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Of matroid polytopes, chow rings and character polynomials

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Graduate Program in Mathematics
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Abstract

Matroids are combinatorial structures that capture various notions of independence. Recently there has been great interest in studying various matroid invariants. In this thesis, we study two such invariants: Volume of matroid base polytopes and the Tutte polynomial. We gave an approach to computing volume of matroid base polytopes using cyclic flats and apply it to the case of sparse paving matroids. For the Tutte polynomial, we recover (some of) its coefficients as degrees of certain forms in the Chow ring of underlying matroid. Lastly, we study the stability of characters of the symmetric group via character polynomials. We show a combinatorial identity in the ring of class functions that implies stability results for certain class of Kronecker coefficients.

Keywords: matroids, matroid base polytopes, chow rings, character polynomials.

Summary for lay audience

This thesis is mainly about a mathematical object called matroid. A matroid can be understood as an abstraction of a matrix (a list of vectors, read point), but without caring specifically about the coordinates of the vectors forming the matrix. What we care about is the dependency relations of these vectors, for example, which among these vectors are collinear, which of them are coplanar et cetera. To a matroid, we can associate a polytope that records that matroid. This is called the base polytope of the underlying matroid. Polytopes are generalizations of polygons to higher dimensions. Like the notion of area is associated to a polygon, the notion of volume is well-defined for a polytope. The problem we tackled in the second chapter is to have a combinatorial approach to computing the volume of the base polytope of an arbitrary matroid. We outlined the previously known methods and contrast them with ours. In the third chapter, we study another invariant of a matroid. Like polytopes, we can also associate a fan to a matroid, called Bergman fan. This fan remembers some properties of the matroid. A famous invariant is the Tutte polynomial of the matroid. We showed that the Bergman fan when decorated with certain weights remembers the Tutte polynomial of the matroid. The fourth chapter is about permutations. We study special functions on the set of permutations on n letters. These functions are of importance in representation theory of symmetric group. We discovered and proved a relation between alternating sums of two classes of such functions. We developed combinatorics of tilings of Young diagrams for its proof. The relation simplifies certain calculations and help us reprove previously known results.

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Lastly and most importantly, I am very lucky and grateful to have such an understanding, supportive and loving family. I owe all of it to them. I dedicate this thesis to my mother and father, who have taught me a lot of meta-lessons. Without their support and encouragement, I would not be able to make it through. My siblings: Affawn, Aisha, Hamza and Hareem were sources of wisdom, humor, perspective and joy.

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Chapter 1

Introduction

This thesis consists of three projects: volumes of matroid polytopes, Chow ring calculations and character polynomials. Each of these projects have a chapter dedicated to it. In this chapter, we provide an overview of the results attained in this thesis. We first state the problems that we are addressing in this thesis and then the results that provide an answer to each one of them.

1.1 Problem statements

There are three problems that we address in this thesis:

1. Does having a universal valutive invariant over space of all matroids simplify volume calculations of matroid base polytopes?
2. Can the coefficients of the Tutte polynomial of a matroid be recovered as degrees of some forms in its Chow ring?
3. Knowing the stable behaviour of characters of the symmetric group, can we compute the stable Kronecker coefficients (at least) for certain shapes?

1.2 Brief overview of results

For each of the problems stated above, we describe the results of this thesis below:

In the second chapter of this thesis, we used Derksen-Fink valutive invariant [DF10] to find volume of any matroid polytope from the lattice of cyclic flats of its respective matroid. In many cases, this approach is more efficient than the previously known formula of Ardila, Benedetti and Doker [ABD10]. As an application, we give a closed formula for the volumes of connected sparse paving matroid polytopes:

Theorem 1.2.1. *Let M_α be a connected sparse paving matroid of rank r with α nontrivial hyperplanes. Let $P(M_\alpha)$ denote the corresponding matroid base polytope. Then*

$$\text{Vol}(P(M_\alpha)) = \frac{1}{(n-1)!} \left(A_{n-1,r-1} - \alpha \binom{n-2}{r-1} \right)$$

where $A_{n-1,r-1}$ is the Eulerian number that counts the number of permutations of $[n-1]$ with $r-1$ descents.

The construction of the Chow ring of a matroid \mathbf{M} is due to Yuzvinsky and Feichtner [FY04a]. Adiprasito, Huh and Katz in [AHK17] showed that one can recover the coefficients of the reduced characteristic polynomial by computing degrees of certain forms. This was crucial to their resolution of Heron-Rota-Welsh conjecture for all matroids. Using the recent Gioan-Las Vergnas [GL18] expression for Tutte polynomial, in the third chapter we showed that its coefficients can also be recovered as degrees of certain forms in the Chow ring:

Theorem 1.2.2. *Let \mathbf{M} be a matroid of rank $r = d + 1$ with Tutte polynomial*

$$t(\mathbf{M}; x, y) = \sum_{i,j} t_{i,j} x^i y^j$$

then for $k = 0, 1, \dots, r - 1$

$$\deg(\alpha^k \cdot \text{csm}^{d-k}(\mathbf{M})) = t_{k+1,0}$$

where $\alpha = \sum_{F \ni i} x_F \in A^1(\mathbf{M})$.

This provides a new proof of Theorem 5.8 of [LRS17].

Lastly, in the fourth chapter we define certain class functions over the symmetric group \mathfrak{S}_n , that counts brick tilings of Young diagrams and are generalizations of permutation character and sign character. Using homology of posets over \mathbb{Z}_2 , we showed that certain alternating sums are equal as class functions:

Theorem 1.2.3. *For positive integers $1 \leq k < n$ and the class functions $\zeta^\lambda, \eta^\lambda : \mathfrak{S}_n \rightarrow \mathbb{N}$ for shape λ , defined as*

$\zeta^\lambda(w) :=$ the number of ordered brick tilings of λ by w

$\eta^\lambda(w) :=$ the number of unordered crackless brick tilings of λ by w

we have

$$\sum_{\mu \vdash k} (-1)^{\ell(\mu)} \zeta^\mu(w) = \sum_{\lambda \vdash k} (-1)^{\ell(\lambda)} \eta^\lambda(w)$$

for all permutations $w \in \mathfrak{S}_n$. The sum on the left is over compositions, while the sum on the right is over partitions of k .

This is then used to prove Rosas' formula [Ros00] for generating function of certain stable Kronecker coefficients.

Chapter 2

Matroid polytope volumes

Ardila, Benedetti and Doker [ABD10] showed that the matroid polytopes are examples of generalized permutohedra. Employing the work of Postnikov [Pos09], they gave an expression of volume of a matroid polytope, in terms of a sum of products of Crapo's beta invariants of certain contractions of the given matroid. Derksen and Fink [DF10] have defined a universal valuative invariant for matroids. Each valuative invariant of a matroid is a specialization of its Derksen-Fink invariant. Recently, motivated by tropical geometry, Hampe [Ham17] invented the notion of matroid intersection ring and showed that it is generated additively by Schubert matroids. More importantly, he showed that the Derksen-Fink invariant is a \mathbb{Z} -module homomorphism from this ring. Using this, along with the Bidkhori-Sullivant formula [Bid12] for the volume of lattice path matroid polytopes, we gave a new approach to find volume of any matroid polytope from lattice of cyclic flats of its underlying matroid. As an application, we give a closed formula for the volumes of connected sparse paving matroid polytopes.

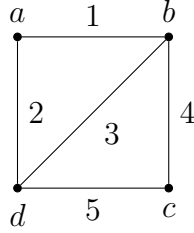
2.1 Introduction

This section covers some basics of matroid theory, and establishes some terminology that persists throughout the thesis. We keep this presentation minimal, in the sense that only those constructs being considered will play some role in the sections to follow.

2.1.1 Matroids

Matroids are combinatorial abstractions of different notions of independence in mathematics. This includes linear independence and algebraic independence. Here we give some basic definitions. To get more details on matroid theory, we suggest the standard reference in this subject [Oxl11] to our readers. There are many equivalent definitions of a matroid and each one of them comes with its own advantages and sophistications. We will start with the basis definition of a matroid:

Definition 2.1.1. *A matroid \mathbf{M} on an underlying set E is a nonempty collection $\mathcal{B} \subseteq 2^E$ of subsets of E , called bases of \mathbf{M} , which satisfies the basis exchange property: for every $A, B \in \mathcal{B}$, and $a \in A \setminus B$, there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.*

Figure 2.1: The graph G_{\square}

In reference to the matroid \mathbf{M} , the set E is called the *ground set* of \mathbf{M} and $\mathcal{B} := \mathcal{B}(\mathbf{M})$ is called the *set of bases* of \mathbf{M} . We denote this by writing $\mathbf{M} = (E, \mathcal{B})$. The operation of deleting an element of $A \setminus B$ from the basis A , and adjoining an element of $B \setminus A$ to A is called *basis exchange*. Two bases that are related by a basis exchange are called *adjacent bases*. This relation is symmetric and therefore gives rise to a graph called the *basis exchange graph* $G(\mathbf{M})$ of \mathbf{M} , where the vertices are bases and the edges are only between adjacent bases.

Example 2.1.2. Let n be a positive integer and $0 \leq r \leq n$. Then all of the size r subsets of $[n] = \{1, 2, \dots, n\}$ are the bases of a matroid on $[n]$, called the uniform matroid $\mathbf{U}_{r,n}$. The basis exchange graph $G(\mathbf{U}_{r,n})$ in this case consists of complete graph $K_{\binom{n}{r}}$ on vertex set $V = \binom{[n]}{r}$. In particular, $\mathbf{U}_{0,n}$ is the matroid on $[n]$, with the only basis \emptyset , and $\mathbf{U}_{n,n}$ is the matroid on $[n]$, with the only basis $[n]$.

Matroids occur naturally in many other areas of mathematics, for example graph theory, algebraic geometry, transversal theory and optimization theory. These areas also serve as rich sources of matroids. We here consider the one coming from graph theory. Given a finite multigraph $G = (V, E)$ on a vertex set V with edge multiset E , it is a result of Whitney [Whi32] that the set of spanning trees of G forms the collection of bases of a matroid on E . Matroids arising from this construction are called *graphic matroids*.

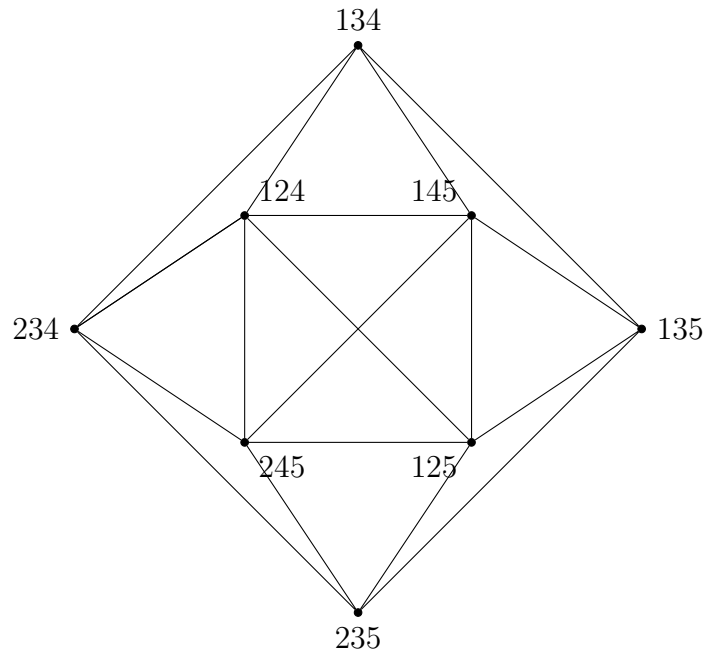
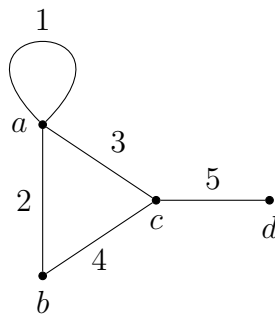
Example 2.1.3. Consider the graph G_{\square} shown in Figure 2.1 on the vertex set $V = \{a, b, c, d\}$ with edge set $\{1, 2, 3, 4, 5\}$. The spanning trees of this graph form bases of the graphic matroid $\mathbf{M}_{\square} := \mathbf{M}(G_{\square})$ associated to G_{\square} . The bases of the matroid \mathbf{M}_{\square} are given by

$$\mathcal{B}(\mathbf{M}_{\square}) = \{\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \\ \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}\}$$

The basis exchange graph for \mathbf{M} is shown in Figure 2.2. For the sake of brevity, we will juxtapose elements of the bases and remove the brackets from each basis. That is, we write

$$\mathcal{B}(\mathbf{M}_{\square}) = \{124, 125, 134, 135, 145, 234, 235, 245\}$$

As long as there is no confusion, we will continue to use this shorthand.

Figure 2.2: The basis exchange graph $G(M_{\square})$ Figure 2.3: The graph G_{01101}

Example 2.1.4. Consider the graph G_{01101} shown in Figure 2.3 on vertex set $V = \{a, b, c, d\}$ with the edge set $\{1, 2, 3, 4, 5\}$. The importance of binary subscript we used for denoting the graph will become evident in later sections. The spanning trees of this graph form bases of the graphic matroid $\mathbf{M}_{01101} := \mathbf{M}(G_{01101})$ associated to G_{01101} . These are given by

$$\mathcal{B}(\mathbf{M}_{01101}) = \{235, 245, 345\}$$

Note that the element 1 is in none of the bases and the element 5 is in each of the bases.

For a matroid $\mathbf{M} = (E, \mathcal{B})$, an element $e \in E$ which is contained in each basis $B \in \mathcal{B}(\mathbf{M})$ is called a *coloop*, and an element $f \in E$ which is not in any of the bases is called a *loop*. So each element in $\mathbf{U}_{0,n}$ is a loop and each element in $\mathbf{U}_{n,n}$ is a coloop. A matroid is called *loopless* if it does not contain any loop.

Whitney in [Whi32] also showed that given a set E of columns (read vectors) of a matrix A with coefficients in a field \mathbb{F} , the set of inclusion-maximal linearly independent subsets of vectors in E forms a collection of bases of a matroid on E . For notational purposes, instead of taking E to be the set of columns, we can take it to be the indexing set of columns and the basis to be the set of indices corresponding to inclusion-maximal linearly independent subsets of vectors. We denote such a matroid by $\mathbf{M}(A)$. These matroids are called *vector matroids*.

Example 2.1.5. Consider the matroid $\mathbf{M}_A := \mathbf{M}(A)$ on the indexing set $E = \{1, 2, 3, 4, 5\}$ obtained from the following matrix with coefficients in \mathbb{Q}

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The collection of bases for this matroid is given by

$$\mathcal{B}(\mathbf{M}_A) = \{124, 125, 134, 135, 145, 234, 235, 245\}$$

This is the same matroid as in Example 2.1.3. Since \mathbf{M}_{\square} is now being realized as a vector matroid. We call \mathbf{M}_{\square} representable over \mathbb{Q} .

Two matroids $\mathbf{M}_1 = (E_1, \mathcal{B}_1)$ and $\mathbf{M}_2 = (E_2, \mathcal{B}_2)$ are called *isomorphic* if there is a bijection $\varphi : E_1 \rightarrow E_2$ that induces a bijection $\varphi_* : \mathcal{B}_1 \rightarrow \mathcal{B}_2$. We denote this by $\mathbf{M}_1 \cong \mathbf{M}_2$.

Example 2.1.6. Let us take the matroid $\mathbf{M}_A := \mathbf{M}(A)$ for

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

over \mathbb{Q} . The collection of bases for this matroid is

$$\mathcal{B}(\mathbf{M}_A) = \{12, 23, 13\}$$

Note that this is precisely the uniform matroid $\mathbf{U}_{2,3}$.

A subset $I \subseteq E$ is called *independent* if $I \subseteq B$ for some $B \in \mathcal{B}(\mathbf{M})$. We denote the collection of independent sets of a matroid \mathbf{M} by $\mathcal{I}(\mathbf{M})$. Matroids can also be defined in terms of axioms on $\mathcal{I}(\mathbf{M})$. We will not delve into it. Any subset of E that is not an independent set is called *dependent*, and minimal dependent sets are called *circuits*. Let \mathbf{M} be a matroid on the ground set E . The collection of independent sets induces the *rank function* $\rho_{\mathbf{M}}$ on the power set 2^E , as follows:

$$\begin{aligned} \rho_{\mathbf{M}} : 2^E &\longrightarrow \mathbb{N} \\ \rho_{\mathbf{M}}(S) &:= \max_{\substack{I \in \mathcal{I} \\ I \subseteq S}} |I| \end{aligned}$$

In other words, $\rho_{\mathbf{M}}(S)$ is the size of an(y) inclusion-maximal independent set contained in S . A matroid \mathbf{M} is said to be of *rank* r if the rank of the underlying set is r . We denote this by $r(\mathbf{M})$.

The operation of basis exchange also induces the following equivalence relation on the elements of E : We write $a \sim b$ if either $a = b$ or there exists bases $A, B \in \mathcal{B}(\mathbf{M})$ such that $(A \setminus \{a\}) \cup \{b\} = B$. In this case, we call a, b *exchangeable*. The equivalence classes with respect to this relation are called the *connected components* of \mathbf{M} . The number of connected components of \mathbf{M} is denoted by $c(\mathbf{M})$, and we call \mathbf{M} *connected* if $c(\mathbf{M}) = 1$.

There are many ways to make new matroids out of old ones. Given a matroid $\mathbf{M} = (E, \mathcal{B})$, we see that if the nonempty family \mathcal{B} satisfies the basis exchange property, then so does the (nonempty) family

$$\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$$

The matroid $\mathbf{M}^* := (E, \mathcal{B}^*)$ is called the *dual* of \mathbf{M} . Given a subset $S \subseteq E$, let J be the maximal independent set contained in S . Then

1. the *deletion* $\mathbf{M} \setminus S$ is the matroid on the ground set $E \setminus S$, where the collection of independent sets is given by

$$\left\{ I \subseteq E \setminus S : I \in \mathcal{I}(\mathbf{M}) \right\} \quad (2.1)$$

2. the *contraction* \mathbf{M} / S is defined as the matroid on the ground set $E \setminus S$, where the collection of independent sets is given by

$$\left\{ I \subseteq E \setminus S : I \cup J \in \mathcal{I}(\mathbf{M}) \right\} \quad (2.2)$$

3. the *restriction* $\mathbf{M} | S$ is defined as the matroid on the ground set S , where the collection of independent sets is given by

$$\left\{ I \subseteq S : I \in \mathcal{I}(\mathbf{M}) \right\} \quad (2.3)$$

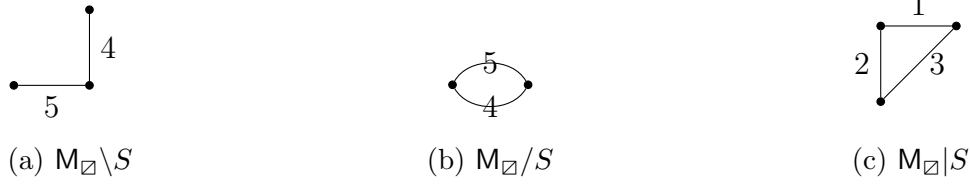


Figure 2.4: Deletion, Contraction and Restriction

If a matroid N is constructed by a sequence of deletion and contraction from a given matroid M , then N is called a *minor* of M . Minors play a major role in the classification of matroids. If M is graphical then these operations corresponds to the usual deletion and contraction of the edges, respectively.

Example 2.1.7. Let us take M_{\square} and take $S = \{1, 2, 3\}$, then Figure 2.4 show the deletion $M_{\square} \setminus S$, the contraction M_{\square} / S and the restriction $M_{\square} | S$, as graphical matroids. Note that

$$M_{\square} \setminus S \cong U_{1,2} \quad (2.4)$$

$$M_{\square} / S \cong U_{1,2} \quad (2.5)$$

$$M_{\square} | S \cong U_{2,3} \quad (2.6)$$

2.1.2 Lattice of flats

The rank function of a matroid M on E induces a closure operator $\text{cl} : 2^E \rightarrow 2^E$ given by

$$\text{cl}(S) = \{x \in E : \rho_M(S) = \rho_M(S \cup x)\}$$

for each $S \subseteq E$. We call cl the *closure operator* of M . A subset $F \subseteq E$ is called a *flat* if it is closed under this closure operator i.e. $\text{cl}(F) = F$. The closure of the empty set is precisely the set of loops of M , and the closure of the ground set is itself. If two elements $e, e' \in E$ belong to each other's closure, that is,

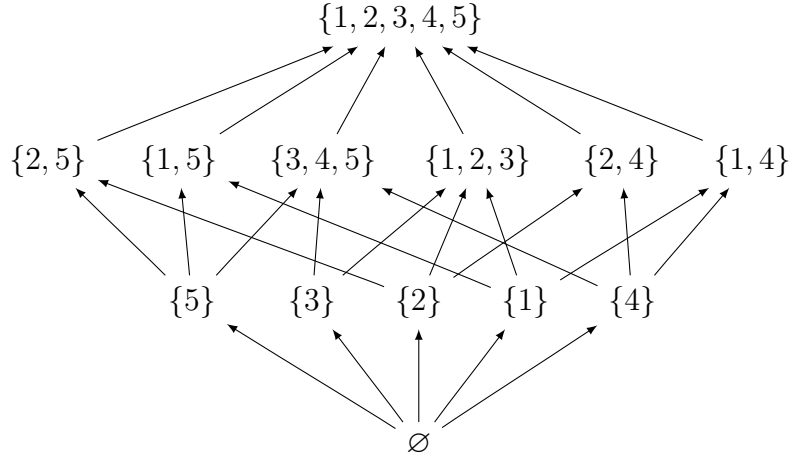
$$e \in \text{cl}(\{e'\}) \text{ and } e' \in \text{cl}(\{e\}) \quad (2.7)$$

then we call e, e' *parallel*. A matroid can equivalently be defined in terms of its flats. The set of flats $L(M)$ of a matroid M is a partially ordered set under set-inclusion. Furthermore, for every $F, G \in L(M)$, we have both a *meet* $F \wedge G$, and a *join* $F \vee G$ given by

$$F \wedge G = F \cap G \quad (2.8)$$

$$F \vee G = \text{cl}(F \cup G) \quad (2.9)$$

Therefore, $L(M)$ is a lattice, and we call it the *lattice of flats* of M . The top element $\hat{1}_{L(M)}$ is given by the underlying set E , and the bottom $\hat{0}_{L(M)}$ is given by the closure of the empty set $\text{cl}(\emptyset)$. In addition, $L(M)$ is a graded lattice where the rank function is

Figure 2.5: Lattice of flats of M_{\square}

given by the rank function of M . Note that the rank function ρ satisfies the semimodular inequality i.e. for $F, G \in L(M)$

$$\rho(F \vee G) + \rho(F \wedge G) \leq \rho(F) + \rho(G)$$

Hence $L(M)$ is *semimodular*. Recall that *atoms* of a lattice are elements covering the bottom $\hat{0}$. Each flat in $L(M)$ is generated as a join of atoms, hence $L(M)$ is *atomic*. Lattices that are atomic and semimodular are called *geometric lattices*. Hence, $L(M)$ is a geometric lattice. It is a theorem due to Birkhoff [Dil90] that every geometric lattice is isomorphic to the lattice of flats of some loopless matroid.

Example 2.1.8. For the uniform matroid $U_{r,n}$ of rank r over a set E of size n , each set of size k where $k < r$ is a flat of rank k , and there is a unique flat of rank r , namely E . Therefore, the lattice of flats of $U_{r,n}$ is a truncated boolean lattice at level $r - 1$, adjoined by E at the top.

Example 2.1.9. For the matroid M_{\square} from Example 2.1.3, the lattice of flats is given in Figure 2.5

There are some natural constructions on matroids. We mention one here. Given two matroids $M_1 = (E_1, \mathcal{B}_1)$, $M_2 = (E_2, \mathcal{B}_2)$, the *direct sum* $M_1 \oplus M_2$ of M_1 and M_2 is defined as a matroid on the disjoint union $E_1 \sqcup E_2$ with bases $\mathcal{B}(M_1 \oplus M_2) = \mathcal{B}_1 \times \mathcal{B}_2$. The lattice of flats $L(M_1 \oplus M_2)$ is isomorphic to $L(M_1) \times L(M_2)$. In general, a matroid M is a direct sum of its connected components. This highlights the importance of determining whether a given matroid is connected or not. Crapo in [Cra67] came up with a criterion to do exactly that in terms of the following invariant.

Definition 2.1.10. The beta invariant $\beta(M)$ of a matroid M is defined as

$$\beta(M) = (-1)^{r(M)} \sum_{X \subseteq E} (-1)^{|X|} \rho_M(X) \quad (2.10)$$

Crapo showed in [Cra67] that $\beta(\mathbf{M})$ is always a non-negative integer and it is positive if and only if $\mathbf{M} = \mathbf{M}(E)$ is connected, provided that the $|E| \geq 2$ (*Crapo's connectivity criterion*). Note that $\beta(\mathbf{M}_{\square}) = 1$, and hence it is connected. On the other hand, the beta invariant $\beta(\mathbf{U}_{n,n}) = 0$ for $n > 1$, and $\beta(\mathbf{M}_{01101}) = 0$, since $\mathbf{U}_{n,n} \cong \mathbf{U}_{1,1}^{\oplus n}$ and $\mathbf{M}_{01101} \cong \mathbf{U}_{0,1} \oplus \mathbf{U}_{2,3} \oplus \mathbf{U}_{1,1}$.

2.1.3 Lattice of cyclic flats

The rank function of a matroid also induces another operator, called the *cyclic operator* $\text{cyc} : 2^E \rightarrow 2^E$ given by

$$\text{cyc}(S) := \{x \in S : \rho_{\mathbf{M}}(S \setminus x) = \rho_{\mathbf{M}}(S)\} \quad (2.11)$$

$$= \bigcup_{\substack{\gamma \text{ circuit of } \mathbf{M} \\ \gamma \subseteq F}} \gamma \quad (2.12)$$

A subset S of E is called *cyclic* if $\text{cyc}(S) = S$, i.e. if it is the union of circuits contained in it. A *cyclic flat* is a flat that is also a cyclic set. The set of all loops of \mathbf{M} , i.e. $\text{cl}(\emptyset)$, is a cyclic flat. If the matroid \mathbf{M} is connected then the ground set E is a cyclic flat. The set of cyclic flats, denoted $\mathcal{Z}(\mathbf{M})$, forms a partially ordered set under set-inclusion. In addition, for every $F, G \in \mathcal{Z}(\mathbf{M})$, we have both a *meet* $F \wedge G$ and a *join* $F \vee G$, given by

$$F \wedge G = \bigcup_{\substack{\gamma \text{ circuit of } \mathbf{M} \\ \gamma \subseteq F \cap G}} \gamma \quad (2.13)$$

$$F \vee G = \text{cl}(F \cup G) \quad (2.14)$$

Brylawski [Bry75] showed that the lattice $\mathcal{Z}(\mathbf{M})$ together with its rank function, determines \mathbf{M} . It is shown by Bonin and De Mier [BM08] that every finite lattice is a lattice of cyclic flats of some bitransversal matroid. This means that the lattice of cyclic flats of a matroid is not restricted to a structural subclass of finite lattices. Given a collection \mathcal{Z} of subsets of a ground set E , and a rank function ρ on \mathcal{Z} , Bonin and De Mier also gave an axiomatic scheme to determine when \mathcal{Z} is the lattice of cyclic flats of a matroid, along with the given rank function ρ . As an example, we show the lattice of flats of $\mathbf{M}(K_4)$ in Figure 2.6, and the lattice of cyclic flats of $\mathbf{M}(K_4)$ in Figure 2.7.

2.2 Matroid polytopes

After going through some basics of matroid theory, we can define the fundamental object of this chapter, the matroid (base) polytope. The vertex picture of a matroid polytope is given by the bases, and the hyperplane picture (with some redundancies) is given by the flats of the underlying matroid. Feichtner and Sturmfels [FS05] gave a description of matroid polytopes as irredundant intersections of half spaces. The facet-defining hyperplanes, in this case, correspond to a special class of flats. Matroid polytopes were also realised by Ardila, Benedetti and Doker [ABD10] as generalized permutohedra. This

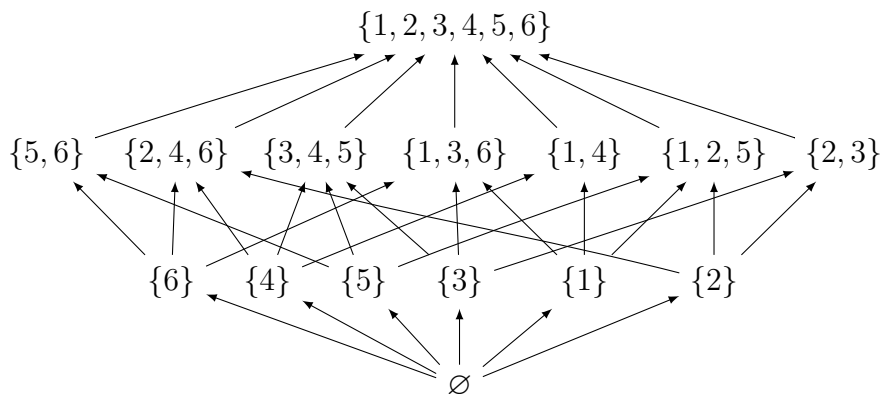


Figure 2.6: Lattice of flats of $M(K_4)$

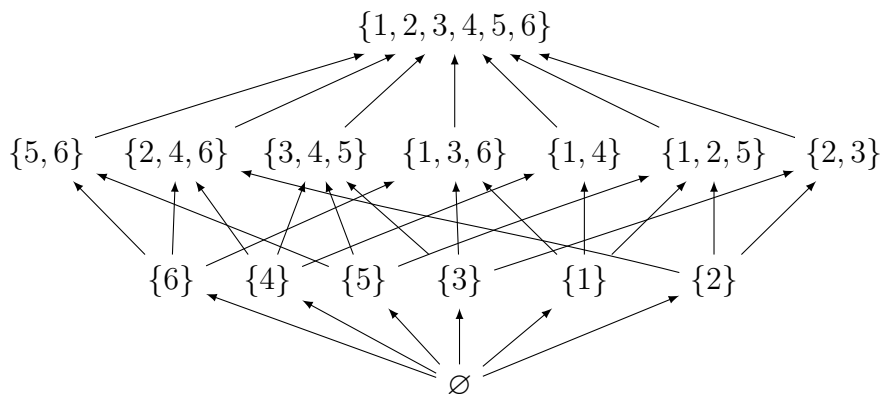
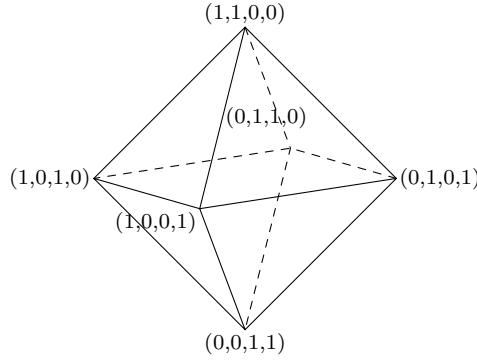


Figure 2.7: Lattice of cyclic flats of $M(K_4)$.

Figure 2.8: The hypersimplex $\Delta_{2,4} = P(\mathbf{U}_{2,4})$

realisation is used to derive their volume from Postnikov's theory of generalized permutohedra [Pos09]. Derksen defined a valuative invariant on matroids [Der09] and conjectured it to be universal. This was proved subsequently by Derksen and Fink in [DF10].

2.2.1 Matroid base polytopes

Let $\mathbf{M} = (E, \mathcal{B})$ be a matroid of rank r , and let \mathbf{e}_i denote the standard basis vector in \mathbb{R}^E . For every base $B \in \mathcal{B}$, the indicator vector $\mathbf{e}_B \in \mathbb{R}^E$ is defined to be

$$\mathbf{e}_B = \sum_{i \in B} \mathbf{e}_i$$

Definition 2.2.1. *The matroid base polytope of \mathbf{M} is defined as the convex hull of incidence vectors of bases of \mathbf{M} , that is,*

$$P(\mathbf{M}) = \text{conv}\{\mathbf{e}_B : B \in \mathcal{B}\} \subseteq \mathbb{R}^E$$

Before going further, let us look at a classical example.

Example 2.2.2. *Consider the uniform matroid $\mathbf{U}_{2,4}$ on $[4] = \{1, 2, 3, 4\}$, then*

$$P(\mathbf{U}_{2,4}) = \text{conv}\{(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), \\ (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1)\}$$

This gives us a regular octahedron embedded in \mathbb{R}^4 as shown in Figure 2.8.

Feichtner and Sturmfels have studied matroid polytopes in [FS05]. They have determined the dimension and a combinatorial description of these polytopes as polyhedra.

Theorem 2.2.3. [FS05] *Let \mathbf{M} be a rank r matroid on the ground set E , then*

$$P(\mathbf{M}) = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^E : \sum_{i \in F} x_i \leq \rho_{\mathbf{M}}(F) \text{ for all flats } F \subseteq E \right\}$$

and $\dim(P(\mathbf{M})) = |E| - c(\mathbf{M})$, where $c(\mathbf{M})$ denotes the number of connected components of \mathbf{M} .

In the above description, each flat of our matroid gives rise to a linear inequality. Some of these linear inequalities might be redundant. One may ask for an irredundant description of $P(M)$, i.e. which flats give rise to facet-determining hyperplanes. Feichtner and Sturmfels have determined them as flats F such that $M|F$ and M/F are both connected.

Example 2.2.4. *Given positive integers $0 < r < n$, the hypersimplex $\Delta_{r,n}$ is defined as the matroid polytope of the uniform matroid $U_{r,n}$. It is classically known (see for example Exercise 4.59(b) in [Sta12]) and can be found in the work of Laplace that*

$$\text{Vol}(\Delta_{r,n}) = \frac{A_{n-1,r-1}}{(n-1)!}$$

where $A_{n-1,r-1}$ are Eulerian numbers, counting the number of permutations $w \in \mathfrak{S}_{n-1}$ with $r-1$ descents.

Gelfand, Goresky, MacPherson and Serganova studied matroid polytopes in their paper [Gel+87]. They proved the following fundamental result:

Theorem 2.2.5. *A convex polytope $P \subseteq \Delta_{r,n}$ with vertices in $\{0,1\}^n$ is the matroid polytope of a matroid of rank r on the ground set E if and only if every edge of P is parallel to $e_i - e_j$ for some $i, j \in E$, $i \neq j$.*

Furthermore, they showed that two vertices e_{B_1} and e_{B_2} for $B_1, B_2 \in \mathcal{B}(M)$ are adjacent if and only if $e_{B_1} - e_{B_2} = e_i - e_j$ for some $i, j \in E$. This implies that the edges of $P(M)$ correspond to basis exchanges in M . Therefore, the vertex-edge graph of $P(M)$ is precisely the basis exchange graph $G(M)$ of the matroid M .

Another implication of the above is that the faces of matroid polytopes are again matroid polytopes. Along these lines, the facial structure of matroid polytopes has been described combinatorially by Ardila and Klivans in [AK06].

This result also implies that the class of matroid polytopes is a subclass of generalized permutohedra. We will see the definition of a generalized permutohedron in the following subsection.

2.2.2 Generalized permutohedra

Recall that given two convex polytopes P and Q in \mathbb{R}^n , the *Minkowski sum* $P + Q$ of P and Q is defined as the set

$$P + Q = \{p + q : p \in P, q \in Q\} \subseteq \mathbb{R}^n$$

We say that P is a *Minkowski summand* of R if there is a polytope Q such that $P + Q = R$. In that case, we call Q the *Minkowski difference* of R and P , and we write $Q = R - P$. A convex polytope P in \mathbb{R}^n is said to be a generalized permutohedron if it satisfies one of the following equivalent conditions (see [Zie95a])

1. Every edge of P is parallel to $e_i - e_j$ for some $1 \leq i < j \leq n$.

2. The normal fan of P in $(\mathbb{R}^n)^*$ is refined by the braid arrangement.
3. The polytope P is a Minkowski summand of a permutohedron.

From Theorem 2.2.5, we see that a matroid polytope $P(\mathbf{M})$ is a generalized permutohedron. Postnikov in [Pos09] showed that every generalized permutohedron can be written uniquely as a signed Minkowski sum of simplices. Set the *signed beta invariant* of \mathbf{M} as

$$\tilde{\beta}(\mathbf{M}) = (-1)^{r(\mathbf{M})+1} \beta(\mathbf{M})$$

Ardila, Benedetti and Doker in [ABD10] gave the following signed Minkowski sum decomposition of $P(\mathbf{M})$

$$P(\mathbf{M}) = \sum_{A \subseteq E} \tilde{\beta}(\mathbf{M}/A) \Delta_{E \setminus A} \quad (2.15)$$

Using the theory developed by Postnikov [Pos09], they have given an elegant formula for volume of the matroid polytope $P(\mathbf{M})$.

Theorem 2.2.6. [ABD10] *Let \mathbf{M} be a connected matroid on a ground set E . Then the volume of the matroid polytope $P(\mathbf{M})$ is given by*

$$\text{Vol}(P(\mathbf{M})) = \frac{1}{(n-1)!} \sum_{(J_1, \dots, J_{n-1})} \tilde{\beta}(\mathbf{M}/J_1) \cdots \tilde{\beta}(\mathbf{M}/J_{n-1}) \quad (2.16)$$

summing over the ordered collections of sets $J_1, \dots, J_{n-1} \subseteq E$ such that if i_1, \dots, i_k are pairwise distinct, then $|J_{i_1} \cap \cdots \cap J_{i_k}| < |E| - k$.

The condition on the ordered collection of sets indexing the terms in the sum above is known as the *dragon marriage condition* [Pos09]. The enumeration of all ordered collections of subsets of $[n]$ satisfying the dragon marriage condition is computationally expensive for large n . The number of such ordered collections for $n = 2, 3, 4, 5, \dots$ is given by the sequence 1, 13, 1009, 354161, \dots [Pos09]. So we here only present a small-scale example.

Example 2.2.7. *Consider the matroid $\mathbf{U}_{2,3}$ on edge set $E = \{1, 2, 3\}$. We have grouped all subsets satisfying the dragon marriage condition according to their cardinality type in Table 2.1. From Table 2.1, we can see that the $\sum_{(J_1, J_2)} \tilde{\beta}(\mathbf{M}/J_1) \tilde{\beta}(\mathbf{M}/J_2)$ can be simplified to*

$$1 - 3 - 3 + 6 = 1$$

This is equal to the Eulerian number $A_{2,1}$. Therefore the volume of $P(\mathbf{U}_{2,3})$ is $\frac{1}{2!}$. The simplification of the sum into the Eulerian number is not evident from the formula itself.

(J_1 , J_2)	count	$\tilde{\beta}(M/J_1)$	$\tilde{\beta}(M/J_2)$	$\sum \tilde{\beta}(M/J_1)\tilde{\beta}(M/J_2)$
(0, 0)	1	-1	-1	1
(1, 0)	3	1	-1	-3
(0, 1)	3	-1	1	-3
(1, 1)	6	1	1	6

Table 2.1: Volume computation for $P(U_{2,3})$

2.2.3 Matroid valuations

Billera, Jia and Reiner [BJR09] introduced the notion of matroid base polytope decomposition. As the name suggests, it is a decomposition of a matroid base polytope into polytopes that in turn are also matroid base polytopes.

Definition 2.2.8. A matroid polytope decomposition of a matroid polytope $P = P(M)$ is a set of matroid polytopes $\{P_1, \dots, P_m\}$ such that

1. $P_1 \cup \dots \cup P_m = P$, and
2. for all $1 \leq i < j \leq m$, the intersection $P_i \cap P_j$ is a face of both P_i and P_j .

For the special case when $m = 2$, i.e. when $P(M)$ has a matroid polytope decomposition into $P(M_1)$ and $P(M_2)$, we call such a decomposition *hyperplane split*. Since the intersections $P(M_i) \cap P(M_j)$ are faces of matroid polytopes, these are themselves matroid polytopes. Given a matroid polytope $P \subseteq \mathbb{R}^n$, we denote by $M(P)$ the matroid whose bases B correspond to vertices e_{BS} of P . Matroid polytope decomposition is analogous to the idea of polytopal subdivision, but now we are restricting ourselves to the specific subclass of matroid polytopes for the subdivision. One can then define the notion of a matroid valuation. We denote the class of all matroids by Mat , and denote the class of all matroids on the ground set $[n]$ by Mat_n .

Definition 2.2.9. Let G be an abelian group. A function $f : \text{Mat} \rightarrow G$ is called a matroid polytope valuation, or valuation for short, if for any matroid polytope decomposition $\{P(M_1), P(M_2), \dots, P(M_m)\}$ of a matroid polytope $P(M)$ of $M \in \text{Mat}$, we have

$$\sum_{A \subseteq [m]} (-1)^{\#A} f\left(M\left(\bigcap_{i \in A} P(M_i)\right)\right) = 0$$

Many examples of matroid valuations were studied and discussed in detail by Ardila, Fink and Rincon in [AFR10]. We mention here some valuations that are relevant in our context. Except for Example 2.2.11, throughout this thesis we take $G = \mathbb{R}$ with addition.

Example 2.2.10. For a positive integer n , the volume function Vol , that gives the $(n-1)$ -dimensional volume of the matroid polytope $P(M)$ of M for each matroid $M \in \text{Mat}_n$ is a matroid polytope valuation.

Example 2.2.11. For a polytope P in \mathbb{R}^d , the Ehrhart function $\text{ehr}(P; n)$ is defined as the number of lattice points contained in the n -th dilate nP of P , that is,

$$\text{ehr}(P; n) = |(nP \cap \mathbb{Z}^d)|$$

This is known to be a polynomial function with rational coefficients [Ehr62] for lattice polytopes. Since matroid polytopes are lattice polytopes, we have a well-defined function

$$\begin{aligned} \text{ehr} : \text{Mat} &\longrightarrow \mathbb{Q}[t] \\ M &\longmapsto \text{ehr}(P(M); t) \end{aligned}$$

The Ehrhart polynomial satisfies the inclusion-exclusion property for lattice polytopes, which implies that the function ehr above is a matroid polytope valuation.

2.2.4 The Derksen-Fink invariant

In this section, we study a valuative invariant of matroids that was first introduced by Derksen in [Der09]. In the literature, it is usually referred to as the \mathcal{G} -invariant. We are opting for the name Derksen-Fink invariant to credit Derksen who defined it in [Der09] and Fink who proved a universality result for it in [DF10] along with Derksen. This also avoids confusion with Speyer's g -invariant [Spe08] in verbal exchanges. Let M be a matroid of rank r on the set $E = \{e_1, \dots, e_n\}$. Each permutation w of the set E gives rise to a flag of sets:

$$S(w) : \emptyset = S_0 \subsetneq S_1 \subsetneq \dots \subsetneq S_n = E$$

where $S_i = \{w(e_1), \dots, w(e_i)\}$ for $i = 1, \dots, n$. The rank sequence $\delta(w) = (\delta_1, \delta_2, \dots, \delta_n)$ of w is defined as

$$\delta_i = \rho_M(S_i) - \rho_M(S_{i-1})$$

Notice that for a matroid, the sequence δ is a binary sequence of length n with r ones. Let $[\delta]$ be a formal symbol, one for each possible rank sequence δ . The Derksen-Fink invariant $\mathcal{G}(M)$ of a matroid M is defined as

$$\mathcal{G}(M) = \sum_{w \in \mathfrak{S}_n} [\delta(w)] = \sum_{\delta \in \{0,1\}^n} g_M(\delta) [\delta] \quad (2.17)$$

where $g_M(\delta)$ is the number of permutations w whose rank sequence is δ . We skip the subscript M whenever the underlying matroid is understood. Note that the invariant $\mathcal{G}(M)$ is originally defined as a quasisymmetric function by Derksen in [Der09]. This definition is equivalent once we identify the formal symbol $[\delta]$ with an appropriate basis of the space of the quasisymmetric functions (for example, fundamental quasisymmetric functions).

Example 2.2.12. For the uniform matroid $U_{r,n}$, all r -sized subsets are bases, and therefore the rank sequence for any permutation is

$$\underbrace{(1 \dots 1)}_r \underbrace{0 \dots 0}_{n-r}$$

This implies

$$\mathcal{G}(\mathbf{U}_{r,n}) = n! \underbrace{[1 \cdots 1]}_r \underbrace{[0 \cdots 0]}_{n-r}$$

Derksen in [Der09] showed that the invariant $\mathcal{G}(\mathbf{M})$ is a valutive invariant, and asked whether it is universal. This was answered affirmatively later by Derksen and Fink [DF10]. A valuation ν is called *universal* if every valuation f can be written as $f = f' \circ \nu$ where f' is a group homomorphism. That is, every valuation factors through ν .

Theorem 2.2.13. *The Derksen-Fink invariant $\mathcal{G}(\mathbf{M})$ is a universal valutive invariant on matroids.*

It is easy to see that the function $b = b(\mathbf{M})$ that assigns the number of bases to each matroid \mathbf{M} is a matroid valuation. Universality of $\mathcal{G}(\mathbf{M})$ implies $b(\mathbf{M})$ should be an evaluation, which in fact it is as the following equality confirms:

$$g(\underbrace{1 \cdots 1}_k \underbrace{0 \cdots 0}_{n-k}) = k!(n-k)!b(\mathbf{M})$$

Speyer [Spe08] proved that the Tutte polynomial is also a matroid valuation. Another polynomial associated to a matroid is the corank-nullity polynomial $S(\mathbf{M}; x, y)$, which is defined as

$$S(\mathbf{M}; x, y) = \sum_{A \subseteq E} x^{r(\mathbf{M}) - \rho(A)} y^{|A| - \rho(A)} \quad (2.18)$$

$$= \sum_{i,j} s_{i,j} x^{r-j} y^{i-j} \quad (2.19)$$

where $s_{i,j}$ is the number of subsets $A \subseteq E$ of size i and rank j . One can show using induction that

$$t(\mathbf{M}; x, y) = S(\mathbf{M}; x-1, y-1)$$

This shows that the corank-nullity polynomial of a matroid \mathbf{M} , determines its Tutte polynomial. The significance of the universality can now be depicted in the following theorem of Derksen that determines the coefficients of the corank-nullity polynomial using the coefficients of the Derksen-Fink invariant.

Theorem 2.2.14. *(Derksen) Given a matroid $\mathbf{M} = (E, \mathcal{B})$ with corank nullity polynomial $S(\mathbf{M}; x, y) = \sum_{i,j} s_{i,j} x^{r-j} y^{i-j}$, the coefficient $s_{i,j}$ equals the following evaluation of the Derksen-Fink invariant $\mathcal{G}(\mathbf{M})$*

$$s_{i,j} = \frac{1}{i!(n-i)!} \sum_{\sum_i \delta = j} g(\delta) \quad (2.20)$$

where the sum on the right is over all $\delta \in \{0, 1\}^n$ with r 1's such that $\sum_i \delta = \delta_1 + \cdots + \delta_i = j$.

Example 2.2.15. *The Derksen-Fink invariant for $\mathbf{M}(K_4)$ is given by*

$$\mathcal{G}(\mathbf{M}(K_4)) = 144[110100] + 576[111000]$$

and the corank-nullity polynomial is given by

$$x^3 + y^3 + 6x^2 + 4xy + 6y^2 + 15x + 15y + 16$$

We illustrate the above theorem by computing these coefficients from the Derksen-Fink invariant in Table 2.2

2.3 Schubert matroids

In this section, we study an important class of matroids called *Schubert matroids*. They have been discovered and rediscovered many times. This (also) explains why we have several alternative names for them in the literature, such as *freedom matroids*, *nested matroids*, *PI-matroids*, *shifted matroids*, *counting matroids* and *generalized Catalan matroids*. Interestingly, these can (all) be realized as lattice path matroids. They played a role in Derksen and Fink's work [DF10]. Schubert matroids also arise as additive bases of Hampe's matroid intersection ring in [Ham17].

2.3.1 The Schubert matroid for a binary sequence

Let $\delta \in \{0, 1\}^n$ be a binary sequence with r 1's. We call such a binary sequence a length n , rank r binary sequence. The *Schubert matroid* \mathbf{R}_δ defined by δ is the matroid of rank r on the set $[n] = \{1, 2, \dots, n\}$ constructed as follows: For each $i = 1$ to n , we do the following:

- if $\delta_i = 0$, put i freely inside $\text{cl}(1, \dots, i-1)$ (for $i = 1$, put 1 as a loop). This operation is called *free extension*.
- if $\delta_i = 1$, put i as a coloop to $\text{cl}(1, \dots, i-1)$ (for $i = 1$, take 1 as a coloop). This operation is called *coloop addition*.

In other words, the Schubert matroid \mathbf{R}_δ is obtained from a loop or a coloop by a sequence of free extensions and coloop additions. If δ has 1's in positions $b_1 < \dots < b_r$ then the Schubert matroid \mathbf{R}_δ has a distinguished flag of flats

$$\text{cl}(\emptyset) = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_r = [n]$$

where $F_i = \{1, 2, \dots, b_{i+1} - 1\}$ is a flat of rank i for $i = 1, \dots, r-1$. As pointed out in [CS05], the independent sets of \mathbf{R}_δ can be constructed from such a flag as

$$\mathcal{I}(\mathbf{R}_\delta) = \{I \subseteq [n] : |I \cap F_i| \leq i \text{ for all } i\}$$

Since the collection of independent sets uniquely defines a matroid, this gives an alternative construction of the Schubert matroid \mathbf{R}_δ . Sometimes, this construction will be more

	1				
		6			
		15	4		
		16	15	6	1

(i, j)	$s_{i,j}$ from Equation 2.20
(0, 0)	$\frac{6!}{144+576} = 1$
(1, 1)	$\frac{5!}{144+576} = 6$
(2, 2)	$\frac{4!2!}{144+576} = 15$
(3, 2)	$\frac{3!3!}{144} = 4$
(3, 3)	$\frac{576}{3!3!} = 16$
(4, 3)	$\frac{4!2!}{144+576} = 15$
(5, 3)	$\frac{5!}{144+576} = 6$
(6, 3)	$\frac{6!}{144+576} = 1$

Table 2.2: Computing coefficients $s_{i,j}$ of corank-nullity polynomial from the Derksen-Fink invariant

convenient, therefore we would like to fix a notation for it. The Schubert matroid coming from applying this construction to a flag of sets

$$F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r = [n]$$

will be denoted as $R[F_0, \dots, F_r]$. Note that this gives rise to a matroid on $[n]$ which is isomorphic to the Schubert matroid

$$R_{\underbrace{0 \dots 0}_{|F_0|} \underbrace{1 \dots 0}_{|F_1 \setminus F_0|} \underbrace{0 \dots 0}_{|F_2 \setminus F_1|} \dots \underbrace{10 \dots 0}_{|F_r \setminus F_{r-1}|}}$$

We have constructed R_δ as a matroid on $[n]$, but such a matroid can be constructed over any finite set E of size n . In what follows, R_δ is considered as a matroid up to isomorphism.

Example 2.3.1. *The uniform matroid $U_{r,n}$ can be constructed by adding first r elements in $[n]$ as coloops and then putting $n - r$ remaining elements freely in the span of the first r elements. This means every r -sized subset is independent. The distinguished flag of flats in this case is*

$$\emptyset \subsetneq \{1\} \subsetneq \cdots \subsetneq \{1, \dots, r-1\} \subsetneq [n]$$

This implies

$$U_{r,n} \cong R_{\underbrace{1 \dots 1}_r \underbrace{0 \dots 0}_{n-r}}$$

Let $V(n, r)$ be the vector space of formal \mathbb{Q} -linear combinations of all symbols $[\delta]$ such that δ is a length n rank r binary sequence. The vector space $V(n, r)$ has dimension $\binom{n}{r}$. One natural choice of a basis for $V(n, r)$ is $\{[\delta] : \delta \in \{0, 1\}^n \text{ such that } \sum_i \delta_i = r\}$. Kung has recently determined another choice of basis for $V(n, r)$ in [Kun17], which makes Schubert matroids R_δ all the more relevant.

Proposition 2.3.2. *The set*

$$\left\{ G(R_\delta) : \delta \in \{0, 1\}^n \text{ such that } \sum_i \delta_i = r \right\}$$

forms a basis for the vector space $V(n, r)$.

In Table 2.3 we show the Derksen-Fink invariant $\mathcal{G}(M)$ for all connected loopless Schubert matroids on $[6] = \{1, 2, 3, 4, 5, 6\}$ of rank 3.

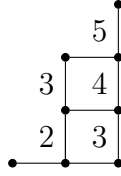
2.3.2 Lattice path matroids

Let J be a set. A J -set system \mathcal{A}_J , or a set system for short, over a set E is a multiset of subsets of E , indexed by the set J . That is,

$$\mathcal{A}_J := \left\{ A_j \subseteq E : j \in J \right\}$$

R_δ	$\mathcal{G}(R_\delta)$
R_{111000}	$720[111000]$
R_{110100}	$36[110100] + 684[111000]$
R_{110010}	$48[101100] + 96[110100] + 576[111000]$
R_{101100}	$48[110010] + 96[110100] + 576[111000]$
R_{101010}	$8[101010] + 40[101100] + 40[110010] + 128[110100] + 504[111000]$
R_{100110}	$36[100110] + 36[101010] + 72[101100] + 72[110010] + 144[110100] + 360[111000]$

Table 2.3: The Derksen-Fink invariants $\mathcal{G}(M)$ of all connected loopless Schubert matroids on $[6]$ of rank 3

Figure 2.9: Lattice-path diagram for $Q = \text{EENNN}$ and $P = \text{ENNEN}$

A set system $\mathcal{B} := \{A_k \subseteq E : k \in K\}$ where $K \subseteq J$ is called a *subsystem* of \mathcal{A}_J . A *transversal* of the set system \mathcal{A}_J is a set $T = \{x_j \in E : j \in J\}$ of $|J|$ distinct elements such that $x_j \in A_j$ for all $j \in J$. A *partial transversal* of \mathcal{A}_J is a transversal of a subsystem of \mathcal{A}_J . We recall the following result due to Edmonds and Fulkerson [EF65].

Theorem 2.3.3. *The partial transversals of a set system over E are the independent sets of a matroid over E .*

A matroid defined in this manner is called a *transversal matroid* and is denoted as $\mathbf{M} = (E, \mathcal{A}_J)$. The set system \mathcal{A}_J is called a *presentation* of \mathbf{M} . We will now define a set system from a pair of lattice paths on $\mathbb{N} \times \mathbb{N}$. We only consider lattice paths that start from $(0, 0)$ and that only use two kinds of steps: *East step* $\mathbf{E} = (1, 0)$ and *North step* $\mathbf{N} = (0, 1)$. For short, we call them \mathbf{E} -step and \mathbf{N} -step, respectively. For us, paths are just words on the alphabet $\{\mathbf{E}, \mathbf{N}\}$. Let $Q = q_1 \cdots q_n$ and $P = p_1 \cdots p_n$ be two paths from $(0, 0)$ to $(n - r, r)$, with Q never going above P . Let p_{i_1}, \dots, p_{i_r} be the north steps of P and let q_{j_1}, \dots, q_{j_r} be the north steps of Q . Define the intervals

$$N_k = [p_{i_k}, q_{j_k}] = \{p_{i_k}, p_{i_k} + 1, \dots, q_{j_k}\}$$

This gives us a set system

$$\mathcal{A} = \left\{ N_k \subseteq [n] : k = 1, 2, \dots, r \right\} \quad (2.21)$$

over the set $[n] = \{1, \dots, n\}$. The corresponding transversal matroid is denoted $\mathbf{M}[P, Q]$. A *lattice path matroid* is a matroid isomorphic to a transversal matroid $\mathbf{M}[P, Q]$. We introduce the following shorthand for the extreme paths: $\lrcorner_{n,r} = \mathbf{E}^{n-r}\mathbf{N}^r$ and $\lrcorner_{n,r} = \mathbf{N}^r\mathbf{E}^{n-r}$.

Example 2.3.4. *Consider the paths $Q = \text{EENNN} = \lrcorner_{5,3}$ and $P = \text{ENNEN}$ shown in Figure 2.9, then $\mathbf{M}[P, Q]$ is a matroid on $[5]$ of rank 3. The collection of bases of $\mathbf{M}[P, Q]$ are given by*

$$\mathcal{B}(\mathbf{M}[P, Q]) = \{235, 245, 345\}$$

This is precisely our matroid \mathbf{M}_{01101} considered in Example 2.1.4.

Bonin and De Mier studied the structural properties of lattice path matroids in [BM06]. Besides other results, they showed the following connectivity criterion for lattice-path matroids:

Theorem 2.3.5. *A lattice path matroid $M[P, Q]$ of rank r on $[n]$ is connected if and only if P and Q intersect only at $(0, 0)$ and $(n - r, r)$.*

This implies that any lattice path matroid of the form $M[\sqcup_{n,r}, P]$ is a connected matroid as long as P is a North-East path whose first step is an N-step and whose last step is an E-step. We do not need to worry about what happens in between the first and the last step for such a path P .

2.3.3 Schubert matroids as lattice path matroids

We would like to realize any Schubert matroid R_δ as a lattice path matroid. Given a $\delta \in \{0, 1\}^n$, we define path $P_\delta \in \{E, N\}^n$ by replacing 0 with E and 1 with N in δ . The following proposition seems to be known, but we are unable to find a reference.

Proposition 2.3.6. *Let δ be a length n rank r binary sequence, then*

$$R_\delta \cong M[P_\delta, \sqcup_{n,r}] \tag{2.22}$$

Proof. Consider the flag of sets of R_δ

$$F_0 \subsetneq \cdots \subsetneq F_r$$

given by

$$\begin{aligned} F_0 &= \{1, \dots, a_1 - 1\} \\ F_i &= F_{i-1} \cup \{a_i, \dots, a_{i+1} - 1\} \\ F_r &= E \end{aligned}$$

where a_i is the position of the i th north step for $i = 1, \dots, r-1$. This gives a distinguished flag of sets, such that the independent sets of $M = M[P_\delta, \sqcup_{n,r}]$ satisfy

$$|(I \cap F_i)| \leq i$$

for all i . So the lattice path matroid $M[P_\delta, \sqcup_{n,r}]$ is isomorphic to $R[F_0, F_1, \dots, F_r]$. Now notice that

$$\begin{aligned} |F_0| &= a_1 - 1 \\ |F_i \setminus F_{i-1}| &= a_{i+1} - a_i - 1 \\ |F_r \setminus F_{r-1}| &= n - a_r + 1 \end{aligned}$$

This is the Schubert matroid

$$R_{\underbrace{0 \dots 0}_{|F_0|} \underbrace{1 \dots 0}_{|F_1 \setminus F_0|} \underbrace{10 \dots 0}_{|F_2 \setminus F_1|} \dots \underbrace{10 \dots 0}_{|F_r \setminus F_{r-1}|}}$$

which is precisely the matroid R_δ . □

Matroid M	$[\lambda]$
R_{111000}	
R_{110100}	
R_{110010}	
R_{101100}	
R_{101010}	
R_{100110}	

Table 2.4: Connected loopless Schubert matroids on $[6]$ of rank 3 along with their respective lattice path diagrams.

Matroid M	Lattice of cyclic flats with ranks
R_{111000}	$\emptyset_{\rho=0} \subsetneq \{1, 2, 3, 4, 5, 6\}_{\rho=3}$
R_{110100}	$\emptyset_{\rho=0} \subsetneq \{1, 2, 3\}_{\rho=2} \subsetneq \{1, 2, 3, 4, 5, 6\}_{\rho=3}$
R_{110010}	$\emptyset_{\rho=0} \subsetneq \{1, 2, 3, 4\}_{\rho=2} \subsetneq \{1, 2, 3, 4, 5, 6\}_{\rho=3}$
R_{101100}	$\emptyset_{\rho=0} \subsetneq \{1, 2\}_{\rho=1} \subsetneq \{1, 2, 3, 4, 5, 6\}_{\rho=3}$
R_{101010}	$\emptyset_{\rho=0} \subsetneq \{1, 2\}_{\rho=1} \subsetneq \{1, 2, 3, 4\}_{\rho=2} \subsetneq \{1, 2, 3, 4, 5, 6\}_{\rho=3}$
R_{100110}	$\emptyset_{\rho=0} \subsetneq \{1, 2, 3\}_{\rho=1} \subsetneq \{1, 2, 3, 4\}_{\rho=2} \subsetneq \{1, 2, 3, 4, 5, 6\}_{\rho=3}$

Table 2.5: Connected loopless Schubert matroids on $[6]$ of rank 3 along with their respective lattices of cyclic flats

The above proposition gives a description of R_δ as a transversal matroid, with the following presentation

$$\left\{ N_k = [p_{i_k}, q_{j_k}] : k \in \{1, 2, \dots, r\} \right\}$$

We have realized R_δ as a lattice path matroid. In Table 2.4, we show the lattice path diagram for all connected loopless Schubert matroids on $[6]$ of rank 3.

This also determines the lattice of cyclic flats of R_δ . This is because of the following structure theorem for lattice path matroids by Bonin.

Proposition 2.3.7. [Bon10] *The lattice $\mathcal{Z}(M)$ of cyclic flats of the lattice-path matroid $M = M[P_\delta, \perp_{n,r}]$ is a chain. Furthermore, the proper non-trivial cyclic flats of the matroid M are given by $F_i = \{1, 2, \dots, c_i\} \subseteq [n]$, where c_i is the i th incident of an E-step that is followed by an N-step.*

So the lattice of cyclic flats of the Schubert matroid R_δ is a chain. This is why they are also known as *nested matroids*.

Example 2.3.8. *As an example, we show all the connected loopless Schubert matroids on $[6] = \{1, 2, 3, 4, 5, 6\}$ of rank 3, and their lattice path diagrams along with the chains of cyclic flats that characterize them in Table 2.5 and Table 2.4.*

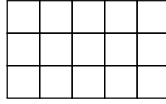


Figure 2.10: Lattice-path diagram for $U_{3,5}$.

Brylawski [Bry75] showed that the cyclic flats together with their ranks uniquely determine the matroid. Therefore, we can reconstruct R_δ from its respective chain of cyclic flats. If R is a Schubert matroid coming from such a chain $F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_k = [n]$ of cyclic flats, with $\rho_R(F_i) = r_i$, then we denote it by $R(F_0^{r_0} \subsetneq F_1^{r_1}, \dots \subsetneq F_k^{r_k})$, or just $R(F_0^{r_0}, F_1^{r_1}, \dots, F_k^{r_k})$.

Example 2.3.9. *The uniform matroid $U_{r,n}$ on $[n]$ of rank r can be realised as a lattice path matroid $M = M[\mathbf{E}^{n-r}\mathbf{N}^r, \mathbf{N}^r\mathbf{E}^{n-r}] = M[\lfloor _ \rfloor_{n,r}, \lceil _ \rceil_{n,r}]$. The bases in this case correspond to all north-east paths from $(0, 0)$ to (n, r) . We show the diagram $[\lambda]$ in Figure 2.10 for the case $U_{3,5}$. The chain of cyclic flats is $\emptyset \subseteq [n]$, that is, $U_{r,n} = R(\emptyset^0, [n]^r)$.*

2.3.4 Hampe’s matroid intersection ring

Let us fix our ground set $E = [n] := \{1, 2, \dots, n\}$ and we will only consider loopless matroids in this section. For $r : 1 \leq r \leq n$, let $\mathcal{C}_{r,n}$ be the set of all chains of subsets of $[n]$ of length r :

$$\emptyset \subsetneq F_1 \subsetneq \dots \subsetneq F_r = [n]$$

We denote by $V_{r,n} = \mathbb{Z}^{\mathcal{C}_{r,n}}$ the free \mathbb{Z} -module whose coordinates are indexed by elements of $\mathcal{C}_{r,n}$. Let $\mathbb{M}_{r,n}^{\text{free}}$ be the free \mathbb{Z} -module with generators the set of all loopless matroids of rank r on the ground set $[n] = \{1, 2, \dots, n\}$. Define a homomorphism

$$\begin{aligned} \Phi_{r,n} : \mathbb{M}_{r,n}^{\text{free}} &\longrightarrow V_{r,n} \\ M &\longmapsto \chi_M \end{aligned}$$

where for each chain C ,

$$(\chi_M)_C := \begin{cases} 1 & \text{if } C \text{ is a chain of flats in } M \\ 0 & \text{otherwise} \end{cases}$$

The \mathbb{Z} -module \mathbb{M}_n is defined by

$$\mathbb{M}_n = \bigoplus_{r=1}^n \mathbb{M}_{r,n}$$

with $\mathbb{M}_{r,n} = \mathbb{M}_{r,n}^{\text{free}} / \ker \Phi_{r,n}$. One way to think of it is to identify matroids with the set of saturated chains of their flats.

Example 2.3.10. For $r = 2$ and $n = 4$, the following identity holds in $\mathbb{M}_{2,4}$:

$$\begin{array}{c}
 \begin{array}{cccc}
 & 1234 & & \\
 & \nearrow \uparrow \nwarrow & & \\
 1 & 2 & 3 & 4 \\
 & \nwarrow \downarrow \nearrow & & \\
 & \emptyset & &
 \end{array}
 +
 \begin{array}{cc}
 1234 & \\
 \nearrow \nwarrow & \\
 14 & 23 \\
 \nwarrow \nearrow & \\
 \emptyset &
 \end{array}
 =
 \begin{array}{c}
 \begin{array}{ccc}
 & 1234 & \\
 & \nearrow \uparrow \nwarrow & \\
 1 & 4 & 23 \\
 & \nwarrow \uparrow \nearrow & \\
 & \emptyset &
 \end{array}
 +
 \begin{array}{ccc}
 & 1234 & \\
 & \nearrow \uparrow \nwarrow & \\
 14 & 3 & 2 \\
 & \nwarrow \uparrow \nearrow & \\
 & \emptyset &
 \end{array}
 \end{array}
 \quad (2.23)$$

where matroids are identified with their respective lattices of flats.

Recall that, given matroids \mathbf{M} and \mathbf{N} on the same ground set E , the *union* $\mathbf{M} \vee \mathbf{N}$ is a matroid on E which is defined in terms of its independent sets

$$\mathcal{I}(\mathbf{M} \vee \mathbf{N}) = \{I \cup J : I \in \mathcal{I}(\mathbf{M}), J \in \mathcal{I}(\mathbf{N})\}$$

The *intersection* $\mathbf{M} \wedge \mathbf{N}$ is defined as

$$\mathbf{M} \wedge \mathbf{N} = (\mathbf{M}^* \vee \mathbf{N}^*)^*$$

where \mathbf{M}^* denote the dual matroid of \mathbf{M} . It is shown by Hampe [Ham17] that \mathbb{M}_n forms a ring under the product defined as

$$\mathbf{M} \cdot \mathbf{N} := \begin{cases} \mathbf{M} \wedge \mathbf{N} & \text{if } \mathbf{M} \wedge \mathbf{N} \text{ is loopfree;} \\ 0 & \text{otherwise,} \end{cases}$$

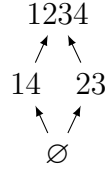
extended to linear combinations of matroids via distributivity. \mathbb{M}_n is called the *intersection ring of matroids* on $[n]$. In our discussion, the multiplicative structure of \mathbb{M}_n is not needed, though it is of independent importance.

For a loopless matroid \mathbf{M} on $[n]$ of rank r , let $\mathcal{Z} = \mathcal{Z}(\mathbf{M})$ be its lattice of cyclic flats. We denote by $\Delta = \Delta(\mathcal{Z})$ the (reduced) order complex of \mathcal{Z} . The complex Δ consists of chains of cyclic flats of \mathbf{M} , without top $[n]^r$ or bottom \emptyset^0 . Let $\mathcal{F} = \mathcal{F}(\Delta)$ be the face lattice of Δ . Let $\Gamma = \Gamma(\mathbf{M})$ be the lattice we get by adjoining a formal bottom to the dual of $\mathcal{F}(\Delta)$, that is, $\Gamma = \mathcal{F}^* \cup \{\hat{0}\}$. The (nonbottom) elements C of Γ are chains of cyclic flats of \mathbf{M} without top E or bottom \emptyset . With each such chain $C = (F_1 \subsetneq \cdots \subsetneq F_{k-1})$, we can associate a Schubert matroid $\mathbf{R}(C) = \mathbf{R}(\emptyset^0 \subsetneq F_1^{r_1} \subsetneq \cdots \subsetneq F_{k-1}^{r_{k-1}} \subsetneq E^r)$, where $r_i = \rho_{\mathbf{M}}(F_i)$ for $i = 1, \dots, r-1$. Again keeping in mind our shorthand, we write $\mathbf{R}(F_1^{r_1}, \dots, F_{k-1}^{r_{k-1}})$ for such a matroid. Note that the top element in $\Gamma(\mathbf{M})$ is the empty face, and the corresponding Schubert matroid for it is the uniform matroid $\mathbf{R}(\emptyset \subseteq [n]^r) = \mathbf{U}_{r,n}$. An important result in [Ham17] is the following:

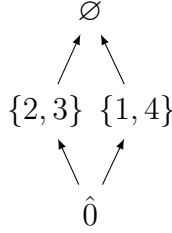
Theorem 2.3.11. [Ham17] *Let \mathbf{M} be a loopfree matroid of rank r on $[n]$. Then the following identity holds in $\mathbb{M}_{r,n}$:*

$$\mathbf{M} = - \sum_{\substack{C \in \Gamma(\mathbf{M}) \\ C \neq \hat{0}}} \mu(\hat{0}, C) \mathbf{R}(C)$$

Example 2.3.12. *The Example 2.3.10 is a special case of the above identity. Let \mathbf{M} be a matroid whose lattice of flats is given by*



Notice that in this case, the lattice of cyclic flats of \mathbf{M} is the same as the lattice of flats, since each flat is cyclic. The lattice $\Gamma(\mathbf{M})$ is given by



This implies that

$$\begin{aligned}
 \mathbf{M} &= \mathbf{R}(\emptyset^0 \subseteq [4]^2) - \mathbf{R}(\emptyset^0 \subsetneq \{2, 3\}^1 \subseteq [4]^2) - \mathbf{R}(\emptyset^0 \subsetneq \{1, 4\}^1 \subseteq [4]^2) \\
 &= \mathbf{R}(\emptyset) - \mathbf{R}(\{2, 3\}^1) - \mathbf{R}(\{1, 4\}^1)
 \end{aligned}$$

which is precisely the equality in Example 2.3.10.

2.4 Volume computations

Recall that the hypersimplex $\Delta_{r,n}$ is the matroid polytope of the uniform matroid $\mathbf{U}_{r,n}$, and its volume is given by $\frac{1}{(n-1)!} A_{n-1,r-1}$ where $A_{n-1,r-1}$ is the Eulerian number that counts the number of permutations of $[n-1]$ with $r-1$ descents. We can ask whether there is a combinatorial formula for the volumes of other matroid polytopes. This is already answered by Ardila, Benedetti and Doker in [ABD10], as we saw in Section 2.2.2. The formula contains a sum whose indexing set are subsets satisfying the dragon marriage condition. We still do not know a lot about this condition. For example, it is not obvious how their formula implies that the volume of a uniform matroid polytope (hypersimplex $\Delta_{r,n}$) is given by $\frac{1}{(n-1)!} A_{n-1,r-1}$. Another class of matroid polytopes for which the answer is known is the lattice-path matroid polytopes. Bidkhori and Sullivant [Bid12] gave an expression for the volume of a lattice-path matroid polytope in terms of the number of standard skew Young tableaux corresponding to certain lattice paths. We have seen in Section 2.3.3 that the Schubert matroid \mathbf{R}_δ is isomorphic to certain lattice-path matroid. This enables us to compute the volume of its base polytope. From Section 2.3.4, we also know that the volume of a matroid polytope can be written as a linear combination of volumes of Schubert matroid polytopes, given by Equation 2.3.11. Combining these two facts, we get a combinatorial algorithm to compute the volume of any matroid polytope. We use this algorithm to give a formula for volumes of connected sparse paving matroid polytopes.

2.4.1 Volume of Schubert matroid

Bidkhorı [Bid12] generalized Stanley’s approach for computing volumes of hypersimplices to all lattice path matroid polytopes. She gave an expression in terms of the sum of numbers of standard skew Young tableaux of certain shapes. These can be computed combinatorially by the Frame-Robinson-Thrall hook-length formula. Bidkhorı decomposed a connected lattice path matroid polytope into a certain type of lattice-path matroid polytopes, called the *border strip matroid polytopes*.

Definition 2.4.1. *Let $M[P, Q]$ be a connected lattice path matroid polytope such that the boxes between P and Q form a border strip λ : that is, if $P = p_1 \cdots p_n$ and $Q = q_1 \cdots q_n$, then $p_n = q_1 = E$, $q_n = p_1 = N$ and $p_i = q_i$ for $1 < i < n$. Let \mathfrak{p} be a path whose vertices are boxes of a border strip λ and whose edges are connected boxes. We call such a matroid $M[P, Q]$ a border strip matroid and denote it by $M[\lambda(\mathfrak{p})]$.*

The border strip λ can be seen as a skew Young diagram coming from p . To highlight this perspective, we denote such a shape by $\lambda(\mathfrak{p})$. The following result of Bidkhorı and Sullivant uses Stanley’s triangulation of a hypersimplex.

Proposition 2.4.2. *[Bid12] The volume of the border strip matroid polytope $P(M[\lambda(\mathfrak{p})])$ is given by the number $f^{\lambda(\mathfrak{p})}$ of the standard skew Young tableaux of shape $\lambda(\mathfrak{p})$.*

For a path p , the number $f^{\lambda(\mathfrak{p})}$ can also be thought of as the number of permutations of the set $\{1, 2, \dots, n-1\}$ which have descents exactly where we have a horizontal step in our path. The number of standard Young tableaux of shape λ can be computed combinatorially by the Frame-Robinson-Thrall hook-length formula [FRT54]. For standard skew Young tableaux of (skew) shape λ , we can compute f^λ using the Naruse’s formula (that generalizes the hook-length formula [Nar14]).

Given a lattice path matroid polytope $M[P, Q]$, Bidkhorı [Bid12] used Alfonsin and Chatelain’s work [CR11] on hyperplane splits to come up with the following matroid polytope decomposition of $M[P, Q]$.

Proposition 2.4.3. *If $M[P, Q]$ is a connected lattice path matroid polytope, then*

$$\left\{ M[\lambda(\mathfrak{p})] : \mathfrak{p} \text{ path of boxes between } P \text{ and } Q \right\}$$

is a matroid polytope decomposition of $M[P, Q]$.

This matroid polytope decomposition arises via iterative hyperplane splits. If we only consider the volume of lattice path matroid polytopes, the above decomposition implies the following corollary.

Corollary 2.4.4. *[Bid12] The volume of a lattice path matroid polytope corresponding to the connected lattice path matroid $M[P] := M[P, \perp_{n,r}]$ on $[n]$ of rank r is given by*

$$\text{Vol}(P(M[P])) = \frac{1}{(n-1)!} \sum_{\mathfrak{p}} f^{\lambda(\mathfrak{p})}$$

where the sum is over all paths \mathfrak{p} under P , and $f^{\lambda(\mathfrak{p})}$ denotes the set of standard skew Young tableaux of shape $\lambda(\mathfrak{p})$.

Matroid M	$\sum f^\lambda$	$\text{Vol}(\mathbb{P}_M)$
R_{111000}	$f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square}$	$\frac{66}{5!}$
R_{110100}	$f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square}$	$\frac{60}{5!}$
R_{110010}	$f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square}$	$\frac{33}{5!}$
R_{101100}	$f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square}$	$\frac{33}{5!}$
R_{101010}	$f^{\square\square\square\square} + f^{\square\square\square\square}$	$\frac{22}{5!}$
R_{100110}	$f^{\square\square\square\square}$	$\frac{6}{5!}$

Table 2.6: Volume of connected loopless Schubert matroid polytopes on $[6]$ of rank 3

Example 2.4.5. *As a special case of the above result, we have*

$$\text{Vol}(\mathbb{P}(R_{\underbrace{1, \dots, 1}_k, \underbrace{0, \dots, 0}_{n-k}})) = \frac{1}{(n-1)!} \sum_{\mathfrak{p}} f^{\lambda(\mathfrak{p})}$$

which translates into

$$A_{n-1, r-1} = \sum_{\mathfrak{p}} f^{\lambda(\mathfrak{p})}$$

The left-hand side counts the number of permutations $w \in \mathfrak{S}_{n-1}$ with $r-1$ descents and the right-hand side counts the same by indexing over all possible $r-1$ descent positions. For example,

$$\begin{aligned} 26 &= f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square} + f^{\square\square\square\square} \\ &= 4 + 9 + 9 + 4 \end{aligned}$$

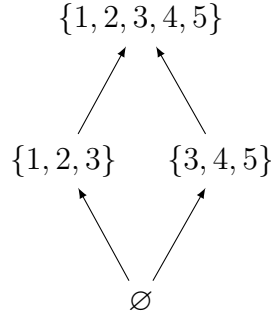
Example 2.4.6. *The volume of all connected loopless Schubert matroids on $[6]$ of rank 3 can then be computed as shown in Table 2.6.*

2.4.2 From the lattice of cyclic flats to the volumes of connected matroid polytopes

In this subsection, we explain how the lattice of cyclic flats of a connected matroid can be used to compute the volume of its matroid polytope. Let M be a matroid of rank r on $[n]$, and let $\mathcal{Z} = \mathcal{Z}(M)$ be its lattice of cyclic flats. We construct the lattice $\Gamma(M)$ as explained in Section 2.3.4, and consider the following relation in $\mathbb{M}_{r,n}$ from Theorem 2.3.11

$$M = - \sum_{\substack{C \in \Gamma(M) \\ C \neq \hat{0}}} \mu(\hat{0}, C) R(C)$$

Since the Derksen-Fink invariant $\mathcal{G} : \mathbb{M}_{r,n} \rightarrow V(n, r)$ is a \mathbb{Z} -module homomorphism, the above can be translated to a relation between Derksen-Fink invariants of matroids. Let

Figure 2.11: Lattice of cyclic flats of M_{\square}

M be a connected matroid, which means that its polytope is $n - 1$ dimensional. Since the $(n - 1)$ -dimensional normalized volume Vol is a matroid invariant and a matroid valuation, and the Derksen-Fink invariant is a universal valuative invariant, this gives that the volume $\text{Vol}(\mathbf{P}(M))$ of the matroid base polytope $\mathbf{P}(M)$ is then an evaluation of the Derksen-Fink invariant of the matroid M . Therefore, we have an equation for $\text{Vol}(\mathbf{P}(M))$ as

$$\text{Vol}(\mathbf{P}(M)) = - \sum_{\substack{C \in \Gamma(M) \\ C \neq \hat{0}}} \mu(\hat{0}, C) \text{Vol}(\mathbf{R}(C))$$

Now the volumes $\text{Vol}(\mathbf{R}(C))$ of Schubert matroids can now be computed by the Bidkhori-Sullivant formula (Corollary 2.4.4). Let us illustrate this for the graphic matroid M_{\square} .

Example 2.4.7. *The lattice of cyclic flats of M_{\square} is given in Figure 2.11. By Theorem 2.3.11, we get*

$$M_{G_{\square}} = -\mathbf{R}(\emptyset) + \mathbf{R}(\{1, 2, 3\}^2) + \mathbf{R}(\{3, 4, 5\}^2)$$

From this it follows that we have

$$\begin{aligned} \mathcal{G}(M_{\square}) &= -\mathcal{G}(\mathbf{R}_{11100}) + 2\mathcal{G}(\mathbf{R}_{11010}) \\ \text{Vol}(\mathbf{P}(M_{\square})) &= -\frac{1}{4!} \left(A_{4,2} - 2 \sum_{\mathfrak{p} \subseteq \boxplus} f^{\lambda(\mathfrak{p})} \right) \\ &= -\frac{1}{4!} (1 \cdot 11 - 2 \cdot 8) \\ &= \frac{1}{4!} (5) \\ &= \frac{5}{4!} \end{aligned}$$

which we already know is the volume of $\mathbf{P}(M_{\square})$.

2.4.3 Volume of connected sparse paving matroid polytopes

In this subsection, we study connected sparse paving matroids and the volume of their base polytopes. A matroid \mathbf{M} of rank r on the ground set E is called *paving* if all circuits have size r or $r + 1$. A hyperplane of size $r - 1$ of \mathbf{M} is called a *trivial hyperplane*, and otherwise it is called nontrivial. The calculation of the Derksen-Fink invariant of a paving matroid is given by the following result due to Tugger [FK17].

Proposition 2.4.8. *Let \mathbf{M} be a rank r paving matroid on $[n]$. The Derksen-Fink invariant $\mathcal{G}(\mathbf{M})$ of a paving matroid \mathbf{M} is given by*

$$\begin{aligned} \mathcal{G}(\mathbf{M}) &= \sum_{H \text{ trivial}} (r-1)!(n-r+1)! [1^r 0^{n-r}] \\ &+ \sum_{H \text{ nontrivial}} \left(\sum_{i=r}^{|H|} \frac{|H|!}{(|H|-i+1)!} (n-|H|)(n-i)! [1^{r-1} 0^{i-r} 10^{n-i}] \right) \end{aligned}$$

A paving matroid is *sparse* if all nontrivial hyperplanes have size r . All uniform matroids are sparse paving, since the hyperplanes are exactly all $r - 1$ subsets of underlying set E , and all of them are trivial. For the Derksen-Fink invariant of sparse paving matroids, the formula in Proposition 2.4.8 can further be simplified.

Corollary 2.4.9. *If \mathbf{M} is a rank- r sparse paving matroid on n elements with α nontrivial hyperplanes, then \mathbf{M} has $\binom{n}{r} - \alpha$ bases. Hence*

$$\mathcal{G}(\mathbf{M}) = \left(\binom{n}{r} - \alpha \right) r!(n-r)! [1^r 0^{n-r}] + \alpha r!(n-r)! [1^{r-1} 0 10^{n-r-1}]$$

The above corollary suggests that we may find a simple formula for the volume of the matroid polytope of a connected sparse paving matroid with a given number of nontrivial hyperplanes. This brings us to our main result.

Theorem 2.4.10. *Let \mathbf{M}_α be a connected sparse paving matroid of rank r with α nontrivial hyperplanes. Let $\mathbf{P}(\mathbf{M}_\alpha)$ denote the corresponding matroid base polytope. Then*

$$\text{Vol}(\mathbf{P}(\mathbf{M}_\alpha)) = \frac{1}{(n-1)!} \left(A_{n-1, r-1} - \alpha \binom{n-2}{r-1} \right)$$

Proof. Notice first that the lattice of cyclic flats $\mathcal{Z} = \mathcal{Z}(\mathbf{M}_\alpha)$ of \mathbf{M}_α is a rank 2 lattice with α atoms which are precisely the hyperplanes of \mathbf{M}_α . Now using the identity from Theorem 2.3.11, we have

$$\begin{aligned} \mathbf{M}_\alpha &= - \left(- \sum_{H \text{ hyperplane}} \mathbf{R}(\emptyset^0 \subsetneq H^{r-1} \subsetneq E^r) + (\alpha - 1) \mathbf{R}(\emptyset^0 \subsetneq E^r) \right) \\ &= - \left(- \sum_{H \text{ hyperplane}} \mathbf{R}_{\underbrace{1 \dots 1}_{r-1} 0 1 \underbrace{0 \dots 0}_{n-r-1}} + (\alpha - 1) U_{r,n} \right) \end{aligned}$$

This implies identity of the Derksen-Fink invariant of respective matroids.

$$\mathcal{G}(\mathbf{M}_\alpha) = - \left(- \sum_{H \text{ hyperplane}} \mathcal{G}(\underbrace{R_1 \dots 1}_{r-1} \ 01 \ \underbrace{0 \dots 0}_{n-r-1}) + (\alpha - 1) \mathcal{G}(U_{r,n}) \right)$$

By the universality of the Derksen-Fink invariant, the above can be translated to an equality of volumes. Appealing to the Bidkhori-Sullivant formula (Corollary 2.4.4), we can simplify the right-hand side

$$\text{Vol}(\mathbf{P}(\mathbf{M}_\alpha)) = - \frac{1}{(n-1)!} \left((\alpha - 1) \sum_{\substack{\mathfrak{p} \subseteq [m^r] \\ \mathfrak{p} \text{ path}}} f^{\lambda(\mathfrak{p})} - \alpha \sum_{\substack{\mathfrak{p} \subseteq [m^r]/[1] \\ \mathfrak{p} \text{ path}}} f^{\lambda(\mathfrak{p})} \right)$$

Let $\overline{\Gamma}$ denote the path $N^r E^{n-r}$. Then we can rewrite

$$\text{Vol}(\mathbf{P}(\mathbf{M}_\alpha)) = - \frac{1}{(n-1)!} \left((\alpha - 1) \left(\sum_{\substack{\mathfrak{p} \subseteq [m^r]/[1] \\ \mathfrak{p} \text{ path}}} f^{\lambda(\mathfrak{p})} + f^{\lambda(\overline{\Gamma})} \right) - \alpha \sum_{\substack{\mathfrak{p} \subseteq [m^r]/[1] \\ \mathfrak{p} \text{ path}}} f^{\lambda(\mathfrak{p})} \right)$$

This can be simplified to

$$\begin{aligned} \text{Vol}(\mathbf{P}(\mathbf{M}_\alpha)) &= - \frac{1}{(n-1)!} \left((\alpha - 1) f^{\lambda(\overline{\Gamma})} - \sum_{\substack{\mathfrak{p} \subseteq [m^r]/[1] \\ \mathfrak{p} \text{ path}}} f^{\lambda(\mathfrak{p})} \right) \\ &= - \frac{1}{(n-1)!} \left(\alpha f^{\lambda(\overline{\Gamma})} - \sum_{\substack{\mathfrak{p} \subseteq [m^r] \\ \mathfrak{p} \text{ path}}} f^{\lambda(\mathfrak{p})} \right) \end{aligned}$$

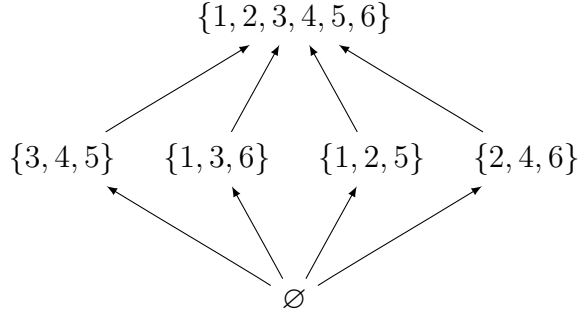
We know $f^{\lambda(\overline{\Gamma})} = \binom{n-2}{r-1}$ by the hook length formula, and from Example 2.4.5, we know the second sum equals the Eulerian number $A_{n-1, r-1}$. Plugging this in gives us the required formula.

$$\text{Vol}(\mathbf{P}(\mathbf{M}_\alpha)) = - \frac{1}{(n-1)!} \left(\alpha \binom{n-2}{r-1} - A_{n-1, r-1} \right)$$

□

We illustrate the above theorem using the graphic matroid $\mathbf{M}(K_4)$ of the complete graph K_4 , which is a connected sparse paving matroid.

Example 2.4.11. Consider $\mathbf{M}(K_4)$, the complete graphic matroid on $\{1, 2, 3, 4, 5, 6\}$. The lattice of cyclic flats of $\mathbf{M}(K_4)$ is given by



By Theorem 2.3.11, this implies that

$$\mathbf{M}(K_4) = -\mathbf{R}(\emptyset) + \mathbf{R}(\{3, 4, 5\}^2) + \mathbf{R}(\{0, 1, 3\}^2) + \mathbf{R}(\{1, 2, 5\}^2) + \mathbf{R}(\{0, 2, 4\}^2)$$

from which it follows that,

$$\begin{aligned} \mathcal{G}(\mathbf{M}(K_4)) &= -3\mathcal{G}(\mathbf{R}_{111000}) + 4\mathcal{G}(\mathbf{R}_{110100}) \\ \text{Vol}(\mathbf{P}(\mathbf{M}(K_4))) &= -\frac{1}{5!} \left(3A_{2,5} - 4 \sum_{p \subseteq \boxplus} f^{\lambda(p)} \right) \\ &= -\frac{1}{5!} (3 \cdot 66 - 4 \cdot 60) \\ &= \frac{42}{5!} \end{aligned}$$

which we know is the volume of $\mathbf{P}(\mathbf{M}(K_4))$ by independent means.

2.5 Conclusion

In this chapter, we have given a new combinatorial approach to finding the volume of matroid polytopes using the cyclic flats of the underlying matroid. The previous answer to this problem is elegant but computationally expensive. Our approach has the benefit of being comparatively efficient whenever the lattice of cyclic flats of the matroid is known. This approach utilizes the work of Hampe [Ham17] on the intersection ring of matroids. In Theorem 2.4.10, we provide an application of this method to compute the volume of a general sparse paving matroid polytope.

Chapter 3

Chow ring calculations

Adiprasito, Huh and Katz [AHK17] proved the Heron-Rota-Welsh conjecture by working with Chow rings of matroids, as defined by Yuzvinsky and Feichtner [FY04a]. The coefficients of the reduced characteristic polynomial occur as degrees of forms in this Chow ring. We recover the coefficients of the Tutte polynomial as degrees of products of csm forms. These are (up to sign) dual to csm classes as constructed by Lopéz de Medrano, Rincón and Shaw in [LRS17]. We also show that these coefficients can also be obtained by counting intersection numbers of certain fans.

3.1 Introduction

To define tropical varieties as balanced polyhedral complexes, we need to review some basic definitions from polyhedral geometry. We refer the reader to [Tho06] for these definitions and basic facts. This section also includes some examples of balanced weights on relevant fans. These examples and definitions can also be found in [MS15].

3.1.1 Polyhedra and polytopes

A *convex combination* of two points $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ is any point of the form

$$\lambda \mathbf{p} + (1 - \lambda) \mathbf{q}$$

where $0 \leq \lambda \leq 1$. The set of all convex combinations of two points is given by the line segment joining these two points, i.e.

$$[\mathbf{p}, \mathbf{q}] := \{\lambda \mathbf{p} + (1 - \lambda) \mathbf{q} : 0 \leq \lambda \leq 1\}$$

A point set Q in \mathbb{R}^n is *convex* if for any two points $\mathbf{p}, \mathbf{q} \in Q$, the interval $[\mathbf{p}, \mathbf{q}]$ is inside Q . A convex set $Q \subseteq \mathbb{R}^n$ is called a *convex cone* if $\lambda Q \subseteq Q$ for any non-negative scalar $\lambda \geq 0$. In this thesis, whenever we refer to a cone, it is implicit that we are talking about a convex cone. A *convex combination* of points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k \in \mathbb{R}^n$, is any point of the form

$$\lambda_1 \mathbf{p}_1 + \lambda_2 \mathbf{p}_2 + \dots + \lambda_k \mathbf{p}_k$$

such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$ and $\lambda_i \geq 0$ for all $i = 1, \dots, k$. For any point set $Q \subseteq \mathbb{R}^n$ the *convex hull* $\text{conv}(Q)$ of Q is defined as the smallest convex subset of \mathbb{R}^n containing Q . Equivalently, it is the set of all convex combinations of points of Q . A *non-negative combination* of points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k \in \mathbb{R}^n$ is any point of the form

$$\lambda_1 \mathbf{p}_1 + \lambda_2 \mathbf{p}_2 + \cdots + \lambda_k \mathbf{p}_k$$

where the coefficients $\lambda_i \geq 0$ for $i = 1, \dots, k$. The *zero cone* $\{\mathbf{0}\}$ is the only bounded cone and it is contained in every other cone. For any set $Q \subseteq \mathbb{R}^n$, the *conical hull* $\text{cone}(Q)$ is defined as the intersection of all cones containing Q . Equivalently, it is the set of all non-negative combinations of points of Q . A cone is called *polyhedral* if it is the conical hull of a finite set of points. That is, for some finite set of points $Q = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$, it is given by

$$\text{cone}(A) := \left\{ \lambda_1 \mathbf{p}_1 + \cdots + \lambda_k \mathbf{p}_k : \lambda_i \geq 0 \text{ for } i = 1, \dots, k \right\}$$

Recall that a *polyhedron* P in \mathbb{R}^n is the intersection of finitely many half-spaces. That is,

$$P = \bigcap_{i=1}^k H_i^{\leq} \quad (3.1)$$

where

$$\begin{aligned} H_i &= \{\mathbf{x} \in \mathbb{R}^n : \omega_i \cdot \mathbf{x} = b_i\} \\ H_i^{\leq} &= \{\mathbf{x} \in \mathbb{R}^n : \omega_i \cdot \mathbf{x} \leq b_i\} \end{aligned}$$

for some vectors $\omega_i \in (\mathbb{R}^n)^*$ for $i = 1, \dots, k$ for some $k \geq 0$. The description given in Equation 3.1 is called the *H-description* of the polyhedron P . We call a halfspace H_j^{\leq} in the *H-description* of P *irredundant* if

$$\bigcap_{i \neq j} H_i^{\leq} \neq P.$$

Putting ω_i 's in rows to form a $k \times n$ matrix A , and writing $\mathbf{b} = (b_1, \dots, b_k)$, we can rewrite the polyhedron P as

$$P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\} \quad (3.2)$$

A polyhedral cone is then a polyhedron for the case when $\mathbf{b} = \mathbf{0}$. The *affine span* of a polyhedron P is the smallest affine subspace containing P , and the *dimension* of P is the dimension of the linear space which is a translate of the affine span of P . A hyperplane H is a *supporting hyperplane* of the polyhedron P if $P \cap H \neq \emptyset$, and either $P \subseteq H^{\leq}$ or $P \subseteq H^{\geq}$. A (nonempty) *face* of P is the intersection of P with one of its supporting hyperplanes. A face of P determined by $\omega \in (\mathbb{R}^n)^*$ is defined to be

$$\text{face}_\omega(P) = \{\mathbf{x} \in P : \omega \cdot \mathbf{x} \leq \omega \cdot \mathbf{y} \text{ for all } \mathbf{y} \in P\}$$

The empty set \emptyset is by convention a face of P . We call it the *empty face* of P . The polyhedron P is itself a face of P . We call it the *full face* of P . Faces of a polyhedron are themselves polyhedra. The k -dimensional faces of P will be referred to as k -*faces* of P . The 0-faces of P are called *vertices* and the 1-faces are called *edges*. Given $\dim(P) = d$, the $(d-1)$ -faces of P are called *facets* and the $(d-2)$ -faces are called *ridges*. The number of k -faces of P is denoted as $f_k(P)$ and are called k -th face numbers of P . The f -vector of a d -dimensional polyhedron P is the vector

$$f(P) := \left(f_0(P), f_1(P), \dots, f_d(P) \right)$$

The faces of a given polyhedron Q are naturally ordered by set inclusion. This give rise to a poset, which is in fact a lattice, called the *face lattice* $\mathcal{F}(Q)$ of Q . We denote the covering relation in this poset by \prec . Two polyhedra P and Q are *combinatorially equivalent* if their face lattices are isomorphic as posets.

A bounded polyhedron is called a *polytope*. A d -dimensional polytope P is called a *simplex* or d -*simplex* if it has $d+1$ vertices. We call a d -dimensional polytope *simplicial* if every facet of P is a d -simplex. We call a d -dimensional polytope *simple* if every vertex of P is the intersection of exactly d facets.

We fix a pair of dual lattices M and N with a pairing denoted by $a \cdot b$ for $a \in M$ and $b \in N$. We should be thinking of M as the character lattice of a torus and N as the lattice of one parameter subgroups. For almost all definitions that follow, considering them as formal lattices isomorphic to \mathbb{Z}^n will suffice. Let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong (\mathbb{R}^n)^*$ be the corresponding dual vector spaces. We will consider polytopes residing in $M_{\mathbb{R}}$, and cones in $N_{\mathbb{R}}$.

3.1.2 Fans

A *polyhedral fan* Σ in \mathbb{R}^n is a finite collection of nonempty polyhedral cones in \mathbb{R}^n such that

- if a cone σ is in Σ , then all of its nonempty faces are also in Σ , and
- the intersection of any two cones in Σ is a face of both.

We call a polyhedral fan Σ *pure* if each maximal cone in Σ has the same dimension. A polyhedron is called *rational* if the matrix A (in Equation 3.2) consists of rational entries (in general this means entries of A are in the underlying lattice N). In the case of a cone, this means all the generators are rational vectors (vectors with rational entries). The union of all the cones in a fan Σ is called the *support* of Σ , and is denoted by $|\Sigma|$. That is,

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$$

Note that different fans can have the same support. We call a fan *complete* if $|\Sigma| = \mathbb{R}^n$. We denote the set of all cones of codimension k of Σ by $\Sigma^{(k)}$. We will only be considering

polyhedral fans in this thesis, so we can skip the adjective. Let P be a full dimensional polytope in \mathbb{R}^n . The *inner normal fan* of P is defined as the polyhedral fan $\mathcal{N}(P)$ consisting of the cones

$$\mathcal{N}(F; P) := \{\omega \in (\mathbb{R}^n)^* : \text{face}_{-\omega}(P) = F\}$$

for each nonempty face F of P . The rays of the inner normal fan $\mathcal{N}(P)$ are the inward normals to the facets of P . Two polytopes P and Q are called *normally equivalent* if they have the same normal fan. Note that two full dimensional polytopes in \mathbb{R}^n are normally equivalent if they are combinatorially equivalent and the corresponding pairs of facets have parallel affine hulls.

Definition 3.1.1. [MS15] *Let Σ be a pure fan of dimension d in $N_{\mathbb{R}} \cong \mathbb{R}^n$. A weight on Σ is a function*

$$c : \Sigma^{(0)} \longrightarrow \mathbb{Z}$$

A fan Σ equipped with a weight c on it is called a weighted fan (Σ, c) .

We say that we are only considering integer weights on the top skeleton of the given fan. The *lineality space* of a polyhedron P is the inclusion-maximal linear subspace $V \subseteq \mathbb{R}^n$ such that for $\mathbf{p} \in P$ and $\mathbf{v} \in V$, we have $\mathbf{p} + \mathbf{v} \in P$. A k -dimensional cone σ in $N_{\mathbb{R}}$ is said to be *unimodular* if there exists lattice vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in N$ such that

$$\sigma = \text{cone}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$$

and $\mathbf{v}_1, \dots, \mathbf{v}_k$ can be extended to a lattice basis for N . A fan Σ is called *unimodular* if all of its cones are unimodular. Given a pure rational fan Σ of dimension d in $N_{\mathbb{R}}$, for a cone $\kappa \in \Sigma$, define the *sublattice* of κ

$$N_{\kappa} := N \cap \text{span}_{\mathbb{R}}(\kappa)$$

Let σ be a d -dimensional cone in the fan Σ , and let τ be a codimension one cone of σ . Then N_{σ}/N_{τ} is a one-dimensional lattice. We denote its primitive generator vector by $\mathbf{u}_{\sigma/\tau}$, which is contained in the image of σ inside $(N_{\sigma}/N_{\tau})_{\mathbb{R}}$.

Definition 3.1.2. *Let Σ be a pure rational fan of dimension d in $N_{\mathbb{R}}$. Given $\tau \in \Sigma^{(1)}$, the weighted fan (Σ, c) is said to be balanced at τ if*

$$\sum_{\sigma: \tau \prec \sigma} c(\sigma) \mathbf{u}_{\sigma/\tau} = 0$$

The weighted fan (Σ, c) is balanced if it is balanced at all $\tau \in \Sigma^{(1)}$.

Now we would like to give a simple example of a balanced weighted fan.

Example 3.1.3. *Let Σ be a one dimensional fan in $N_{\mathbb{R}}$ with three cones given by*

$$\begin{aligned} \sigma_1 &= \mathbb{R}_{\geq 0}(1, 0) \\ \sigma_2 &= \mathbb{R}_{\geq 0}(0, 1) \\ \sigma_3 &= \mathbb{R}_{\geq 0}(-1, -1) \end{aligned}$$

Then the weight function $c : \Sigma^{(0)} \longrightarrow \mathbb{Z}$ given by $c(\sigma_1) = c(\sigma_2) = c(\sigma_3) = 1$ is balanced.

Example 3.1.4. Let $\Sigma^{(d-1)}$ be the set of rays of the normal fan $\Sigma = \mathcal{N}(\mathbf{P})$ for a full dimensional lattice polytope $\mathbf{P} \subseteq N_{\mathbb{R}}$. Then $\Sigma^{(d-1)}$ is a 1-dimensional fan with a natural weight function that assigns to each ray the normalized volume of the dual facet in its respective affine hull. The one dimensional skeleton $\Sigma^{(d-1)}$ with this weight function is a balanced fan.

Example 3.1.5. Let $\Sigma^{(1)}$ be the set of $d - 1$ dimensional cones in the normal fan $\Sigma = \mathcal{N}(\mathbf{P})$ of a full dimensional lattice polytope \mathbf{P} in $N_{\mathbb{R}}$. Then $\Sigma^{(d-1)}$ is a $(d - 1)$ -dimensional fan with a natural weight function that assigns to each cone the edge length of the dual edge. The fan $\Sigma^{(1)}$ with this weight function is a balanced fan.

Let $E = \{0, 1, 2, \dots, n\}$, and let \mathbf{M} be a matroid of rank $r = d + 1$. Let N be the lattice given by

$$N = \mathbb{Z}^E / \mathbb{Z}(1, \dots, 1) \quad (3.3)$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$, the standard basis unit vectors be coordinates of N , and let $\mathbf{e}_0 = -\mathbf{e}_1 - \dots - \mathbf{e}_n$. For a subset $S \subseteq E$, let

$$\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$$

in $N_{\mathbb{R}}$. A *flag of flats* \mathcal{F} of \mathbf{M} is a set of subsets of E of the form

$$\mathcal{F} := \left\{ F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \right\}$$

where each F_i is a flat of \mathbf{M} . Moreover, if each F_i is a nonempty proper flat, then we call \mathcal{F} a *flag of nonempty proper flats* of \mathbf{M} . The *Bergman fan* of a matroid \mathbf{M} , denoted by $\Sigma(\mathbf{M})$, is the fan in $N_{\mathbb{R}}$ whose cones are given by

$$\sigma_{\mathcal{F}} := \text{cone}(\mathbf{e}_{F_1}, \mathbf{e}_{F_2}, \dots, \mathbf{e}_{F_k})$$

for every flag of nonempty proper flats $\mathcal{F} = \left\{ F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \right\}$ of \mathbf{M} . This definition is due to Ardila and Klivans from [AK06]. The rays of the Bergman fan $\Sigma(\mathbf{M})$ are $\mathbb{R}_{\geq 0}\mathbf{e}_F$ for each flat F of \mathbf{M} . This fan was studied by Fiechtner and Sturmfels in [FS05]. It is shown that $\Sigma(\mathbf{M})$ is a unimodular pure d -dimensional fan. This implies that the associated toric variety $X(\Sigma(\mathbf{M}))$ is smooth (see [CLS11]). Note that $\Sigma(\mathbf{M})$ is not necessarily a complete fan.

Example 3.1.6. Let $\mathbf{M} = \mathbf{U}_{2,3}$ on the set $E = \{0, 1, 2\}$. In this case the Bergman fan $\Sigma(\mathbf{U}_{2,3})$ is given by three 1-dimensional cones

$$\mathbb{R}_{\geq 0}\mathbf{e}_1, \mathbb{R}_{\geq 0}\mathbf{e}_2, \mathbb{R}_{\geq 0}\{-\mathbf{e}_1 - \mathbf{e}_2\}$$

all of which have a common face, the zero cone.

Example 3.1.7. Let $\mathbf{M} = \mathbf{U}_{n+1-r, n+1}$ be the uniform matroid on the set $E = \{0, 1, 2, \dots, n\}$ of rank $n + 1 - r$. The cones of the Bergman fan $\Sigma = \Sigma(\mathbf{U}_{n+1-r, n+1})$ are of the form

$$\sigma_{\mathcal{F}} = \text{cone}\{\mathbf{e}_{F_1}, \dots, \mathbf{e}_{F_{n-r}}\}$$

for each flag of sets $\emptyset \subsetneq F_1 \subseteq F_2 \subsetneq \dots \subsetneq F_{n-r}$ with the property that $|F_i| = i$ for $i = 1, \dots, n - r$. For this fan, the support $|\Sigma|$ is the set of points in \mathbb{R}^n such that the minimum of $\{0, x_1, \dots, x_n\}$ is achieved at least $r + 1$ times.

Example 3.1.8. Let $\Sigma^{(1)}$ be the d dimensional cones in the Bergman fan $\Sigma(\mathbf{M})$. Having a constant weight on $\Sigma^{(1)}$ make $\Sigma(\mathbf{M})$ into a balanced fan. This fact is equivalent to the matroid axioms for flats, as shown in [Kat16]. Furthermore, the only weights that balance $\Sigma(\mathbf{M})$ are constant weights. Usually when we take $\Sigma(\mathbf{M})$ as weighted fan, we take weights to be 1 on each maximal cone.

A polyhedral complex Δ in \mathbb{R}^n is a finite collection of nonempty polyhedra in \mathbb{R}^n such that

- if a polyhedron Q is in Δ , then all of its nonempty faces are also in Δ , and
- the intersection of any two polyhedra in Δ is a face of both.

So a fan is a special case of a polyhedral complex, where each polyhedron is a cone.

3.2 Tropical varieties and Minkowski weights

In this section, we give a brief overview of what a tropical cycle is and how we can compute intersection numbers for two tropical cycles of complementary dimensions. This will be used later when we give an expression of certain coefficients of the Tutte polynomial as intersection numbers of certain tropical cycles. The tropical cycles we are looking at are those defined in [LRS17] as Minkowski weights on a Bergman fan.

3.2.1 Tropical varieties

The tropical semiring is the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ together with the following addition \oplus and multiplication \otimes

$$\begin{aligned} a \oplus b &= \min(a, b) \\ a \otimes b &= a + b \end{aligned}$$

Note that \oplus is associative, and \otimes is distributive over \oplus . The additive identity is ∞ and the multiplicative identity is 0. The semiring of tropical polynomials $\overline{\mathbb{R}}[x_1, \dots, x_n]$ consists of polynomials with coefficients in $\overline{\mathbb{R}}$ and tropical operations. Given a Laurent polynomial $f \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ such that

$$f = \sum c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$$

where \mathbb{K} is a field with a valuation val , the *tropicalization* $\text{trop}(f) : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as

$$\text{trop}(f)(\mathbf{w}) = \min(\text{val}(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u})$$

The *tropicalization* of the hypersurface $V(f)$ is the tropical hypersurface defined by

$$\left\{ \mathbf{w} \in \mathbb{R}^n \text{ where the min is achieved at least twice} \right\}$$

Note that

$$\text{trop}(\mathbf{x}^{\mathbf{u}}f) = \text{trop}(\mathbf{x}^{\mathbf{u}}) + \text{trop}(f)$$

as functions over \mathbb{R}^n , which means $\text{trop}(\mathbf{x}^{\mathbf{u}}f)(\mathbf{w}) = \mathbf{w} \cdot \mathbf{u} + \text{trop}(f)(\mathbf{w})$. This implies $\text{trop}(V(\mathbf{x}^{\mathbf{u}}f)) = \text{trop}(V(f))$; therefore we only consider varieties in $(\mathbb{K}^{\times})^n$. Let $Y = V(I) \subseteq (\mathbb{K}^{\times})^n$, where I is an ideal in the Laurent polynomial ring $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The *tropicalization of Y* is defined as

$$\text{trop}(Y) = \bigcap_{f \in I} \text{trop}(V(f)) \quad (3.4)$$

The following structure theorem [MS15] relates weighted fans with tropical varieties

Theorem 3.2.1. [MS15] *Let \mathbb{K} be a field with the trivial valuation and let $X \subseteq (\mathbb{K}^{\times})^n$ be an irreducible d -dimensional variety. Then the tropical variety $\text{trop}(X)$ is the support of a balanced polyhedral fan of dimension d .*

This motivates the definition of the following equivalence relation on weighted balanced fans. Let (Σ, c) and (Σ', c') be two weighted d -dimensional fans. We say $(\Sigma, c) \sim (\Sigma', c')$ if $|\Sigma| = |\Sigma'|$ (see Section 3.1.2), and the weight functions c and c' induce the same weight function on each of the common subdivisions. We call the equivalence classes for this relation *tropical cycles* (see [KK18]).

Given two tropical cycles $\mathcal{T} = (\Sigma, c)$ and $\mathcal{T}' = (\Sigma', c')$, their *stable intersection* can be defined (see [JY16]) as the tropical cycle with support

$$\lim_{\epsilon \rightarrow 0} |\Sigma| \cap |(\Sigma' + \epsilon \mathbf{v})| \quad (3.5)$$

for a generic \mathbf{v} , and the weight (or *intersection multiplicities*) given by

$$m(\gamma) = \sum_{\substack{\sigma \in \Sigma \\ \sigma' \in \Sigma'}} c(\sigma)c(\sigma')[N : N_{\sigma} + N_{\sigma'}] \quad (3.6)$$

where the sum is over all $\gamma \subseteq \sigma, \sigma'$ and $\sigma \cap (\sigma' + \mathbf{v}) \neq \emptyset$. In the case of complementary dimensional tropical cycles, the stable intersection will be a set of points with multiplicities. The sum of these multiplicities will be referred to as *intersection number*, and denoted by $i(\mathcal{T} \cdot \mathcal{T}')$.

3.2.2 Minkowski weights

Let Σ be a rational fan in the latticed vector space $N_{\mathbb{R}}$. A *Minkowski weight* of codimension k on Σ is a balanced weight on its codimension k skeleton $\Sigma^{(k)}$. We have already seen examples of a Minkowski weight of codimension $d - 1$ (Example 3.1.4) and Minkowski weight of codimension 1 (Example 3.1.5) on the normal fan $\Sigma(P)$ of a full dimensional lattice polytope. The set of k -dimensional Minkowski weights on Σ forms an additive group that we denote by $\text{MW}_k(\Sigma)$. The *group of Minkowski weights* on Σ is defined as

$$\text{MW}_*(\Sigma) := \bigoplus_{k=0}^d \text{MW}_k(\Sigma)$$

Given a rational fan Σ in the latticed vector space $N_{\mathbb{R}}$, if the toric variety $X(\Sigma)$ associated to Σ is smooth then the Chow cohomology $A^*(X)$ has a combinatorial description given by [FS97]. They showed that $A^k(X)$ is canonically isomorphic to $\text{MW}_k(\Sigma)$. Note that $\text{MW}_0(\Sigma) \cong \mathbb{Z}$, and if Σ is complete then $\text{MW}_n(\Sigma) \cong \mathbb{Z}$. We give two examples of Minkowski weights.

Example 3.2.2. Let \mathbf{P} be a full dimensional rational polytope and let $\Sigma = \mathcal{N}(\mathbf{P})$ be its outer normal fan. Then the function

$$\begin{aligned} \omega_k : \Sigma^{(k)} &\longrightarrow \mathbb{R} \\ \sigma_F &\longmapsto \text{Vol}_k(F) \end{aligned}$$

where σ_F is normal cone to a k -dimensional face F of \mathbf{P} and $\text{Vol}_k(F)$ denote the k dimensional volume form. It is a classically known fact (due to Minkowski) that ω is real-valued Minkowski weight of codimension k .

Example 3.2.3. Let \mathbf{M} be a matroid and let $\Sigma(\mathbf{M})$ be its Bergman fan. López de Medrano, Rincón and Shaw recently showed in [LRS17] that the function

$$\begin{aligned} \text{csm}_k : \Sigma^{(k)} &\longrightarrow \mathbb{N} \\ \sigma_{\mathcal{F}} &\longmapsto \beta(\mathcal{F}) := \prod_{i=0}^k \beta(\mathbf{M}|_{F_{i+1}/F_i}) \end{aligned}$$

where $\mathcal{F} = (\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E)$, and

$$\sigma_{\mathcal{F}} = \text{cone}\{\mathbf{e}_{F_1}, \dots, \mathbf{e}_{F_k}\}$$

is a Minkowski weight of codimension $d - k$. Note that we have considered here the unsigned version of the one defined in [LRS17].

3.3 Forms in the Chow ring

The compactifications of hyperplane arrangement complements were constructed in [DP95] by de Concini and Procesi. These compactifications are now known as “wonderful compactifications”. Feichtner and Yuzvinsky in [FY04b] gave a combinatorial description of their Chow rings. This description is quite formal and can be defined for any geometric lattice. Adiprasito, Huh and Katz in [AHK17] worked with this Chow ring for a matroid to give a proof of the Heron-Rota-Welsh conjecture.

3.3.1 The (reduced) characteristic polynomial

Let \mathbf{M} be a rank $r := d + 1$ matroid on the set $E = \{1, \dots, n\}$, with the rank function $\rho_{\mathbf{M}} : 2^E \rightarrow \mathbb{N}$. The *characteristic polynomial* of M is defined by

$$\chi(\mathbf{M}; t) = \sum_{F \in L(\mathbf{M})} \mu(\hat{0}, F) t^{r - \rho(F)}$$

where $L(\mathbf{M})$ is the lattice of flats and μ is its Möbius function. Let μ_i denote the unsigned coefficient of t^{r-i} in $\chi(\mathbf{M}; t)$. That is,

$$\chi(\mathbf{M}; t) = \mu_0 t^r - \mu_1 t^{r-1} + \dots + (-1)^r \mu_r = \sum_{i=0}^r (-1)^i \mu_i t^{r-i}$$

Note that, by definition of the Möbius function we have

$$\sum_{F \in L(\mathbf{M})} \mu(\hat{0}, F) = 0$$

This implies $\chi(\mathbf{M}; t)$ has a root at 1. This motivates the following definition:

Definition 3.3.1. *The reduced characteristic polynomial of a matroid \mathbf{M} is defined as*

$$\bar{\chi}(\mathbf{M}; t) = \frac{\chi(\mathbf{M}; t)}{t - 1}$$

Let $\bar{\mu}_i$ denote the unsigned coefficients of t^{d-i} of $\bar{\chi}(\mathbf{M}; t)$, i.e.

$$\bar{\chi}(\mathbf{M}; t) = \sum_{i=0}^d (-1)^i \bar{\mu}_i t^{d-i}$$

Let $\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k$ be a k -step flag of flats in \mathbf{M} . We call such a flag *initial* if $\rho(F_i) = i$ for each i , and we call such a flag *descending* if

$$\min(F_1) > \min(F_2) > \dots > \min(F_k) > 0$$

The following counting interpretation of $\bar{\mu}_k$ is due to Huh and Katz [HK12], which trace it back to [Bjo92].

Proposition 3.3.2. *Let k be a positive integer and let D_k denote the set of initial descending k -step flags of flats. Then we have*

$$\bar{\mu}_k = |D_k| \tag{3.7}$$

3.3.2 Bjorner chains

Let $(P, <)$ be a graded poset. For $x, y \in P$, we say that x is covered by y , or y covers x , if $x < y$ and there exists no $z \in P$ such that $x < z < y$. We denote this by writing $x \triangleleft y$. Let $\text{Cov}(P)$ denote the set of covering relations, that is

$$\text{Cov}(P) = \{(x, y) \in P \times P : x \triangleleft y\}$$

Given a labelling function $\lambda : \text{Cov}(P) \rightarrow \mathbb{N}$ on the covering relations, there is an induced labelling on chains in P : If $C : x_1 \triangleleft x_2 \triangleleft \cdots \triangleleft x_{k-1} \triangleleft x_k$ is a chain in P , then the induced label on C is given by

$$\lambda(C) = (\lambda(x_1, x_2), \dots, \lambda(x_{k-1}, x_k))$$

We call it the λ -label of the chain C . A labelling function

$$\lambda : \text{Cov}(P) \rightarrow \mathbb{N}$$

is called an *EL labelling* if for each interval $[x, y]$ in P it satisfies the following two conditions:

1. there is a unique maximal chain $C_{xy} : x < \cdots < y$ in $[x, y]$ with increasing λ -label.
2. the λ -label of C_{xy} is lexicographically least among all saturated chains in $[x, y]$.

For the lattice of flats of a loopless matroid \mathbf{M} on a ground set E along with a total ordering ω on the elements of E , Björner showed in [Bjo92] that the labelling given by

$$\lambda_\omega(F \triangleleft G) = \min_\omega(G \setminus F)$$

is an EL-labelling. We call the unique maximal chain in $[F, G]$ with increasing λ_ω -label, the *Björner chain of interval* $[F, G]$. The unique maximal chain in $[\hat{0}, \hat{1}]$ with increasing λ_ω -label will be referred simply as *Björner chain of \mathbf{M}* .

3.3.3 The Chow ring of a matroid

The following definition of the Chow ring of a matroid \mathbf{M} is due to Yuzvinsky and Feichtner [FY04a]. This ring is extensively used in the proof of the Rota-Heron-Welsh-Brylawski conjecture.

Let \mathbf{M} be a loopless matroid on the ground set $E = \{1, 2, \dots, n\}$ with rank $r = d + 1$, and let $L(\mathbf{M})$, denote the associated lattice of flats. The Chow ring of \mathbf{M} is defined to be

$$A^*(\mathbf{M}) := \frac{\mathbb{Z}[x_F : F \text{ nonempty proper flat in } L(\mathbf{M})]}{I + J} \quad (3.8)$$

where I is the Stanley-Reisner ideal for the (reduced) order complex of $L(\mathbf{M})$ given by

$$I := (x_F x_G : F, G \text{ are incomparable in } L(\mathbf{M}))$$

and J is the linear ideal given by

$$J := \left(\sum_{F \ni i} x_F - \sum_{F \ni j} x_F : i \neq j \text{ for } i, j \in E \right)$$

The ring $A^*(\mathbf{M})$ is a graded ring with generators $x_F \in A^1(\mathbf{M})$. It is shown in [AHK17], that there is an isomorphism of abelian groups

$$\begin{aligned} \deg_{\mathbf{M}} : A^d(\mathbf{M}) &\longrightarrow \mathbb{Z} \\ x_{F_1} x_{F_2} \cdots x_{F_d} &\longmapsto 1 \end{aligned}$$

whenever $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_d$ is a maximal flag of non-empty proper flats. In this ring we have some special elements $\alpha, \beta \in A^1(\mathbf{M})$

$$\begin{aligned} \alpha &= \alpha_i = \sum_{F \ni i} x_F \\ \beta &= \beta_i = \sum_{F \not\ni i} x_F \end{aligned}$$

These are well-defined elements in $A^*(\mathbf{M})$. Given a flag of flats

$$\mathcal{F} := \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E\}$$

We use the notation

$$x_{\mathcal{F}} := \prod_{\substack{F \in \mathcal{F} \\ F \neq \emptyset, E}} x_F$$

Adiprasito, Huh and Katz [AHK17] showed that

$$\beta^k = \sum_{\mathcal{F}} x_{\mathcal{F}} \in A^*(\mathbf{M}) \tag{3.9}$$

where the sum is over all descending k -step flags of nonempty proper flats of \mathbf{M} . This is then used to show

$$\deg(\alpha^{d-k} \beta^k) = \bar{\mu}_k$$

Given a proper flag \mathcal{F} of flats

$$\mathcal{F} := \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E\}$$

we call it *increasing* if the sequence $\min_{\omega}(F_i \setminus F_{i-1})$ is increasing with i , that is,

$$\min_{\omega}(F_1 \setminus F_0) < \min_{\omega}(F_2 \setminus F_1) < \cdots < \min_{\omega}(F_{k+1} \setminus F_k)$$

We now prove the first new result of this section:

Lemma 3.3.3. *In $A^*(M)$*

$$\alpha^k = \sum_{\mathcal{F}} x_{\mathcal{F}} \quad (3.10)$$

where the sum is over the set Ω_k of all increasing proper flags \mathcal{F} of flats of length $k + 1$.

Proof. Let us consider the base case $k = 1$. All proper flags of length 2 are of the form

$$\emptyset \subsetneq F \subsetneq E$$

These will be increasing if $\min_{\omega}(F) < \min_{\omega}(E \setminus F)$, which is true if and only if $0 \in F$. This gives

$$\alpha = \sum_{\substack{F \ni 0 \\ F \neq \emptyset, E}} x_F = \sum_{F \in \Omega_1} x_F$$

This just follows from the definition of α . Now let us assume the equality is true for $2 < k < r$, and consider

$$\begin{aligned} \alpha^k \alpha &= \left(\sum_{\mathcal{F}} x_{\mathcal{F}} \right) \alpha \\ &= \sum_{\mathcal{F}} \alpha x_{\mathcal{F}} \end{aligned}$$

for each increasing proper flag \mathcal{F} of length $k + 1$. The idea is to give a map from Ω_k to Ω_{k+1} for $k < r := r(M)$. We consider a single term in this sum, say $\alpha x_{\mathcal{F}}$, where $\mathcal{F} \in \Omega_k$ is given by

$$\mathcal{F} := \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E\}$$

Let $j = \min(E \setminus F_k)$. Note that

- If $F_k \triangleleft E$, then $\alpha_j x_{\mathcal{F}} = 0$. This is because $j \notin F_k$ and hence $j \notin F_i$ for each $i \leq k$. Therefore, this is just an implication of the Stanley-Reisner relations.
- Otherwise, if F_k is not covered by E , then we have

$$\alpha_j x_{\mathcal{F}} = \left(\sum_{\substack{F \ni j \\ F \neq \emptyset, E}} x_F \right) x_{\mathcal{F}}$$

Since F_i does not contain j for all $i \leq k$, this implies we can simplify the sum as

$$\alpha_j x_{\mathcal{F}} = \left(\sum_{\substack{F \ni j \\ F_k \subsetneq F \subsetneq E}} x_F \right) x_{\mathcal{F}}$$

Since $j \in F$, $\min_{\omega}(F \setminus F_k) = \min_{\omega}(E \setminus F_k) = j < \min_{\omega}(E \setminus F)$, so once expanded, each summand still corresponds to a proper increasing flag of flats, now of length $k + 2$.

Now, we would like to show that each flag in Ω_{k+1} can be obtained from a flag in Ω_k in this manner. Consider a flag $\mathcal{G} \in \Omega_{k+1}$ for $k+1 < r$, as

$$\mathcal{G} := \{\emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_k \subsetneq G_{k+1} \subsetneq G_{k+2} = E\}$$

We remove G_{k+1} from \mathcal{G} to get

$$\mathcal{G}' := \{\emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_k \subsetneq G_{k+2} = E\}$$

We already know that

$$\min_{\omega}(G_k \setminus G_{k-1}) < \min_{\omega}(G_{k+1} \setminus G_k) < \min_{\omega}(E \setminus G_{k+1})$$

and since $E \setminus G_k = (E \setminus G_{k+1}) \sqcup (G_{k+1} \setminus G_k)$, it follows that

$$\min_{\omega}(E \setminus G_k) = \min_{\omega}(\min_{\omega}(G_{k+1} \setminus G_k), \min_{\omega}(E \setminus G_{k+1})) = \min_{\omega}(G_{k+1} \setminus G_k)$$

which implies that

$$\min_{\omega}(G_k \setminus G_{k-1}) < \min_{\omega}(E \setminus G_k)$$

Hence \mathcal{G}' is an increasing flag of flats of length $k+1$, and therefore it is in Ω_k . Furthermore, $x_{\mathcal{G}}$ gives rise to the summand $\alpha x_{\mathcal{G}'}$ by the procedure above. For that, we will take $j = \min_{\omega}(E \setminus G_k)$ and since $\min_{\omega}(E \setminus G_k) = \min_{\omega}(G_{k+1} \setminus G_k)$, $x_{\mathcal{G}}$ will occur as one of the summands when expanding $\alpha_j x_{\mathcal{G}'}$.

Now the last point to show is that, each flag $\mathcal{G} \in \Omega_{k+1}$ comes from a unique flag $\mathcal{F} \in \Omega_k$. Let us assume otherwise. Suppose we have two flags $\mathcal{F}, \mathcal{F}'$ in Ω_k , such that both of them give rise to the same flag $\mathcal{G} \in \Omega_{k+1}$, via the above construction. Let

$$\begin{aligned} \mathcal{F} &:= \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E\} \\ \mathcal{F}' &:= \{\emptyset = F'_0 \subsetneq F'_1 \subsetneq \cdots \subsetneq F'_k \subsetneq F'_{k+1} = E\} \end{aligned}$$

But $\mathcal{F}, \mathcal{F}'$ should agree with \mathcal{G} at all spots except the $(k+1)$ -th spot. Therefore, we already have $F_i = F'_i$ for $i \leq k$ which means $\mathcal{F} = \mathcal{F}'$. □

Example 3.3.4. *Let us do an example of the graphical matroid \mathbf{M}_W of rank 3 matroid on $\{0, 1, 2, 3, 4, 5\}$ corresponding to the graph W given in the figure below. In this case,*

$$\begin{aligned} \alpha &= \sum_{\substack{0 \ni F \\ F \neq \emptyset, E}} x_F \\ \alpha^2 &= x_0 x_{01} + x_0 x_{0123} + x_0 x_{0145} + x_{01} x_{0123} \\ \alpha^3 &= x_0 x_{01} x_{0123} \end{aligned}$$

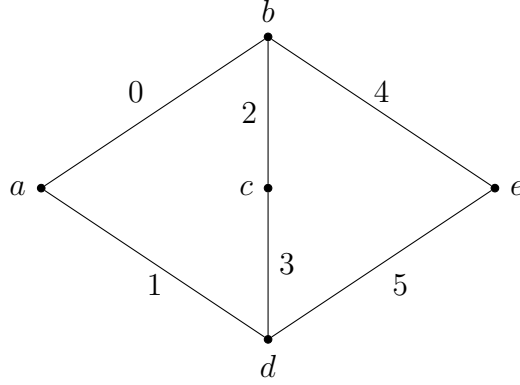


Figure 3.1: Graph W

An implication of the above lemma is that

$$\alpha^d = x_{\mathcal{H}} \tag{3.11}$$

where \mathcal{H} is the Björner chain of the matroid \mathbf{M} . Since the computation in the Chow ring $A^*(\mathbf{M})$ does not depend on the total order on the ground E . Therefore, by picking a suitable total order on E , we can make any maximal chain in $L(\mathbf{M})$ the Björner chain, and hence

$$\alpha^d = x_{\mathcal{F}} \tag{3.12}$$

where \mathcal{F} is any maximal chain. This is also in accordance with Proposition 5.8 in [AHK17].

Motivated by the Minkowski weight csm_k for the Bergman fan of a matroid \mathbf{M} , Ardila, Denham and Huh [ADH18] defined the form $\text{csm}^{d-k}(\mathbf{M}) \in A^{d-k}(\mathbf{M})$ by the formula

$$\text{csm}^{d-k}(\mathbf{M}) = \sum_{\mathcal{F}} \beta(\mathcal{F}) x^{\mathcal{F}}$$

where $x^{\mathcal{F}}$ is Poincaré dual to $x_{\mathcal{F}}$, and the sum is over all flags of flats

$$\mathcal{F} = (\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E)$$

and the beta invariant $\beta(\mathcal{F})$ of the flag \mathcal{F} is given by the product

$$\beta(\mathcal{F}) := \prod_{i=1}^k \beta(\mathbf{M}|F_{i+1}/F_i)$$

We will see in the next section how this can be used to recover certain coefficients of the Tutte polynomial as degrees of forms in $A^*(\mathbf{M})$.

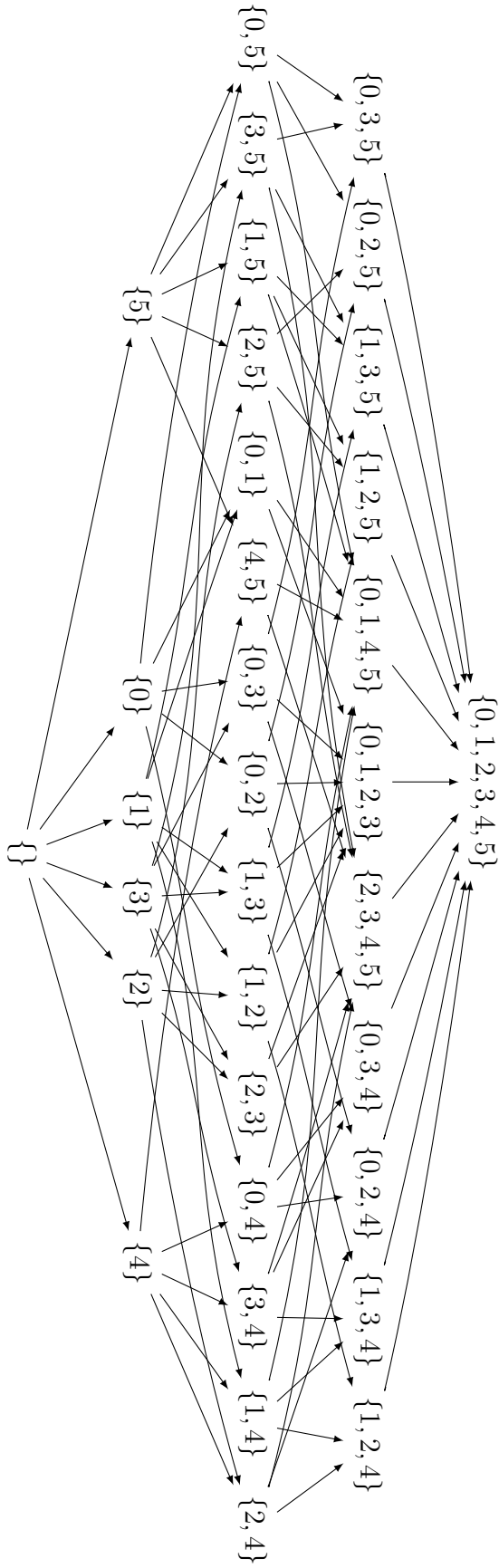


Figure 3.2: Lattice $L(M_w)$ for Example 3.3.4

3.4 The Tutte polynomial

In this section, we introduce the Tutte polynomial of a matroid \mathbf{M} as the generating function of activities over bases of \mathbf{M} . The main reference for this section is [Oxl11]. Given a basis B and an element $e \in E \setminus B$ there is a unique circuit contained in $B \cup e$, called the *fundamental circuit of e with respect to B* . It is given by

$$\gamma(e; B) = \{x \in E : (B \cup e) \setminus x \in \mathcal{B}\}$$

Similarly, given a basis B and an element $b \in B$, there is a unique cocircuit disjoint from $B \setminus b$, called the *fundamental cocircuit of b with respect to B* . It is given by

$$\gamma^*(b; B) = \{x \in E : (B \cup x) \setminus b \in \mathcal{B}\}$$

Note that the cocircuit $\gamma^*(b; B)$ in \mathbf{M} equals the circuit $\gamma(E \setminus B, b)$ in the dual \mathbf{M}^* . We denote this fact by

$$\gamma(b; B, \mathbf{M}) = \gamma(b; E \setminus B, \mathbf{M}^*)$$

Let ω be a linear order on the ground set E . For a basis B , define the set

$$EA(B) := \{e \in E \setminus B : \min_{\omega}(\gamma(e; B)) = e\} \quad (3.13)$$

The elements of $EA(B)$ are called *externally active* with respect to B . In other words, e is externally active with respect to a basis B if it is the ω -smallest element within its fundamental circuit with respect to B . The elements of $E \setminus B$, that are not externally active with respect to B are called *externally passive* with respect to B , and the set of such elements is denoted by $EP(B)$. By definition,

$$EA(B) \sqcup EP(B) = E \setminus B \quad (3.14)$$

Dually, for a basis B , define the set

$$IA(B) = \{b \in B : \min_{\omega}(\gamma^*(b; B)) = b\} \quad (3.15)$$

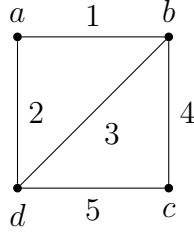
The elements of $IA(B)$ are called *internally active* with respect to B . In other words, b is internally active with respect to a basis B if it is the ω -smallest element within its fundamental cocircuit with respect to B . The elements of B that are not internally active with respect to B are called *internally passive* with respect to B , and the set of such elements is denoted by $IP(B)$. By definition,

$$IA(B) \sqcup IP(B) = B \quad (3.16)$$

Also note that

$$IA(B; \mathbf{M}) = EA(E \setminus B; \mathbf{M}^*) \quad (3.17)$$

$$EA(B; \mathbf{M}) = IA(E \setminus B; \mathbf{M}^*) \quad (3.18)$$

Figure 3.3: The graph G_{\square}

bases	$IA(B)$	$EA(B)$
$\{1, 2, 4\}$	$\{1, 2, 4\}$	\emptyset
$\{1, 2, 5\}$	$\{1, 2\}$	\emptyset
$\{1, 3, 4\}$	$\{1, 4\}$	\emptyset
$\{1, 3, 5\}$	$\{1\}$	\emptyset
$\{1, 4, 5\}$	$\{1\}$	$\{3\}$
$\{2, 3, 4\}$	$\{4\}$	$\{1\}$
$\{2, 3, 5\}$	\emptyset	$\{1\}$
$\{2, 4, 5\}$	\emptyset	$\{1, 3\}$

Table 3.1: Activities on M_{\square}

The Tutte polynomial can be defined as the activity generating function of M

$$t(M; x, y) = \sum_{B \in \mathcal{B}} x^{\iota(B)} y^{\epsilon(B)} \quad (3.19)$$

$$= \sum_{i,j} t_{i,j} x^i y^j \quad (3.20)$$

where the sum is over $\mathcal{B}(M)$, the set of all bases of M . The coefficient $t_{i,j}$ of the Tutte polynomial is the number of bases with internal activity i and the external activity j .

Example 3.4.1. We consider the graphic matroid $M_{\square} = M(G_{\square})$ whose graph is given in Figure 3.3. Its bases and activities are given in table above. The Tutte polynomial is therefore given by

$$\begin{aligned} t(M_{\square}; x, y) &= x^3 + x^2 + x^2 + x + xy + xy + y + y^2 \\ &= x^3 + 2x^2 + x + 2xy + y + y^2 \end{aligned}$$

3.4.1 Gioan-Las Vergnas expression

Given a matroid M on the set E . A flag of subsets of E of the form

$$\emptyset = F'_\epsilon \subsetneq \cdots \subsetneq F'_0 = F_c = F_0 \subsetneq \cdots \subsetneq F_\iota = E$$

is called *connected (ι, ϵ) -filtration* if all of the following conditions are satisfied

1. for every $0 \leq k \leq \iota$, the subset F_k is a flat of \mathbf{M} .
2. for every $0 \leq j \leq \epsilon$, the subset $E \setminus F'_j$ is a flat of \mathbf{M}^* .
3. the subset F_c is a cyclic-flat of \mathbf{M} .
4. the sequence $\min(F_k \setminus F_{k-1})$, $1 \leq k \leq \iota$ is increasing with k .
5. the sequence $\min(F'_{j-1} \setminus F'_j)$, $1 \leq j \leq \epsilon$ is increasing with j .
6. for $1 \leq k \leq \iota$, the minor $\mathbf{M}|F_k/F_{k-1}$ is connected and is not a loop.
7. for $1 \leq j \leq \epsilon$, the minor $\mathbf{M}|F'_{j-1}/F'_j$ is connected and is not an isthmus.

The recent theorem of Gioan and Las Vergnas [GL18] gave an expression of coefficients of the Tutte polynomial as a sum of products of beta invariants of certain minors associated with connected filtrations.

Theorem 3.4.2. *Let \mathbf{M} be a matroid on the set E , then the Tutte polynomial of \mathbf{M} has the following form*

$$t(\mathbf{M}; x, y) = \sum \left(\prod_{1 \leq k \leq \iota} \beta(\mathbf{M}|F_k/F_{k-1}) \right) \left(\prod_{1 \leq j \leq \epsilon} \beta^*(\mathbf{M}|F'_{j-1}/F'_j) \right) x^\iota y^\epsilon$$

where the sum is over all connected filtrations.

Gioan and Las Vergnas call such flags *connected filtrations*. We are only interested in the case of $t(\mathbf{M}; x, 0)$, i.e.

$$t(\mathbf{M}; x, 0) = \sum \left(\prod_{1 \leq j \leq \iota} \beta(\mathbf{M}|F_j/F_{j-1}) \right) x^\iota \quad (3.21)$$

where the sum is over all proper increasing flags of flats

$$\mathcal{F} := \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{\iota-1} \subsetneq F_\iota = E\}$$

of length ι such that for $1 \leq j \leq \iota$, the minor $\mathbf{M}|F_j/F_{j-1}$ is connected and is not a loop. Note that this connectivity condition on minors is redundant as minors that do not satisfy this condition have beta invariant 0. Equation 3.21 is then equivalent to

$$t_{\iota+1,0} = \sum \beta(\mathcal{F}) \quad (3.22)$$

where the sum is over all increasing flags of flats of length $\iota + 1$. What we aim to show now is the following formula that also implies Proposition 8.3. of [ADH18].

Theorem 3.4.3. *For a simple matroid \mathbf{M} , in $A^*(\mathbf{M})$ we have*

$$\deg_{\mathbf{M}}(\alpha^k \cdot \text{csm}^{d-k}(\mathbf{M})) = t_{k+1,0}$$

Proof. We use the definition of $\text{csm}^{d-k}(\mathbf{M})$ and Lemma 3.3.3

$$\deg(\alpha^k \cdot \text{csm}^{d-k}(\mathbf{M})) = \deg\left(\sum_{\mathcal{F}} x_{\mathcal{F}} \cdot \sum_{\mathcal{F}} \beta(\mathcal{F}) x^{\mathcal{F}}\right)$$

where both the sums are over proper flags of flats of length $k+1$. Now, by the Poincaré duality of $A^*(\mathbf{M})$ (as shown in [AHK17]) we have

$$\deg(\alpha^k \cdot \text{csm}^{d-k}(\mathbf{M})) = \sum \beta(\mathcal{F})$$

where the sum is over all increasing flags of flats of length $k+1$. By Equation 3.22, which is essentially due to Gioan and Las Vergnas, we have

$$\deg(\alpha^k \cdot \text{csm}^{d-k}(\mathbf{M})) = t_{k+1,0}$$

□

3.4.2 Counting stable intersection

Here we would like to give certain coefficients of the Tutte polynomial as stable intersection numbers. This is part of joint work with Spencer Backman. We would like to show that

Theorem 3.4.4. *Let \mathbf{M} be a matroid and let $\Sigma(\mathbf{U}_{n+1-k, n+1})$ denote the Bergman fan of $\mathbf{U}_{n+1-k, n+1}$ with weight 1 on each maximal cone. Then*

$$i(\text{csm}_k(\mathbf{M}) \cdot \Sigma(\mathbf{U}_{n+1-k, n+1})) = t_{k+1,0} \quad (3.23)$$

To show this we stably intersect $\text{csm}_k(\mathbf{M})$ and $\mathcal{B}(\mathbf{U}_{n+1-k, n+1})$. Note that our definition of $\text{csm}_k(\mathbf{M})$ differs from that in [LRS17]. The proof goes exactly as the proof of Lemma 5.3 in [HK12]. To the author, the argument was pointed out by Spencer Backman.

Proof. Let $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ be a generic vector such that $v_1 > v_2 > \dots > v_n$. For a flag of nonempty proper flats \mathcal{F} , we consider $\sigma_{\mathcal{F}}$ in $\text{csm}_k(\mathbf{M})$, and consider a point $\mathbf{p} \in |\Sigma(\mathbf{U}_{n+1-k, n+1})| \cap (\sigma_{\mathcal{F}} - \epsilon \mathbf{v})$ for sufficiently small $\epsilon > 0$. This implies we can write $\mathbf{p} = \mathbf{x} - \epsilon \mathbf{v}$ such that

$$x = t_1 \mathbf{e}_{F_1} + \dots + t_k \mathbf{e}_{F_k}$$

for $t_i \geq 0$. Any flag of flats

$$\emptyset = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_k \subsetneq F_{k+1} = E$$

induces a partition of E given by

$$E = \bigsqcup_{j=1}^{k+1} F_j \setminus F_{j-1}$$

For each element $e \in E$, let j_e be such that $e \in F_{j_e} \setminus F_{j_e-1}$. Then for any $i \in E$

$$x_i = \begin{cases} t_{j_i} + \cdots + t_{j_0-1} & \text{if } j_i < j_0 \\ 0 & \text{if } j_i = j_0 \\ -t_{j_0} - \cdots - t_{j_i-1} & \text{if } j_i > j_0 \end{cases}$$

Since we also have that $\mathbf{p} \in |\Sigma(\mathbf{U}_{n+1-k, n+1})|$, this means that

$$\min\{0, x_1 + \epsilon v_1, x_2 + \epsilon v_2, \dots, x_n + \epsilon v_n\}$$

should be achieved at least $k+1$ times. First note that

$$\min_{i \in F_{j_i} \setminus F_{j_i-1}} \left\{ x_i + \epsilon v_i \right\} = x_{\min(F_{j_i} \setminus F_{j_i-1})} + \epsilon v_{\min(F_{j_i} \setminus F_{j_i-1})}$$

Since \mathbf{v} is generic enough, the minimum is achieved uniquely for each $j = 1, \dots, k$. Therefore the minimum is achieved at least k times. For it to be achieved $k+1$ times, 0 has to be minimum, so 0 must not be in any of the F_i 's. Furthermore, since 0 has to be achieved $k+1$ times, this implies

$$\min_{i \in F_{j_i} \setminus F_{j_i-1}} \left\{ x_i + \epsilon v_i \right\} = x_{\min(F_{j_i} \setminus F_{j_i-1})} + \epsilon v_{\min(F_{j_i} \setminus F_{j_i-1})} = 0$$

Let $r_i = \min(F_i \setminus F_{i-1})$. Then this implies

$$v_{r_1} = \frac{1}{\epsilon} x_{r_1} < \cdots < v_{r_k} = \frac{1}{\epsilon} x_{r_k}$$

This implies $x_{r_1} < \cdots < x_{r_k}$, which in turn implies $r_1 > \cdots > r_k > 0$. Hence we have a flag with the property that

$$\min(F_1 \setminus F_0) > \cdots > \min(F_k \setminus F_{k+1}) > \min(F_{k+1} \setminus F_k)$$

The proof that this is the only point where these two cones intersect is exactly the one in [HK12], and furthermore the span of their lattices generate N . This implies that the intersection number is

$$i(\text{csm}_k(\mathbf{M}) \cdot \Sigma(\mathbf{U}_{n+1-k, n+1})) = \sum_{\mathcal{F}} \beta(\mathcal{F})$$

where the sum is over flags of flats with the property that

$$\min(F_1 \setminus F_0) > \cdots > \min(F_k \setminus F_{k+1}) > \min(F_{k+1} \setminus F_k)$$

By changing the total order on the underlying set, this is the same as the sum over increasing filtrations. Hence we have the desired result. \square

3.5 Conclusion

In this chapter, we have given some results concerning the coefficients of the Tutte polynomial of an arbitrary matroid. We gave two expressions of these coefficients: One as degrees of forms in the matroid Chow ring and the other as intersection numbers of tropical cycles. Both of these are new results. The first result is due to the author and the second is part of a collaboration with Spencer Backman. The first result also fits nicely with the ongoing proof of log-concavity of the h -vector in [ADH18].

Chapter 4

Character Polynomials

Character polynomials are polynomials in variables c_k s where the c_k s are class functions on \mathfrak{S}_n that count cycles of length k , and carry information about the irreducible characters of \mathfrak{S}_n . These polynomials were studied by Frobenius [Fro04], Murnaghan [Mur37], Macdonald [Mac79] and many others. More recent among these is the work of Garsia and Goupil [GG09], and Orellana and Zabrocki [OZ15]. We define class functions that can be represented as polynomials in c_k s and have a combinatorial interpretation. Using these we study character polynomials in the special case of two row partitions and hook partitions. As an application, we give a new proof of Rosas' rational expression for the generating function of hook partition stable Kronecker coefficients.

4.1 Introduction

Representation theory of the symmetric group \mathfrak{S}_n employs combinatorics of (integer) partitions of n . The irreducible representations of \mathfrak{S}_n are indexed by partitions of n . One way to generate these irreducible representations is through constructing a vector space M^λ generated by equivalence classes of tableaux, called *tabloids* of shape λ , and then showing that each M^λ contains an irreducible representation of \mathfrak{S}_n as a subspace. The number of tabloids of shape λ can also be viewed as counting certain tilings of the Young diagram of shape λ . This motivates us to introduce class functions over \mathfrak{S}_n that count certain tilings which we call *brick tilings*. In this section, we review the representation theory of \mathfrak{S}_n , and define these class functions. We also recall Doubilet's inversion formula, and the formula for face numbers of permutohedron, which will be used in the later sections.

4.1.1 Partitions and compositions

A *partition* λ of a positive integer n , denoted by $\lambda \vdash n$, is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of positive integers adding up to n . The positive integers λ_i are called *parts* of λ , and the number of parts is called the *length* of λ , denoted as $\ell(\lambda)$. If we want to emphasize that $\ell(\lambda) = r$, we write $\lambda \vdash_r n$. In relation to λ , the integer n is called the *weight* of λ , and it is denoted by $|\lambda|$. It is also useful to write $\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$,

where m_i denote the multiplicity of i in the partition λ . Given a partition $\lambda \vdash n$, the *reduced partition* $\langle \lambda \rangle$ is a partition of $n - \lambda_1$ defined as

$$\langle \lambda \rangle := (\lambda_2, \dots, \lambda_r) \vdash |\lambda| - \lambda_1 \quad (4.1)$$

Similarly, for a partition $\lambda \vdash k$, and a positive integer $n \geq k + \lambda_1$, the *augmented partition* $\lambda[n]$ is a partition of n defined as

$$\lambda[n] := (n - k, \lambda_1, \dots, \lambda_r) \quad (4.2)$$

In other words, the *reduced partition* $\langle \lambda \rangle$ is a partition we get by removing the first part of λ , while the augmented partition is a partition we get by appending a suitable first part to λ . We identify a partition λ with its Young diagram, which is a finite collection of unit cells arranged in left justified rows with λ_i cells in the i th row. A Young tableau of shape λ is a labeling of the cells of the Young diagram of λ with integers $1, 2, \dots, n$, with each number occurring exactly once.

Example 4.1.1. *Let $\lambda = (2, 2, 1)$ be a partition of 5. The multiplicity notation for λ would be $(1, 2^2)$. The partition λ gives rise to the reduced partition $(2, 1)$ of 3, and the augmented partition $(n - 5, 2, 2, 1)$ of $n \geq 7$.*

A composition μ of a positive integer n is a sequence $(\mu_1, \mu_2, \dots, \mu_r)$ of positive integers adding to n . This is denoted as $\mu \vDash n$. We extend the definitions and notations introduced above for partitions to compositions. Given a composition μ , we denote by $\tilde{\mu}$ the partition obtained by rearranging the parts of μ in weakly decreasing order. For any positive integer n , let $\mathbf{Comp}(n)$ denote the set of all compositions of n .

We define a partial order on $\mathbf{Comp}(n)$ in the following manner: Given two compositions $\nu = (\nu_1, \dots, \nu_s)$ and $\mu = (\mu_1, \dots, \mu_t)$ in $\mathbf{Comp}(n)$, we say that μ covers ν , and write $\nu \prec \mu$ if $\ell(\mu) = \ell(\nu) + 1$ and there exists a unique j such that

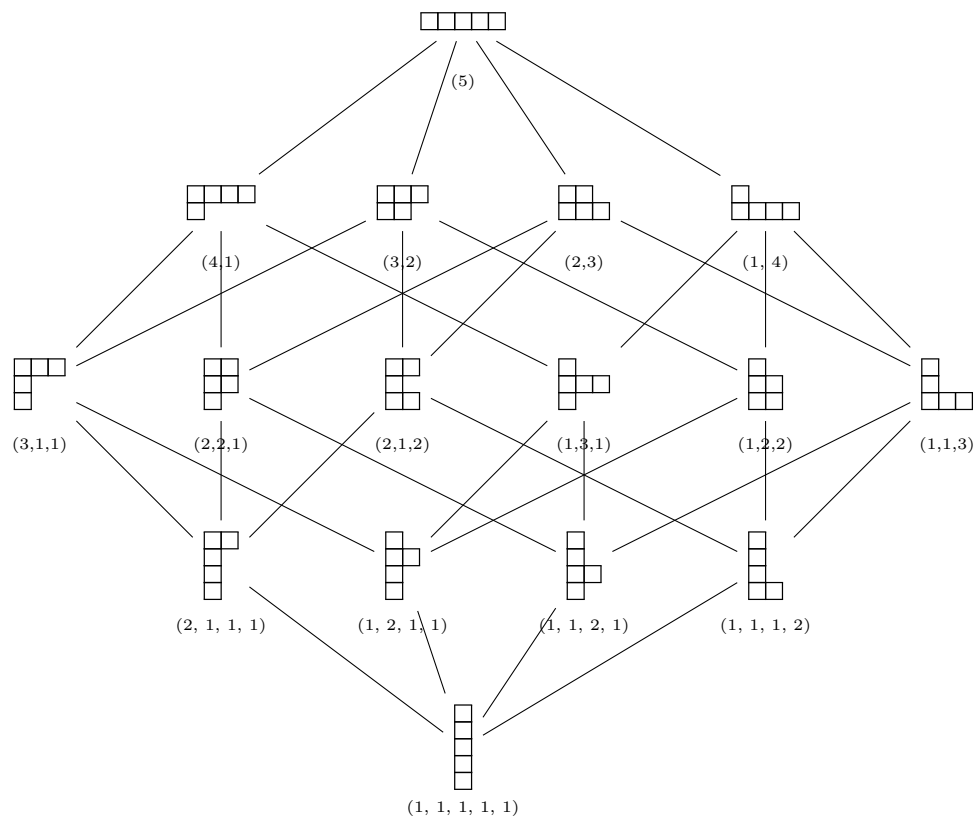
$$\mu_i = \begin{cases} \nu_i & \text{for } i < j \\ \nu_i + \nu_{i+1} & \text{for } i = j \\ \nu_{i+1} & \text{for } i > j \end{cases}$$

i.e. the covering relations are given by adding adjacent entries. The partial order on $\mathbf{Comp}(n)$ induced by this relation is denoted by \leq . For instance, in Figure 4.1 we draw the Hasse diagram of this partial order on $\mathbf{Comp}(5)$. For $n > 1$, there is an anti-isomorphism of posets between $\mathbf{Comp}(n)$ and $B(n - 1)$, the boolean poset of order $n - 1$. This map is explicitly given by

$$\begin{aligned} f : \mathbf{Comp}(n) &\longrightarrow B(n - 1) \\ (c_1, c_2, \dots, c_k) &\mapsto \{a_1, \dots, a_{k-1}\} \end{aligned}$$

where $a_j = \sum_{i=1}^j c_i$ are the partial sums of compositions.

We identify the abstract group \mathfrak{S}_n with the group of permutations on the set $[n] = \{1, 2, \dots, n\}$. Let w be an element of \mathfrak{S}_n , then w can be written as a product of pairwise

Figure 4.1: Poset of composition $\text{Comp}(5)$

disjoint cycles, called *cyclic factors* of w . Let r denote the number of these cyclic factors including the fixed points (1-cycles). Let μ_i be their lengths for $i = 1, \dots, r$. By choosing an element j_i for the i -th cyclic factor, we can write

$$w = \prod_{i=1}^r (j_i, wj_i, \dots, w^{\mu_i-1}j_i)$$

This notation is not unique. We can make this unique by choosing j_i such that for all positive integers m ,

$$j_i \geq w^m j_i$$

and for all $i = 1, 2, \dots, r-1$, by taking

$$j_i < j_{i+1}$$

Such a unique decomposition is called the *canonical cycle decomposition* of w . This plays an important role in Foata's first fundamental bijection [FS78]. Note that $\mu := (\mu_1, \mu_2, \dots, \mu_r)$ is a composition of n . The underlying partition $\tilde{\mu}$ is called the *cycle type* of w and is denoted as $\text{cyc}(w)$. We know that two permutations u, w belong to the same conjugacy class in \mathfrak{S}_n if and only if $\text{cyc}(u) = \text{cyc}(w)$.

Example 4.1.2. If $w = 947213865 \in \mathfrak{S}_9$, then we have the canonical cycle decomposition

$$w = (4, 2)(8, 6, 3, 7)(9, 5, 1)$$

with $\text{cyc}(w) = (4, 3, 2) \vdash 9$.

4.1.2 Representation theory of the symmetric group

A tabloid $[t]$ of shape λ is an equivalence class of Young tableaux of shape λ , where we consider two tableaux t and t' equivalent if the entries in each row of t agree with the entries in the corresponding row of t' . Given the set $T(\lambda)$ of all Young tableaux of shape λ , there is a natural action of \mathfrak{S}_n on $T(\lambda)$ by just permuting the labels of the tableaux. This induces an action on tabloids. Given a Young tableau t , the polytabloid e_t associated to t is defined as the linear combination

$$e_t := \sum_{\pi \in C_t} \text{sign}(\pi) \pi[t]$$

where C_t is the column group associated to t , i.e. the subgroup of \mathfrak{S}_n consisting of permutations that only permute elements within each column of t . For each partition $\lambda \vdash n$, the vector space of \mathbb{C} -linear combinations of polytabloids of shape λ gives an irreducible representation of \mathfrak{S}_n over \mathbb{C} . This is referred to as the Specht module S^λ corresponding to $\lambda \vdash n$ in the literature [Ste12]. Let $\text{Cl}(\mathfrak{S}_n)$ denote the vector space of class functions on the group \mathfrak{S}_n over \mathbb{C} . The characters of Specht modules, $(\chi^\lambda)_{\lambda \vdash n}$, gives a basis for $\text{Cl}(\mathfrak{S}_n)$. There is a scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{S}_n}$ on $\text{Cl}(\mathfrak{S}_n)$ defined as

$$\langle \chi^\lambda, \chi^\mu \rangle = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma) \chi^\mu(\sigma)$$

and extended linearly. The decomposition of the permutation character in terms of the irreducible character basis $\{\chi^\mu\}$ is given by Young's rule, which gives:

$$\zeta^\lambda = \sum_{\mu \triangleright \lambda} K_{\mu\lambda} \chi^\mu \quad (4.3)$$

where the coefficients $K_{\mu\lambda}$, are the Kostka numbers, and the sum is over all partitions μ which are greater than or equal to λ in the *dominance order*.

4.1.3 Brick tilings

A *brick* of length j is a labelled horizontal array of j unit cells. We will view it as a $1 \times j$ rectangle. To each j -cycle $a = (a_1 a_2 \cdots a_j)$ in the canonical cycle decomposition of w , we can associate a brick of length j with the i th square labelled a_i . Given $w \in \mathfrak{S}_n$ where w has cycle type $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, we denote by B_w the set of associated bricks of length $\lambda_1, \dots, \lambda_r$ corresponding to each cyclic factor in the canonical cycle decomposition of w . Note that the 1-cycles correspond to 1×1 square bricks.

For a composition $\lambda \vDash n$, a *tiling* of λ by a set of bricks B is a covering of the Young diagram of λ with bricks from B with no overlaps such that no brick is used twice and each cell of λ is covered by some brick from B . We often refer to it as tiling of *shape* λ . An *ordered brick tiling* of $\lambda \vDash k$ by $w \in \mathfrak{S}_n$ is a tiling of the Young diagram of λ by bricks from B_w , where no brick is in more than one row and the order of the bricks in a row is irrelevant. To be more precise, the ordered brick tiling of shape $\lambda = (\lambda_1, \dots, \lambda_r)$ with $w \in \mathfrak{S}_n$ is an ordered tuple (S_1, S_2, \dots, S_r) of disjoint subsets of B_w such that

$$\left| \bigcup_{U \in S_j} U \right| = \lambda_j$$

for all $j = 1, \dots, r$. The set of brick tilings of $\lambda \vDash n$ by $w \in \mathfrak{S}_n$ is denoted by $B_w(\lambda)$. So for each element (S_1, S_2, \dots, S_r) of $B_w(\lambda)$ with $\ell(\lambda) = r$, the set S_i represents the set of bricks used to tile the i -th row of the Young diagram of λ . Notice, since the order of the tiles in a row does not matter, there is no ambiguity in this notation. If need be, we write the tiles S_i explicitly using the canonical cycle decomposition.

We can define an equivalence relation among brick tilings of shape λ as follows: Two brick tilings $S = (S_1, S_2, \dots, S_r)$ and $T = (T_1, T_2, \dots, T_r)$ of a partition $\lambda \vDash_r n$, are equivalent if one is a permutation of the other, i.e.

$$(S_1, \dots, S_r) = (T_{\pi(1)}, \dots, T_{\pi(r)}) \text{ for some } \pi \in \mathfrak{S}_r$$

We refer to these equivalence classes of tilings as *unordered brick tilings* of λ by w , and we denote the set of these equivalence classes by $\tilde{B}_w(\lambda)$.

We say a brick tiling is *crackless* whenever we have exactly one tile in each row. Otherwise, we say it is *cracked*. A *crack* in a brick tiling is the occurrence of two tiles in one row of a Young diagram. If a row contains c many tiles, we say it has $c - 1$ cracks, and the number of cracks in a brick tiling is the sum of the number of cracks in its rows.

For T in $\tilde{B}_w(\lambda)$ (resp. $B_w(\lambda)$), we call λ the *shape of* T and denote it by $\text{sh}(T)$. Furthermore, for any subset $A = \{b_1, b_2, \dots, b_r\}$ of B_w , the *shape of* A , denoted as $\text{sh}(A)$ is the sequence of lengths of bricks b_i in decreasing order.

Example 4.1.3. Consider $\lambda = (2, 1, 2) \vdash 5$ and let

$$\begin{aligned} u &= (3, 1)(4)(5, 2) \\ w &= (2)(3, 1)(4)(5) \end{aligned}$$

be permutations in \mathfrak{S}_5 . We have

$$\begin{aligned} B_u &= \left\{ \begin{bmatrix} 3 & 1 \\ 4 \\ 5 & 2 \end{bmatrix} \right\} \\ B_w &= \left\{ \begin{bmatrix} 2 \\ 3 & 1 \\ 4 \\ 5 \end{bmatrix} \right\} \end{aligned}$$

The diagram of λ is given by $\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$. The ordered and unordered tilings of λ by u and w are given below:

$$\begin{aligned} \tilde{B}_u(\lambda) &= \left\{ \begin{bmatrix} 3 & 1 & 5 & 2 \\ 4 & & 4 & \\ 5 & 2 & 3 & 1 \end{bmatrix} \right\} \\ \tilde{B}_w(\lambda) &= \left\{ \begin{bmatrix} 3 & 1 & 3 & 1 & 3 & 1 & 2 & 4 & 2 & 5 & 4 & 5 \\ 5 & & 4 & & 2 & & 5 & & 4 & & 2 & \\ 2 & 4 & 2 & 5 & 4 & 5 & 3 & 1 & 3 & 1 & 3 & 1 \end{bmatrix} \right\} \end{aligned}$$

and

$$\begin{aligned} B_u(\lambda) &= \left\{ \begin{bmatrix} 3 & 1 \\ 4 \\ 5 & 2 \end{bmatrix} \right\} \\ B_w(\lambda) &= \left\{ \begin{bmatrix} 3 & 1 \\ 5 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 4 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 \\ 4 & 5 \end{bmatrix} \right\}. \end{aligned}$$

Note that all the elements of $\tilde{B}_u(\lambda)$ are crackless, but all the elements of $\tilde{B}_w(\lambda)$ are cracked with one crack each.

Note that a partition is just a non-increasing composition. Therefore, the sets of ordered and unordered brick tilings are well-defined for partitions. The ordered brick tilings are also called ordered brick tabloids in the literature [MR15]. These were first defined and studied in [ER91] (pointed out to the author by one of the referees).

4.1.4 Tiling class functions

In this section, we define class functions on \mathfrak{S}_n that count the number of different types of brick tilings. Let k be a positive integer less than or equal to n and let $\lambda \vdash k$ (or $\lambda \vDash k$). We define functions $\zeta^\lambda, \xi^\lambda, \eta^\lambda : \mathfrak{S}_n \rightarrow \mathbb{N}$ as follows

$$\begin{aligned} \zeta^\lambda(w) &:= \text{the number of ordered brick tilings of } \lambda \text{ by } w, \text{ i.e. } |\tilde{B}_w(\lambda)| \\ \xi^\lambda(w) &:= \text{the number of unordered brick tilings of } \lambda \text{ by } w, \text{ i.e. } |B_w(\lambda)| \\ \eta^\lambda(w) &:= \text{the number of unordered crackless brick tilings of } \lambda \text{ by } w \end{aligned}$$

$\zeta^\lambda(u) = 2$	$\zeta^\lambda(w) = 6$
$\xi^\lambda(u) = 1$	$\xi^\lambda(w) = 3$
$\eta^\lambda(u) = 1$	$\eta^\lambda(w) = 0$

Table 4.1: Combinatorial class functions for Example 4.1.3

These are class functions on \mathfrak{S}_n . What that means is that they are constant on each conjugacy class of \mathfrak{S}_n . Note that the notion of a tiling of $\lambda \vdash k$ with $w \in \mathfrak{S}_n$ when $k \neq n$ still makes sense. For $k > n$, all of the above class functions are identically zero, so we keep the condition $k \leq n$.

Example 4.1.4. *Going back to Example 4.1.3, for the respective partition λ and the permutations u, w , we show the value of these class functions in Table 4.1.*

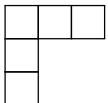
In the example above, the size k of the composition λ is the same as the length n of the permutations u and v . In order to highlight the idea that the above mentioned class functions are well-defined for permutations of any length $k \geq n$, we give another example.

Example 4.1.5. *We give another example where $k < n$. Consider $\lambda = (3, 1, 1) \vDash 5$, and let*

$$\begin{aligned} u &= (4, 3, 2, 1)(7, 6, 5)(9, 8)(10)(11) \\ w &= (3, 2, 1)(6, 5, 4)(9, 8, 7)(11, 10) \end{aligned}$$

be permutations in \mathfrak{S}_{11} . We have

$$\begin{aligned} B_u &= \left\{ \begin{array}{|c|c|c|c|} \hline 4 & 3 & 2 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 7 & 6 & 5 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 9 & 8 \\ \hline \end{array}, \begin{array}{|c|} \hline 10 \\ \hline \end{array}, \begin{array}{|c|} \hline 11 \\ \hline \end{array} \right\} \\ B_w &= \left\{ \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 6 & 5 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 9 & 8 & 7 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 11 & 10 \\ \hline \end{array} \right\} \end{aligned}$$

The diagram of λ is given by . The ordered and unordered tilings of λ by u

and w are given below:

$$\begin{aligned} \tilde{B}_u(\lambda) &= \left\{ \begin{array}{|c|c|c|} \hline 7 & 6 & 5 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 7 & 6 & 5 \\ \hline \end{array} \right\} \\ \tilde{B}_w(\lambda) &= \emptyset \end{aligned}$$

and

$$\begin{aligned} B_u(\lambda) &= \left\{ \begin{array}{|c|c|c|} \hline 7 & 6 & 5 \\ \hline 10 & & \\ \hline 11 & & \\ \hline \end{array} \right\} \\ B_w(\lambda) &= \emptyset \end{aligned}$$

$\zeta^\lambda(u) = 2$	$\zeta^\lambda(w) = 0$	$\zeta^\mu(u) = 2$	$\zeta^\mu(w) = 2$
$\xi^\lambda(u) = 1$	$\xi^\lambda(w) = 0$	$\xi^\mu(u) = 2$	$\xi^\mu(w) = 2$
$\eta^\lambda(u) = 1$	$\eta^\lambda(w) = 0$	$\eta^\mu(u) = 1$	$\eta^\mu(w) = 2$

Table 4.2: Combinatorial class functions for Example 4.1.5

Note that if we take the the partition $\mu = (3, 2)$, i.e. the shape $\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}$, then we have

$$\begin{aligned}\tilde{B}_u(\mu) &= \left\{ \begin{array}{|c|c|c|} \hline 7 & 6 & 5 \\ \hline 11 & 10 & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 7 & 6 & 5 \\ \hline 9 & 8 & \\ \hline \end{array} \right\} \\ \tilde{B}_w(\mu) &= \left\{ \begin{array}{|c|c|c|} \hline 6 & 5 & 4 \\ \hline 11 & 10 & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 9 & 8 & 7 \\ \hline 11 & 10 & \\ \hline \end{array} \right\}\end{aligned}$$

and

$$\begin{aligned}B_u(\mu) &= \left\{ \left[\begin{array}{|c|c|c|} \hline 7 & 6 & 5 \\ \hline 11 & 10 & \\ \hline \end{array} \right], \left[\begin{array}{|c|c|c|} \hline 7 & 6 & 5 \\ \hline 9 & 8 & \\ \hline \end{array} \right] \right\} \\ B_w(\mu) &= \left\{ \left[\begin{array}{|c|c|c|} \hline 6 & 5 & 4 \\ \hline 11 & 10 & \\ \hline \end{array} \right], \left[\begin{array}{|c|c|c|} \hline 9 & 8 & 7 \\ \hline 11 & 10 & \\ \hline \end{array} \right] \right\}\end{aligned}$$

These calculations give us the values of class functions defined above on the permutation u, w as shown in Table 4.2.

The interesting case is when $k = n$. This means that all of the bricks from B_w are utilized to cover a diagram $\lambda \vdash n$. So the ordered brick tilings correspond precisely to tabloids, and hence we have the following result.

Proposition 4.1.6. *For a partition $\lambda \vdash n$, where $n \geq 1$*

1. *The function ζ^λ is the character corresponding to the permutation representation M^λ of \mathfrak{S}_n . Furthermore,*

$$\xi^\lambda = \frac{1}{\lambda!} \zeta^\lambda \quad (4.4)$$

where $\lambda! := m_1! m_2! \cdots$ for $\lambda = (1^{m_1}, 2^{m_2}, \dots)$.

2. *The function η^λ is the indicator function of cycle structure. That is,*

$$\eta^\lambda(w) = \begin{cases} 1 & \text{if } \text{cyc}(w) = \lambda \\ 0 & \text{otherwise} \end{cases} \quad (4.5)$$

Proof. For a partition $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and for $w \in \mathfrak{S}_n$ with cycle type $\mu = (\mu_1, \dots, \mu_r)$, it is classically known (due to Frobenius [Fro04]) that the character of \mathfrak{S}_n corresponding to M^λ is the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_\ell^{\lambda_\ell}$ in the product

$$\prod_{i=1}^r (x_1^{\mu_i} + \cdots + x_\ell^{\mu_i})$$

This precisely counts the ordered brick tilings of w by λ as interpreted in [ER91], or follows directly from definition. Furthermore, to each unordered brick tiling of λ we get $\lambda! = m_1!m_2!\cdots$ ordered brick tilings by permuting the tilings in the parts of the same size (and vice versa). For the second claim, a crackless tiling of λ is only possible if each part of $\text{cyc}(w)$ fits (perfectly) in a unique part of λ . This is just another way of saying that $\text{cyc}(w) = \lambda$. □

Lemma 4.1.7. *If $k \leq n$ are positive integers. Then*

1. *for any $\mu \vDash k$ the following holds as identities of class functions on \mathfrak{S}_n .*

$$\zeta^\mu = \zeta^{\bar{\mu}} \quad , \quad \xi^\mu = \xi^{\bar{\mu}} \quad , \quad \eta^\mu = \eta^{\bar{\mu}}$$

2. *for any $\lambda \vdash n$ then the following holds as an identity of class functions on \mathfrak{S}_n .*

$$\zeta^\lambda = \zeta^{\langle \lambda \rangle}$$

where $\langle \lambda \rangle$ is the reduced partition associated to λ . Moreover, if $\lambda_1 > \lambda_2$, then $\xi^\lambda = \xi^{\langle \lambda \rangle}$ and $\eta^\lambda = \eta^{\langle \lambda \rangle}$.

Proof. The first statement is just a consequence of the fact that the number of brick tilings of any type does not depend on the relative order of parts of λ . The second says that if the weight of λ and the weight of $\text{cyc}(w)$ are equal, then to determine a tiling of λ by w , we just need to determine a tiling of $\langle \lambda \rangle$ by w . Because whatever tiles are not being used will have to fit in λ_1 , and there is only one way of doing that, the order of tiles in rows of λ does not matter. □

An important thing to note for the second claim above is that when $\lambda \vdash n$ and we take ζ^λ as a class function on \mathfrak{S}_n , then $\zeta^\lambda = \zeta^{\langle \lambda \rangle}$, but once we have reduced λ to $\langle \lambda \rangle$, this is no longer a partition of n and therefore we cannot apply the reduction again.

Example 4.1.8. *Let $\lambda = (2, 2, 1) \vdash 5$ and $u = (3, 1)(4)(5, 2)$ and $w = (2)(3, 1)(4)(5)$. Recall*

$$\begin{aligned} \zeta^\lambda(u) &= |\tilde{B}_u(\lambda)| = 2 \\ \zeta^\lambda(w) &= |\tilde{B}_w(\lambda)| = 6 \end{aligned}$$

In this case, $\langle \lambda \rangle = (2, 1) \vdash 3$, the diagram of $\langle \lambda \rangle$ is given by

, and we have

$$\begin{aligned} \tilde{B}_u(\langle \lambda \rangle) &= \left\{ \begin{array}{|c|c|} \hline 5 & 2 \\ \hline 4 & \\ \hline \end{array} , \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 4 & \\ \hline \end{array} \right\} \\ \tilde{B}_w(\langle \lambda \rangle) &= \left\{ \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array} , \begin{array}{|c|c|} \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array} , \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 2 & \\ \hline \end{array} , \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & \\ \hline \end{array} , \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 4 & \\ \hline \end{array} , \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 2 & \\ \hline \end{array} \right\} \end{aligned}$$

which is in accordance with Table 4.1.

Let $T \subseteq B_w$ be a set of bricks coming from $w \in \mathfrak{S}_n$. We denote by ξ_T^λ the number of tilings of $\lambda \vdash n$ with brick set given by T . Then counting tilings with respect to the set of bricks they employ, we have

$$\xi^\lambda(w) = \sum_{T \subseteq B_w} \xi_T^\lambda \quad (4.6)$$

and

$$\eta^\lambda(w) = \sum_{\substack{T \subseteq B_w \\ \text{sh}(T)=\lambda}} 1 \quad (4.7)$$

Furthermore, since for the row shape $(k) \vdash k$, we can count the tilings of all shapes, which gives

$$\zeta^{(k)} = \xi^{(k)} = \sum_{\mu \vdash k} \eta^\mu \quad (4.8)$$

These are some identities that will come in handy later to prove our main result.

4.1.5 Doubilet's inversion formula

Recall Young's rule which states that

$$M^\lambda = \bigoplus_{\mu \geq \lambda} K_{\mu\lambda} S^\mu$$

where $K_{\mu\lambda}$ are the Kostka numbers. If we denote by χ^λ the irreducible character of \mathfrak{S}_n corresponding to the Specht module S^λ , then this implies

$$\zeta^\lambda = \sum_{\mu \geq \lambda} K_{\mu\lambda} \chi^\mu$$

We also know that for partitions $\lambda, \mu \vdash n$, the coefficients $K_{\mu\lambda} \neq 0$ if and only if $\mu \leq \lambda$. Furthermore, $K_{\lambda\lambda} = 1$. This implies that the Kostka matrix $K = (K_{\mu\lambda})$ is invertible. Therefore, we can write the irreducible character χ^λ as a linear combination of ζ^λ 's. Such an inversion formula was given by Doubilet in [Dou73], which states

$$\chi^\lambda = \sum_{\substack{\sigma \in \mathfrak{S}_n: \\ \sigma\lambda \vdash n}} \text{sign}(\sigma) \zeta^{\widetilde{\sigma\lambda}}$$

where $\sigma\lambda$ is the sequence defined as

$$\sigma\lambda = (\lambda_1 + \sigma(1) - 1, \lambda_2 + \sigma(2) - 2, \dots, \lambda_n + \sigma(n) - n)$$

$(\sigma\lambda)_i = \lambda_i + \sigma(i) - i$ for all $i = 1, 2, \dots, n$ and $\widetilde{\sigma\lambda}$ is the rearrangement of this sequence in a weakly decreasing order. Since $\lambda_i = 0$ for $\ell(\lambda) < i \leq n$, this implies that for $\widetilde{\sigma\lambda}$ to be a partition, we must have

$$\sigma(i) - i \geq 0$$

equivalently $\sigma(i) \geq i$ for all $\ell(\lambda) < i \leq n$. This implies $\sigma(n) = n$. Since σ is a bijection and hence injective, $\sigma(n-1) = n-1$. Inductively, we then see that $\sigma(i) = i$ for $\ell(\mu) < i < n$. Let $\widetilde{\mathfrak{S}}_{\ell(\lambda)} \cong \mathfrak{S}_{\ell(\lambda), 1, \dots, 1}$ be the Young subgroup of permutations that pointwise fix all the elements from $\ell(\lambda) + 1$ to n . Then, the above observation implies that the sum on the right of the inversion formula can be taken over $\sigma \in \widetilde{\mathfrak{S}}_{\ell(\lambda)}$ such that $\widetilde{\sigma\lambda} \vdash n$. Hence we can restate the inversion formula as:

Lemma 4.1.9. (*Doubilet's inversion formula*) *Let $\lambda \vdash n$. Then keeping the notation of this section, we have*

$$\chi^\lambda = \sum_{\substack{\sigma \in \widetilde{\mathfrak{S}}_{\ell(\lambda)} \\ \widetilde{\sigma\lambda} \vdash n}} \text{sign}(\sigma) \zeta^{\widetilde{\sigma\lambda}}$$

For our purposes later, we will be able to reduce this sum to a smaller indexing set using the condition that $\sigma\lambda$ needs to be a composition of n . This implies each part of $\sigma\lambda$ should be greater than or equal to 1. For the time being, we show a simple example.

Example 4.1.10. *For a positive integer $n \geq 1$, consider the partition $\lambda = (n-1, 1) \vdash n$. Then Doubilet's inversion formula implies*

$$\chi^{(n-1,1)} = \zeta^{(n-1,1)} - \zeta^{(n)}$$

which reflects the fact that $M^{(n-1,1)} = S^{(n-1,1)} \oplus M^{(n)}$, where $S^{(n-1,1)}$ is the regular representation and $M^{(n)}$ is the trivial representation of \mathfrak{S}_n .

4.1.6 Face numbers of the permutohedron

The standard permutohedron is an example of a convex polytope associated to permutations. To each permutation $w \in \mathfrak{S}_n$, we associate a point in \mathbb{R}^n

$$p_w = (w(1), w(2), \dots, w(n)) \in \mathbb{R}^n$$

The *standard permutohedron* Π_n is defined to be the convex hull of these points, given by

$$\Pi_n = \text{conv}\{p_w : w \in \mathfrak{S}_n\}$$

Note that for each p_w , the sum of all the coordinates equals $1 + 2 + \dots + n$,

$$\sum_{i=1}^n p_{w,i} = 1 + 2 + \dots + n = \binom{n+1}{2}$$

This means that Π_n lies in an affine hyperplane in \mathbb{R}^n and consequently $\dim(\Pi_n) \leq n-1$. Furthermore, for each p_w , the sum of any k coordinates is at least $1 + 2 + \dots + k$, i.e. for any $I \subset [n]$

$$\sum_{i \in I} p_{w,i} \geq 1 + 2 + \dots + |I| = \binom{|I|+1}{2}$$

It is known classically that these inequalities are enough to describe Π_n . Recall that the Stirling number of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ counts the number of partitions of $[n]$ into k blocks. The number of k -dimensional faces are given in terms of Stirling numbers by the following well-known theorem in polytope theory (see [Zie95b]).

Theorem 4.1.11. [Zie95b] *The standard permutohedron Π_n is a simple $(n-1)$ -dimensional convex polytope given by*

$$\Pi_n = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = n \quad \text{and} \quad \sum_{i \in I} x_i \geq \binom{|I|+1}{2}, \quad \text{for all } I \subset [n] \right\}$$

with face numbers given by

$$f_k(\Pi_n) = (n-k)! \left\{ \begin{smallmatrix} n \\ n-k \end{smallmatrix} \right\}$$

for $k = 0, \dots, n-1$.

Since Π_n is a simple convex polytope, its dual Π_n^* is a simplicial convex polytope. We can consider the simplicial complex $\Delta_{n-1} := \partial\Pi_n^*$ (the boundary complex of Π_n^*). Its face numbers are given by

$$f_{k-1}(\Delta_{n-1}) = (k+1)! \left\{ \begin{smallmatrix} n \\ k+1 \end{smallmatrix} \right\}$$

for $k = 0, 1, \dots, n-1$. This implies that the reduced Euler characteristic of Δ_{n-1} is given by

$$\begin{aligned} \sum_{k=0}^{n-1} (-1)^{k-1} f_{k-1}(\Delta_{n-1}) &= \sum_{k=0}^{n-1} (-1)^{k-1} (k+1)! \left\{ \begin{smallmatrix} n \\ k+1 \end{smallmatrix} \right\} \\ &= \sum_{k=1}^n (-1)^{k-1} k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \end{aligned}$$

Since Δ_{n-1} is homeomorphic to the $(n-2)$ -dimensional sphere S^{n-2} , its reduced Euler characteristic is equal to $(-1)^{n-2}$. That is,

$$\sum_{k=1}^n (-1)^k k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = (-1)^n \tag{4.9}$$

We will use this identity in a later section.

4.2 Combinatorics of tiling functions

Characters of the symmetric group \mathfrak{S}_n are examples of class functions. That is, they are constant over conjugacy classes in \mathfrak{S}_n . This implies that their value over a permutation

only depends on the cycle structure of that permutation. Simple examples of class functions are c_i s which count the number of cycles of length i in a permutation. It is a result of Frobenius [Fro04] that every irreducible character of \mathfrak{S}_n is a polynomial function of c_i 's, called *character polynomials*. They were also studied later by Murnaghan [Mur51] and Specht [Spe60]. Macdonald mentions them in [Mac79] and attributes them to Frobenius. Garsia and Goupil [GG09] gave an umbral construction of these polynomials. Kerber also studied them in his book [Ker99] on group actions. Recently, they have reoccurred in the context of representation stability [CEF15].

We study our tiling class functions as polynomials in c_i s. The main result in this section is the identity in Theorem 4.2.5 which equates two alternating sums of class functions. We provide two proofs of it: one using homology on the poset of brick tilings, and the other using the reduced Euler characteristic of the boundary complex of the dual polytope to the permutohedron.

4.2.1 Character polynomials

The class functions $\{c_i\}_{i \in \mathbb{N}}$ are defined as

$$\begin{aligned} c_i : \mathfrak{S}_n &\longrightarrow \mathbb{N} \\ w &\longmapsto \text{number of cycles of } w \text{ of length } i \end{aligned}$$

Let $\mathbb{Q}[c_1, c_2, \dots]$ be the ring of polynomials in c_i 's with rational coefficients. We call a polynomial $q(c_1, c_2, \dots) \in \mathbb{Q}[c_1, c_2, \dots]$ a *class polynomial*. Frobenius [Fro04] showed the following:

Theorem 4.2.1. [Mac79] *Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and let $w \in \mathfrak{S}_n$ be a permutation of cycle type $\text{cyc}(w) = \mu = (\mu_1, \dots, \mu_r) \vdash n$, then $\chi^\lambda(w)$ is equal to the coefficient of $x_1^{\lambda_1 + \ell - 1} x_2^{\lambda_2 + \ell - 2} \dots x_\ell^{\lambda_\ell}$ in the expansion of*

$$\prod_{1 \leq i < j \leq \ell} (x_i - x_j) \prod_{i=1}^r (x_1^{\mu_i} + x_2^{\mu_i} + \dots + x_\ell^{\mu_i})$$

The *character polynomial* $q_\lambda(c_1, c_2, \dots)$ is defined as the *unique* polynomial in $\mathbb{Q}[c_1, c_2, \dots]$ such that for all partitions $\lambda \vdash k$ and $n \geq \lambda_1 + k$, we have

$$\chi^{\lambda[n]}(w) = q_\lambda(c_1(w), c_2(w), \dots) \quad (4.10)$$

for all $w \in \mathfrak{S}_n$. Note that the character polynomial q_λ as a polynomial does not depend on n , and hence gives a uniform description of irreducible characters corresponding to the augmented partition $\lambda[n]$ for all symmetric groups at the same time (with $n \geq \lambda_1 + k$). This is quite remarkable.

Example 4.2.2. *The character polynomial for the empty partition is the constant function 1, which corresponds to the trivial representation. The character polynomial for $\lambda = (1)$ is given by*

$$q_{(1)} = c_1 - 1$$

This is because of the fact that $\lambda[n] = (n-1, 1)$ corresponds to the regular representation.

For $\lambda \vdash k$, there is a special family of class polynomials called *binomial class polynomials* $\binom{c}{\lambda}$ defined to be

$$\binom{c}{\lambda} := \prod_{i=1}^n \binom{c_i}{m_i(\lambda)}$$

where $m_i(\lambda)$ is the multiplicity of i in λ . As an example, for $\lambda = (4, 2, 2, 1, 1)$, we have

$$\binom{c}{\lambda} := \binom{c_4}{1} \binom{c_2}{2} \binom{c_1}{2}$$

Notice that for $\lambda \vdash k$ and $w \in \mathfrak{S}_n$ with $n \geq k$, having an unordered crackless tiling of λ by w is equivalent to choosing $m_i(\lambda)$ bricks from all $c_i(w)$ bricks of w of length i , for each $i = 1, \dots, n$. This implies

$$\eta^\lambda = \binom{c}{\lambda} \quad (4.11)$$

as class functions on \mathfrak{S}_n . This also means that its generating function is given by

$$\sum_{k=0}^{\infty} \left(\sum_{\mu \vdash k} \eta^\mu \right) t^k = \prod_{i=1}^{\infty} (1 + t^i)^{c_i} \quad (4.12)$$

in the ring of formal power series $\mathbb{Q}[c_1, c_2, \dots][[t]]$. Note that in the above equality what we really mean is that the coefficients of t^n are equal as polynomial functions of c_i s.

4.2.2 Characters of two row partitions

We consider the case of $\lambda = (k)$ to generalize Example 4.1.10. In this case, we are looking at the augmented partition $\lambda[n] = (n - k, k) \vdash n$ for $n \geq 2k$. Then Doubilet's Inversion Formula 4.1.9 implies as class functions on \mathfrak{S}_n

$$\begin{aligned} \chi^{\lambda[n]} &= \sum_{\sigma \in \mathfrak{S}_2} \text{sign}(\sigma) \zeta^{\widetilde{\sigma\lambda[n]}} \\ &= \zeta^{(n-k, k)} - \zeta^{(n-k+1, k-1)} \\ &= \zeta^{(k)} - \zeta^{(k-1)} \\ &= \sum_{\mu \vdash k} \eta^\mu - \sum_{\mu \vdash k-1} \eta^\mu \end{aligned}$$

Therefore from Equation 4.12 we have the following identity in $\mathbb{Q}[c_1, c_2, \dots][[t]]$

$$\sum_{k=0}^{\infty} q_{(k)} t^k = (1 - t) \prod_{i=1}^{\infty} (1 - t^i)^{c_i} \quad (4.13)$$

Again by equality, we mean that the coefficients of t^n are equal as polynomial functions of c_i s.

Example 4.2.3. *We can find the first few character polynomials from the above equality. A table of them can also be found in Kerber's book [Ker99].*

$$\begin{aligned} q_{\emptyset} &= 1 \\ q_{(1)} &= \binom{c_1}{1} - 1 \\ q_{(2)} &= \binom{c_2}{1} + \binom{c_1}{2} - \binom{c_1}{1} \\ q_{(3)} &= \binom{c_3}{1} + \binom{c_1}{1} \binom{c_2}{1} + \binom{c_1}{3} - \binom{c_2}{1} - \binom{c_1}{2} \end{aligned}$$

4.2.3 Characters of hook partitions

Let us consider $\lambda = (1^k) \vdash k$. The augmented partition in this case is the hook partition $\lambda[n] = (n - k, 1^k) \vdash n$, for $n \geq 2k$. Recall that the Doubilet Inversion Formula 4.1.9 says

$$\chi^{\lambda[n]} = \sum_{\sigma \in \overline{\mathfrak{S}}_{k+1}} \text{sign}(\sigma) \zeta^{\sigma \widetilde{\lambda[n]}} \quad (4.14)$$

where $\overline{\mathfrak{S}}_{k+1}$ denotes the subgroup of \mathfrak{S}_n consisting of permutations that fixes all the elements from $k + 2$ to n . The sum can further be restricted to those $\sigma \in \overline{\mathfrak{S}}_k$ for which $\sigma(\lambda[n])$ is a weak composition of n , which is to say

$$(\sigma(\lambda[n]))_i = \lambda_i + \sigma(i) - i \geq 0$$

For $i = 2, \dots, k + 1$, $\lambda_i = 1$, we are therefore looking for $\sigma \in \overline{\mathfrak{S}}_{k+1}$ such that $\sigma(i) \geq i - 1$. Note that there is no condition on $\sigma(1)$ because $n \geq 2k$. Therefore, $\sigma(i) \geq i - 1$ for all $i = 1, \dots, k + 1$. This implies such a permutation is uniquely determined by the set

$$S(\sigma) = \{(i, \sigma(i)) : \sigma(i) \geq i\}$$

Fixing $j = k + 1 - \sigma(1)$, we see that there are 2^{j-1} such permutations and each such permutation gives a composition $\mu \vDash j$. On the other hand, given a composition $\mu \vDash j$, we can construct σ by taking

$$\begin{aligned} S(\sigma) = \{(1, k + 1 - j), (k + 1 - j + 1, k + 1 - j + \mu_1), \\ (k + 1 - j + \mu_1 + 1, k + 1 - j + \mu_1 + \mu_2), \dots\} \end{aligned} \quad (4.15)$$

And for such a σ , we have

$$\begin{aligned} \sigma \lambda[n] &= (n - \sigma(1), \mu_1, \mu_2, \dots, \mu_\ell) \\ \langle \sigma \lambda[n] \rangle &= (\mu_1, \dots, \mu_\ell) \vDash j \end{aligned}$$

Combining these we can prove the following proposition:

Proposition 4.2.4. *For $\lambda = (1^k) \vdash k$ and $n \geq 2k$, we have the following equality of class functions*

$$\chi^{\lambda[n]} = \sum_{j=0}^k (-1)^{j+1} \left(\sum_{\mu \neq j} (-1)^{\ell(\mu)} \zeta^\mu \right)$$

Proof. Due to the observations above, we can simplify Equation 4.14 as follows: Fix $\sigma(1) = k + 1 - j$. Now each permutation gives a unique composition μ of j , so we sum over all compositions of j . And we do this over all possible images $\sigma(1)$, which can be any integer from 1 to $k + 1$. This translates to j varying from 0 to k .

$$\chi^{\lambda[n]} = \sum_{\sigma \in \widetilde{\mathfrak{S}}_{k+1}} \text{sign}(\sigma) \zeta^{\sigma\tilde{\lambda}}$$

Since $\zeta^{\sigma\tilde{\lambda}} = \zeta^{\langle \sigma\tilde{\lambda} \rangle}$ by Lemma 4.1.7,

$$\chi^{\lambda[n]} = \sum_{\sigma \in \widetilde{\mathfrak{S}}_{k+1}} \text{sign}(\sigma) \zeta^{\langle \sigma\tilde{\lambda} \rangle} \quad (4.16)$$

Now using Doubilet's Inversion Formula (Lemma 4.1.9) along with our above mentioned observation

$$\chi^{\lambda[n]} = \sum_{j=0}^k (-1)^{k+1-j} \left(\sum_{\mu \neq j} (-1)^{\sum \mu+1} \zeta^\mu \right) \quad (4.17)$$

where we have used the notation

$$\begin{aligned} \sum(\mu + 1) &:= \left(\sum_{i=1}^{\ell(\mu)} (\mu_i + 1) \right) \\ &= \ell(\mu) + j \end{aligned}$$

Simplifying the right hand side of Equation 4.17, we get

$$\chi^{\lambda[n]} = \sum_{j=0}^k (-1)^{k+1-j} \left(\sum_{\mu \neq j} (-1)^{\ell(\mu)+j} \zeta^\mu \right) \quad (4.18)$$

$$= (-1)^{k+1} \sum_{j=0}^k \left(\sum_{\mu \neq j} (-1)^{\ell(\mu)} \zeta^\mu \right) \quad (4.19)$$

This completes the proof. □

Now we move to an interesting observation that simplifies further the calculation of $\chi^{\lambda[n]}$ for $\lambda = (1^k) \vdash k$ where $n \geq 2k$. This result should be thought of as the main result of this chapter.

Theorem 4.2.5. *For a positive integer $j \leq n$, we have the following equality of class functions*

$$\sum_{\mu \vDash j} (-1)^{\ell(\mu)} \zeta^\mu = \sum_{\lambda \vDash j} (-1)^{\ell(\lambda)} \eta^\lambda \quad (4.20)$$

on \mathfrak{S}_n

The left-hand side is an alternating sum over compositions of j , while the right-hand side is an alternating sum over partitions of j . It is worth noticing that not only has the sum on the right got fewer terms but even each term is *smaller*. In the sections to follow, we provide two proofs of this result. After submission, it is pointed out by Mike Zabrocki to the author that both sides of the equation are known to have Frobenius image the symmetric function $h_{n-j}e_j$, where h_i denote the complete homogeneous symmetric function and e_i denote the elementary symmetric function.

4.2.4 The tiling poset

Fix a positive integer $j \leq n$ and $w \in \mathfrak{S}_n$. Then we can consider the set $\text{Til}(w; j)$ of all ordered brick tilings of all compositions of j by w . That is

$$\text{Til}(w; j) := \bigcup \left\{ \tilde{B}_w(\lambda) : \lambda \vDash j \right\}$$

We equip this set with a partial order that is induced by the following covering relations: For two ordered brick tilings $A' = (A'_1, \dots, A'_{k+1})$ and $A = (A_1, \dots, A_k)$ in $\text{Til}(w; j)$, we say A covers A' , and write $A' \triangleleft A$ if both of the following conditions are satisfied:

- $\text{sh}(A') \triangleleft \text{sh}(A)$ in $\text{Comp}(j)$.
- There exists a unique $t \in \{1, \dots, k\}$ such that for all $i = 1, \dots, k$

$$A_i = \begin{cases} A'_i & \text{for } i < t \\ A'_i \cup A'_{i+1} & \text{for } i = t \\ A'_{i+1} & \text{for } i > t \end{cases}$$

Example 4.2.6. *For $w = (31)(4)(5, 2)$, and $j = 5$, we show the Hasse diagram of the poset $\text{Til}(w; j)$ in Figure 4.2.*

Notice that in general $\text{Til}(w; j)$ can be thought of as disjoint union of smaller posets, that is,

$$\text{Til}(w; j) = \bigsqcup_{T \subseteq B_w} \text{Til}(T; j) \quad (4.21)$$

where $\text{Til}(T; j)$ is the set of all ordered brick tilings of all compositions of j , whose brick set is T . From the poset $\text{Til}(w; j)$, we define the following chain complex over the field with 2 elements \mathbb{Z}_2

$$0 \longrightarrow C_{j-1} \longrightarrow C_{j-2} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

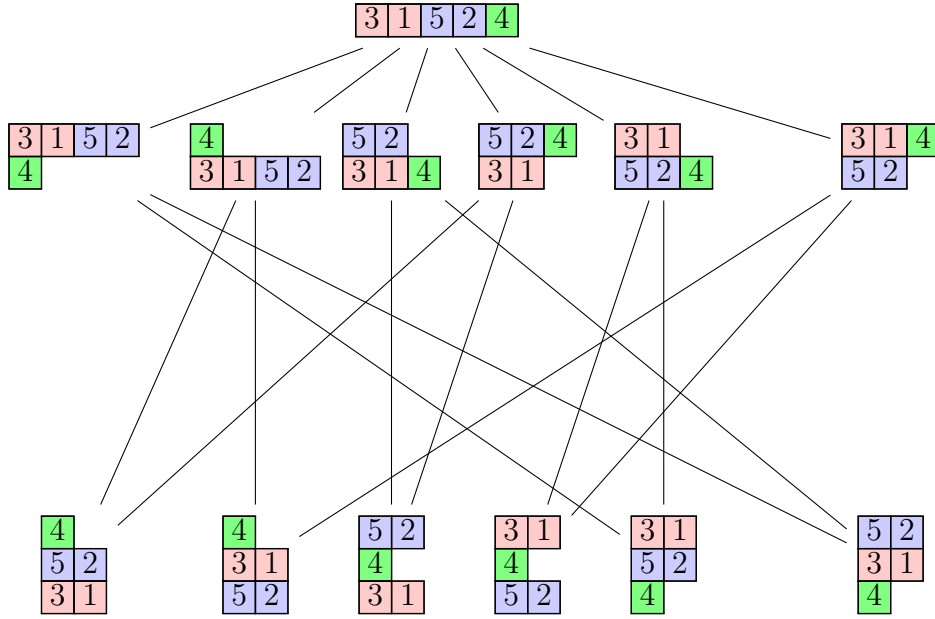


Figure 4.2: Poset $\text{Til}(w; j)$ for $w = (3, 1)(4)(5, 2)$ and $j = 5$

where the chains are defined to be

$$C_i = \mathbb{Z}_2 \{e_A : A \in \text{Til}(w; j) \text{ such that } \text{sh}(A) \vDash_{j-i} j\}$$

where e_T are just formal symbols. The differentials are induced by the covering relations of the poset, i.e. for $i = 1, \dots, j - 1$

$$\begin{aligned} \partial_i : C_i &\longrightarrow C_{i-1} \\ e_A &\longmapsto \sum_{A' \triangleleft T} e_{A'} \end{aligned}$$

Lemma 4.2.7. *For a given $w \in \mathfrak{S}_n$ and a positive integer j , the complex $(C_\bullet, \partial_\bullet)$ is a chain complex.*

Proof. To show we indeed have a chain complex we still need to show that $\partial^2 = 0$. Without loss of generality, assume that there exist some $A'' < A$ such that $[A'', A]$ is an interval of length 2. Then let

$$\partial_{i-1} \circ \partial_i(e_A) = \sum \alpha_{AA''} e_{A''}$$

where the sum is over all A'' for which there exists A' such that $A'' \triangleleft A' \triangleleft A$ and the coefficients are given by

$$\alpha_{AA''} = |\{A' : \text{such that } A'' \triangleleft A' \triangleleft A\}| \pmod{2}$$

Let $\beta_{AA''} = |\{A' : \text{such that } A'' \triangleleft A' \triangleleft A\}|$. We claim that for every pair (A'', A) such that there exists A' with $A'' \triangleleft A' \triangleleft A$, we have $\beta_{AA''} = 2$. Let $A'' = (A''_1, A''_2, \dots, A''_{k+2})$ and $A = (A_1, A_2, \dots, A_k)$. Then we can have one of the following two situations:

1. We have two indices a, b , with $a < b$, such that

$$A_i = \begin{cases} A_i'' & \text{if } i < a \\ A_i'' \cup A_{i+1}'' & \text{if } i = a \\ A_{i+1}'' & \text{if } a < i < b \\ A_{i+1}'' \cup A_{i+2}'' & \text{if } i = b \\ A_{i+2}'' & \text{if } b < i \leq k \end{cases}$$

In this case, there are two possibilities for A' : Either we have $A' = (A_1'', \dots, A_a'' \cup A_{a+1}'', \dots, A_{k+2}'')$, or $A' = (A_1'', \dots, A_b'' \cup A_{b+1}'', \dots, A_{k+2}'')$. That is to say, either we get A' by joining tiles of the a th and $(a+1)$ th rows, or the b th and $(b+1)$ th rows of A'' . This implies $\beta_{AA''} = 2$.

2. We have an index a such that

$$A_i = \begin{cases} A_i'' & \text{if } i < a \\ A_i'' \cup A_{i+1}'' \cup A_{i+2}'' & \text{if } i = a \\ A_{i+2}'' & \text{if } a < i \leq k \end{cases}$$

In this case, $A' = (A_1'', \dots, A_a'' \cup A_{a+1}'', \dots, A_{k+2}'')$ or $A' = (A_1'', \dots, A_{a+1}'' \cup A_{a+2}'', \dots, A_{k+2}'')$, and we also have that $\beta_{AA''} = 2$.

Since we are dealing with coefficients mod 2, in both cases $\alpha_{AA''} = 0$. This proves that $\partial^2 = 0$, and so we indeed have a chain complex. \square

To illustrate the idea concretely in detail, we show an example.

Example 4.2.8. For $w = (3, 1)(5, 2)(4)$, we get the following for $j = 5$

$$\begin{aligned}
 \partial \begin{array}{|c|c|c|c|c|} \hline 3 & 1 & 5 & 2 & 4 \\ \hline \end{array} &= \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 \\ \hline 4 & \end{array} + \begin{array}{|c|c|c|} \hline 5 & 2 & 4 \\ \hline 3 & 1 & \end{array} + \begin{array}{|c|c|c|c|} \hline 3 & 1 & 5 & 2 \\ \hline 4 & \end{array} + \begin{array}{|c|c|c|c|} \hline 4 & \end{array} + \begin{array}{|c|c|c|} \hline 3 & 1 & 4 \\ \hline 5 & 2 & \end{array} + \begin{array}{|c|c|c|} \hline 5 & 2 & \\ \hline 3 & 1 & 4 \\ \hline \end{array} \\
 \partial \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 \\ \hline 4 & \end{array} &= \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 \\ \hline 4 & \end{array} + \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 4 & \\ \hline 5 & 2 \\ \hline \end{array} \\
 \partial \begin{array}{|c|c|c|} \hline 5 & 2 & 4 \\ \hline 3 & 1 & \end{array} &= \begin{array}{|c|c|} \hline 5 & 2 \\ \hline 4 & \\ \hline 3 & 1 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 4 & \\ \hline 5 & 2 \\ \hline 3 & 1 \\ \hline \end{array} \\
 \partial \begin{array}{|c|c|c|c|} \hline 3 & 1 & 5 & 2 \\ \hline 4 & \end{array} &= \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 \\ \hline 4 & \end{array} + \begin{array}{|c|c|} \hline 5 & 2 \\ \hline 3 & 1 \\ \hline 4 & \end{array} \\
 \partial \begin{array}{|c|c|c|c|} \hline 4 & \end{array} &= \begin{array}{|c|c|} \hline 4 & \\ \hline 3 & 1 \\ \hline 5 & 2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 4 & \\ \hline 5 & 2 \\ \hline 3 & 1 \\ \hline \end{array} \\
 \partial \begin{array}{|c|c|c|} \hline 3 & 1 & 4 \\ \hline 5 & 2 & \end{array} &= \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 4 & \\ \hline 5 & 2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 4 & \\ \hline 3 & 1 \\ \hline 5 & 2 \\ \hline \end{array} \\
 \partial \begin{array}{|c|c|} \hline 5 & 2 \\ \hline 3 & 1 \\ \hline 4 & \end{array} &= \begin{array}{|c|c|} \hline 5 & 2 \\ \hline 3 & 1 \\ \hline 4 & \end{array} + \begin{array}{|c|c|} \hline 5 & 2 \\ \hline 4 & \\ \hline 3 & 1 \\ \hline \end{array} \\
 \partial \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 \\ \hline 4 & \end{array} &= \partial \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 4 & \\ \hline 5 & 2 \\ \hline \end{array} = \partial \begin{array}{|c|c|} \hline 5 & 2 \\ \hline 3 & 1 \\ \hline 4 & \end{array} = \partial \begin{array}{|c|c|} \hline 5 & 2 \\ \hline 4 & \\ \hline 3 & 1 \\ \hline \end{array} = \partial \begin{array}{|c|c|} \hline 4 & \\ \hline 3 & 1 \\ \hline 5 & 2 \\ \hline \end{array} = \partial \begin{array}{|c|c|} \hline 4 & \\ \hline 5 & 2 \\ \hline 3 & 1 \\ \hline \end{array} = 0
 \end{aligned}$$

An element $J \in C_i$ for some positive integer $i < j$, can be written as

$$J = \sum_{T \in L} e_T$$

The set L is called the *support* of J , and we denote it by $\text{supp}(J)$. For a subspace V of the chain complex, let \bar{V} be the subspace of V generated by elements in V whose support is crackless brick tilings, and let \widetilde{V} be the subspace of V generated by elements in V whose support is cracked brick tilings. An interesting consequence of Equation 4.21 is the following decomposition:

$$\ker \partial_i = \widetilde{\ker \partial_i} \oplus \overline{\ker \partial_i} \quad \text{and} \quad \text{im} \partial_{i+1} = \widetilde{\text{im} \partial_{i+1}} \oplus \overline{\text{im} \partial_{i+1}}$$

One inclusion is obvious. The other inclusion is the consequence of the fact that a cracked brick tiling and a crackless brick tiling with $j - i$ parts cannot have the same set of bricks for all $i = 1, \dots, j - 1$. Therefore the decomposition follows from the partition given in Equation 4.21. Hence, we can write the homology of the above chain complex as

$$\begin{aligned}
 H_i(C) &= \frac{\ker \partial_i}{\text{im} \partial_{i+1}} \\
 &= \frac{\widetilde{\ker \partial_i} \oplus \overline{\ker \partial_i}}{\widetilde{\text{im} \partial_{i+1}} \oplus \overline{\text{im} \partial_{i+1}}}
 \end{aligned}$$

Notice that in proving $\partial^2 = 0$, we concluded something more than that. We proved that each subinterval of rank 2 in $\widetilde{\text{Til}(w; j)}$ is isomorphic to the boolean interval of length $B(2)$ as posets, which implies $\widetilde{\text{im}\partial_{i+1}} = \widetilde{\text{ker}\partial_i}$. This simplifies the above expression to

$$H_i(C) = \frac{\overline{\text{ker}\partial_i}}{\text{im}\partial_{i+1}}$$

Now notice that every crackless brick tiling with $k - i$ parts is in the kernel of ∂_i , hence

$$H_i(C) = \frac{\overline{C_i}}{\text{im}\partial_{i+1}}$$

We know that $\text{im}\partial_{i+1}$ is generated by $\partial_i e_A$ for $e_A \in C_{i+1}$. The crackless part of $\text{im}\partial_{i+1}$ is generated by the image of those $e_A \in C_{i+1}$ which have exactly one crack, and for each such A we have

$$\partial_{i+1} e_A = e_B + e_{B'}$$

where B and B' are obtained by “cracking” A in one of the two possible ways. This implies we can think of $H_i(C)$ as

$$H_i(C) = \frac{\overline{C_i}}{(e_B = e_{B'})}$$

since this says $e_B = e_{B'}$, whenever the Young diagram of B differs from B' by a simple transposition. Now notice that if $e_A \in \overline{C_i}$, then for every permutation A' of A , $e_{A'} \in \overline{C_i}$. But the relation $e_B = e_{B'}$ identifies all of them since all permutations are generated by transpositions. This implies $H_i(C)$ is generated by e_A , where A is an unordered crackless brick tiling with $j - i$ parts. Hence

$$\dim H_i(C) = \sum_{\substack{\lambda \vdash j \\ \ell(\lambda) = j - i}} \eta^\lambda(w) \quad (4.22)$$

Now using this, we provide a proof for Theorem 4.2.5.

Proof. Given a positive integer $j \leq n$, for any permutation $w \in \mathfrak{S}_n$, we construct the poset $\text{Til}(w; j)$, and consider the chain complex associated to it as defined above. The construction implies that $\dim(C_i)$ is given by

$$\dim(C_i) = \sum_{\substack{\mu \vDash j \\ \ell(\mu) = j - i}} \zeta^\mu(w)$$

Now, we can compute the Euler characteristic of this chain complex as an alternating sum of these dimensions, which gives

$$\begin{aligned} \sum_{i=0}^{j-1} (-1)^i \dim(C_i) &= \sum_{i=0}^{j-1} (-1)^i \sum_{\substack{\mu \vDash j \\ \ell(\mu) = j - i}} \zeta^\mu(w) \\ &= \sum_{\mu \vDash j} (-1)^{j - \ell(\mu)} \zeta^\mu(w) \end{aligned}$$

Computing the Euler characteristic via the alternating sum of dimensions of homologies (employing Equation 4.22) gives

$$\begin{aligned} \sum_{i=0}^{j-1} (-1)^i \dim(H_i) &= \sum_{i=0}^{j-1} (-1)^i \sum_{\substack{\lambda \vdash j \\ \ell(\lambda)=j-i}} \eta^\lambda(w) \\ &= \sum_{\lambda \vdash j} (-1)^{j-\ell(\lambda)} \eta^\lambda(w) \end{aligned}$$

Since both of the them are Euler characteristic of the same complex, they are equal for each $w \in \mathfrak{S}_n$. This implies that the equality holds as an equality of class functions on \mathfrak{S}_n . \square

4.2.5 Counting proof

The proof in the previous section is an application of homology to combinatorics. We also provide another combinatorial proof below.

Proof. Since $\zeta^\lambda = \zeta^\mu$ for $\mu \vdash j$ and $\lambda \vDash j$ such that $\tilde{\lambda} = \mu$, we can write

$$\sum_{\mu \vDash j} (-1)^{\ell(\mu)} \zeta^\mu = \sum_{\mu \vdash j} (-1)^{\ell(\mu)} \sum_{\tilde{\lambda}=\mu} \zeta^\lambda$$

The sum $\sum_{\tilde{\lambda}=\mu} \zeta^\lambda(w)$ counts the number of ordered tilings by w of all compositions λ whose underlying partition is μ . This can also be regarded as counting all permutations (of parts) of unordered brick tilings of μ . Therefore, we can rewrite

$$\sum_{\mu \vDash j} (-1)^{\ell(\mu)} \sum_{\tilde{\lambda}=\mu} \zeta^\lambda(w) = \sum_{\mu \vdash j} (-1)^{\ell(\mu)} \ell(w)! \xi^\mu(w)$$

where $\xi^\mu(w)$ is the number of unordered tilings of μ by w . For a subset of bricks $T \subseteq B_w$, let ξ_T^μ be the number of unordered brick tilings of μ by T . Then we can count the sum over T 's

$$\begin{aligned} \sum_{\mu \vDash j} (-1)^{\ell(\mu)} \ell(\mu)! \xi^\mu(w) &= \sum_{\mu \vdash j} (-1)^{\ell(\mu)} \ell(\mu)! \sum_{T \subseteq B_w} \xi_T^\mu \\ &= \sum_{T \subseteq B_w} \sum_{\mu \vdash j} (-1)^{\ell(\mu)} \ell(\mu)! \xi_T^\mu \\ &= \sum_{T \subseteq B_w} \sum_{k=1}^j \sum_{\substack{\mu \vdash j \\ \ell(\mu)=k}} (-1)^{\ell(\mu)} \ell(\mu)! \xi_T^\mu \\ &= \sum_{T \subseteq B_w} \sum_{k=1}^j (-1)^k k! \sum_{\substack{\mu \vdash j \\ \ell(\mu)=k}} \xi_T^\mu \end{aligned}$$

Since the set of tilings counted in $\sum_{\ell(\mu)=k}^{\mu \vdash j} \xi_T^\mu$ is in one to one correspondence with unordered partitions of T into k sets, these are counted by Stirling numbers of the second kind $\left\{ \begin{matrix} \ell(\text{sh}(T)) \\ k \end{matrix} \right\}$.

$$\sum_{T \subseteq B_w} \sum_{k=1}^j (-1)^k k! \sum_{\substack{\mu \vdash j \\ \ell(\mu)=k}} \xi_T^\mu(\pi) = \sum_{T \subseteq B_w} \sum_{k=1}^j (-1)^k k! \left\{ \begin{matrix} \ell(\text{sh}(T)) \\ k \end{matrix} \right\}$$

We have encountered the inner alternating sum before, and from Equation 4.9 it simplifies to $(-1)^{\ell(\text{sh}(T))}$. Hence,

$$\begin{aligned} \sum_{\mu \vdash j} (-1)^{\ell(\mu)} \zeta^\mu &= \sum_{T \subseteq B_w} (-1)^{\ell(\text{sh}(T))} \\ &= \sum_{\lambda \vdash j} \sum_{\substack{T \subseteq B_w \\ \text{sh}(T)=\lambda}} (-1)^{\ell(\lambda)} \\ &= \sum_{\lambda \vdash j} (-1)^{\ell(\lambda)} \sum_{\substack{T \subseteq B_w \\ \text{sh}(T)=\lambda}} 1 \end{aligned}$$

Now the interior sum counts the number of unordered tilings of μ by B_w as indicated by Equation 4.7. Therefore

$$\sum_{\mu \vdash j} (-1)^{\ell(\mu)} \zeta^\mu = \sum_{\lambda \vdash j} (-1)^{\ell(\lambda)} \eta^\lambda(w)$$

□

Let us illustrate the Theorem 4.2.5 using a small example:

Example 4.2.9. *In the Table 4.3, we take the case of $j = 4$. From the Table 4.3, we can see that $\sum_{\mu \vdash 4} (-1)^{\ell(\mu)} \zeta^\mu$ is equal to*

$$-\eta^{(4)} + \eta^{(3,1)} + \eta^{(2,2)} - \eta^{(2,1,1)} + \eta^{(1,1,1,1)}$$

4.3 Applications

In order to continue our study of generating functions of character polynomials and apply our results to get some previously known identities, we define the notion of stability in the first subsection. The generating function relevant to our discussion is the cycle-index generating function. We go back to our main theorem (Theorem 4.2.5) and use it to derive Goupil's generating function identity [Gou99] for hook partitions. Lastly, combining this identity with stability of the cycle-index generating function, we were able to provide a new proof of Rosas' formula [Ros00].

$$\begin{array}{rcl}
\zeta^{(4)} & = & \eta^{(4)} + \eta^{(3,1)} + \eta^{(2,2)} + \eta^{(2,1,1)} + \eta^{(1,1,1,1)} \\
\zeta^{(3,1)} & = & \eta^{(3,1)} + 2\eta^{(2,1,1)} + 4\eta^{(1,1,1,1)} \\
\zeta^{(1,3)} & = & \eta^{(3,1)} + 2\eta^{(2,1,1)} + 4\eta^{(1,1,1,1)} \\
\zeta^{(2,2)} & = & 2\eta^{(2,2)} + 2\eta^{(2,1,1)} + 12\eta^{(1,1,1,1)} \\
\zeta^{(2,1,1)} & = & 2\eta^{(2,1,1)} + 6\eta^{(1,1,1,1)} \\
\zeta^{(1,2,1)} & = & 2\eta^{(2,1,1)} + 6\eta^{(1,1,1,1)} \\
\zeta^{(1,1,2)} & = & 2\eta^{(2,1,1)} + 6\eta^{(1,1,1,1)} \\
\zeta^{(1,1,1,1)} & = & 24\eta^{(1,1,1,1)}
\end{array}$$

Table 4.3: Example for $j = 4$

4.3.1 Stability for sequences of polynomials and power series

We say a sequence $(f_n)_{n \in \mathbb{N}}$ of polynomials $f_n \in \mathbb{Q}[x]$ *stabilizes* to $f \in \mathbb{Q}[[x]]$ if, for each positive integer k , there exists a positive integer N_k such that, for all $m, n > N_k$, we can find $f \in \mathbb{Q}[x]$ satisfying

$$[x^k]f_m = [x^k]f_n = [x^k]f$$

where $[x^k]f$ denotes the coefficient of x^k in f . If f exists, then it is seen to be unique. This definition can also be generalized to the multivariable case: We say a sequence $(f_n)_{n \in \mathbb{N}}$ of multivariable polynomials $f_n \in \mathbb{Q}[x_1, x_2, \dots, x_r]$ *stabilizes* to $f \in \mathbb{Q}[[x_1, \dots, x_r]]$ if for each sequence of positive integers $k = (k_1, \dots, k_r)$, there exists a positive integer N_k such that, for all $m, n > N_k$, we can find $f \in \mathbb{Q}[[x_1, \dots, x_r]]$ satisfying

$$[x_1^{k_1} \cdots x_r^{k_r}]f_m = [x_1^{k_1} \cdots x_r^{k_r}]f_n = [x_1^{k_1} \cdots x_r^{k_r}]f$$

Example 4.3.1. *The sequence $f_n(x) = x^n + x^{n-1} + \cdots + x + 1$ stabilizes to $f(x) = \frac{1}{1-x}$. On the other hand the sequence $f_n(x) = x^n$ does not stabilize. An example of the multivariable case would be the sequence*

$$g_n(x, y) = \sum_{i+j=n} x^i y^j$$

This sequence stabilizes to

$$g(x, y) = \frac{1}{1-x} \frac{1}{1-y}$$

For a commutative ring R and an element $f \in R[[t]]$, we say the *coefficients of f stabilize* if there exists a least integer k such that for all $n > k$, we have

$$[t^n]f = [t^{n+1}]f$$

This is called the *coefficient of stabilization*, and such a k is called the *point of stability*.

Example 4.3.2. *Given a polynomial $p(t) = \sum a_i t^i \in \mathbb{Q}[t]$, the coefficients of the power series*

$$\frac{p(t)}{1-t} \in \mathbb{Q}[[t]]$$

stabilize. The coefficient of stabilization is $p(1) = \sum a_i$ and the point of stability is $\deg(p) + 1$.

We require a generalization of the above notion for the ring $\mathbb{Q}[c_1, c_2, \dots][[t]]$ of formal power series. Consider an element $f \in \mathbb{Q}[c_1, c_2, \dots][[t]]$, given by

$$f = \sum_{n=0}^{\infty} f_n t^n$$

where $f_n \in \mathbb{Q}[c_1, c_2, \dots]$ are polynomials in c_i s. We say f stabilizes if the sequence of polynomials $(f_n)_{n \in \mathbb{N}}$ stabilizes to some $g \in \mathbb{Q}[[c_1, \dots, c_r]]$. In that case, there exists a sequence (m_1, \dots, m_r) such that we have the equality

$$[c_1^{k_1} \cdots c_r^{k_r}] f = [c_1^{k_1} \cdots c_r^{k_r}] \frac{g}{1-t}$$

for all $k_i > m_i$ where $i = 1, 2, \dots, r$.

Example 4.3.3. Consider the element $f \in \mathbb{Q}[x][[t]]$ given by

$$f = 1 + (1+x)t + (1+x+x^2)t + \cdots$$

In this case, $g = \frac{1}{1-x}$, and therefore f stabilizes to

$$\frac{1}{1-x} \frac{1}{1-t}$$

4.3.2 The cycle index generating function

Recall the *cycle index* of the symmetric group \mathfrak{S}_n which is defined as

$$Z_n := Z(\mathfrak{S}_n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \prod_{i=1}^n x_i^{c_i(w)}$$

It is well known, for example see [Cam99], that the generating function Γ of cycle indices of \mathfrak{S}_n , defined as

$$\Omega = 1 + \sum_{n=1}^{\infty} Z_n t^n,$$

can also be written in the form

$$\Omega = \exp \left(\sum_{i=1}^{\infty} \frac{x_i t^i}{i} \right) \quad (4.23)$$

This is a very useful result for computing certain statistics over \mathfrak{S}_n as indicated by the following example:

Example 4.3.4. Consider the formal equality

$$1 + \sum_{n=1}^{\infty} Z_n t^n = \exp \left(\sum_{i=1}^{\infty} \frac{x_i t^i}{i} \right)$$

Evaluating the formal partial derivative $\frac{\partial}{\partial x_k}$ of both sides at $x_i = 1$ for all $i = 1, 2, \dots$, gives the following identity

$$\sum_{n=1}^{\infty} \left(\frac{1}{n!} \sum_{w \in \mathfrak{S}_n} c_k(w) \right) t^n = \frac{1}{k} \frac{t^k}{1-t}$$

This says that the expected number of k cycles in a permutation of \mathfrak{S}_n for $n \geq k$ equals $\frac{1}{k}$. This is a classical result in combinatorial probability theory [Bón12].

Some other identities we would like to highlight here are

$$\Omega[x_i \leftarrow x^i] = \frac{1}{1-xt} \tag{4.24}$$

and

$$\Omega[x_i \leftarrow 1-x^i] = \frac{1-xt}{1-t} \tag{4.25}$$

where the notation $\Omega[x_i \leftarrow f(x_i)]$ means substituting $f(x_i)$ for x_i in the expression for Ω .

4.3.3 Goupil's generating function identity

Going back to our main theorem (Theorem 4.2.5), we consider the class function

$$\sum_{\lambda \vdash j} (-1)^{\ell(\lambda)+1} \eta^\lambda$$

From the identity 4.12, we have the following generating function

$$\sum_{j=0}^{\infty} \left(\sum_{\lambda \vdash j} (-1)^{\ell(\lambda)+1} \eta^\lambda \right) t^j = \prod_{i=1}^{\infty} \left(1 - (-t)^i \right)^{c_i}$$

Using this, and Proposition 4.2.4 we have another proof of the following identity of Goupil [Gou99] for the generating function for hook characters

Theorem 4.3.5. For $\lambda = (1^k) \vdash k$, we have the following generating function identity

$$\sum_{k=0}^{\infty} q_{(1^k)} t^k = \frac{1}{1+t} \prod_{i=1}^{\infty} \left(1 - (-t)^i \right)^{c_i} \tag{4.26}$$

Proof. From Proposition 4.2.4, we have

$$\chi^{\lambda[n]} = \sum_{j=0}^k (-1)^{j+1} \left(\sum_{\mu \models j} (-1)^{\ell(\mu)} \zeta^{\mu} \right)$$

which corresponds to an alternating sum of the class functions for which the character polynomial has the generating function given by Equation 4.26. Now recall that the generating function for the alternating sum of a sequence can be constructed by multiplying the generating function by $\frac{1}{1+t}$. This gives the required result. \square

Example 4.3.6. *From the above expression we derive expressions for first few character polynomials. A table of these can also be found in [Ker99]*

$$\begin{aligned} q_{()} &= 1 \\ q_{(1)} &= \binom{c_1}{1} - 1 \\ q_{(1^2)} &= \binom{c_1}{2} - \binom{c_2}{1} - \binom{c_1}{1} + 1 \\ q_{(1^3)} &= \binom{c_1}{2} - \binom{c_1}{1} \binom{c_2}{1} + \binom{c_3}{1} - \binom{c_1}{2} + \binom{c_2}{1} + \binom{c_1}{1} - 1 \end{aligned}$$

(Compare this with Example 4.2.3).

4.3.4 Rosas' formula for certain Kronecker coefficients

Let λ, μ and ν be partitions of n . The Kronecker coefficients $g_{\mu\nu}^{\lambda}(\mathfrak{S}_n)$ are defined as the coefficient of χ^{λ} in the expansion of $\chi^{\mu}\chi^{\nu}$ into irreducible characters:

$$g_{\mu\nu}^{\lambda}(\mathfrak{S}_n) = \langle \chi^{\lambda}, \chi^{\mu}\chi^{\nu} \rangle_{\mathfrak{S}_n} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^{\lambda}(\sigma) \chi^{\mu}(\sigma) \chi^{\nu}(\sigma)$$

Using Equation 4.26 for the generating function of irreducible hook characters, here we will derive the formula for Kronecker coefficients indexed by hooks given by Rosas [Ros00], which says:

Theorem 4.3.7. *Let $\bar{g}_{k_1 k_2 k_3}$ be the reduced Kronecker coefficients corresponding to the triple $((n - k_1, 1^{k_1}), (n - k_2, 1^{k_2}), (n - k_3, 1^{k_3}))$. Then*

$$\sum_{k_1, k_2, k_3} \bar{g}_{k_1 k_2 k_3} x^{k_1} y^{k_2} z^{k_3} = \frac{1 + xyz}{(1 - xy)(1 - yz)(1 - xz)}$$

Proof. We start with the substitution $x_i \leftarrow (1 - (-x)^i)$ in the cycle index function

$$\begin{aligned} Z_n[x_i \leftarrow (1 - (-x)^i)] &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \prod_{i=1}^n (1 - (-x)^i)^{a_i(w)} \\ [x^k] \frac{1}{1+x} Z_n[x_i \leftarrow (1 - (-x)^i)] &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi^{(1^k)[n]}(w) \end{aligned}$$

Let

$$U_n = Z_n[x_i \leftarrow (1 - (-x)^i)(1 - (-y)^i)(1 - (-z)^i)] \quad (4.27)$$

This implies

$$\begin{aligned} [x^{k_1}y^{k_2}z^{k_3}] \frac{1}{(1+x)(1+y)(1+z)} U_n &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi^{(1^{k_1})[n]}(w) \chi^{(1^{k_2})[n]}(w) \chi^{(1^{k_3})[n]}(w) \\ &= g_{k_1 k_2 k_3} \end{aligned}$$

On the other hand defining

$$U := \sum_{n=0}^{\infty} U_n t^n \quad (4.28)$$

we have

$$U = \Omega[x_i \leftarrow (1 - (-x)^i)(1 - (-y)^i)(1 - (-z)^i)] \quad (4.29)$$

$$= \exp\left(\sum_{i=1}^{\infty} \frac{x_i}{i} t^i\right) [x_i \leftarrow (1 - (-x)^i)(1 - (-y)^i)(1 - (-z)^i)] \quad (4.30)$$

$$= \frac{1}{1-t} \frac{(1+xyz)(1+xt)(1+yt)(1+zt)}{(1-xyt)(1-yzt)(1-xzt)} \quad (4.31)$$

Define

$$\begin{aligned} V &:= \frac{1}{1-t} \frac{1}{(1-xyt)(1-yzt)(1-xzt)} \\ &= \left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^n \sum_{a+b+c=j} (xy)^a (yz)^b (xz)^c \right) t^n \right) \end{aligned}$$

Note that V as a member of $\mathbb{Q}[x, y, z][[t]]$ stabilizes to

$$\frac{1}{1-t} \frac{1}{(1-xy)(1-yz)(1-xz)}$$

This implies that U stabilizes to

$$\frac{1}{1-t} \frac{(1+xyz)(1+x)(1+y)(1+z)}{(1-xy)(1-yz)(1-xz)}$$

Therefore,

$$\frac{1}{(1+x)(1+y)(1+z)} U \quad (4.32)$$

which is generating function for Kronecker coefficients $g_{k_1 k_2 k_3}$ stabilizes to

$$\frac{(1+xyz)}{(1-xy)(1-yz)(1-xz)} \quad (4.33)$$

□

4.4 Conclusion

In this chapter, we have defined three types of class functions on the symmetric group \mathfrak{S}_n , that generalize well known characters. These count various brick tilings of partition diagrams by cycles of permutations. We then proved the main theorem (Theorem 4.2.5). This helps us to determine the generating function for sequences of hook-partition characters and two-row-partition characters. As an application, we use this character to find the rational expression for the generating function of stable Kronecker coefficients. This provides a new proof of Rosas' formula [Ros00] for these coefficients.

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