Completely real? A critical note on the claims by Colbeck and Renner

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Abstract

In a series of papers Colbeck and Renner (2011, 2015a,b) claim to have shown that the quantum state provides a complete description for the prediction of future measurement outcomes. In this paper I argue that thus far no solid satisfactory proof has been presented to support this claim. Building on the earlier work of Leifer (2014), Landsman (2015) and Leegwater (2016), I present and prove two results that only partially support this claim. I then discuss the arguments by Colbeck, Renner and Leegwater concerning how these results are to generalize to the full claim. This argument turns out to hinge on the implicit use of an assumption concerning the way unitary evolution is to be represented in any possible completion of quantum mechanics. I argue that this assumption is unsatisfactory and that possible attempts to validate it based on measurement theory also do not succeed.

1   Introduction

Can quantum-mechanical description of physical reality be considered complete? In the famous paper with this title, Einstein, Podolsky, and Rosen (1935) argued that the question should be answered in the negative. It was one of several arguments that Einstein devised and although it was presumably among his least favorites (Fine, 2017), it is still the most widely known.

Around the same time, von Neumann (1927, 1932) presented a formal argument towards the opposite conclusion. Apart from starting from different assumptions[1], the adopted notions of completeness are also quite distinct (Elby, Brown, and Foster, 1993). While Einstein was concerned with whether the quantum mechanical description sufficed to give a physical *explanation* of the phenomena predicted, von Neumann adopted a more operational approach concerning the question whether the addition of hidden variables could allow for deviating *predictions* for the phenomena. More precisely, the following question was considered. If we consider an ensemble of systems E described by a pure quantum state ψ, is it possible to decompose this ensemble into sub-ensembles $E_1, E_2, \ldots$ such that the predictions for the sub-ensembles are not equal to the predictions for the total ensemble? If not, then quantum mechanics may be considered complete.[2]

Colbeck and Renner's completeness claim (Colbeck and Renner, 2011, 2015a,b) alludes to von Neumann's notion of completeness. In their own words, they show that "[u]nder the assumption that measurements can be chosen freely [...] no extension of quantum theory can give more information about the outcomes of future measurements than quantum theory itself" (Colbeck and Renner, 2011, p. 1). In a follow-up paper Colbeck and Renner (2012) extended their proof to obtain a ψ-ontology theorem.[3] Making use of the same "free choice" assumption, they conclude that "a system's wave function is in one-to-one correspondence with its elements of reality."

The assumption of "free choice" has since been identified as the conjunction of two more familiar assumptions: parameter independence and setting independence[4] (Ghirardi and Romano, 2013; Vona and Liang, 2014). Despite this clarification, there has been confusion about whether these two assumptions suffice, or if more assumptions that rely on the specific mathematical structure of quantum mechanics are needed. Since then, Leifer (2014), relying on (Colbeck and Renner, 2017), presented a rigorous proof for the second claim by Colbeck and Renner that does not rely on the validity of the completeness claim. Landsman (2015) gave a critical assessment of the completeness claim, arguing that on top of these explicit assumptions, the proof for ψ-completeness relies on no less than four further rather technical assumptions. A far more friendly conclusion was

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[1]See (Dieks, 2017) for a comprehensive account of von Neumann's proof.

[2]Actually, von Neumann was specifically considering the possibility of dispersion-free sub-ensembles.

[3]The first and most famous ψ-ontology theorem is the PBR theorem (Pusey, Barrett, and Rudolph, 2012). See (Leifer, 2014) for an overview of ψ-ontology theorems.

[4]This second assumption states that the settings for a measurement are independent of the state of the system to be measured. It also appears in the literature as measurement independence, λ-independence, no conspiracy, free will, and presumably under several other names.

reached by Leegwater (2016) who gave a thorough reworking of Colbeck and Renner's original proof. However, Landsman's worries were not explicitly addressed by Leegwater, and the proof is not transparent enough to easily assess whether Landsman's criticism was indeed moot.

In this paper I argue that the general conclusion drawn by Colbeck and Renner is currently unwarranted. To this end a formal statement of their two claims in terms of the ontic models framework is given in section 3 after motivating the use of this framework in section 2. In section 3 it is also shown that the completeness claim is logically stronger than the ψ-ontology claim, thus showing that the issues of Landsman (2015) are not simply resolved by reworking the proof of Leifer (2014). The general strategy for proving the completeness claim is to start with proving it for the special case of two systems in a maximally entangled state. This case is discussed in section 4 as well as its generalization to arbitrary entangled states. In section 5 I discuss how this result is supposed to generalize to the case of individual systems. The crucial step needed to make that generalization is then scrutinized and criticized in section 6 and I conclude that no satisfactory proof of the completeness claim is currently available.

2   The use of ontic models

Ontic models are a useful tool for studying non-classical features of quantum mechanics.[5] They mimic the structure of a "classical" theory by associating with every system a set Λ of possible ontic states for the system. These ontic states determine how the system is to respond in the case of a measurement, in the sense that with each possible measurement procedure the state $\lambda \in \Lambda$ associates a probability distribution over possible measurement outcomes. These probability distributions may be thought of as representing, possibly dispositional, properties of the system. Possible preparations of the system are associated with probability distributions over the state space (thus Λ is assumed to be a measurable space). These may be thought of as representing ignorance concerning the "true" state of the system λ.

The use of ontic models is not trivially innocuous. Colbeck and Renner themselves use a framework that is more akin to the use of causal networks such as in the work of Wood and Spekkens (2015), so some notes are in order.[6]

A possible reason for disliking ontic models, is that they seem to come equipped with metaphysical baggage. It is common, as suggested by their name, to interpret ontic states as representing the physical properties of the system and with a slight abuse of language these states may be called "real".[7] Colbeck and Renner indeed seem hesitant to embrace this kind of terminology as indicated for example by their short discussion of the PBR theorem (Colbeck and Renner, 2015a, p. 68). But no such interpretation is forced upon us. If one wants, one can simply think of ontic states as mathematical objects competing with quantum states for being "the best" description of the system. The interpretation of ontic states for an ontic model is then as much open to debate as the interpretation of quantum states.

Another possible reason for not wanting to use ontic models is that they may not be general enough. Indeed, setting independence is an implicit assumption of this framework since probability distributions over ontic states are taken to not depend on which measurements are or are not performed on the system. This is in fact a common loophole in no-go theorems that may be exploited in, for example, retrocausal approaches (Friederich and Evans, 2019). Although there are ways to generalize the framework to try to accommodate for this loophole[8], there is no need to go into this issue since setting independence is accepted as part of the assumptions in Colbeck and Renner's claims.

A reason for preferring ontic models is that it is conceptually clearer. In a causal network approach, all variables are treated on a par. Thus all the probabilities derive from a single general probability distribution on some space that is large enough to model all relevant variables as random variables. This means that probability distributions not only specify the probabilities for states and measurement outcomes, but also for which measurements are to be performed. The peculiarity of such a state of affairs has also been considered troublesome by Seevinck and Uffink (2011, p. 438):

> Even quantum mechanics leaves the question what measurement is going to be performed on a system as one that is decided *outside* the theory, and does not specify how much more probable one measurement is than another. It thus seems reasonable not to require from the candidate theories that they describe such probabilities.

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[5]The framework was first introduced in (Spekkens, 2005). See also (Harrigan, Rudolph, and Aaronson, 2007).

[6]It deserves to be noted though that the causal arrows in the approach of Colbeck and Renner have a meaning that is quite distinct from the one adopted in the theory of causal networks.

[7]See (Halvorson, 2019) for further discussion.

[8]See for example (Hermens, 2019).

But perhaps more objectionable than demanding the well-definedness of probabilities for measurement settings, is that such an approach invites ambiguity concerning the role of probability. In the ontic models there is a separate role for both epistemic and ontological probabilities, but these get intertwined in a causal network approach when the measurement choices correspond to nodes in the network.

The prime constraint for ontic models, is that they can reproduce the quantum mechanical predictions for measurements on any quantum system. Here, with a quantum system we associate a finite-dimensional Hilbert space $\mathcal{H}$ together with a triple of sets $(\mathcal{P}, \mathcal{M}, \mathcal{T})$ denoting sets of preparations, measurements and transformations respectively. It is assumed that every preparation $P \in \mathcal{P}$ can be represented by a density operator ρ, every measurement $M \in \mathcal{M}$ can be represented by a self-adjoint operator A (i.e., only PVMs are considered), and every preparation $T \in \mathcal{T}$ can be represented by a unitary operator U. If ρ is pure, it will often be represented by a unit vector ψ that satisfies $\rho = [\psi]$, where $[\psi]$ is the 1-dimensional projection on the line spanned by ψ. Taking into account the work of Spekkens (2005), the triple $(\mathcal{P}, \mathcal{M}, \mathcal{T})$ is expected to be contextual, i.e, the mappings $M \mapsto A$, $P \mapsto \rho$, $T \mapsto U$ will in general be many-to-one. For the sake of simplicity, it is assumed that the mappings are also onto. Thus every quantum state can be prepared, every self-adjoint operator can be measured, and every unitary operation can be brought about by means of a transformation.[9] The probabilities for measurement outcomes are given by the Born rule. That is, when P, T, M are represented by ρ, U, A respectively, then

$$\mathbb{P}(a|M, T, P) = \text{Tr}\left(U \rho U^* P_a^A\right), \tag{1}$$

where P_a^A is the projection onto the eigenspace of A corresponding to the eigenvalue a.

An ontic model for a quantum system consists of a measurable space of ontic states (Λ, Σ), where Λ is the set of ontic states and Σ is a σ-algebra of subsets of Λ. Every measurement $M \in \mathcal{M}$, is associated with a Markov kernel p_M, called a response function, that associates with every $\lambda \in \Lambda$ a probability distribution $\text{p}_M(.|\lambda)$ over the possible measurement outcomes. Following Leegwater (2016), these probabilities will be called λ-probabilities. Every preparation $P \in \mathcal{P}$ is associated with a probability measure μ_P over the ontic states and every transformation $T \in \mathcal{T}$ is associated with a Markov kernel γ_T from Λ to itself. On average, the predictions of quantum mechanics are required to be reproduced:

$$\mathbb{P}(a|M, P) = \int \text{p}_M(a|\lambda) \text{d}\mu_P(\lambda),$$
$$\mathbb{P}(a|M, T, P) = \iint \text{p}_M(a|\lambda) \gamma_T(\text{d}\lambda|\tilde{\lambda}) \text{d}\mu_P(\tilde{\lambda}). \tag{2}$$

Often, when there is no cause for confusion, the use of P, T, M will be replaced by their quantum mechanical representatives, resulting in more transparent equations like

$$\iint \text{p}_A(a|\lambda) \gamma_U(\text{d}\lambda|\tilde{\lambda}) \text{d}\mu_\psi(\tilde{\lambda}) = \langle U\psi | P_a^A | U\psi \rangle. \tag{3}$$

In other cases, the quantum representatives will be added as subscripts. So M_A denotes a measurement procedure represented by the self-adjoint operator A in quantum mechanics.

It is worth noting that transformations can be reconsidered to be part of either the preparation procedure or the measurement procedure. Specifically, for any M_A, T_U there is a response function $\text{p}_{M_A \circ T_U}$ defined by

$$\text{p}_{M_A \circ T_U}(a|\lambda) := \int \text{p}_{M_A}(a|\tilde{\lambda}) \gamma_{T_U}(\text{d}\tilde{\lambda}|\lambda), \tag{4}$$

which corresponds to an operational procedure for a measurement that is represented quantum mechanically by the operator U^*AU. Likewise, for any T_U, P_ρ there is a probability distribution $\mu_{T_U \circ P_\rho}$ defined by

$$\mu_{T_U \circ P_\rho}(\Delta) := \int \gamma_{T_U}(\Delta|\lambda) \text{d}\mu_{P_\rho}(\lambda), \ \Delta \in \Sigma_{\mathcal{H}}, \tag{5}$$

which corresponds to an operational procedure for a preparation of the state represented by $U\rho U^*$. Finally, any two transformations $\gamma_{T_1}, \gamma_{T_2}$ may be stringed together to give the transformation "T_2 after T_1" given by

$$\gamma_{T_2 \circ T_1}(\Delta|\lambda) := \int \gamma_{T_2}(\Delta|\tilde{\lambda}) \gamma_{T_1}(\text{d}\tilde{\lambda}|\lambda). \tag{6}$$

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[9]If one is interested in the possibility of experimental tests, a focus on theorems that can be formualted for finite triples is preferable (as for example in Bell's theorem), since the validity of quantum mechanics can only be investigated for at most a finite number of predictions. In this paper the issue is rather the validity of a particular claim. So it makes sense to work with one of the weakest formulations of the claim relying on the validity of quite a lot of quantum mechanics.

3 ψ-completeness and ψ-ontology

A straightforward ontic model for quantum mechanics is the one by Beltrametti and Bugajski (1995), which is basically quantum mechanics itself. The set of ontic states Λ is taken to be the set of one-dimensional projections. The λ-probabilities for a measurement M_A are given by the Born rule, i.e., for each $[\psi] \in \Lambda$

$$\mathrm{p}_{M_A}(a|[\psi]) = \langle \psi | P_a^A | \psi \rangle \tag{7}$$

A preparation P_ψ of a pure state corresponds to the Dirac-distribution centered on $[\psi]$, while a preparation of a mixed state corresponds to an appropriate convex combination of such Dirac-distributions. So in general there are multiple distinct distributions μ_ρ corresponding to the same ρ. The quantum dynamics are just copied, i.e,

$$\gamma_U(\Delta|[\psi]) = \begin{cases} 1 & [U\psi] \in \Delta, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

This model may rightfully be said to be trivial. In particular because it has the property that the λ-probabilities coincide with the quantum probabilities. This is the key idea to the formal notion of triviality and ψ-completeness.

Definition 1. An ontic model for a quantum system is said to be *trivial* with respect to a set of measurements $\mathcal{M}' \subset \mathcal{M}$ and preparations $\mathcal{P}' \subset \mathcal{P}$ if for every preparation $P_\rho \in \mathcal{P}'$ and every measurement $M_A \in \mathcal{M}'$ the λ-probabilities μ_{P_ρ}-almost surely coincide with the quantum mechanical probabilities:

$$\int \left| \mathrm{p}_{M_A}(a|\lambda) - \mathrm{Tr}\left(\rho P_a^A\right) \right| \mathrm{d}\mu_{P_\rho}(\lambda) = 0 \tag{9}$$

or, equivalently,

$$\mu_{P_\rho}\left(\left\{\lambda \in \Lambda \mid \mathrm{p}_{M_A}(a|\lambda) = \mathrm{Tr}\left(\rho P_a^A\right)\right\}\right) = 1. \tag{10}$$

If the ontic model is trivial with respect to all measurements and preparations, then it is called *ψ-complete*.

Colbeck and Renner's completeness claim may now be formulated as follows:

Claim 1. *For any quantum system every ontic model that satisfies parameter independence is ψ-complete.*

It is worth pointing out that the notion of ψ-completeness was already introduced by Harrigan and Spekkens (2010) with a different, but formally related, meaning. They call an ontic model ψ-complete if the set of ontic states is isomorphic to the set of ontic states of the Beltrametti-Bugajski model and preparations of pure quantum states correspond to the appropriate Dirac distributions. It follows that if an ontic model is ψ-complete in the sense of Harrigan and Spekkens, it is automatically ψ-complete in the sense of Definition 1.

In the same paper, Harrigan and Spekkens introduced the distinction between ψ-ontic and ψ-epistemic ontic models. Roughly, the idea is that whenever two preparations corresponding to two distinct pure quantum states, then in a ψ-ontic model these preparations will yield distinct ontic states for the system. The following defintion is the now standard way to make this idea precise.

Definition 2. An ontic model for a quantum system is said to be *ψ-ontic* if for any pair of preparations P_1, P_2 represented by distinct pure quantum states the variational distance between their corresponding probability measures μ_{P_1}, μ_{P_2} is one, i.e.,

$$\sup_{\Delta \in \Sigma_{\mathcal{H}}} |\mu_{P_1}(\Delta) - \mu_{P_2}(\Delta)| = 1. \tag{11}$$

Colbeck and Renner's ψ-ontology claim may now be formulated as follows:

Claim 2. *For any quantum system every ontic model that satisfies parameter independence is ψ-ontic.*

This claim and Claim 1 above are intended to be paraphrases of Claim 1 and Claim 2 in (Colbeck and Renner, 2015b) adapted to the framework of ontic models. But it deserves to be noted that instead of looking at models for specific systems, Colbeck and Renner are looking at an "alternative theory that is compatible with quantum theory". In the next section it will become clear that this is not a moot point since their argument relies on relating several quantum systems, which will then yield a slight modification of the two claims.

In the papers (Colbeck and Renner, 2012, 2015b), Colbeck and Renner proved Claim 2 as part of a corollary of a proof for Claim 1. An independent proof for Claim 2 was given in (Colbeck and Renner, 2017) which was also refered to in (Colbeck and Renner, 2015a) and served as the main inspiration for Leifer's formulation and proof of the Colbeck-Renner ψ-ontology theorem in (Leifer, 2014). The following proposition establishes the relation between the two claims more directly.

Proposition 1. *Any ontic model for a quantum system that is ψ-complete is also ψ-ontic.*

Proof. Suppose a quantum system is given together with a ψ-complete ontic model. Let P_1, P_2 be any two preparations corresponding to distinct pure quantum states ψ_1, ψ_2. Then there exists a measurement $M \in \mathcal{M}$ corresponding to a self-adjoint operator A with eigenvalue a such that

$$\langle \psi_1 | P_a^A | \psi_1 \rangle \neq \langle \psi_2 | P_a^A | \psi_2 \rangle. \tag{12}$$

Because the model is ψ-complete it follows that

$$\mu_{P_1} \left(\left\{ \lambda \in \Lambda_{\mathcal{H}} \mid \mathrm{p}_M(a|\lambda) = \langle \psi_1 | P_a^A | \psi_1 \rangle \right\} \right) = 1, \tag{13}$$

while

$$\mu_{P_2} \left(\left\{ \lambda \in \Lambda_{\mathcal{H}} \mid \mathrm{p}_M(a|\lambda) = \langle \psi_1 | P_a^A | \psi_1 \rangle \right\} \right) = 0 \tag{14}$$

so (11) holds. $\qquad\square$

Claim 2 is thus significantly weaker than Claim 1. As far as I am concerned, Leifer (ibid.) gave a rigorous proof of Claim 2 within the ontic models framework. My focus is therefore on Claim 1, which Landsman (2015) has criticized but which was endorsed by Leegwater (2016) after he clarified some of the technicalities behind the proof. I shall argue that a rigorous proof for Claim 1 is currently lacking. But before that, I shall discuss and prove some weaker versions of Claim 1 as a partial endorsement of Leegwater's conclusion.

4 Partial proofs for the completeness claim

4.1 The equiprobability theorem

The simplest partial proof for Claim 1 concerns local measurements on a pair of d-level quantum systems in a maximally entangled state. The question is then of course how to generalize this to arbitrary states, arbitrary measurements and arbitrary systems. But first an unambiguous formulation and proof of the partial claim is required.

The scenario is the familiar EPR setup with associated Hilbert space $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$, and a preparation of the system in the state

$$\psi_d := \sum_{i=1}^{d} \frac{1}{\sqrt{d}} e_i \otimes e_i, \tag{15}$$

where $(e_i)_{i=1}^d$ is an arbitrary orthonormal basis for $\mathbb{C}^d$. One of the subsystems is then send to Alice, and the other to Bob, who are assumed to be space-like separated.

Now let $\mathcal{M}_A \subset \mathcal{M}$ denote the set of possible measurements where only Alice performs a measurement (locally). These can be represented by self-adjoint operators of the form $A \otimes \mathbb{1}$. Symmetrically, let $\mathcal{M}_B$ to be the possible measurements where only Bob performs a measurements, which can thus be represented by operators of the form $\mathbb{1} \otimes B$. Finally, take $\mathcal{M}_{\mathrm{LOC}}$ to be the set of measurements where either Alice or Bob performs a measurement or both. So one may take $\mathcal{M}_{\mathrm{LOC}} \simeq \mathcal{M}_A \times \mathcal{M}_B$. Parameter independence can now be formulated as

$$\mathrm{p}_{M_{A \otimes \mathbb{1}}}(a|\lambda) = \sum_b \mathrm{p}_{M_{A \otimes B}}(a, b|\lambda), \; \mathrm{p}_{M_{\mathbb{1} \otimes B}}(b|\lambda) = \sum_a \mathrm{p}_{M_{A \otimes B}}(a, b|\lambda), \tag{16}$$

for all $M_{A \otimes \mathbb{1}} \in \mathcal{M}_A, M_{\mathbb{1} \otimes B} \in \mathcal{M}_B, M_{A \otimes B} \in \mathcal{M}_{\mathrm{LOC}}$. Here the local operational procedures $M_{A \otimes \mathbb{1}}$ and $M_{\mathbb{1} \otimes B}$ should be the same as those represented by $M_{A \otimes B}$. One then has the following theorem.

Theorem 1 (Equiprobability theorem). *Any ontic model for the pair of d-level systems that satisfies parameter independence must be trivial w.r.t. $\mathcal{M}_{\mathrm{LOC}}$ and all preparations that are represented by ψ_d.*

The basic structure of the proof for this theorem is due to Colbeck and Renner (2011).[10] A more rigorous proof can be found in the proofs of Theorems 10.4 and 10.7 in (Leifer, 2014). For completeness a proof is given in Appendix A, which will also serve as a stepping stone for the proof of a generalized version of the theorem below.

Although the equiprobability theorem is still far removed from Claim 1, it deserves to be noted that the result is an improvement on earlier work. Stairs (1983) showed that for any pair in a maximally entangled d-level systems with $d \geq 3$, there is no value definite ontic model that satisfies parameter independence.[11] In

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[10]See also (Barrett, Kent, and Pironio, 2006) and references therein for the use of chained Bell inequalities.
[11]This result later became known as the free will theorem by Conway and Kochen (2006, 2009).

the present formulation, value definiteness can be understood as the assumption that the λ-probabilities are 0,1-valued. The work by Colbeck and Renner improves on this result on two accounts: the result also holds for the case $d = 2$ and, not only must ontic models be probabilistic, they must even follow the quantum probability rule.

Like the proof of Stairs' theorem, the proof of the equiprobability theorem relies heavily on the special properties of maximally entangled states. Because of the symmetry in the maximally entangled state, it assigns equal probabilities to all outcomes for a non-disperse observable. This implies that it suffices to show that all λ-probabilities should be equal. Thus no direct relation to the Born rule needs to be established. The proof that all the λ-probabilities are equal in turn relies heavily on the perfect correlations of the maximally entangled state.

The extension of the equiprobability theorem still relies on perfect correlations, but relaxes the symmetry property. Specifically, one makes the generalization to arbitrary entangled states, but does so by restricting to perfectly correlated measurements, i.e., measurements whose corresponding operator is diagonal in the Schmidt basis. To obtain this result, one further needs to extend the framework of ontic models. This is done in the next subsection.

4.2   The extended equiprobability theorem

The scenario of the extended equiprobability theorem is again a pair of d-level quantum systems, but now prepared in the state

$$\psi_d^S := \sum_{i=1}^{d} c_i e_i \otimes e_i, \tag{17}$$

where $(e_i)_{i=1}^d$ is an arbitrary orthonormal basis for $\mathbb{C}^d$. Instead of arbitrary local measurements, one now only considers measurements whose corresponding operator is diagonal with respect to this basis. The strategy to prove that the λ-probabilities for these measurements should equal the quantum probabilities relies on coupling the pair of systems to a pair of D-level systems. The combined system is then brought to a state that is approximately maximally entangled and then the equiprobability theorem is applied so probabilities for local measurements should then approximately equal $\frac{1}{D}$. These probabilities are then related to the λ-probabilities for the initial pair of systems, which can then be approximated to have the value $\frac{n_i}{D} \approx |c_i|^2$.

To deal with the notion of coupling one system to another, one needs a way to translate the ontic models for two (or more) quantum systems to the ontic model for the joint quantum system. Here enters the idea for what will be called a *complete ontic model*.[12]

Definition 3. A *complete ontic model* for quantum mechanics is a collection of ontic models, with one ontic model for every finite-dimensional quantum system. For every pair of quantum systems with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and for any system with Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ there is for each preparation P_2 of the second system a Markov kernel γ_{P_2} from the ontic model (Λ_1, Σ_1) for the first system to the ontic model $(\Lambda_{12}, \Sigma_{12})$ for the joint system such that for every preparation P_1 of the first system the probability measure $\mu_{P_1 P_2}$ defined as

$$\mu_{P_1 P_2}(\Delta) := \int \gamma_{P_2}(\Delta | \lambda) \mathrm{d}\mu_{P_1}(\lambda) \tag{18}$$

corresponds to some preparation $P_1 P_2$ of the joint system such that if P_1 is represented by the quantum state ρ_1 and P_2 by ρ_2, then $P_1 P_2$ is represented by $\rho_1 \otimes \rho_2$.

A complete ontic model is said to satisfy *parameter independence* if all its ontic models satisfy parameter independence.

One can think of the Markov kernel γ_{P_2} as modeling the act of appending an ancilla prepared according to P_2 to the initial system. It provides a translation from the ontic model for the initial system to the ontic model for the joint system. It seems natural to demand that the description of the initial system in the ontic model for the joint system, now considered as a subsystem, should be at least as rich as its description in the initial ontic model. The assumption of ancilla independence captures this intuition.

Definition 4. A complete ontic model is said to satisfy *ancilla independence* when λ-probabilities for an individual system arise as averages of λ-probabilities for local measurements on a joint system. Specifically, for

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[12]This definition is based on Leifer's proposal for how to deal with appending ancillas in ontic models (Leifer, 2014, §8.2), which is also relevant for his proof of Claim 2.

every response function p_{M_A} in the initial ontic model there is a response function $\mathrm{p}_{M_{A\otimes 1}}$ in the ontic model for the joint system such that for every preparation P_2 for a second system

$$\mathrm{p}_{M_A}(a|\lambda) = \int \mathrm{p}_{M_{A\otimes 1}}(a|\tilde{\lambda})\gamma_{P_2}(\mathrm{d}\tilde{\lambda}|\lambda) \tag{19}$$

for all possible measurement outcomes a and all ontic states $\lambda \in \Lambda_1$.

At first sight the notion of ancilla independence may be reminiscent of preparation independence as adopted in the PBR theorem, since the validity of (19) is independent of the choice of P_2. Note though that unlike preparation independence, which relies on the Cartesian product assumption (Leifer, 2014, p. 100), ancilla independence does not have direct implications for the structure of the set of ontic states for the joint system and how these ontic states relate to the ontic states for the individual systems. It seems to me that ancilla independence is a reasonable assumption. Either way, it is needed to prove the extended equiprobability theorem.

Theorem 2 (Extended equiprobability theorem). *In a complete ontic model that satisfies ancilla independence and parameter independence, every ontic model for the pair of d-level systems with $d \geq 2$ must be trivial with respect to all preparations that are represented by ψ_d^S and all local measurements whose corresponding operator is diagonal in the orthonormal basis $(e_i)_{i=1}^{d}$.*

A proof for this theorem is presented in Appendix B. As far as I know it is the first fully rigorous proof of this type of theorem. Of course, the theorem as presented here is strictly speaking new simply because it is formulated in the language of ontic models. It also does not map directly onto earlier theorems and proof because the strategy for purported proofs for Claim 1 have variations. Nevertheless, some general remarks can be made.

The criticism of Landsman (2015) pertaining to earlier proofs by Colbeck and Renner can be traced to the fact that the proof makes use of three limiting operations that are intertwined and that Colbeck and Renner appear to rely on special mathematical properties of the formalism of quantum mechanics to deal with these operations. Landsman partly deals with these issues by resorting to strong assumptions, which he takes to be part of the analysis of Colbeck and Renner, but only now are being made explicit. For example, his assumption of continuity of probabilities relates to the fact that when the initial system is related to a pair of D-level systems, the state of the latter merely approximates a maximally entangled state. A naieve direct application of the equiprobability theorem is thus not available. Leegwater (2016) at first sight seems to avoid this issue by working in the limit, but whether this is legitimate is not obvious since the proof of the equiprobability theorem itself relies on a limiting operation: the chained Bell inequalities (this is the second limiting operation). The final limiting operation is that of approximating the quantum probabilities $|c_i|^2$ with rational numbers $\frac{n_i}{D}$ where the approximation becomes better as $D \to \infty$. Here again Leegwater is not very clear about whether the limit $D \to \infty$ can directly be applied to a result that itself relies on a limiting operation. These issues are avoided in the proof in Appendix B, where all limiting operations are postponed to the final steps, where it also becomes clear that in fact these operations are intertwined.

5 Completeness for individual systems

The idea of a completeness proof for individual quantum systems seems peculiar given that non-trivial ontic models for arbitrary d-level quantum systems have been around since the work of Bell (1966) and Gudder (1970). Trivially, these models may be assumed to satisfy parameter independence since there is no second system in play with which it could interact. So strictly speaking these models are counterexamples to Claim 1.

The lack of a description of interaction in these models may be considered to be a serious deficiency. In fact, in light of the extended equiprobability theorem, it seems that Claim 1 should at least be reformulated to

Claim 3. *In any complete ontic model that satisfies ancilla independence and parameter independence every ontic model is ψ-complete.*

In principle, one can imagine that under certain minimal assumptions on interactions, parameter independence becomes applicable and ontic models necessarily have to become trivial. This seems at least to be the aim of Colbeck and Renner, and so it is useful to investigate how their strategy is supposed to work.

Consider an individual quantum system with Hilbert space $\mathcal{H}_1 = \mathbb{C}^d$ and a preparation corresponding to the pure state

$$\psi_1 = \sum_{i=1}^{d} c_i e_i, \tag{20}$$

for some orthonormal basis $\{e_1, \ldots, e_d\}$. The focus is on a possible measurement represented by a complete self-adjoint operator $A = \sum_{i=1}^{d} a_i[e_i]$. It is straightforward to devise an ontic model for this system that is non-trivial with respect to this measurement for this state. That is, a model in which

$$\int \left| p_A(a_i|\lambda) - |c_i|^2 \right| d\mu_{\psi_1}(\lambda) \neq 0. \tag{21}$$

This is in striking contrast to the consequence of extended equiprobability theorem which says that for any ontic model for a pair of such systems (that is part of an appropriate complete ontic model) one has

$$\int \left| p_{A \otimes \mathbb{1}}(a_i|\lambda) - |c_i|^2 \right| d\mu_{\psi_d^S}(\lambda) = 0 \tag{22}$$

with ψ_d^S as in (17). Could it be possible to argue from this that any ontic model for the individual system in which (21) is the case should be rejected?

The step needed to make the connection is most explicit in the work of Leegwater (2016, §8).[13] Instead of looking at $A \otimes \mathbb{1}$ the focus is on $\mathbb{1} \otimes B$ where $B = \sum_{i=1}^{d} b_i[e_i]$. By Theorem 1 the relevant analogue of (22) also holds for measurements represented by this operator. The step is then that since "by definition"[14]

$$\mathbb{P}(a_i|A, \psi_1) = \mathbb{P}(b_i|\mathbb{1} \otimes B, \psi_d^S) \tag{23}$$

"the same relation holds when considering λ-probabilities:"

$$p_A^{\psi_1}(a_i|, \lambda) = p_{\mathbb{1} \otimes B}^{\psi_d^S}(b_i|\lambda). \tag{24}$$

Presumably, (24) is supposed to imply that (21) cannot hold in any ontic model that is part of a complete ontic model that satisfies ancilla independence and parameter independence. Whether this is the case of course depends on what (24) exactly expresses. This is not entirely trivial as a formal definition of the expression is lacking.

As a first step in fleshing out what (24) means, assume that the inference from (23) to (24) is valid if and only if the inference from

$$\mathbb{P}(a_i|A, \psi_1) = \mathbb{P}(a_i|A \otimes \mathbb{1}, \psi_d^S) \tag{25}$$

to

$$p_A^{\psi_1}(a_i|\lambda) = p_{A \otimes \mathbb{1}}^{\psi_d^S}(a_i|\lambda) \tag{26}$$

is valid. This is reasonable since if one inference holds, then, by the extended equiprobability theorem, the other holds as well.

In quantum mechanics (25) holds as a consequence of the mathematical structure of the theory and how it deals with composing joint systems out of individual systems. But it is important to note that the equation expresses a numerical equivalence of two probabilities that are defined in separate models. The objects A and ψ_1 belong to the quantum model for the individual system while $A \otimes \mathbb{1}$ and ψ_d^S belong to the pair model. The fact that the same symbol $\mathbb{P}$ is used on both sides of the equation does not mean that the equation establishes the equality of two values a single function takes on for two distinct arguments; the two functions are distinct.

To have a similar link between λ-probabilities of bipartite systems and λ-probabilities of their parts, one has to suppose that the ontic models for all relevant systems are part of (something like) a complete ontic model. But even then it is not clear what it means for the same λ to appear on both sides of the equation, just as it wouldn't make sense to have the qame quantum state on both sides of the equation in (25). These are just some preliminary considerations for tackling the real issue at hand, namely, that the objects on both sides of (26) also are not well-defined. As these issues are intertwined I will deal with them simultaneously.

A part of the problem stems from trying to analyze (26) in the language of ontic models, while Colbeck, Renner and Leegwater take a different approach. On their approach, both ψ_1 and λ are possible values for random variables, as is the measurement setting and the measurement outcome. Then $p_A^{\psi_1}(a_i|\lambda)$ expresses the probability of obtaining the outcome a_i given that the quantum state is ψ_1 and the measurement is A and something else, dubbed λ. It may be possible that they intend the random variable that determines the quantum state to not just range over states for a specific Hilbert space, but also across different Hilbert spaces. Then both ψ_1 and ψ_d^S are possible values and so there is a well-defined probability that the system will be a

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[13]In (Colbeck and Renner, 2011) the relevant step connects to assumption QMb in the supplementary material and in (Colbeck and Renner, 2015b) it is discussed in section 6.

[14]The notation of the equations has been adjusted to fit the notation in this paper.

single d-level system and a well-defined probability that it will be a pair, determined by the single function p. But this idea will be avoided here.

Another important distinction is that Colbeck, Renner and Leegwater do not presuppose that λ provides all relevant information concerning the system and so "adding" ψ to λ may give more information about the possible outcomes. This explains the occurrence of quantum states in (26). Informally speaking, the quantum state may also be taken to give relevant information within the use of an ontic model. As it specifies the preparation of the system, it may indicate what behavior of λ may be considered to be "typical". So the notation in (26) may be interpreted as providing a convenient way of specifying a constraint on the possible λ's. A way to make this precise, is to assume (like Landsman (2015)) that λ-probabilities with quantum states attached to them express equations that hold μ_{ψ_1}-almost surely, i.e.,

$$\mathrm{p}_A^{\psi_1}(a_i|\lambda) = f(\lambda) \iff \int |\mathrm{p}_A(a_i|\lambda) - f(\lambda)|\,\mathrm{d}\mu_{\psi_1}(\lambda) = 0. \tag{27}$$

This helps in clarifying how to interpret both sides of (26). It does not help yet in interpreting what equality between the two means. A problem with (26) is that it refers to two distinct quantum states (never mind that they also belong to distinct Hilbert spaces). Landsman (ibid.) doubles down on the almost surely interpretation here and proposes the definition

$$\mathrm{p}_A^{\psi_1}(a_i|\lambda) = \mathrm{p}_{A\otimes\mathbb{1}}^{\psi_d^S}(a_i|\lambda) \iff \forall f : \mathrm{p}_A^{\psi_1}(a_i|\lambda) = f(\lambda) \text{ iff } \mathrm{p}_{A\otimes\mathbb{1}}^{\psi_d^S}(a_i|\lambda) = f(\lambda). \tag{28}$$

A first problem here is that, in the current setting, this is still not well-defined as the λ's belong to distinct ontic models and so it is not clear what kind of f could even have the proper domain. But even if that issue can be resolved, (28) is an unreasonably strong assumption, as it implies that if (26) holds, then $p_A(a_i|\lambda) = p_{A\otimes\mathbb{1}}(a_i|\lambda)$ both μ_{ψ_1}-almost surely and $\mu_{\psi_d^S}$-almost surely. Moreover, distributions for μ_{ψ_1} and $\mu_{\psi_d^S}$ completely overlap on the region of Λ where $\mathrm{p}_A(a_i|\lambda)$ is non-zero.[15] This is in strong tension with results like the BCLM theorem (Barrett, Cavalcanti, et al., 2014), which demonstrate that overlaps should become arbitrarily small as d increases.

A solution to the problem can be found in the work of Leegwater (2016), and it overlaps with the solution to the double use of λ for physically distinct systems. In the discussion of some of the notation used that is similar to that occurring in (26), Leegwater notes

> here λ still refers to the variable assigned to system A when it was in the state $|\psi\rangle_A$ [...] λ always refers to the original system A, and there is only one measure $\mu(\lambda)$ that is considered. [p.21]

The measure μ refers to a probability distribution over Λ that may depend on $|\psi\rangle_A$ as well as other factors. Given that there is only one measure considered, which is related to a particular quantum state, all other quantum states should be understood as being arrived at after interactions with a system prepared according to $|\psi\rangle_A$.

Translating this to the present discussion, it implies that $\mathrm{p}_{A\otimes\mathbb{1}}^{\psi_d^S}(a_i|\lambda)$ refers to a single d-level system that was initially prepared in the quantum state ψ_1 and having ontic state λ, then was coupled to a second d-level system with unknown state and then a transformation on the joint system was performed yielding the quantum state ψ_d^S. Within a complete ontic model this process can be modeled. For sake of definiteness, assume that the second system was prepared according to the quantum state ϕ. Appending this system to the first one is then modeled by a Markov kernel γ_ϕ from the initial ontic model to the model for the joint system. Now let U be a unitary operator such that $U(\psi_1 \otimes \phi) = \psi_d^S$ and let γ_U be a Markov kernel that models it. A proper reformulation of (26) is then that[16]

$$\mathrm{p}_A(a_i|\lambda) = \iint \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\tilde{\lambda})\gamma_U(\mathrm{d}\tilde{\lambda}|\lambda')\gamma_\phi(\mathrm{d}\lambda'|\lambda) \tag{29}$$

holds μ_{ψ_1}-almost surely.

This reconstruction seems to be on the right track. If the inference from (25) to (29) is valid, then the extended equiprobability theorem can be used to prove the completeness claim. Since $\mu_{T_U \circ T_\phi \circ P_{\psi_1}}$ corresponds

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[15]This may be seen as follows. The first claim follows from evaluating (28) for the choice $f(\lambda) = p_A(a_i|\lambda)$. The second claim follows with a proof from contradiction. If there exists a $\Delta \subset \Lambda$ such that $\mu_{\psi_d^S}(\Delta) > 0 = \mu_{\psi_1}(\Delta)$ and on which $\mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda)$ is non-zero, then for any f such that $\mathrm{p}_A^{\psi_1}(a_i|\lambda) = f(\lambda)$ and $\mathrm{p}_{A\otimes\mathbb{1}}^{\psi_d^S}(a_i|\lambda) = f(\lambda)$ one can define an f' that differs from f only on Δ in such a way that $\mathrm{p}_{A\otimes\mathbb{1}}^{\psi_d^S}(a_i|\lambda) = f'(\lambda)$ no longer holds. So (26) would fail.

[16]Note that both integrals are over the ontic states for the joint system, while λ is an ontic state for the initial system.

to a preparation that can be represented by the state ψ_d^S, it follows that $\mathrm{p}_{A\otimes\mathbb{1}}(a_i|\tilde\lambda)$ is almost surely equal to the quantum probability $|c_i|^2$ with respect to this measure. It then follows from (29) that $\mathrm{p}_A(a_i|\lambda)$ is equal to the quantum probability μ_{ψ_1}-almost surely. The remaining question is then why the inference from (25) to (29) should be valid. In the next section I argue that no uncontroversial arguments are available.

6 Unitary processes and measurements

It is not obvious from the works of Colbeck, Renner and Leegwater what should be taken to be the main argument for the inference from (25) to (29). To get a handle on what it takes for (29) to hold, I show that it can be derived with the help of two assumptions. The first assumption is ancilla independence as introduced in section 4. The second I call unitary invariance. In spirit it is similar to the assumption of unitary invariance adopted by Landsman (2015), but it avoids problems similar to those related to (28).

Definition 5. An ontic model is said to satisfy *unitary invariance* if for every ρ, U, A, a

$$\mathrm{Tr}\left(\rho P_a^A\right) = \mathrm{Tr}\left(U\rho U^* P_a^A\right) \implies \mathrm{p}_{MA}(a|\lambda) = \int \mathrm{p}_{MA}(a|\tilde\lambda)\gamma_{T_U}(\mathrm{d}\tilde\lambda|\lambda)\ \mu_{P_\rho}\text{-a.s.} \tag{30}$$

With these assumptions (29) can indeed be derived:

$$\int\left|\mathrm{p}_A(a_i|\lambda) - \iint \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\tilde\lambda)\gamma_{U\circ\phi}(\mathrm{d}\tilde\lambda|\lambda)\right|\mathrm{d}\mu_{\psi_1}(\lambda)$$

$$= \int\left|\int \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda')\gamma_\phi(\mathrm{d}\lambda'|\lambda) - \iint \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\tilde\lambda)\gamma_{U\circ\phi}(\mathrm{d}\tilde\lambda|\lambda)\right|\mathrm{d}\mu_{\psi_1}(\lambda)$$

$$\leq \iint\left|\mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda') - \int \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\tilde\lambda)\gamma_U(\mathrm{d}\tilde\lambda|\lambda')\right|\gamma_\phi(\mathrm{d}\lambda'|\lambda)\mathrm{d}\mu_{\psi_1}(\lambda) \tag{31}$$

$$= \int\left|\mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda) - \int \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\tilde\lambda)\gamma_U(\mathrm{d}\tilde\lambda|\lambda)\right|\mathrm{d}\mu_{\psi_1\otimes\phi}(\lambda) = 0.$$

However, it is not trivial to come up with a motivation for unitary invariance. This is problematic, as this assumption has immediate consequences that are quite crucially related to the notion of completeness. Note that completeness boils down to the idea that response functions are dispersion free under preparations of pure quantum states. Now consider the example where $\langle\psi|P_a^A|\psi\rangle = \langle U\psi|P_a^A|U\psi\rangle$. Unitary invariance then has the following consequence for the dispersion of any corresponding response function:

$$\mathrm{Var}_\psi(\mathrm{p}_A(a|\,.\,)) = \int \mathrm{p}_A(a|\lambda)^2\mathrm{d}\mu_\psi(\lambda) - \left(\int \mathrm{p}_A(a|\lambda)\mathrm{d}\mu_\psi(\lambda)\right)^2$$

$$= \int\left(\int \mathrm{p}_A(a|\tilde\lambda)\gamma_U(\mathrm{d}\tilde\lambda|\lambda)\right)^2\mathrm{d}\mu_\psi(\lambda) - \left(\iint \mathrm{p}_A(a|\tilde\lambda)\gamma_U(\tilde\lambda|\lambda)\mathrm{d}\mu_\psi(\lambda)\right)^2 \tag{32}$$

$$\leq \iint \mathrm{p}_A(a|\tilde\lambda)^2\gamma_U(\mathrm{d}\tilde\lambda|\lambda)\mathrm{d}\mu_\psi(\lambda) - \left(\iint \mathrm{p}_A(a|\tilde\lambda)\gamma_U(\tilde\lambda|\lambda)\mathrm{d}\mu_\psi(\lambda)\right)^2$$

$$= \mathrm{Var}_{U\psi}(\mathrm{p}_A(a|\,.\,))$$

It follows that if one can show that for some quantum state $U\psi$ the response function is dispersion free for a particular measurement, then the response function must also be dispersion free for any other quantum state ψ whenever the operational probabilities for the measurement are equal. Since this gets at the heart of the Colbeck-Renner completeness claim, this implicit assumption should have at least been explicitly stated and preferably be well-motivated. Neither appears to be the case though.

A possible explanation is that perhaps the inference from (25) to (29) is not assumed to hold in general, but only under circumstances that make it more plausible. An appeal to measurement procedures could perhaps do the trick. According to quantum theory, instead of directly measuring A on the initial system in the state ψ_1, one may equivalently first entangle it with a second system to obtain the joint state ψ_d^S and measure instead B on the second system. Unitary invariance comes down to the idea that this is also always a legitimate procedure for the ontic model. But one may instead focus on the case where the second system is a measurement device designed to measure A. This is indeed the scenario within which the crucial inference is considered in (Colbeck and Renner, 2011, p.4-5), (Colbeck and Renner, 2015b, p.515) and (Leegwater, 2016, §8).

Arguably, an appeal to what actually happens in a measurement process is not a very elegant strategy in a theory that is infamous for its measurement problem. The transition from ψ_1 to ψ_d^S is of course well-known as

part of the von Neumann measurement scheme.[17] The usual problem is then that the state ψ_d^S, on its own, does not signify a situation in which a measurement outcome is obtained. Single world unitary quantum mechanics without hidden variables is not compatible with the assumption that measurements have definite outcomes.

Ontic models are usually not designed to resolve this problem, as the Beltrametti-Bugajski model nicely illustrates. More generally, in any ontic model probabilities for outcomes of measurements are well-defined without any mentioning of how the relevant system would interact with a measurement apparatus. Ideally, this peculiarity is to be solved in a complete ontic model. The λ-probability $\mathrm{p}_M(a|\lambda)$ then encodes the probability with which the system, upon interaction with an appropriate measurement apparatus, would evolve towards a joint state in which the measurement apparatus can be taken to be in a well-defined pointer state displaying the outcome of the measurement. Explicitly, if Δ_a^M denotes the set of ontic states of the joint system in which the apparatus displays the outcome a, then ideally there is a Markov kernel γ_M modelling the interaction between system and apparatus such that

$$\mathrm{p}_M(a|\lambda) = \gamma_M\left(\Delta_a^M|\lambda\right). \tag{33}$$

A motivation for the inference from (25) to (29) in the context of measurements then should boil down to an assumption concerning γ_M.

The question is how the measurement process γ_M relates to the von Neumann-type process $\gamma_U \circ \gamma_\phi$. And this may of course depend on how the complete ontic model is supposed to solve the measurement problem.[18] In a spontaneous collapse theory, the macroscopic superposition ψ_d^S is extremely unlikely to obtain and γ_M contains a collapse way before the unitary evolution U is completed. Consequently, an appeal to measurements is irrelevant for the question whether Claim 3 applies to such theories and a general assumption like unitary invariance is required instead.

To be fair, in the first paper Colbeck and Renner (2011) did assume that "all processes within quantum theory can be considered as unitary evolutions". Although this explicit statement is dropped in any of the following papers, a charitable reading is that the completeness claim was never meant to apply to collapse theories. But even if it is accepted that γ_M should correspond to a unitary process, it is not obvious that it should be operationally equivalent to $\gamma_U \circ \gamma_\phi$ (i.e., can be represented by the same unitary operator). For that, one needs to assume that every operational procedure to measure the observable A requires an interaction that yields the state ψ_d^S. But even in quantum theory it is not obvious that this is *necessarily* the case. As far as quantum theory is concerned, it would be perfectly okay if the process γ_M would yield a state of the form $\sum_{i=1}^d c_i \chi \otimes e_i$ instead.[19]

Still, Claim 3 may pertain just to ontic models for unitary quantum mechanics where measurement processes necessarily lead to entanglement with perfect correlations. And the claim is then that such models must be trivial or violate parameter independence. To see if at least this claim is valid, it is worthwhile to quickly recap what the argument for it has become. The procedure to measure A on a system, involves hooking it up to a measurement device, evolving the joint system to ψ_d^S and then reading of the value of B from the device. Reading of this value constitutes a measurement of B on the measurement device. Because in a no-collapse theory this is roughly what it means to measure A, the outcome statistics for B on the joint system should equal the outcome statistics for A. But, if in addition the criteria for the extended equiprobability theorem are in place, then the outcome statistics for B should be equal to the quantum statistics and so also the statistics for A should equal the quantum statistics.

Applicability of the extended equiprobability theorem requires parameter independence. Thus it has to be assumed that it is possible to postpone the B-measurement until system and apparatus are again spatially separated. Moreover, at this stage it should also be possible to refrain from performing a B-measurement, and instead perform any other measurement on the apparatus and, in addition, to perform any other measurement on the system as well. If not, the chained Bell-inequalities upon which the proof for Theorem 2 rests cannot come of the ground.

This scenario seems quite unlikely when one thinks about how, for example, spin measurements work in Bohmian mechanics (Norsen, 2014). Even though this theory violates parameter independence, it may serve as a source of inspiration to see if the logic behind the measurement-based argument for Claim 3 is sound. In the Stern-Gerlach setup, the spin degree of freedom becomes encoded in the trajectory of the silver atom. A position measurement then yields the outcome of the spin measurement. But there is of course no sense in

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[17]Of course, this scheme is highly idealized, as also recognized by Colbeck, Renner and Leegwater. But although they also consider more general standard measurement procedures, these considerations are not relevant to the objections I raise.

[18]It is not entirely clear if Colbeck and Renner expect the alternative theory to solve the measurement problem. Doing so would seem unreasonable as it sets a higher standard for the alternative theory than adopted for quantum mechanics. But not doing so makes the appeal to measurement processes quite awkward. Regardless, the following arguments do not rely on whether or not γ_M actually satisfies (33). It just denotes the measurement process in the ontic model, whatever it may be.

[19]I picked up this idea from (Wallace, 2019), which contains more insightful discussion of what "orthodox quantum mechanics" should be taken to mean.

which at that point one can still also measure the spin in another direction whilst not affecting the particle position. Perhaps then the right system to figure as apparatus is the screen. While the particles of the screen become entangled with the silver atom, it is hard to imagine how they may be separated again from the silver atom in such a way that still different spin measurements can be performed on the atom, as well as different measurements on the screen. One may complaint that in principle the separation of the two systems once in the state ψ_d^S should be possible and that in principle the required measurements are possible. But I do not see how that argument would amount to anything more substantial than the idea that in principle unitary invariance should just be true.

What lies at the heart of the problem here, is that in Bohmian mechanics, particles do not have spin properties in any classical sense. Necessarily so, given the Kochen-Specker theorem. Instead, spin properties correspond to dispositional properties and the physical quantities are only meaningful in the context of an experimental setup that can be used to reveal them (Goldstein, 2017, §10). This is also the way to think about response functions in ontic models: they encode dispositional properties that can come to the for in a measurement scenario. But once brought about, it is not per se meaningful to say that dispositional properties before the measurement encoded by other response functions are still there. This idea is of course very Bohrian in spirit. In fact, Bohr's response to Einstein, Podolsky and Rosen, when taken out of context, seems strangely apt to qualify the issue regarding the stage when ψ_d^S obtains, if one changes just one word, and it is perfectly fit to serve as the main conclusion of this paper:

> But even at this stage there is essentially the question of *an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system.* Since these conditions constitute an inherent element of the description of any phenomenon to which the term "physical reality" can properly be attached, we see that the argumentation of the mentioned authors does not justify their conclusion that the quantum-mechanical description is essentially [complete]. (Bohr, 1935, p.700)

Acknowledgments

A   Proof of the equiprobability theorem

In this appendix the equiprobability theorem is proven. For convenience the theorem is formulated again here.

Theorem. *Consider the quantum system of a pair of d-level systems. Any ontic model for this system that satisfies parameter independence is trivial w.r.t. $\mathcal{M}_{\mathrm{LOC}}$ and all preparations represented by the state*

$$\psi_d = \sum_{i=1}^{d} \frac{1}{\sqrt{d}} e_i \otimes e_i.$$

Proof. Throughout, assume that μ_{ψ_d} is an arbitrary, but fixed, probability measure corresponding to a preparation of ψ_d. First consider a complete local measurement M_A with possible outcomes $a_1, a_2, \ldots, a_d$ corresponding to the operator $A \otimes \mathbb{1}$. For the state ψ_d, the probability for each possible outcome is $\frac{1}{d}$. To show that the λ-probabilities are identical, it suffices to show that the λ-probabilities for all possible outcomes are equal:

$$\int \left| \mathrm{p}_{M_A}(a_i|\lambda) - \tfrac{1}{d} \right| \mathrm{d}\mu_{\psi_d}(\lambda) = \frac{1}{d} \int \left| d\mathrm{p}_{M_A}(a_i|\lambda) - \sum_{j=1}^{d} \mathrm{p}_{M_A}(a_j|\lambda) \right| \mathrm{d}\mu_{\psi_d}(\lambda)$$

$$\leq \frac{1}{d} \sum_{j=1}^{d} \int |\mathrm{p}_{M_A}(a_i|\lambda) - \mathrm{p}_{M_A}(a_j|\lambda)| \, \mathrm{d}\mu_{\psi_d}(\lambda). \tag{34}$$

So it remains to be shown that

$$\int |\mathrm{p}_{M_A}(a_i|\lambda) - \mathrm{p}_{M_A}(a_j|\lambda)| \, \mathrm{d}\mu_{\psi_d}(\lambda) = 0 \tag{35}$$

for all i, j.

Without loss of generality, it may be assumed that each e_i is an eigenstate for A for the eigenvalue a_i. Let $B = \sum_{i=1}^d b_i[e_i]$ be a second arbitrary complete self-adjoint operator and let M_B be a local measurement corresponding to the operator $\mathbb{1} \otimes B$.

Now let i, j be given. If $i = j$, then (35) holds trivially, so assume $i \neq j$. For $\theta \in [0, \frac{\pi}{2}]$ the unitary operator U_θ is defined by the actions

$$U_\theta e_i = \cos \theta e_i + \sin \theta e_j, \ U_\theta e_j = \cos \theta e_j - \sin \theta e_i, \ U_\theta e_k = e_k \text{ when } k \neq i, j. \tag{36}$$

Further define

$$A_\theta := U_\theta A U_\theta^*, \ B_\theta := U_\theta B U_\theta^* \tag{37}$$

and let M_{A_θ} and M_{B_θ} denote local measurements corresponding to the operators $A_\theta \otimes \mathbb{1}$ and $\mathbb{1} \otimes B_\theta$ respectively. For simplicity, it may be assumed that $M_{A_0} = M_A$, and the M will be dropped in the notation.

For every $N \in \mathbb{N}$ the angle between $[0, \frac{\pi}{2}]$ can be divided in $2N + 1$ equal sized smaller angles. For $n \leq N$ set

$$A_{N,n} = A_{\frac{2n\pi}{(2N+1)2}}, \ B_{N,n} = B_{\frac{2(n+1)\pi}{(2N+1)2}}. \tag{38}$$

This is used to make the estimate

$$\int |\mathrm{p}_A(a_i|\lambda) - \mathrm{p}_A(a_j|\lambda)| \, \mathrm{d}\mu_{\psi_d}(\lambda)$$

$$\leq \sum_{n=0}^N \int \left| \mathrm{p}_{A_{N,n}}(a_i|\lambda) - \mathrm{p}_{B_{N,n}}(b_i|\lambda) \right| \mathrm{d}\mu_{\psi_d}(\lambda) + \sum_{n=0}^{N-1} \int \left| \mathrm{p}_{B_{N,n}}(b_i|\lambda) - \mathrm{p}_{A_{N,n+1}}(a_i|\lambda) \right| \mathrm{d}\mu_{\psi_d}(\lambda)$$

$$+ \int \left| \mathrm{p}_{B_{\frac{\pi}{2}}}(b_i|\lambda) - \mathrm{p}_A(a_j|\lambda) \right| \mathrm{d}\mu_{\psi_d}(\lambda). \tag{39}$$

The final term in (39) equals zero. To show this, parameter independence is used to make the estimate

$$\mathrm{p}_{B_{\frac{\pi}{2}}}(b_i|\lambda) - \mathrm{p}_A(a_j|\lambda) = \sum_{k=1}^d \mathrm{p}_{A \otimes B_{\frac{\pi}{2}}}(a_k, b_i|\lambda) - \sum_{l=1}^d \mathrm{p}_{A \otimes B_{\frac{\pi}{2}}}(a_j, b_l|\lambda)$$

$$= \sum_{k \neq j} \mathrm{p}_{A \otimes B_{\frac{\pi}{2}}}(a_k, b_i|\lambda) - \sum_{l \neq i} \mathrm{p}_{A \otimes B_{\frac{\pi}{2}}}(a_j, b_l|\lambda) \tag{40}$$

$$\leq \sum_{k \neq j} \mathrm{p}_{A \otimes B_{\frac{\pi}{2}}}(a_k, b_i|\lambda) + \sum_{l \neq i} \mathrm{p}_{A \otimes B_{\frac{\pi}{2}}}(a_j, b_l|\lambda).$$

Using this, it follows that

$$\int \left| \mathrm{p}_{B_{\frac{\pi}{2}}}(b_i|\lambda) - \mathrm{p}_A(a_j|\lambda) \right| \mathrm{d}\mu_{\psi_d}(\lambda) \leq \sum_{k \neq j} \int \mathrm{p}_{A \otimes B_{\frac{\pi}{2}}}(a_k, b_i|\lambda) \mu_{\psi_d}(\lambda) + \sum_{l \neq i} \int \mathrm{p}_{A \otimes B_{\frac{\pi}{2}}}(a_j, b_l|\lambda) \mu_{\psi_d}(\lambda)$$

$$= \sum_{k \neq j} \langle \psi_d | [e_k \otimes U_{\frac{\pi}{2}} e_i] | \psi_d \rangle + \sum_{l \neq i} \langle \psi_d | [U_{\frac{\pi}{2}} e_j \otimes e_l] | \psi_d \rangle \tag{41}$$

$$= \sum_{k \neq j} \langle \psi_d | [e_k \otimes e_j] | \psi_d \rangle + \sum_{l \neq i} \langle \psi_d | [e_i \otimes e_l] | \psi_d \rangle = 0,$$

where the final step follows from the perfect correlations in ψ_d.

For the other $2N + 1$ terms in (39) the correlations are not perfect. But with parameter independence one still has for each term the estimate

$$\int \left| \mathrm{p}_{A_\theta}(a_i|\lambda) - \mathrm{p}_{B_\phi}(b_i|\lambda) \right| \mathrm{d}\mu_{\psi_d}(\lambda) \leq \sum_{k \neq i} \int \mathrm{p}_{A_\theta \otimes B_\phi}(a_i, b_k|\lambda) \mu_{\psi_d}(\lambda) + \sum_{l \neq i} \int \mathrm{p}_{A_\theta \otimes B_\phi}(a_l, b_i|\lambda) \mu_{\psi_d}(\lambda)$$

$$= \sum_{k \neq i} \langle \psi_d | [U_\theta e_i \otimes U_\phi e_k] | \psi_d \rangle + \sum_{l \neq i} \langle \psi_d | [U_\theta e_l \otimes U_\phi e_i] | \psi_d \rangle \tag{42}$$

$$= \langle \psi_d | [U_\theta e_i \otimes U_\phi e_j] | \psi_d \rangle + \langle \psi_d | [U_\theta e_j \otimes U_\phi e_i] | \psi_d \rangle .$$

And for the expectation values one has

$$
\begin{aligned}
\langle \psi_d | [U_\theta e_i \otimes U_\phi e_j] | \psi_d \rangle &= \frac{1}{d} \sum_{k=i,j} \sum_{l=i,j} \langle e_k \otimes e_k | [U_\theta e_i \otimes U_\phi e_j] | e_l \otimes e_l \rangle \\
&= \frac{1}{d} \sum_{k=i,j} \sum_{l=i,j} \langle e_k | U_\theta e_i \rangle \langle e_k | U_\phi e_j \rangle \langle U_\theta e_i | e_l \rangle \langle U_\phi e_j | e_l \rangle \\
&= \frac{1}{d} \sum_{k=i,j} \langle e_k | U_\theta e_i \rangle \langle e_k | U_\phi e_j \rangle (\cos\theta\sin\phi - \sin\theta\cos\phi) \\
&= \frac{1}{d} (\cos\theta\sin\phi - \sin\theta\cos\phi)^2 = \frac{1}{d}\sin^2(\phi - \theta).
\end{aligned}
\tag{43}
$$

Making use of the fact that for all the $2N+1$ terms in (39) $|\theta - \phi| = \frac{\pi}{(2N+1)2}$:

$$
\int |\mathrm{p}_A(a_i | \lambda) - \mathrm{p}_A(a_j | \lambda)| \, \mathrm{d}\mu_{\psi_d}(\lambda) \le (2N+1)\frac{2}{d}\sin^2\left(\frac{\pi}{(2N+1)2}\right) \le \frac{\pi^2}{2d(2N+1)}.
\tag{44}
$$

Since this holds for all $N \in \mathbb{N}$, the proof for (35) is complete.

Finally, consider the case where A is a non-maximal operator. Assume again that each e_i is an eigenstate for A. Let a_k be any eigenvalue for A with degeneracy n_k and complete set of eigenstates $e_{i_1}, \dots, e_{i_{n_k}}$. Let B be a complete self-adjoint operator as before. Then

$$
\int \left| \mathrm{p}_A(a_k | \lambda) - \frac{n_k}{d} \right| \mathrm{d}\mu_{\psi_d}(\lambda) \le \int \left| \mathrm{p}_A(a_k | \lambda) - \sum_{j=1}^{n_k} \mathrm{p}_B(b_{i_j} | \lambda) \right| \mathrm{d}\mu_{\psi_d}(\lambda) + \int \left| \sum_{j=1}^{n_k} \left(\mathrm{p}_B(b_{i_j} | \lambda) - \frac{1}{d} \right) \right| \mathrm{d}\mu_{\psi_d}(\lambda).
\tag{45}
$$

The first term equals zero because of the perfect correlations between the outcomes and that the second term equals zero follows by applying (35) with A replaced by B. $\qquad\square$

B Proof of the extended equiprobability theorem

Theorem 3. *Consider the quantum system of a pair of d-level systems and the entangled state*

$$
\psi_d^S := \sum_{i=1}^d c_i e_i \otimes e_i \in \mathcal{H}_A \otimes \mathcal{H}_B \simeq \mathbb{C}^d \otimes \mathbb{C}^d.
\tag{46}
$$

In a complete ontic model that satisfies ancilla independence and parameter independence, every ontic model for the system pair must be trivial with respect to all preparations that are represented by ψ_d^S and all local measurements whose corresponding operator is diagonal in the orthonormal basis $(e_i)_{i=1}^d$.

Proof.

Setup The main strategy of the proof is to reshape the situation such that the proof strategy of the equiprobability theorem can be adopted. There are then two entwined issues that need to be faced.

First, the proof of Theorem 1 relies on the state being maximally entangled. To solve this issue, one uses the method of embezzlement (van Dam and Haden, 0203). The system is coupled to a second pair of systems prepared in a special entangled state. Then, by applying only local unitary operations, one can transform the initial state to a state that is arbitrarily close to any desired entangled state.

The second problem is that the equiprobability theorem yields equal probabilities, whereas now arbitrary Born probabilities with values $|c_i|^2$ are to be derived. This is solved by approximating these probabilities with sums of equal probabilities, which now pertain to yet another pair of systems coupled to the initial pair.

Without loss of generality, it may be assumed that the basis vectors e_i are such that all the c_i are positive reals. Now let M_A be a complete local measurement with corresponding operator $A \otimes \mathbb{1}$ and $A = \sum_{i=1}^d a_i[e_i]$. The aim is to show that for any $\mathcal{E} > 0$

$$
\int \left| \mathrm{p}_A(a_i | \lambda) - c_i^2 \right| \mathrm{d}\mu_{\psi_d^S}(\lambda) < \mathcal{E}
\tag{47}
$$

for all $i = 1, \dots, d$.

The first step is to approximate the Born probabilities with rational numbers. For given $\epsilon > 0$, choose natural numbers $n_1^\epsilon, \ldots, n_d^\epsilon$ such that[20]

$$\frac{c_i^2 d^\epsilon}{n_i^\epsilon} \in (1 - \epsilon, 1 + \epsilon), \text{ with } d^\epsilon := \sum_{i=1}^d n_i^\epsilon, \tag{48}$$

for all $i = 1, \ldots, d$. The system is then coupled to a pair of systems that is big enough to harbor this approximation. The Hilbert spaces $\mathcal{H}_A^\epsilon, \mathcal{H}_B^\epsilon$ for these systems are copies of $\mathbb{C}^{m^\epsilon}$ with $m^\epsilon := \max_i n_i^\epsilon$ and for each an orthonormal basis $f_1, f_2, \ldots, f_{m^\epsilon}$ is fixed. So the idea is that because $\sum_{j=1}^{n_i^\epsilon} \frac{1}{d^\epsilon} \approx c_i^2$, the initial state $\psi_d^S \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be mimicked by the maximally entangled state

$$\psi_{d^\epsilon} := \sum_{i=1}^d \sum_{j=1}^{n_i^\epsilon} \frac{1}{\sqrt{d^\epsilon}} f_j \otimes f_j \otimes e_i \otimes e_i, \tag{49}$$

which is an element of $\mathcal{H}_A^\epsilon \otimes \mathcal{H}_B^\epsilon \otimes \mathcal{H}_A \otimes \mathcal{H}_B$.

For any results for the extended system to be relevant for the initial system, it has to evolve towards a state close to ψ_{d^ϵ}. This is accomplished by an interaction with a third pair of systems with corresponding Hilbert spaces $\mathcal{H}_A^N, \mathcal{H}_B^N$ that are copies of $\mathbb{C}^D$ where

$$D := N \prod_{i=1}^d n_i^\epsilon \tag{50}$$

and N is a natural number that in the end will be taken to be large enough to have the right relation to $\mathcal{E}$.

For both spaces an orthonormal basis $g_1, \ldots, g_D$ is fixed and it is assumed that the pair is initially prepared in the state

$$\psi_D := C_N \sum_{k=1}^D \frac{1}{\sqrt{k}} g_k \otimes g_k \in \mathcal{H}_A^N \otimes \mathcal{H}_B^N, \tag{51}$$

where C_N is a normalization constant. The initial state for the totality of six systems is then taken to be

$$\Psi_I := C_N \sum_{k=1}^D \sum_{i=1}^d \frac{c_i}{\sqrt{k}} g_k \otimes g_k \otimes f_1 \otimes f_1 \otimes e_i \otimes e_i. \tag{52}$$

Note that although Ψ_I depends on the parameters ϵ, N, this is suppressed in the notation, as these parameters are taken to be fixed for the larger part of the proof.

Throughout, the bases will be fixed and a considerable chunk of the proof reduces to manipulations of the indices of these basis vectors. Therefore it is useful to adopt a notation that highlights the indices:

$$\left| k, \tilde{k} \middle| j, \tilde{j} \middle| i, \tilde{i} \right\rangle := g_k \otimes g_{\tilde{k}} \otimes f_j \otimes f_{\tilde{j}} \otimes e_i \otimes e_{\tilde{i}}. \tag{53}$$

So (52) becomes

$$\Psi_I = C_N \sum_{k=1}^D \sum_{i=1}^d \frac{c_i}{\sqrt{k}} |k, k|1, 1|i, i\rangle. \tag{54}$$

In a similar fashion a more compact notation for products of projection operators is also adopted:

$$\left(P_1, P_2 | P_3, P_4 | P_5, P_6 \right) := P_1 \otimes P_2 \otimes P_3 \otimes P_4 \otimes P_5 \otimes P_6. \tag{55}$$

Coupling the systems   The first step is to use the fact that the ontic models are part of a complete ontic model. So for every N, ϵ there exists a Markov kernel $\gamma_{N,\epsilon}$ that translates the probability measure $\mu_{\psi_d^S}$ for the initial ontic model to a probability measure

$$\mu_{\Psi_I}(\Delta) := \int \gamma_{N,\epsilon}(\Delta|\lambda) d\mu_{\psi_d^S}(\lambda) \tag{56}$$

---

[20]How ϵ is to relate to $\mathcal{E}$ is made clear at the end of the proof.

for the six systems, that corresponds to a preparation of the state Ψ_{I}. Moreover, because of ancilla independence, the initial response function p_A should correspond to some response function $\tilde{\mathrm{p}}_A$ in the larger ontic model such that

$$\mathrm{p}_A(a_i|\lambda) = \int \tilde{\mathrm{p}}_A(a_i|\tilde{\lambda})\gamma_{N,\epsilon}(\mathrm{d}\tilde{\lambda}|\lambda). \tag{57}$$

One then has the estimate

$$\int \left|\mathrm{p}_A(a_i|\lambda) - c_i^2\right| \mathrm{d}\mu_{\psi_d^S}(\lambda) = \int \left|\int \tilde{\mathrm{p}}_A(a_i|\tilde{\lambda})\gamma_{N,\epsilon}(\mathrm{d}\tilde{\lambda}|\lambda) - c_i^2\right| \mathrm{d}\mu_{\psi_d^S}(\lambda) \leq \int \left|\tilde{\mathrm{p}}_A(a_i|\tilde{\lambda}) - c_i^2\right| \mathrm{d}\mu_{\psi_{\mathrm{I}}}(\tilde{\lambda}). \tag{58}$$

So the remainder of the proof focuses entirely on the large system.

Local measurements on the total system The main estimate is obtained by chaining Bell inequalities for local measurements on the total system. The local measurements needed for the proof, however, can be conveniently defined in terms of the bases for $\mathcal{H}_A^\epsilon, \mathcal{H}_B^\epsilon, \mathcal{H}_A, \mathcal{H}_B$ alone. Consider measurements $M_{\overline{A}}, M_{\overline{B}}$ corresponding to the self-adjoint operators

$$\begin{aligned}
\overline{A} &:= \sum_{i=1}^{d} \sum_{j=1}^{n_i^\epsilon} a_{i,j} U^* P_{i,j}^A U, \text{ where } P_{i,j}^A := \Big(\mathbb{1}, \mathbb{1}|[f_j], \mathbb{1}|[e_i], \mathbb{1} \Big) , \\
\overline{B} &:= \sum_{i=1}^{d} \sum_{j=1}^{n_i^\epsilon} b_{i,j} V^* P_{i,j}^B V, \text{ where } P_{i,j}^B := \Big(\mathbb{1}, \mathbb{1}|\mathbb{1}, [f_j]|\mathbb{1}, [e_i] \Big) .
\end{aligned} \tag{59}$$

Here U and V are local unitary operators, i.e., U acts as the unit on the Hilbert spaces $\mathcal{H}_B^N, \mathcal{H}_B^\epsilon, \mathcal{H}_B$ and V acts as the unit on the Hilbert spaces $\mathcal{H}_A^N, \mathcal{H}_A^\epsilon, \mathcal{H}_A$. They are further assumed to have the following action on specific basis vectors:[21]

$$U \left|k, k|1, 1|i, i\right\rangle = \left|\left\lceil \tfrac{k}{n_i^\epsilon} \right\rceil, k\Big|j_{i,k}, 1\Big|i, i\right\rangle, \ V \left|k, k|1, 1|i, i\right\rangle = \left|k, \left\lceil \tfrac{k}{n_i^\epsilon} \right\rceil\Big|1, j_{i,k}\Big|i, i\right\rangle, \tag{60}$$

where

$$j_{i,k} := k - n_i^\epsilon \left(\left\lceil \tfrac{k}{n_i^\epsilon} \right\rceil - 1 \right) \tag{61}$$

takes on values in the range $\{1, 2, \ldots, n_i^\epsilon\}$. It is further convenient to introduce the state

$$\Psi_{\mathrm{F}} := UV\Psi_{\mathrm{I}} = C_N \sum_{k=1}^{D} \sum_{i=1}^{d} \frac{c_i}{\sqrt{k}} \left|\left\lceil \tfrac{k}{n_i^\epsilon} \right\rceil, \left\lceil \tfrac{k}{n_i^\epsilon} \right\rceil\Big|j_{i,k}, j_{i,k}\Big|i, i\right\rangle. \tag{62}$$

The theory of embezzlement states that this state becomes a better approximation of $\psi_D \otimes \psi_{d^\epsilon}$ as N becomes larger and ϵ becomes smaller. This is, on a formal level, what makes the proof work.

The next step is to make a further estimate picking up again from (58):

$$\begin{aligned}
\int \left|\tilde{\mathrm{p}}_A(a_i|\lambda) - c_i^2\right| \mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) &\leq \int \left|\tilde{\mathrm{p}}_A(a_i|\lambda) - \sum_{j=1}^{n_i^\epsilon} \mathrm{p}_{\overline{B}}(b_{i,j}|\lambda)\right| \mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) \\
&+ \int \left|\sum_{j=1}^{n_i^\epsilon} \mathrm{p}_{\overline{B}}(b_{i,j}|\lambda) - \frac{n_i^\epsilon}{d^\epsilon}\right| \mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) + \int \left|\frac{n_i^\epsilon}{d^\epsilon} - c_i^2\right| \mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda).
\end{aligned} \tag{63}$$

By (48) the final term is smaller than ϵ. The first term will be shown to equal zero. Making use of parameter independence and the fact that V is a local operator, the standard estimate gives

$$\begin{aligned}
\int &\left|\tilde{\mathrm{p}}_A(a_i|\lambda) - \sum_{j=1}^{n_i^\epsilon} \mathrm{p}_{\overline{B}}(b_{i,j}|\lambda)\right| \mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) \\
&\leq \sum_{\tilde{\imath}\neq i} \sum_{j=1}^{n_i^\epsilon} \int \mathrm{p}_{A,\overline{B}}(a_i, b_{\tilde{\imath},j}|\lambda) \mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) + \sum_{\tilde{\imath}'\neq i} \sum_{j=1}^{n_i^\epsilon} \int \mathrm{p}_{A,\overline{B}}(a_{\tilde{\imath}'}, b_{i,j}|\lambda) \mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) \\
&= \sum_{\tilde{\imath}\neq i} \sum_{j=1}^{n_i^\epsilon} \left\langle V\Psi_{\mathrm{I}}\right| \Big(\mathbb{1}, \mathbb{1}|\mathbb{1}, [f_j]|[e_i], [e_{\tilde{\imath}}] \Big) \left|V\Psi_{\mathrm{I}}\right\rangle + \sum_{\tilde{\imath}'\neq i} \sum_{j=1}^{n_i^\epsilon} \left\langle V\Psi_{\mathrm{I}}\right| \Big(\mathbb{1}, \mathbb{1}|\mathbb{1}, [f_j]|[e_{\tilde{\imath}'}], [e_i] \Big) \left|V\Psi_{\mathrm{I}}\right\rangle.
\end{aligned} \tag{64}$$

---

[21] Here $\lceil x \rceil$ denotes the smallest natural number n such that $n \geq x$.

Then each term in the final sum equals zero:

$$
\begin{aligned}
\Big\langle V\Psi_{\mathrm{I}}\Big| & \Big(\mathbb{1},\mathbb{1}\big|\mathbb{1},[f_l]\big|[e_m],[e_n]\Big)\Big|V\Psi_{\mathrm{I}}\Big\rangle \\
=& C_N^2 \sum_{k,\tilde{k}=1}^{D}\sum_{i,\tilde{i}=1}^{d}\sum_{j=1}^{n_i^\epsilon}\sum_{\tilde{j}=1}^{n_{\tilde{i}}^\epsilon}\frac{c_i c_{\tilde{i}}}{\sqrt{k\tilde{k}}}\,\langle g_k|g_{\tilde{k}}\rangle\,\Big\langle g_{\lceil\frac{k}{n_i^\epsilon}\rceil}\Big|g_{\lceil\frac{\tilde{k}}{n_{\tilde{i}}^\epsilon}\rceil}\Big\rangle\,\langle f_1|f_1\rangle\,\Big\langle f_{j_{i,k}}\Big|[f_l]\Big|f_{j_{\tilde{i},\tilde{k}}}\Big\rangle\,\langle e_i|[e_m]|e_{\tilde{i}}\rangle\,\langle e_i|[e_n]|e_{\tilde{i}}\rangle \\
=& 0,
\end{aligned}
\tag{65}
$$

where the final equality follows because $m\neq n$.

The conclusion of this step is then that

$$
\int\big|\tilde{\mathrm{p}}_A(a_i|\lambda)-c_i^2\big|\,\mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda)<\int\bigg|\sum_{j=1}^{n_i^\epsilon}\mathrm{p}_{\overline{B}}(b_{i,j}|\lambda)-\frac{n_i^\epsilon}{d^\epsilon}\bigg|\,\mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda)+\epsilon.
\tag{66}
$$

Applying chained Bell inequalities   The next step is to find an estimate for the second term on the right hand side of (63). This is where the strategy for the proof of Theorem 1 is copied. First note that

$$
\begin{aligned}
\int\bigg|\sum_{j=1}^{n_i^\epsilon}\mathrm{p}_{\overline{B}}(b_{i,j}|\lambda)-\frac{n_i^\epsilon}{d^\epsilon}\bigg|\,\mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda)\leq & \sum_{j=1}^{n_i^\epsilon}\int\bigg|\mathrm{p}_{\overline{B}}(b_{i,j}|\lambda)-\frac{1}{d^\epsilon}\bigg|\,\mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) \\
\leq & \frac{1}{d^\epsilon}\sum_{j=1}^{n_i^\epsilon}\sum_{\tilde{i}=1}^{d}\sum_{\tilde{j}=1}^{n_{\tilde{i}}^\epsilon}\int\big|\mathrm{p}_{\overline{B}}(b_{i,j}|\lambda)-\mathrm{p}_{\overline{B}}(b_{\tilde{i},\tilde{j}}|\lambda)\big|\,\mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda).
\end{aligned}
\tag{67}
$$

Next, for fixed values of $i,j,\tilde{i},\tilde{j}$ and for $\theta\in[0,\pi/2]$, two local unitary operators U_θ and V_θ are defined with the following actions:[22]

$$
\begin{aligned}
U_\theta\,|k_1,k_2|j,j_2|i,i_2\rangle &= \cos\theta\,|k_1,k_2|j,j_2|i,i_2\rangle+\sin\theta\,|k_1,k_2|\tilde{j},j_2|\tilde{i},i_2\rangle, \\
U_\theta\,|k_1,k_2|\tilde{j},j_2|\tilde{i},i_2\rangle &= \cos\theta\,|k_1,k_2|\tilde{j},j_2|\tilde{i},i_2\rangle-\sin\theta\,|k_1,k_2|j,j_2|i,i_2\rangle
\end{aligned}
\tag{68}
$$

and

$$
\begin{aligned}
V_\theta\,|k_1,k_2|j_1,j|i_1,i\rangle &= \cos\theta\,|k_1,k_2|j_1,j|i_1,i\rangle+\sin\theta\,|k_1,k_2|j_1,\tilde{j}|i_1,\tilde{i}\rangle, \\
V_\theta\,|k_1,k_2|j_1,\tilde{j}|i_1,\tilde{i}\rangle &= \cos\theta\,|k_1,k_2|j_1,\tilde{j}|i_1,\tilde{i}\rangle-\sin\theta\,|k_1,k_2|j_1,j|i_1,i\rangle,
\end{aligned}
\tag{69}
$$

for arbitrary values of k_1,k_2,j_1,j_2,i_1,i_2 and U_θ and V_θ act as the unit operator on all other basis vectors.

For given $i,j,\tilde{i},\tilde{j},\theta$, two local measurements $M_{\overline{A}_\theta}$ and $M_{\overline{B}_\theta}$ are introduced corresponding to the operators

$$
\overline{A}_\theta:=\sum_{\iota=1}^{d}\sum_{\eta=1}^{n_\iota^\epsilon}a_{\iota,\eta}U^*U_\theta^* P_{\iota,\eta}^A U_\theta U,\quad \overline{B}_\theta:=\sum_{\iota=1}^{d}\sum_{\eta=1}^{n_\iota^\epsilon}b_{\iota,\eta}V^*V_\theta^* P_{\iota,\eta}^B V_\theta V.
\tag{70}
$$

Without loss of generality, it may be assumed that $M_{\overline{B}_0}$ and $M_{\overline{B}}$ refer to the same measurement.

For any $L\in\mathbb{N}$ the angle from 0 to $\frac{\pi}{2}$ can be divided into $2L+1$ equal smaller angles so as to obtain the inequality

$$
\begin{aligned}
\int\big|\mathrm{p}_{\overline{B}}(b_{i,j}|\lambda)-\mathrm{p}_{\overline{B}}(b_{\tilde{i},\tilde{j}}|\lambda)\big|\,\mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda)\leq & \sum_{l=0}^{L}\int\Big|\mathrm{p}_{\overline{B}_{L,l}}(b_{i,j}|\lambda)-\mathrm{p}_{\overline{A}_{L,l}}(a_{i,j}|\lambda)\Big|\,\mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) \\
& +\sum_{l=0}^{L-1}\int\Big|\mathrm{p}_{\overline{A}_{L,l}}(a_{i,j}|\lambda)-\mathrm{p}_{\overline{B}_{L,l+1}}(b_{i,j}|\lambda)\Big|\,\mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) \\
& +\int\Big|\mathrm{p}_{\overline{A}_{L,L}}(a_{i,j}|\lambda)-\mathrm{p}_{\overline{B}}(b_{\tilde{i},\tilde{j}}|\lambda)\Big|\,\mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda),
\end{aligned}
\tag{71}
$$

where

$$
\overline{B}_{L,l}:=\overline{B}_{\frac{2l\pi}{(2L+1)2}}\quad\text{and}\quad \overline{A}_{L,l}:=\overline{A}_{\frac{(2l+1)\pi}{(2L+1)2}}.
\tag{72}
$$

---

[22]For notational convenience the variables $i,j,\tilde{i},\tilde{j}$ are suppressed in the expression for these unitary operators.

Evaluating each of these terms is somewhat tedious. Therefore the calculations to obtain the following results are transported to the next section. For the final term one simply has

$$\int \left| \mathrm{p}_{\overline{A}_{\frac{\pi}{2}}}(a_{i,j}|\lambda) - \mathrm{p}_{\overline{B}}(b_{\tilde{i},\tilde{j}}|\lambda) \right| \mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) = 0. \tag{73}$$

For the other terms one obtains the expression

$$\int \left| \mathrm{p}_{\overline{A}_{\theta}}(a_{i,j}|\lambda) - \mathrm{p}_{\overline{B}_{\phi}}(b_{i,j}|\lambda) \right| \mathrm{d}\mu_{\Psi_{\mathrm{I}}}(\lambda) \leq \frac{C_N^2}{2} \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \left(\frac{c_i^2(2\sin^2(\theta-\phi) + \sin 2\theta \sin 2\phi)}{k_{m,i,j}} - \frac{c_i c_{\tilde{i}} \sin 2\theta \sin 2\phi}{\sqrt{k_{m,i,j} k_{m,\tilde{i},\tilde{j}}}} \right)$$
$$+ \frac{C_N^2}{2} \sum_{m=1}^{\frac{D}{n_{\tilde{i}}^\epsilon}} \left(\frac{c_{\tilde{i}}^2(2\sin^2(\theta-\phi) + \sin 2\theta \sin 2\phi)}{k_{m,\tilde{i},\tilde{j}}} - \frac{c_i c_{\tilde{i}} \sin 2\theta \sin 2\phi}{\sqrt{k_{m,i,j} k_{m,\tilde{i},\tilde{j}}}} \right), \tag{74}$$

where

$$k_{m,i,j} := j + (m-1)n_i^\epsilon. \tag{75}$$

Obtaining the final estimate   The sum over the angles in (71) is now postponed to first perform the sums over j and $\tilde{j}$ from (67). For two of the four terms in (74) this summation easily yields a nice result:

$$\sum_{j=1}^{n_i^\epsilon} \sum_{\tilde{j}=1}^{n_{\tilde{i}}^\epsilon} \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{1}{k_{m,i,j}} = \sum_{\tilde{j}=1}^{n_{\tilde{i}}^\epsilon} \sum_{k=1}^{D} \frac{1}{k} = \frac{n_{\tilde{i}}^\epsilon}{C_N^2}, \quad \sum_{j=1}^{n_i^\epsilon} \sum_{\tilde{j}=1}^{n_{\tilde{i}}^\epsilon} \sum_{m=1}^{\frac{D}{n_{\tilde{i}}^\epsilon}} \frac{1}{k_{m,\tilde{i},\tilde{j}}} = \sum_{j=1}^{n_i^\epsilon} \sum_{k=1}^{D} \frac{1}{k} = \frac{n_i^\epsilon}{C_N^2}. \tag{76}$$

For the other two terms one has the estimate

$$\sum_{j=1}^{n_i^\epsilon} \sum_{\tilde{j}=1}^{n_{\tilde{i}}^\epsilon} \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{1}{\sqrt{k_{m,i,j} k_{m,\tilde{i},\tilde{j}}}} \geq \sum_{j=1}^{n_i^\epsilon} \sum_{\tilde{j}=1}^{n_{\tilde{i}}^\epsilon} \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{1}{\sqrt{mn_i^\epsilon mn_{\tilde{i}}^\epsilon}} = \sqrt{n_i^\epsilon n_{\tilde{i}}^\epsilon} \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{1}{m} = \sqrt{n_i^\epsilon n_{\tilde{i}}^\epsilon} \left(\frac{1}{C_N^2} - \sum_{m=\frac{D}{n_i^\epsilon}+1}^{D} \frac{1}{m} \right)$$
$$\geq \sqrt{n_i^\epsilon n_{\tilde{i}}^\epsilon} \left(\frac{1}{C_N^2} - \int_{\frac{D}{n_i^\epsilon}}^{D} \frac{1}{x} \mathrm{d}x \right) = \sqrt{n_i^\epsilon n_{\tilde{i}}^\epsilon} \left(\frac{1}{C_N^2} - \log(n_i^\epsilon) \right) \tag{77}$$

and similarly

$$\sum_{j=1}^{n_i^\epsilon} \sum_{\tilde{j}=1}^{n_{\tilde{i}}^\epsilon} \sum_{m=1}^{\frac{D}{n_{\tilde{i}}^\epsilon}} \frac{1}{\sqrt{k_{m,i,j} k_{m,\tilde{i},\tilde{j}}}} \geq \sqrt{n_i^\epsilon n_{\tilde{i}}^\epsilon} \left(\frac{1}{C_N^2} - \log(n_{\tilde{i}}^\epsilon) \right). \tag{78}$$

Combining these estimates with the expression (74) and making use of (48) gives

$$\sum_{j=1}^{n_i^\epsilon} \sum_{\tilde{j}=1}^{n_{\tilde{i}}^\epsilon} \int \left| \mathrm{p}_{\overline{A}_\theta}(a_{i,j}|\lambda) - \mathrm{p}_{\overline{B}_\phi}(b_{i,j}|\lambda) \right| \mathrm{d}\mu_{\Psi_I^\epsilon}(\lambda)$$
$$\leq \frac{c_i^2 n_{\tilde{i}}^\epsilon}{2}(2\sin^2(\theta-\phi) + \sin 2\theta \sin 2\phi) - \frac{c_i c_{\tilde{i}} \sqrt{n_i^\epsilon n_{\tilde{i}}^\epsilon}}{2} \left(1 - C_N^2 \log(n_i^\epsilon) \right) \sin 2\theta \sin 2\phi$$
$$+ \frac{c_{\tilde{i}}^2 n_i^\epsilon}{2}(2\sin^2(\theta-\phi) + \sin 2\theta \sin 2\phi) - \frac{c_i c_{\tilde{i}} \sqrt{n_i^\epsilon n_{\tilde{i}}^\epsilon}}{2} \left(1 - C_N^2 \log(n_{\tilde{i}}^\epsilon) \right) \sin 2\theta \sin 2\phi \tag{79}$$
$$< \frac{n_i^\epsilon n_{\tilde{i}}^\epsilon (1+\epsilon)}{d^\epsilon} \left(2\sin^2(\theta-\phi) + \sin 2\theta \sin 2\phi \right) - \frac{n_i^\epsilon n_{\tilde{i}}^\epsilon (1-\epsilon)}{d^\epsilon} \left(1 - \frac{1}{2} C_N^2 \log(n_i^\epsilon n_{\tilde{i}}^\epsilon) \right) \sin 2\theta \sin 2\phi$$
$$< 2n_{\tilde{i}}^\epsilon \sin^2(\theta-\phi) + 2n_{\tilde{i}}^\epsilon \epsilon + n_{\tilde{i}}^\epsilon C_N^2 \log(d^\epsilon).$$

Finally, combining this result with (66), (67) and (71), one obtains

$$\int \left| \tilde{\mathrm{p}}_A(a_i|\lambda) - c_i^2 \right| \mathrm{d}\mu_{\psi_{\mathrm{I}}}(\lambda) < \epsilon + \frac{1}{d^\epsilon} \sum_{\tilde{i}=1}^{d} (2L+1)n_{\tilde{i}}^\epsilon \left(2\sin^2\left(\frac{\pi}{2(2L+1)}\right) + 2\epsilon + C_N^2 \log(d^\epsilon) \right)$$
$$< \frac{\pi^2}{2(2L+1)} + (4L+3)\epsilon + \frac{(2L+1)\log(d^\epsilon)}{\log(1 + N \prod_{i=1}^{d} n_i^\epsilon)}. \tag{80}$$

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Now choose L such that $\frac{\pi^2}{2(2L+1)} < \frac{1}{3}\mathcal{E}$, then choose ϵ such that $(4L+3)\epsilon < \frac{1}{3}\mathcal{E}$, and finally choose N such that $\frac{(2L+1)\log(d^\epsilon)}{\log(1+N\prod_{i=1}^{d} n_i^\epsilon)} < \frac{1}{3}\mathcal{E}$. Plugging these choices into (80) one obtains (47).

$\square$

C   Some remaining calculations

Proof for (73)   The equality follows with the use of parameter independence to get an estimate in terms of quantum probabilities.

$$
\int \left| \mathrm{p}_{\overline{A}_{\frac{\pi}{2}}}(a_{i,j}|\lambda) - \mathrm{p}_{\overline{B}}(b_{\tilde{\imath},\tilde{\jmath}}|\lambda) \right| \mathrm{d}\mu_{\Psi_\mathrm{I}}(\lambda)
$$

$$
\leq \sum_{(r,s)\neq(\tilde{\imath},\tilde{\jmath})} \int \mathrm{p}_{\overline{A}_{\frac{\pi}{2}}\overline{B}}(a_{i,j}b_{r,s}|\lambda)\mathrm{d}\mu_{\Psi_\mathrm{I}}(\lambda) + \sum_{(r,s)\neq(i,j)} \int \mathrm{p}_{\overline{A}_{\frac{\pi}{2}}\overline{B}}(a_{r,s}b_{\tilde{\imath},\tilde{\jmath}}|\lambda)\mathrm{d}\mu_{\Psi_\mathrm{I}}(\lambda)
$$

$$
= \sum_{(r,s)\neq(\tilde{\imath},\tilde{\jmath})} \left\langle \Psi_\mathrm{F} \middle| U_{\frac{\pi}{2}}^* P_{i,j}^A U_{\frac{\pi}{2}} P_{r,s}^B \middle| \Psi_\mathrm{F} \right\rangle + \sum_{(r,s)\neq(i,j)} \left\langle \Psi_\mathrm{F} \middle| U_{\frac{\pi}{2}}^* P_{r,s}^A U_{\frac{\pi}{2}} P_{\tilde{\imath},\tilde{\jmath}}^B \middle| \Psi_\mathrm{F} \right\rangle \qquad (81)
$$

$$
= \sum_{(r,s)\neq(\tilde{\imath},\tilde{\jmath})} \left\langle \Psi_\mathrm{F} \middle| P_{\tilde{\imath},\tilde{\jmath}}^A P_{r,s}^B \middle| \Psi_\mathrm{F} \right\rangle + \sum_{(r,s)\neq(\tilde{\imath},\tilde{\jmath})} \left\langle \Psi_\mathrm{F} \middle| P_{r,s}^A P_{\tilde{\imath},\tilde{\jmath}}^B \middle| \Psi_\mathrm{F} \right\rangle = 0,
$$

where the final step follows because of the perfect correlations in Ψ_F.

Proof for (74)   The first step is again to make use of parameter independence to get an estimate in terms of quantum probabilities.

$$
\int \left| \mathrm{p}_{\overline{A}_\theta}(a_{i,j}|\lambda) - \mathrm{p}_{\overline{B}_\phi}(b_{i,j}|\lambda) \right| \mathrm{d}\mu_{\Psi_\mathrm{I}}(\lambda)
$$

$$
\leq \sum_{(r,s)\neq(i,j)} \int \mathrm{p}_{\overline{A}_\theta \overline{B}_\phi}(a_{i,j}b_{r,s}|\lambda)\mathrm{d}\mu_{\Psi_\mathrm{I}}(\lambda) + \sum_{(r,s)\neq(i,j)} \int \mathrm{p}_{\overline{A}_\theta \overline{B}_\phi}(a_{r,s}b_{i,j}|\lambda)\mathrm{d}\mu_{\Psi_\mathrm{I}}(\lambda)
$$

$$
= \sum_{(r,s)\neq(i,j)} \left\langle U_\theta V_\phi \Psi_\mathrm{F} \middle| P_{i,j}^A P_{r,s}^B \middle| U_\theta V_\phi \Psi_\mathrm{F} \right\rangle + \sum_{(r,s)\neq(i,j)} \left\langle U_\theta V_\phi \Psi_\mathrm{F} \middle| P_{r,s}^A P_{i,j}^B \middle| U_\theta V_\phi \Psi_\mathrm{F} \right\rangle. \qquad (82)
$$

The next step is to find a better expression for the first sum, and then use the similarity between the two sums to also obtain an expression for the second sum. For this look at the action of the projection operators:

$$
P_{i,j}^A U_\theta P_{r,s}^B V_\phi \Psi_\mathrm{F} = C_N \sum_{k=1}^{D} \sum_{\iota=1}^{d} \frac{c_\iota}{\sqrt{k}} P_{i,j}^A U_\theta P_{r,s}^B V_\phi \left| \left\lceil \frac{k}{n_\iota^\epsilon} \right\rceil, \left\lceil \frac{k}{n_\iota^\epsilon} \right\rceil \middle| j_{\iota,k}, j_{\iota,k} \middle| \iota, \iota \right\rangle. \qquad (83)
$$

The action of $P_{i,j}^A U_\theta$ on each component in the sum yields a non-zero result only if $(\iota, j_{\iota,k}) = (i, j)$ or $(\iota, j_{\iota,k}) = (\tilde{\imath}, \tilde{\jmath})$. For $\iota = i$, $j_{\iota,k}$ takes on the value j exactly $\frac{D}{n_i^\epsilon}$ times in the sum over k. On these occasions, k takes on the values $j, j+n_i^\epsilon, \ldots, j+(\frac{D}{n_i^\epsilon}-1)n_i^\epsilon$. For this reason the expression $k_{m,i,j} = j + (m-1)n_i^\epsilon$ is introduced. The case for $\iota = \tilde{\imath}$ is similar. Thus performing the sum over ι in (83) gives

$$
P_{i,j}^A U_\theta P_{r,s}^B V_\phi \Psi_\mathrm{F} = C_N \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{c_i \cos\theta}{\sqrt{k_{m,i,j}}} P_{r,s}^B V_\phi \left| m, m|j, j|i, i \right\rangle - C_N \sum_{m=1}^{\frac{D}{n_{\tilde{\imath}}^\epsilon}} \frac{c_{\tilde{\imath}} \sin\theta}{\sqrt{k_{m,\tilde{\imath},\tilde{\jmath}}}} P_{r,s}^B V_\phi \left| m, m|j, \tilde{\jmath}|i, \tilde{\imath} \right\rangle \qquad (84)
$$

Because of the action of the projection operator $P_{r,s}^B$, the sum over $(r,s)\neq(i,j)$ only picks up a term when $(r,s) = (\tilde{\imath},\tilde{\jmath})$. So

$$
\sum_{(r,s)\neq(i,j)} P_{i,j}^A U_\theta P_{r,s}^B V_\phi \Psi_\mathrm{F} = C_N \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{c_i \cos\theta}{\sqrt{k_{m,i,j}}} P_{\tilde{\imath},\tilde{\jmath}}^B V_\phi \left| m, m|j, j|i, i \right\rangle - C_N \sum_{m=1}^{\frac{D}{n_{\tilde{\imath}}^\epsilon}} \frac{c_{\tilde{\imath}} \sin\theta}{\sqrt{k_{m,\tilde{\imath},\tilde{\jmath}}}} P_{\tilde{\imath},\tilde{\jmath}}^B V_\phi \left| m, m|j, \tilde{\jmath}|i, \tilde{\imath} \right\rangle
$$

$$
= C_N \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{c_i \cos\theta \sin\phi}{\sqrt{k_{m,i,j}}} \left| m, m|j, \tilde{\jmath}|i, \tilde{\imath} \right\rangle - C_N \sum_{m=1}^{\frac{D}{n_{\tilde{\imath}}^\epsilon}} \frac{c_{\tilde{\imath}} \sin\theta \cos\phi}{\sqrt{k_{m,\tilde{\imath},\tilde{\jmath}}}} \left| m, m|j, \tilde{\jmath}|i, \tilde{\imath} \right\rangle. \qquad (85)
$$

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Bringing also the other rotations in (82) to the right gives

$$
\sum_{(r,s)\neq(i,j)} U_\theta^* P_{i,j}^A U_\theta V_\phi^* P_{r,s}^B V_\phi \Psi_F = C_N \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{c_i \cos\theta \sin\phi}{\sqrt{k_{m,i,j}}} \left(\cos\theta V_\phi^* \left|m,m|j,\tilde{\jmath}|i,\tilde{\imath}\right\rangle - \sin\theta V_\phi^* \left|m,m|\tilde{\jmath},\tilde{\jmath}|\tilde{\imath},\tilde{\imath}\right\rangle \right)
$$
$$
+ C_N \sum_{m=1}^{\frac{D}{n_{\tilde{\imath}}^\epsilon}} \frac{c_{\tilde{\imath}} \sin\theta \cos\phi}{\sqrt{k_{m,\tilde{\imath},\tilde{\jmath}}}} \left(\sin\theta V_\phi^* \left|m,m|\tilde{\jmath},\tilde{\jmath}|\tilde{\imath},\tilde{\imath}\right\rangle - \cos\theta V_\phi^* \left|m,m|j,\tilde{\jmath}|i,\tilde{\imath}\right\rangle \right). \tag{86}
$$

And finally

$$
\sum_{(r,s)\neq(i,j)} U_\theta^* P_{i,j}^A U_\theta V_\phi^* P_{r,s}^B V_\phi \Psi_F
$$
$$
= C_N \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{c_i \cos\theta \sin\phi}{\sqrt{k_{m,i,j}}} \left(\cos\theta\cos\phi \left|m,m|j,\tilde{\jmath}|i,\tilde{\imath}\right\rangle + \cos\theta\sin\phi \left|m,m|j,j|i,i\right\rangle \right.
$$
$$
\left. - \sin\theta\cos\phi \left|m,m|\tilde{\jmath},\tilde{\jmath}|\tilde{\imath},\tilde{\imath}\right\rangle - \sin\theta\sin\phi \left|m,m|\tilde{\jmath},j|\tilde{\imath},i\right\rangle \right)
$$
$$
+ C_N \sum_{m=1}^{\frac{D}{n_{\tilde{\imath}}^\epsilon}} \frac{c_{\tilde{\imath}} \sin\theta \cos\phi}{\sqrt{k_{m,\tilde{\imath},\tilde{\jmath}}}} \left(\sin\theta\cos\phi \left|m,m|\tilde{\jmath},\tilde{\jmath}|\tilde{\imath},\tilde{\imath}\right\rangle + \sin\theta\sin\phi \left|m,m|\tilde{\jmath},j|\tilde{\imath},i\right\rangle \right.
$$
$$
\left. - \cos\theta\cos\phi \left|m,m|j,\tilde{\jmath}|i,\tilde{\imath}\right\rangle - \cos\theta\sin\phi \left|m,m|j,j|i,i\right\rangle \right) \tag{87}
$$

Because of the perfect correlations in Ψ_F, only the four terms with i,i and $\tilde{\imath},\tilde{\imath}$ in the final expression contribute to the sum over expectation values in (82). This results in

$$
\sum_{(r,s)\neq(i,j)} \left\langle U_\theta V_\phi \Psi_F \left| P_{i,j}^A P_{r,s}^B \right| U_\theta V_\phi \Psi_F \right\rangle
$$
$$
= C_N^2 \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{c_i^2}{k_{m,i,j}} \cos^2\theta \sin^2\phi - C_N^2 \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{c_i c_{\tilde{\imath}}}{\sqrt{k_{m,i,j} k_{m,\tilde{\imath},\tilde{\jmath}}}} \cos\theta \sin\theta \cos\phi \sin\phi
$$
$$
+ C_N^2 \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{c_i^2}{k_{m,\tilde{\imath},\tilde{\jmath}}} \sin^2\theta \cos^2\phi - C_N^2 \sum_{m=1}^{\frac{D}{n_i^\epsilon}} \frac{c_i c_{\tilde{\imath}}}{\sqrt{k_{m,i,j} k_{m,\tilde{\imath},\tilde{\jmath}}}} \cos\theta \sin\theta \cos\phi \sin\phi. \tag{88}
$$

Making use of the trigonometric expression $\sin^2(\theta - \phi) = \cos^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi - 2\cos\theta\sin\theta\cos\phi\sin\phi$, the desired result (74) follows.

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