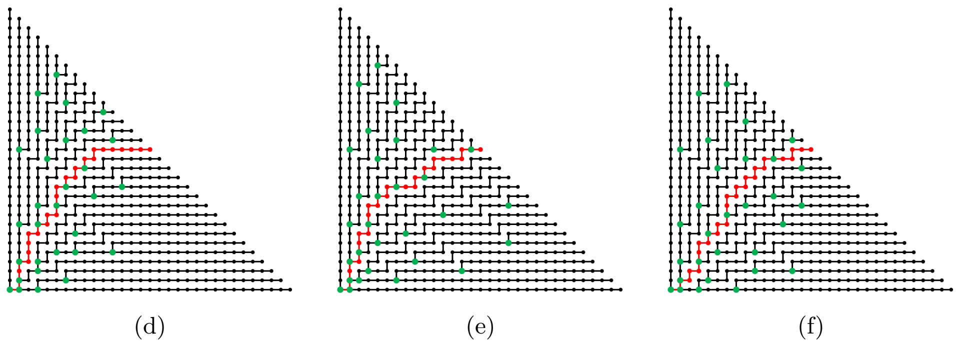

Figure 6 CDRs for (d) $y = a\tilde{\sin}x$ $(0 \leq x \leq \pi/2)$, (e) $y = a\sigma(x)(x \leq 6)$, and (f) $y = a\log(x + 1)$. Green nodes are branching nodes. Red paths are the digital curves towards $p = (15, 15)$.

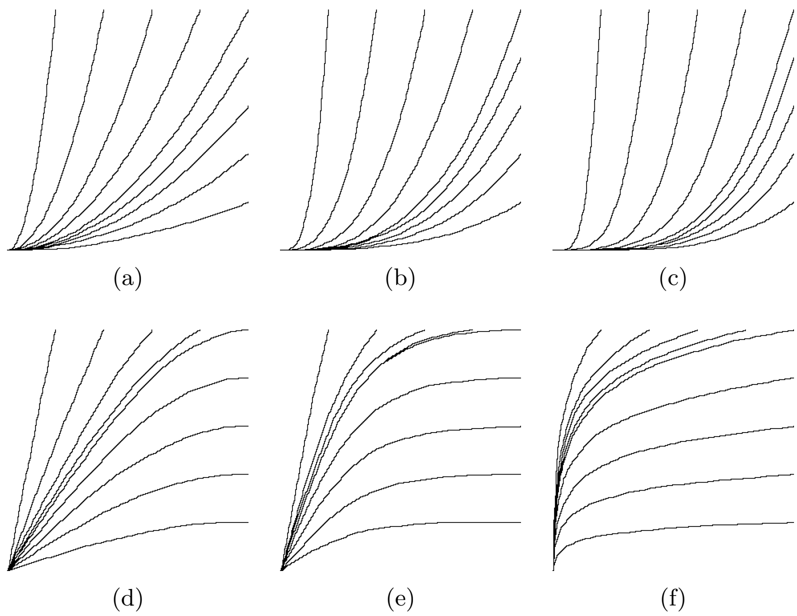

Figure 7 Sampled Curves of CDRs for (a) $y = ax^2$, (b) $y = ax^3$, (c) $y = ax^4$, (d) $y = a\tilde{\sin}x$ $(0 \leq x \leq \pi/2)$, (e) $y = a\sigma(x)(x \leq 6)$, and (f) $y = a\log(x + 1)$ in the 300×300 grid.

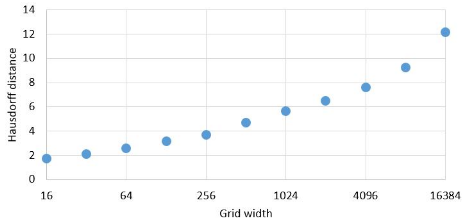

Figure 8 The largest distance from a parabola and the corresponding digital ray in T_{para}.

6 Concluding remarks

The experimental result suggests that our $O(\sqrt{n}\log n)$ bound seems to be loose. Although currently the lower bound mentioned for Lemma 10 prevents us to improve it beyond $O(\sqrt{n})$, recent progress on low-discrepancy sequences [16] might be applied.

For the line segments, a construction of consistent digital segments (CDS) is known [4] with $O(\log n)$ distance error bound. Although we have a generalization of CDS to handle some families of curves, there are a lot of questions to invest further: For example, we do not know how to handle the set of all axis parallel parabolas.

References

1 M. Chiu, M. Korman, M. Sutherland, and T. Tokuyama. Distance bounds for 3-D consistent digital rays and 2-D partially-consistent digital rays. In preparation.
2 I. Chowdhury and M Gibson. A Characterization of Consistent Digital Line Segments in $\mathbb{Z}^2$. In *Proc. 23rd ESA*, pages 337–348, 2015.
3 I. Chowdhury and M Gibson. Constructing Consistent Digital Line Segments. In *Proc. 12th LATIN*, pages 263–274, 2016.
4 T. Christ, D. Pálvölgyi, and M. Stojaković. Consistent Digital Line Segment. *Discrete & Computational Geometry*, 47-4:691–710, 2012.
5 J. Chun, M. Korman, M. Nöllenburg, and T. Tokuyama. Consistent Digital Rays. *Discrete & Computational Geometry*, 42-3:359–378, 2009.
6 F.P.Preparata and M.I.Shamos. *Computational Geometry – an Introduction.* Springer Verlag, 1985.
7 D. Greene and F. Yao. Finite Resolutional Computational Geometry. In *Proc. 27th IEEE FOCS*, pages 143–152, 1986.
8 D. Halperin and E. Packer. Iterated Snap Rounding. *Comput. Geom. Theor. Appl.*, 23:209–225, 2002.
9 J. Hershberger. Stable Snap Rounding. *Comput. Geom. Theor. Appl.*, 46:403–416, 2013.
10 J. Hobby. Practical Segment Intersection with Finite Precision Output. *Comput. Geom. Theor. Appl.*, 13:199–214, 1999.
11 R. Klette and A. Rosenfeld. Digital straightness – a review. *Discrete Applied Math.*, 139:197–230, 2004.
12 J. Matousěk. *Geometric Discrepancy.* Springer Verlag, 1991.
13 K. Mehlhorn and S. Näher. *LEDA: a platform for combinatorial and geometric computation.* Cambridge University Press, 1999.
14 M.G.Lubby. Grid geometries which preserve properties of Euclidean geometry: A study of graphics line drawing algorithms. In *NATO Conference on Graphics/CAD*, pages 397–432, 1987.
15 V. Milenkovic. Double Precision Geometry: A General Technique for Calculating Lines and Segment Intersections using Rounding Arithmetic. In *Proc. 30th IEEE FOCS*, pages 500–505, 1989.
16 H. Niederreiter. Recent constructions of low-discrepancy sequences. *Mathematics and Computers in Simulation*, 135:18–27, 2017.