VNIVERSITAT DE VALÈNCIA

HARMONIC ANALYSIS IN SPACES OF MATRICES AND OPERATORS

Ismael García Bayona

Supervisor: Óscar Francisco Blasco de la Cruz

Facultat de Ciències Matemàtiques
Programa de Doctorat en Matemàtiques

This dissertation is submitted for the degree of

PhD in Mathematics

May 2019

A mis padres.

I declare that this dissertation entitled *Harmonic analysis in spaces of matrices and operators* and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a degree of PhD in Mathematics at Universitat de València.
- Where I have consulted the works of others, this is clearly attributed.
- Where I have quoted from the works of others, the source is always given. With the exception of such quotations, the dissertation is entirely my own work.
- I have acknowledged all main sources of help.

Valencia, May 17th, 2019.

Ismael García Bayona

I declare that this dissertation presented by **Ismael García Bayona** entitled *Harmonic analysis in spaces of matrices and operators* has been done under my supervision at Universitat de València. I also state that this work corresponds to the thesis project approved by this institution and it satisfies all the requisites to obtain the degree of PhD in Mathematics.

Valencia, May 17th, 2019.

Óscar Francisco Blasco de la Cruz

Agradecimientos

"La aritmética más difícil de dominar es aquella que nos capacita para contar nuestras bendiciones."

—Eric Hoffer

En primer lugar, quiero dar las gracias a mi director de tesis, Óscar. Por tu guía y apoyo a lo largo de estos años, sin los que esta tesis no habría podido existir. Por ser siempre accesible y comprensivo. Por haber sido compañero de teoremas y de forma. Para mí, sin duda un referente tanto en lo profesional como en lo personal.

También quiero expresar mi agradecimiento a Liviu Marcoci, Anca Marcoci, Nicolae Popa y toda la gente del "Simion Stoilow Institute of Mathematics" de Rumanía, donde hice mi estancia del doctorado. Era un experiencia frente a la que sentía bastante vértigo, pero ellos consiguieron con su trato amable y cercano que en poco tiempo verdaderamente me sintiese como en casa, y quien me conoce sabe que eso no es fácil. Los meses allí fueron una experiencia que no olvidaré.

Gracias a todos los profesores que me han enseñado matemáticas en la Universitat de València, y también en etapas anteriores de mi formación. En especial, a Juanjo Isach, uno de los mejores profesores que he conocido, y que en su momento eliminó cualquier duda que pudiese tener a la hora de elegir la carrera de matemáticas.

Además, quiero dar las gracias a todos los miembros del Departamento de Análisis Matemático. También a los estudiantes de doctorado de la Facultad de Matemáticas, geniales compañeros de viaje y muy buena gente.

A Karn, Manowar y Dany, por esas sesiones de risas que siempre alegran y ayudan a olvidar las penas.

A Arantxa, por compartir ilusiones y preocupaciones aún en la distancia.

Y finalmente, pero no por ello menos importante, gracias a mi familia. Especialmente a mis padres, los pilares de mi vida. Por vuestro apoyo. Por saber escuchar en vez de oír. Por ser esa luz que ilumina cuando llega la oscuridad. Por ser esa mano que te agarra con fuerza y te ayuda a levantarte cuando te caes. Gracias por enseñarme lo que es verdaderamente importante. Gracias por no fallarme nunca.

Esta memoria ha sido redactada en el Departament d'Anàlisi Matemàtica de la Facultat de Ciències Matemàtiques de la Universitat de València, durante el período de disfrute de un contrato predoctoral FPU, con referencia FPU14/01032, otorgado por el Ministerio de Ciencia, Innovación y Universidades.

Esta investigación ha sido parcialmente financiada por el proyecto "Análisis Vectorial, Multilineal y Aplicaciones - MTM2014-53009-P". Asimismo, dicho proyecto también financió parcialmente la estancia de investigación de 3 meses de duración que el doctorando disfrutó en el "Simion Stoilow Institute of Mathematics of the Romanian Academy", del 1 de septiembre al 30 de noviembre de 2018.

According to the normative of Universitat de València, this PhD dissertation starts with three extended abstracts. The first two are in the official languages of Valencian Community, Spanish and Valencian, whereas the last one is written in English. The thesis really begins at page 1, after the abstracts.

Also, I would like to point out that in each chapter we have described the methodologies, objectives, results and conclusions of this work.

Resumen

"Todo gran cometido debe tener un principio, pero es en la continuidad hasta el final, hasta que se ha acabado totalmente, donde está la verdadera gloria."

—Sir Francis Drake.

Esta tesis está dedicada al estudio de problemas en las áreas de medibilidad y análisis armónico matricial en el contexto de espacios de matrices y operadores. Más concretamente, el objetivo es extender conceptos y resultados de esas teorías del contexto escalar a un marco de trabajo más general donde las funciones toman valores en espacios de operadores, o las matrices tienen como entradas a su vez a otras matrices.

En primer lugar, se comenzará con una sección introductoria donde trataremos aspectos relativos a notación, algunas definiciones y herramientas que aparecerán a lo largo del trabajo, y daremos una idea general del tipo de resultados que uno puede esperar encontrar en cada capítulo.

El Capítulo 1 trata sobre cuestiones de medibildiad para funciones con valores en operadores $f : \Omega \to \mathcal{L}(E_1, E_2)$, donde (Ω, Σ) es un espacio medible (i.e. Σ es una σ-álgebra sobre Ω) y $\mathcal{L}(E_1, E_2)$ denota el espacio de operadores lineales y acotados entre dos espacios de Banach E_1 y E_2. En $\mathcal{L}(E_1, E_2)$, se emplearán las tres topologías fundamentales, a saber, la topología de la norma $\tau_{\|\cdot\|}$, la topología fuerte de operadores τ_{SOT} y la topología débil

de operadores τ_{WOT}. La notación $\mathcal{N}_{\|\cdot\|}$, $\mathcal{N}_{SOT}$ y $\mathcal{N}_{WOT}$ se utilizará para las bases de las correspondientes topologías dadas por

$$B(T,\varepsilon) = \{S \in \mathcal{L}(E_1, E_2); \|S - T\| < \varepsilon\},$$

$$N(T; \mathbf{x}, \varepsilon) = \{S \in \mathcal{L}(E_1, E_2) : \max_{1\leq j\leq n} \|(S - T)(x_j)\| < \varepsilon\}$$

y

$$N(T; \mathbf{x}, \mathbf{y}^*, \varepsilon) = \{S \in \mathcal{L}(E_1, E_2) : \max_{1\leq j\leq n} |\langle (S - T)x_j, y_j^*\rangle| < \varepsilon\},$$

donde $\varepsilon > 0$, $T \in \mathcal{L}(E_1, E_2)$, $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in (E_1)^n$, $\mathbf{y}^* = (y_1^*, y_2^*, \cdots, y_n^*) \in (E_2^*)^n$ y $n \in \mathbb{N}$, respectivamente.

Claramente, $\mathcal{N}_{SOT(\mathbb{K},E_2)} = \mathcal{N}_{\|\cdot\|}$, y denotaremos $\mathcal{N}_{weak} = \mathcal{N}_{WOT(\mathbb{K},E_2)}$. Se observa que cuando el segundo espacio es el cuerpo, resulta que $\mathcal{N}_{SOT(E_1,\mathbb{K})} = \mathcal{N}_{WOT(E_1,\mathbb{K})}$ y en este caso utilizaremos la notación $\mathcal{N}_{weak^*}$. En el caso en que E_2 sea un espacio dual, $E_2 = F^*$, como $\mathcal{L}(E_1, E_2)$ es un espacio dual en sí mismo, además de $\mathcal{N}_{WOT}$ y $\mathcal{N}_{weak}$, también tendremos $\mathcal{N}_{weak^*}$, correspondiente a la base dada por

$$N(T; \mathbf{x}, \mathbf{y}, \varepsilon) = \{S \in X : \max_{1\leq j\leq n} |\langle (S - T)x_j, y_j\rangle| < \varepsilon\}.$$

donde $\mathbf{x} = (x_1, \cdots, x_n) \in (E_1)^n$, $\mathbf{y} = (y_1, \cdots, y_n) \in (E_2)^n$ y $n \in \mathbb{N}$.

La noción de función medible depende fuertemente de la formulación que se adopte tanto cuando se trata con funciones que toman valores en espacios de Banach (ver [20, 23, 33]) como cuando se trabaja con espacios de operadores (ver [4, 23, 33, 37, 47]). La considerable variedad de nociones que pueden encontrarse en el literatura, junto con el hecho de que habitualmente las cuestiones sobre medibilidad son tratadas con la intención de desarrollar una teoría de integración (y por tanto no recibiendo el tratamiento detallado que merecen por derecho propio) puede hacer que un primer acercamiento a la materia resulte algo confuso. Por tanto, en este primer capítulo introduciremos una terminología, trabajando únicamente con espacios medibles sin ninguna medida subyacente, que creemos que permite diferenciar las correspondientes definiciones de medibilidad y desentrañar las relaciones entre ellas de una forma sistemática.

Manejaremos dos conceptos de medibilidad diferentes pero, como se verá, relacionados. Uno de ellos se define por medio de las correspondientes bases de cada topología, mientras que el otro utiliza la noción de aproximabilidad. Sea (Ω, Σ) un espacio medible, y (Y, τ) un espacio vectorial topológico con base $\beta \subseteq \tau$.

- Una función $f : \Omega \to Y$ se dice que es β-**medible** cuando $f^{-1}(A) \in \Sigma$ para todo $A \in \beta$.

- Una función $f : \Omega \to Y$ se dice que es τ-**aproximable** cuando f es límite puntual de funciones finitamente valuadas. En otras palabras, cuando $s_N \to f$ en la topología τ, donde $s_N = \sum_{k=1}^{N} y_k \chi_{A_k}$ con $y_k \in Y$ y A_k conjuntos disjuntos tales que $\cup_{k=1}^{N} A_k = \Omega$.

A continuación recordamos algunos resultados básicos que conectan estas nociones, adaptados a nuestra terminología. Probablemente el más famoso de ellos es el Teorema de medibilidad de Pettis.

Teorema *(ver [20, Capítulo 2, Teorema 2]) Teorema de medibilidad de Pettis. Sea (Ω, Σ, μ) un espacio de medida finita completo y sea E un espacio de Banach. Entonces $f : \Omega \to E$ es $\|\cdot\|$-aproximable μ-a.e. (esto es, salvo un conjunto de medida nula) si y sólo si f es $\mathcal{N}_{weak}$-medible y $f(\Omega)$ es esencialmente separable, es decir, $f(\Omega \setminus A)$ es separable para algún $A \in \Sigma$ con $\mu(A) = 0$.*

Un corolario de Dunford nos dice que para $E = \mathcal{L}(E_1, E_2)$, $f : \Omega \to \mathcal{L}(E_1, E_2)$ satisface que f_x es $\|\cdot\|$-aproximable μ-a.e para todo $x \in E_1$ si y sólamente si f es $\mathcal{N}_{WOT}$-medible y $f_x(\Omega)$ es esencialmente separable para todo $x \in E_1$, donde $f_x(\omega) = f(\omega)(x)$.

Teorema *(ver [33, Teorema 3.5.5]) Teorema de medibilidad de Dunford. Sea (Ω, Σ, μ) un espacio de medida finita completo y sean E_1, E_2 espacios de Banach. Entonces $f : \Omega \to \mathcal{L}(E_1, E_2)$ es $\|\cdot\|$-aproximable μ-a.e. si y sólo si f es $\mathcal{N}_{WOT}$-medible y $f(\Omega)$ es esencialmente separable en $\mathcal{L}(E_1, E_2)$.*

Teorema *(ver [37, Teorema 2]) Teorema de medibilidad de Johnson. Sea* (Ω, Σ, μ) *un espacio de medida finita completo y sea* H *un espacio de Hilbert separable. Entonces* $f : \Omega \to \mathcal{L}(H, H)$ *satisface que* f_x *es* $\|\cdot\|$*-aproximable* μ*-a.e. para todo* $x \in H$ *si y sólo si* f *es* $\mathcal{N}_{SOT}$*-medible.*

En la Sección 1.2 comenzamos analizando la noción de medibilidad respecto a bases de diferentes topologías. Se empieza recordando un hecho bien conocido en el contexto de espacios métricos.

Proposición *Sea* (Y, d) *un espacio métrico separable. Entonces* $f : \Omega \to Y$ *es* $\mathcal{N}_d$*-medible si y sólo si* f *es* τ_d*-medible. En particular, si* E *es un espacio de Banach,* $f : \Omega \to E$ *es* $\mathcal{N}_{\|\cdot\|}$*-medible y* $f(\Omega)$ *es separable entonces* f *es* $\tau_{\|\cdot\|}$*-medible.*

Al tratar con functiones $f : \Omega \to \mathcal{L}(E_1, E_2)$, en ocasiones la siguiente notación puede resultar útil: para $x \in E_1$ e $y^* \in E_2^*$, denotamos $f_x(w) = f(w)(x)$ y $f_{x,y^*}(w) = \langle y^*, f(w)(x)\rangle$. La siguiente proposición da uso a dicha notación y proporciona una caracterización de algunas nociones de β-medibilidad.

Proposición *Sean* $X = \mathcal{L}(E_1, E_2)$ *y* $f : \Omega \to X$*. Entonces*

(i) f *es* $\mathcal{N}_{\|\cdot\|}$*-medible* $\iff \|f(\cdot) - T\|$ *es medible para todo* $T \in X$*.*

(ii) f *es* $\mathcal{N}_{SOT}$*-medible* $\iff f_x$ *es* $\mathcal{N}_{\|\cdot\|}$*-medible para todo* $x \in E_1$*.*

(iii) f *es* $\mathcal{N}_{WOT}$*-medible* $\iff f_{x,y^*}$ *es medible para todo* $x \in E_1$ *e* $y^* \in E_2^*$*.*

Es claro que $\mathcal{N}_{WOT} \subset \mathcal{N}_{weak}$, y por tanto la $\mathcal{N}_{weak}$-medibilidad implica la $\mathcal{N}_{WOT}$-medibilidad. Por otra parte, se tiene que $\tau_{weak} \subset \tau_{\|\cdot\|}$ en cualquier espacio de Banach y también $\tau_{WOT} \subset \tau_{SOT} \subset \tau_{\|\cdot\|}$ para $X = \mathcal{L}(E_1, E_2)$. Por lo tanto, funciones que son $\tau_{\|\cdot\|}$-medibles son también $\mathcal{N}_{weak}$-medibles y $\mathcal{N}_{SOT}$-medibles. Bajo ciertas condiciones de separabilidad, se pueden recuperar versiones de medibilidad más fuertes a partir de otras más débiles, como muestran las siguientes proposiciones, que cierran la Sección 1.2.

Proposición *Sea* $f : \Omega \to X = \mathcal{L}(E_1, E_2)$*, donde* E_1 *es un espacio de Banach separable. Si* f *es* $\mathcal{N}_{SOT}$*-medible, entonces* f *es también* $\mathcal{N}_{\|\cdot\|}$*-medible.*

Proposición *Sea* $f : \Omega \to \mathcal{L}(E_1, E_2)$ *con* E_2 *separable. Entonces* f *es* $\mathcal{N}_{WOT}$*-medible si y sólo si* f *es* $\mathcal{N}_{SOT}$*-medible.*

El estudio de las nociones de medibilidad en términos de aproximación está cubierto por la Sección 1.3. El concepto de función simple y numerablemente valorada es importante y recordamos su definición.

Definición *En un espacio topológico Hausdorff* (X, τ)*, decimos que una función* $f : \Omega \to X$ *es simple (respectivamente contablemente valorada) si existe un conjunto finito (respectivamente sucesión)* $(x_n)_n \subset X$ *y una partición finita (respectivamente una partición numerable) de conjuntos* $(A_n)_n \subset \Sigma$ *disjuntos dos a dos, tales que* $\Omega = \cup_k A_k$ *y* $f = \sum_n x_n \chi_{A_n}$.

La $\|\cdot\|$-, *weak*-, SOT- y WOT-aproximabilidad de una función $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ está determinada por la existencia de una sucesión de funciones simples $s_n : \Omega \to X$ con valores en operadores tal que

$$\|s_n(\omega) - f(\omega)\| \xrightarrow[n\to\infty]{} 0, \quad \forall \omega \in \Omega,$$

$$\langle s_n(\omega), T^* \rangle \xrightarrow[n\to\infty]{} \langle f(\omega), T^* \rangle, \quad \forall T^* \in X^*, \quad \forall \omega \in \Omega,$$

$$\lim_n \|s_n(\omega)(x) - f(\omega)(x)\| = 0, \quad \forall x \in E_1, \quad \forall \omega \in \Omega$$

o bien

$$\langle s_n(\omega)(x), y^* \rangle \xrightarrow[n\to\infty]{} \langle f(\omega)(x), y^* \rangle, \quad \forall x \in E_1, \quad \forall y^* \in E_2^*, \quad \forall \omega \in \Omega,$$

respectivamente.

Las conexiones entre la aproximabilidad, la medibilidad con respecto a bases y la separabilidad del rango en la topología correspondiente comienzan a brillar en el resto de la sección. Por ejemplo, tenemos el siguiente teorema.

Teorema *Sea* (Y, d) *un espacio métrico y* $f : \Omega \to Y$*. Las siguientes afirmaciones son equivalentes:*

(i) f *es d-aproximable.*

(ii) f *es* τ_d*-medible y* $f(\Omega)$ *es d-separable.*

(iii) f *es* $\mathcal{N}_d$*-medible y* $f(\Omega)$ *es d-separable.*

Como una consecuencia del teorema previo y del Teorema de Banach-Alaoglu, obtendremos el siguiente resultado.

Proposición *Sea* E *un espacio de Banach separable,* $X = E^*$ *y sea* $f : \Omega \to X$ *una función acotada. Entonces* f *es weak***-aproximable si y sólo si* f *es* $\mathcal{N}_{weak^*}$*-medible.*

Es bastante sorprendente que, para cualquier espacio de Banach, las nociones de $\|\cdot\|$-aproximabilidad y aproximabilidad débil coinciden.

Proposición *Sea* X *un espacio de Banach y* $f : \Omega \to X$*. Entonces,* f *es* $\|\cdot\|$*-aproximable si y sólo si* f *es weak-aproximable.*

Estudiaremos algunos ejemplos para diferenciar ciertas nociones de medibilidad, con lo que cerraremos la sección y pasaremos a la Sección 1.4. En ella, el primer objetivo será dar una prueba del Teorema de medibilidad de Dunford (y por tanto también del de Pettis).

Teorema *Sea* $f : \Omega \to X = \mathcal{L}(E_1, E_2)$*. Las siguientes afirmaciones son equivalentes:*

(i) f *es* $\|\cdot\|$*-aproximable.*

(ii) f *es* $\tau_{\|\cdot\|}$*-medible y* $f(\Omega)$ *es separable en* X*.*

(iii) f *es* $\mathcal{N}_{\|\cdot\|}$*-medible y* $f(\Omega)$ *es separable en* X*.*

(iv) f *es* $\mathcal{N}_{SOT}$*-medible y* $f(\Omega)$ *es separable en* X*.*

(v) f *es* $\mathcal{N}_{WOT}$*-medible y* $f(\Omega)$ *es separable en* X*.*

Nuestra principal contribución en esta sección, y en el capítulo, es la versión del Teorema de medibilidad de Pettis para la topología SOT, en el caso en que E_1 es un espacio de Banach separable.

Teorema *Sea $f : \Omega \to X = \mathcal{L}(E_1, E_2)$, donde E_1 es separable. Las siguientes afirmaciones son equivalentes.*

(i) f es SOT-aproximable

(ii) f es WOT-aproximable.

(iii) f es $\mathcal{N}_{WOT}$-medible y $f(\Omega)$ es WOT-separable.

Para terminar el capítulo, aplicaremos este teorema para construir algunos ejemplos naturales de funciones con valores en operadores que son SOT-aproximables.

Los resultados que aparecen en este capítulo se encuentran publicados en el siguiente artículo:

> Blasco, O.; García-Bayona, I., Remarks on Measurability of Operator-valued Functions, *Mediterr. J. Math.* **13** (2016), 5147–5162. DOI: 10.1007/s00009-016-0798-1.

En el Capítulo 2, pasamos al área del análisis armónico matricial. El capítulo comienza con una sección introductoria en la cual, tras recordar algunos aspectos y resultados clásicos del caso escalar, como el Teorema de Toeplitz o el Teorema de Bennett, comenzamos definiendo algunos de los elementos más importantes en los cuales trabajaremos a lo largo de este capítulo y de algunos de los siguientes. En primer lugar, tenemos el espacio $\mathcal{B}(\ell^2(H))$.

Definición *Dada una matriz $\mathbf{A} = (T_{kj})$ con entradas $T_{kj} \in \mathcal{B}(H)$ y $\mathbf{x} \in c_{00}(H)$, denotamos $\mathbf{A}(\mathbf{x})$ a la sucesión $(\sum_{j=1}^{\infty} T_{kj}(x_j))_k$. Diremos que $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ si la aplicación $\mathbf{x} \to \mathbf{A}(\mathbf{x})$ extiende a un operador lineal y acotado en $\ell^2(H)$, esto es*

$$\Big(\sum_{k=1}^{\infty} \|\sum_{j=1}^{\infty} T_{kj}(x_j)\|^2\Big)^{1/2} \leq C\Big(\sum_{j=1}^{\infty} \|x_j\|^2\Big)^{1/2}.$$

La norma en este espacio viene dada de la siguiente manera:

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{A}\mathbf{x}\|_{\ell^2(H)} \leq C\|\mathbf{x}\|_{\ell^2(H)}\}.$$

La principal operación que estudiaremos será una versión del clásico producto de Schur (producto de matrices entrada a entrada) en el contexto de matrices cuyas entradas son operadores, y lo definimos como sigue.

Definición *Sean* $\mathbf{A} = (T_{kj})$ *y* $\mathbf{B} = (S_{kj})$ *matrices con* $T_{kj}, S_{kj} \in \mathcal{B}(H)$. *Definimos su producto de Schur como*

$$\mathbf{A} * \mathbf{B} = (T_{kj} S_{kj}),$$

donde $T_{kj} S_{kj}$ *es la composición de los operadores* T_{kj} *y* S_{kj}.

Un concepto muy importante que se comenzará a estudiar en este capítulo es el de multiplicador de Schur. Los multiplicadores para nuestro producto tienen una definición similar a la de los multiplicadores en el contexto escalar. Sin embargo, dado que nuestro producto no es conmutativo, es necesario definir multiplicadores a derecha y a izquierda.

Definición *Dada una matriz* $\mathbf{A} = (T_{kj})$, *se dice que* $\mathbf{A}$ *es un multiplicador de Schur a derecha (respectivamente multiplicador de Schur a izquierda), y lo denotamos por* $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ *(respectivamente* $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ *), siempre que* $\mathbf{B} * \mathbf{A} \in \mathcal{B}(\ell^2(H))$ *(respectivamente* $\mathbf{A} * \mathbf{B} \in \mathcal{B}(\ell^2(H))$ *) para cualquier matriz* $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. *Las expresiones de la norma en estos espacios son*

$$\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{B} * \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq C \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\}$$

y

$$\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{A} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \leq C \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\}.$$

La Sección 2.2 incluye algunas nociones básicas sobre sucesiones y funciones con valores vectoriales que serán utilizadas en el resto del capítulo. En particular, se estudiarán espacios como $\ell^2(\mathbb{N}, \mathcal{B}(H))$, $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ y $\ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$, y exploraremos relaciones entre ellos, proporcionando ejemplos y contraejemplos cuando sea necesario.

El producto tensorial proyectivo es una herramienta que aparecerá en varias ocasiones a lo largo de las pruebas. Aún más intenso es el uso de ciertas ideas de la teoría de medidas vectoriales, y por esta razón hemos incluido la Sección 2.3. En ella, recordamos tres identificaciones entre operadores y medidas, que mencionamos aquí brevemente. La primera de ellas nos dice que $\mathfrak{M}(\mathbb{T}, E)$ puede identificarse con el espacio de los operadores débilmente compactos $T_\mu : C(\mathbb{T}) \to E$ y que $\|T_\mu\| = \|\mu\|$ (ver [20, Capítulo 6]). En el caso de espacios duales $E = F^*$, el Teorema de Singer (ver [53, 54, 32])) asegura que $M(\mathbb{T}, E) = C(\mathbb{T}, F)^*$. En otras palabras, que existe un operador lineal y acotado $\Psi_\mu : C(\mathbb{T}, F) \to \mathbb{C}$ con $\|\Psi_\mu\| = |\mu|$ tal que $\Psi_\mu(y\phi) = T_\mu(\phi)(y)$, $\phi \in C(\mathbb{T})$, $y \in F$. Además, en el contexto de operadores, aún hay una tercera posibilidad a considerar, utilizando la topología fuerte de operadores, y es $\Phi_\mu : C(\mathbb{T}, X) \to Y^*$, definido por $\Phi_\mu(f)(y) = \Psi_\mu(f \otimes y)$, $f \in C(\mathbb{T}, X)$, $y \in Y$, donde $f \otimes y(t) = f(t) \otimes y$.

Describiremos diferentes tipos de espacios de medidas vectoriales, y de especial importancia será el espacio $M_{SOT}(\mathbb{T}, \mathcal{B}(H))$, compuesto por medidas $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ tales que $\mu_x \in M(\mathbb{T}, H)$ para cualquier $x \in H$. La norma en este espacio se expresa como

$$\|\mu\|_{SOT} = \sup\{|\mu_x| : x \in H, \|x\| = 1\}.$$

Además, la llamada "medida adjunta" también jugará un papel importante.

Definición *Sea* $\mu : \mathfrak{B}(\mathbb{T}) \to \mathcal{L}(X, Y^*)$ *una medida vectorial. Definimos la "medida adjunta"* $\mu^* : \mathfrak{B}(\mathbb{T}) \to \mathcal{L}(Y, X^*)$ *mediante la fórmula*

$$\mu^*(A)(y)(x) = \mu_x(A)(y), \quad A \in \mathfrak{B}(\mathbb{T}), x \in X, y \in Y.$$

Con el concepto de medida adjunta en mente, el espacio $M_{SOT}(\mathbb{T}, \mathcal{B}(H))$, muy relacionado con los multiplicadores de Schur como se verá en secciones posteriores, puede ser descrito mediante operadores como sigue.

Proposición *Sea* $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. *Entonces* $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ *si y sólo si* $\Phi_{\mu^*} \in$

$\mathcal{L}(C(\mathbb{T},H),H)$. *Es más,* $\|\mu\|_{SOT} = \|\Phi_{\mu^*}\|$.

En la Sección 2.4 presentamos condiciones suficientes y necesarias para que una matriz pertenezca a $\mathcal{B}(\ell^2(H))$, y algunas involucran condiciones sobre las filas y columnas relacionadas con los espacios vistos en la Sección 2.2. Uno de los resultados más importantes de la Sección 2.4 es la versión del Teorema de Schur en el marco de las matrices con entradas en operadores, esto es, demostraremos que el espacio $\mathcal{B}(\ell^2(H))$ define un álgebra de Banach con el producto de Schur. Sin embargo, para que un producto de matrices caiga en el espacio $\mathcal{B}(\ell^2(H))$, no es necesario que ambas matrices estén en dicho espacio, por supuesto. De hecho, en la subsección 2.4.2, encontramos el siguiente teorema, donde comprobamos que es suficiente con que una de las matrices esté en $\mathcal{B}(\ell^2(H))$ y que la otra matriz cumpla ciertas condiciones sobre sus filas relacionadas con espacios de sucesiones con valores en operadores vistos en la Sección 2.2.

Teorema *Sean* $\mathbf{A} = (T_{k,j})$ *y* $\mathbf{B} = (S_{k,j})$ *matrices con entradas en* $\mathcal{B}(H)$. *Si* $\mathbf{B} \in B(\ell^2(H))$ *y* $\mathbf{A} \in \ell^\infty(\mathbb{N}, \ell^2(\mathcal{B}(H))) \cup \ell^\infty\left(\mathbb{N}, \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))\right)$, *entonces* $\mathbf{A} * \mathbf{B} \in B(\ell^2(H))$, *esto es,* $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$.

La Sección 2.5 es la última del capítulo y contiene algunos de los resultados más importantes acerca de matrices de Toeplitz (espacio de matrices constantes por diagonales, denotado por $\mathcal{T}$). El primero de estos resultados es la generalización del Teorema de Toeplitz, que proporciona una condición suficiente y necesaria en relación a la función asociada a una matriz de Toeplitz para que dicha matriz pertenezca al espacio $\mathcal{B}(\ell^2(H))$. El espacio de medidas vectoriales $V^\infty(\mathbb{T}, \mathcal{B}(H))$, definido en la Sección 2.3, hace su aparición como el sustituto de $L^\infty(\mathbb{T})$ en esta versión generalizada.

Teorema *Sea* $\mathbf{A} = (T_{kj}) \in \mathcal{T}$. *Entonces,* $A \in \mathcal{B}(\ell^2(H))$ *si y sólamente si existe una medida* $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ *tal que* $T_{kj} = \hat{\mu}(j-k)$ *para todo* $k, j \in \mathbb{N}$. *Además,* $\|\mathbf{A}\| = \|\mu\|_\infty$.

La sección continúa presentando algunas condiciones suficientes para que una matriz sea un multiplicador de Schur en nuestro contexto, y termina con un par de resultados que tratan de generalizar el Teorema de Bennett a nuestro marco de trabajo.

Teorema *Si* $\mu \in M(\mathbb{T}, \mathcal{B}(H))$ *y* $\mathbf{A} = (T_{kj}) \in \mathcal{T}$ *con* $T_{kj} = \hat{\mu}(j-k)$ *para* $k, j \in \mathbb{N}$ *entonces* $\mathbf{A} \in \mathcal{M}_l(\ell^2(H)) \cap \mathcal{M}_r(\ell^2(H))$. *Además,*

$$\max\{\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}, \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\} \leq |\mu|.$$

Teorema *Sea* $\mathbf{A} = (T_{kj}) \in \mathcal{T} \cap \mathcal{M}_r(\ell^2(H))$. *Entonces, existe* $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ *tal que* $T_{kj} = \hat{\mu}(j-k)$ *para todo* $k, j \in \mathbb{N}$. *Es más,*

$$\|\mu\|_{SOT} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}.$$

Los resultados de este capítulo se encuentran publicados en el artículo:

> Blasco, O.; García-Bayona, I., Schur Product with Operator-valued Entries, *Taiwanese J. Math.*, advance publication, 30 November 2018. DOI:10.11650/tjm/181110. https://projecteuclid.org/euclid.twjm/1543546839.

El Capítulo 3 está dedicado principalmente al estudio de un tipo particular de matrices, llamadas "matrices continuas". Este espacio se denotará por $C(\ell^2(H))$, y es el espacio de matrices que pueden ser aproximadas en la norma de operadores por matrices con un número finito de diagonales no nulas, o más precisamente, "matrices polinomiales". El espacio de las matrices continuas con entradas escalares fue introducido por Barza, Persson y Popa (ver [6]).

Decimos que una matriz $\mathbf{A} = (T_{kj})$ con entradas $T_{kj} \in \mathcal{B}(H)$ es una "matriz polinomial", de manera abreviada $\mathbf{A} \in \mathcal{P}(\ell^2(H))$, cuando se satisfacen dos condiciones: que $\sup_{k,j} \|T_{kj}\| < \infty$ y que existan $N, M \in \mathbb{N}$ tales que $\mathbf{A}$ puede ser escrita como una suma finita de diagonales $\mathbf{A} = \sum_{l=-N}^{M} \mathbf{D_l}$.

La primera sección del capítulo introduce los conceptos necesarios y la notación que se utilizará. Principalmente se trata de una continuación natural de lo que se trató en el Capítulo 2. Un nuevo tipo de matrices que aparece por primera vez es el de las matrices M_μ. Dada $\mu \in M(\mathbb{T})$, denotaremos $\mathbf{M}_\mu$ a la matriz de Toeplitz definida por

$$\mathbf{M}_\mu = (\hat{\mu}(j-k)Id)_{k,j} \in \mathcal{T}.$$

La siguiente fórmula, que es válida por ejemplo para $\mu \in M(\mathbb{T})$ y $f \in L^1(\mathbb{T})$ (donde $\mu * f(t) = \int_0^{2\pi} f(e^{i(t-s)})d\mu(s)$ es la convolución entre funciones y medidas en $\mathbb{T}$) sugiere la relación entre multiplicadores de Fourier y multiplicadores de Schur, y da una idea de la importancia de este nuevo tipo de matrices definido anteriormente.

$$\mathbf{M}_\mu * \mathbf{M}_f = \mathbf{M}_f * \mathbf{M}_\mu = \mathbf{M}_{\mu * f}.$$

En el proceso de exploración de las conexiones entre el análisis de Fourier y el análisis matricial (ver [44]) la siguiente función con valores en matrices tendrá un papel importante.

Definición *Sea $\mathbf{A} = (T_{kj})$ con $T_{kj} \in \mathcal{B}(H)$ para $k, j \in \mathbb{N}$. Definimos*

$$f_{\mathbf{A}}(t) = \mathbf{M}_t * \mathbf{A} = (e^{i(j-k)t}T_{kj}), \quad t \in [0, 2\pi].$$

En el caso de matrices triangulares superiores (denotadas por $\mathcal{U}$) trabajaremos con

$$F_{\mathbf{A}}(z) = (z^{(j-k)}T_{kj}), \quad |z| < 1.$$

Después de la sección preliminar, la Sección 3.2 presenta ejemplos particulares de matrices con entradas en operadores, y algunos procedimientos para construirlas. Destacamos el siguiente, que permite incluir el caso escalar en el contexto de operadores, generando sencillos ejemplos de matrices en nuestro marco de trabajo.

Ejemplo *Sean* $A = (a_{k,j}) \in \mathcal{B}(\ell^2)$ *y* $T \in \mathcal{B}(H)$. *Entonces*

$$\mathbf{A} = (a_{k,j}T) \in \mathcal{B}(\ell^2(H)) \ \ y \ \ \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|A\|_{\mathcal{B}(\ell^2)}\|T\|_{\mathcal{B}(H)}.$$

Además, definiremos una versión matricial del álgebra de Wiener. La Sección 3.3 trata en su totalidad sobre multiplicadores de Schur. Daremos condiciones necesarias para que una matriz sea un multiplicador de Schur, y presentaremos la versión de multiplicadores del ejemplo anterior para construir de manera sencilla ejemplos de multiplicadores con entradas en operadores. Utilizando núcleos de sumabilidad, definimos las matrices $M_n(\mathbf{A})$ como $M_n(\mathbf{A}) = \mathbf{M}_{K_n} * \mathbf{A}$, y demostramos la siguiente proposición, que muestra su importancia, pues permiten determinar cuándo una matriz define un operador o un multiplicador.

Proposición *Sean* $\mathbf{A}$ *una matriz con entradas en* $\mathcal{B}(H)$ *y* $\{k_n\}$ *un núcleo de sumabilidad, y denotemos* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$. *Entonces:*

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H)) \Leftrightarrow \sup_n\|M_n(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} < \infty$.

(ii) $\mathbf{A} \in \mathcal{M}_r(\ell^2(H)) \Leftrightarrow \sup_n\|M_n(\mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} < \infty$.

(iii) $\mathbf{A} \in \mathcal{M}_l(\ell^2(H)) \Leftrightarrow \sup_n\|M_n(\mathbf{A})\|_{\mathcal{M}_l(\ell^2(H))} < \infty$.

La Sección 3.4 comienza haciendo uso de la función $f_{\mathbf{A}}$ definida arriba para mostrar la conexión entre este nuevo espacio de matrices y el espacio de las funciones continuas. Un primer resultado sencillo pero útil muestra que esta función es una isometría entre los espacios correspondientes. La sección avanza presentando varios resultados entre los cuales queremos destacar un par de ellos por su importancia y utilidad. El primero de ellos es una caracterización de las matrices de $C(\ell^2(H))$ que nos muestra diferentes formas en las que se puede ver este espacio y la relación con las funciones continuas.

Teorema *Sea* $\mathbf{A} = (T_{k,j})_{k,j}$ *una matriz con entradas en* $\mathcal{B}(H)$, *cumpliendo la condición* $\sup_{k,j}\|T_{k,j}\| < \infty$. *Las siguientes afirmaciones son equivalentes:*

1) $\mathbf{A} \in C(\ell^2(H))$.

2) $\lim_{n\to\infty} M_n(\mathbf{A}) = \mathbf{A}$ *en* $\mathcal{B}(\ell^2(H))$ *donde* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$ *y* $\{k_n\} \subseteq L^1(\mathbb{T})$ *es un núcleo de sumabilidad.*

3) $\lim_{n\to\infty} \sigma_n(\mathbf{A}) = \mathbf{A}$ *en* $\mathcal{B}(\ell^2(H))$.

4) $t \to f_{\mathbf{A}}(t)$ *es una función continua con valores en* $\mathcal{B}(\ell^2(H))$.

El segundo resultado muestra que, cuando trabajamos con multiplicadores de Schur, es suficiente con considerar aquellos que mandan $C(\ell^2(H))$ en sí mismo. Más precisamente, tenemos:

Teorema $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ *(respectivamente* $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$*) si y sólamente si* $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_l$ *(respectivamente* $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_r$*).*

Pasamos entonces a la Subsección 3.4.1, centrada en matrices de Toeplitz. De esta sección, destacamos también dos resultados. El primero de ellos presenta otra caracterización que deja clara la total conexión entre el espacio $C(\ell^2(H))$ y el espacio de las funciones continuas.

Teorema *Sea* $(T_l)_{l\in\mathbb{Z}}$ *una sucesión de operadores en* $\mathcal{B}(H)$ *y sea* $\mathbf{A} = (T_{j-k})_{k,j}$*. Entonces,* $\mathbf{A} \in C(\ell^2(H))_{\mathcal{T}}$ *si y sólo si existe* $g_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(H))$ *tal que* $\widehat{g_{\mathbf{A}}}(l) = T_l$*. Además,* $\|g_{\mathbf{A}}\|_{C(\mathbb{T},\mathcal{B}(H))} = \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}$.

El segundo resultado, ya visto en el anterior capítulo, es la caracterización de los multiplicadores Toeplitz en términos de medidas SOT. Sin embargo, en esta ocasión, la prueba alternativa que proporcionamos utiliza técnicas y resultados de este capítulo, sin depender de medidas vectoriales.

La subsección final, 3.4.2, introduce una versión matricial del álgebra del disco, para matrices triangulares superiores. Asumiendo que una matriz $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{U}$ satisface $\sup_{k,j}\|T_{k,j}\| < \infty$, es claro que

$$F_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} \mathbf{D_l} z^l \in \mathcal{H}(\mathbb{D}, \mathcal{B}(\ell^2(H)))$$

es una función holomorfa bien definida. Probamos el siguiente resultado.

Teorema *Sea* $\mathbf{A} = (T_{kj}) \in \mathcal{U}$ *satisfaciendo* $\sup_{k,j} \|T_{k,j}\| < \infty$.

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *si y sólo si* $F_{\mathbf{A}} \in H^\infty(\mathbb{D}, \mathcal{B}(\ell^2(H)))$. *Es más,* $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|F_{\mathbf{A}}\|_{H^\infty(\mathbb{D},\mathcal{B}(\ell^2(H)))}$.

(ii) $\mathbf{A} \in C(\ell^2(H))$ *si y sólo si* $F_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(\ell^2(H)))$.

Los resultados de este capítulo se encuentran recogidos en el siguiente artículo:

> Blasco, O.; García-Bayona, I., New spaces of matrices with operator entries, *Quaest. Math*, 2019. DOI: 10.2989/16073606.2019.1605416.

El enfoque del Capítulo 4 está en sintonía con el que empleamos en el capítulo previo, pero en esta ocasión el énfasis sobre los multiplicadores es mayor. Consideraremos la clase de los multiplicadores de Schur que pueden aproximarse en la norma de multiplicadores por matrices polinomiales. También, como es habitual, daremos un tratamiento especial al caso de matrices de Toeplitz y triangulares superiores.

La primera sección del capítulo es de carácter preliminar, y hace hincapié en algunos de los conceptos que serán necesarios a lo largo del mismo. También, se presenta la nueva clase de matrices que será objeto de estudio en el capítulo: el espacio de las "matrices integrables", denotadas por $\mathcal{L}^1(\ell^2(H))$.

Definición *Definimos* $\mathcal{L}^1_l(\ell^2(H))$ *(respectivamente* $\mathcal{L}^1_r(\ell^2(H))$*) como la clausura de* $\mathcal{P}(\ell^2(H))$ *en* $\mathcal{M}_l(\ell^2(H))$ *(respectivamente* $\mathcal{M}_r(\ell^2(H))$*). Utilizaremos la notación siguiente:* $\mathcal{L}^1(\ell^2(H)) = \mathcal{L}^1_l(\ell^2(H)) \cap \mathcal{L}^1_r(\ell^2(H))$.

Cuando estudiamos las matrices continuas vimos que este nombre era razonable para esa clase de matrices, debido a las propiedades que las relacionan con el espacio de funciones continuas. El significado del nombre "matrices integrables" será también ampliamente justificado en el capítulo por los resultados que presentaremos. La Sección 4.2 comienza estudiando el nuevo espacio $\mathcal{L}^1(\ell^2(H))$ y proporcionando algunos ejemplos de matrices en él. Además, se obtendrá una formulación equivalente utilizando el producto de Schur con matrices de Toeplitz dadas por núcleos de sumabilidad en la línea de la obtenida para el espacio $C(\ell^2(H))$.

Teorema *Sea* $\mathbf{A} = (T_{k,j})_{k,j}$ *una matriz con entradas en* $\mathcal{B}(H)$ *satisfaciendo* $\sup_{k,j} \|T_{k,j}\| < \infty$. *Las siguientes afirmaciones son equivalentes:*

1) $\mathbf{A} \in \mathcal{L}_r^1(\ell^2(H))$.

2) $\lim_{n\to\infty} M_n(\mathbf{A}) = \mathbf{A}$ *en* $\mathcal{M}_r(\ell^2(H))$ *donde* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$ *y* $\{k_n\} \subseteq L^1(\mathbb{T})$ *es un núcleo de sumabilidad.*

3) $\lim_{n\to\infty} \sigma_n(\mathbf{A}) = \mathbf{A}$ *en* $\mathcal{M}_r(\ell^2(H))$.

4) $\lim_{r\to 1} P_r(\mathbf{A}) = \mathbf{A}$ *en* $\mathcal{M}_r(\ell^2(H))$.

Estudiaremos también matrices columna y matrices diagonales en $\mathcal{L}_l^1(\ell^2(H))$, comprobando, en particular, que una versión del lema de Riemann-Lebesgue se cumple en nuestro contexto.

Proposición *(Lema de Riemann-Lebesgue) Si* $A = \sum_l \mathbf{D_l} \in \mathcal{L}_r^1(\ell^2(H))$, *entonces*

$$\|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} \xrightarrow[|l|\to\infty]{} 0.$$

La Sección 4.3 comienza analizando la conexión entre el espacio $\mathcal{L}^1(\ell^2(H))$ y las funciones integrables por medio de la función $f_{\mathbf{A}}$ vista como una función con valores en multiplicadores. Las matrices en $\mathcal{L}_l^1(\ell^2(H))$ pueden caracterizarse de la siguiente manera.

Teorema *Sea* $\mathbf{A}$ *una matriz con entradas en* $\mathcal{B}(H)$. *Entonces* $\mathbf{A} \in \mathcal{L}_l^1(\ell^2(H))$ *si y sólo si la función* $t \to f_{\mathbf{A}}(t)$ *es una función continua con valores en* $\mathcal{M}_l(\ell^2(H))$.

En la Subsección 4.3.1 prestamos una atención especial al caso de matrices de Toeplitz, el cual, como es habitual, demuestra ser el puente más directo que conecta el mundo de las matrices con el mundo de las funciones/medidas. Utilizando varios resultados intermedios, obtenemos la caracterización del espacio $\mathcal{L}_r^1(\ell^2(H)) \cap \mathcal{T}$ como sigue.

Teorema

$$\mathcal{L}_r^1(\ell^2(H)) \cap \mathcal{T} = \tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H)),$$

donde el espacio $\tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H))$ es la clausura de los polinomios en la norma $\|\cdot\|_{L^1_{SOT}}$ dada por

$$\|P\|_{L^1_{SOT}} = \sup_{\|x\|=1} \int_0^{2\pi} \left\| \sum_l T_l(x) e^{ilt} \right\| \frac{dt}{2\pi},$$

con $P \in P(\mathbb{T}, \mathcal{B}(H))$, $P(t) = \sum_l T_l e^{ilt}$, $(T_l)_{l\in\mathbb{Z}} \in c_{00}(\mathcal{B}(H))$.

El capítulo se cierra con la Subsección 4.3.2, que completa los resultados vistos al final del Capítulo 3 relacionados con funciones holomorfas con valores en operadores y matrices triangulares. Por ejemplo, vemos el siguiente teorema.

Teorema *Sea* $\mathbf{A} = (T_{j-k}) \in \mathcal{U} \cap \mathcal{T}$ *con* $\sup_{l\geq 0} \|T_l\| < \infty$, *y consideremos* $G_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} T_l z^l, |z| < 1$.

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *si y sólo si* $G_{\mathbf{A}} \in H^\infty(\mathbb{D}, \mathcal{B}(H))$.

(ii) $\mathbf{A} \in C(\ell^2(H))$ *si y sólo si* $G_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(H))$.

(iii) Si $G_{\mathbf{A}} \in H^1(\mathbb{D}, \mathcal{B}(H))$ *entonces* $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$.

(iv) Si $G_{\mathbf{A}} \in H^1(\mathbb{T}, \mathcal{B}(H))$ *entonces* $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$.

Los resultados del capítulo se encuentran recogidos en el siguiente artículo:

> Blasco, O.; García-Bayona, I., A class of Schur multipliers of matrices with operator entries, *Mediterr. J. Math*, to appear.

El Capítulo 5 es el capítulo final de la tesis. En los anteriores tres capítulos se ha estudiado un producto de Schur basado en la composición de operadores. Este capítulo introduce otra versión de producto tipo Schur para matrices con entradas en operadores, denotado $\circledast$, donde la operación entre las entradas de la matriz es el producto de Schur. También con la misma idea, un producto de tipo Kronecker, $\boxtimes$, será definido en este contexto.

La Sección 5.2 explora algunas propiedades que este nuevo producto de tipo Schur satisface. El resultado más importante, probado en varios pasos, muestra que los operadores lineales y acotados, provistos de dicho producto, forman una estructura de álgebra de Banach conmutativa. En el proceso de probar este teorema, nos daremos cuenta de una relación existente entre las matrices escalares y las matrices con entradas en operadores que nos permitirá obtener algunas aplicaciones, pues hay un modo de calcular la norma de operador o multiplicador de matrices con entradas en operadores en términos de matrices escalares.

La primera aplicación proporciona un método para obtener multiplicadores para el nuevo producto (un espacio denotado por $\mathcal{M}^{\circledast}$) en términos de multiplicadores para el producto clásico de Schur, y viceversa.

Teorema *(i) Sea $A = (a_{i,j})_{i,j}$ una matriz de $\mathcal{M}(\ell^2)$. Entonces, dado $n \geq 1$, la matriz $\mathbf{A}^{n)}$ formada tomando bloques de tamaño $n \times n$ en A es una matriz con entradas en operadores que define un elemento de $\mathcal{M}^{\circledast}\left(\mathcal{B}\left(\ell^2\left(\ell^2_n(\mathbb{C})\right)\right)\right)$. Obsérvese que en el caso en que A es Toeplitz, $\mathbf{A}^{n)}$ también lo es.*

(ii) Sea $\mathbf{A}^{n)}$ una matriz cuyas entradas son matrices de tamaño $n \times n$ con $\mathbf{A}^{n)}$ en el espacio $\mathcal{M}^{\circledast}\left(\mathcal{B}\left(\ell^2\left(\ell^2_n(\mathbb{C})\right)\right)\right)$. Entonces, la matriz A de entradas escalares obtenida liberando las entradas de $\mathbf{A}^{n)}$, define un elemento de $\mathcal{M}(\ell^2)$. Además, si $\mathbf{A}^{n)}$ es Toeplitz, A no necesariamente lo es.

La última aplicación de la sección es un método para construir una cantidad numerable de elementos pertenecientes a diferentes espacios de medidas vectoriales a partir de un único elemento de $L^\infty(\mathbb{T})$.

Teorema *Sea $f(t) := \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{ikt} \in L^\infty(\mathbb{T})$. Entonces, dado $N \in \mathbb{N}$, tenemos que la distribución*

$$f_N(t) \sim \sum_{k=-\infty}^{\infty} T_k^{N)} e^{ikt}$$

pertenece a $V^\infty\left(\mathbb{T}, \mathcal{B}(\ell^2_N(\mathbb{C}))\right)$, donde $T_k^{N)}$ es una matriz de Toeplitz dada por la sucesión $(\widehat{f}(Nk+j))_{j=-N+1}^{j=N-1}$.

La Sección 5.3 tiene como objeto de estudio a las matrices bloque finitas, y por comodidad se empleará la notación $\mathcal{M}_N(\mathcal{M}_n) := \mathcal{M}_{N\times N}(\mathcal{M}_{n\times n}(\mathbb{R}))$. Introduciremos el nuevo producto de tipo Kronecker $\boxtimes$ para matrices bloque mencionado anteriormente, también basado en el producto de Schur clásico. El propósito de esta sección es estudiar trazas de matrices bloque junto con estos dos nuevos productos. Recordamos que el operador traza para matrices bloque, $\mathrm{tr} : \mathcal{M}_N(\mathcal{M}_n) \to \mathbb{R}$, actúa como sigue: dada $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n)$, entonces

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{N} \mathrm{tr}(T_{i,i}) = \sum_{i=1}^{N}\sum_{l=1}^{n} T_{i,i}(l,l),$$

donde la traza que aparece tras la primera igualdad es la traza usual para matrices con entradas escalares.

Estudiamos algunas igualdades y desigualdades en relación con este operador y los productos $\circledast$ y $\boxtimes$. Comprobaremos que la traza no es submultiplicativa para ninguno de ellos. Sin embargo, introduciremos dos espacios en el contexto de los cuales esto puede cambiar: los espacios $\mathcal{M}_N^S(\mathcal{M}_n)$ y $\mathcal{M}_N^+(\mathcal{M}_n)$.

Definición *Dados $N, n \in \mathbb{N}$, definimos los siguientes subconjuntos de $\mathcal{M}_N(\mathcal{M}_n)$:*

$$\mathcal{M}_N^S(\mathcal{M}_n) := \{(T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n) \;/\; \sum_{k=1}^{N} T_{k,k}(l,l) \geq 0,\ \forall\, 1 \leq l \leq n\},$$

$$\mathcal{M}_N^+(\mathcal{M}_n) := \{(T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n) \;/\; T_{k,k}(l,l) \geq 0,\ \forall\, 1 \leq k \leq N,\ \forall\, 1 \leq l \leq n\}.$$

En efecto, se tiene el resultado siguiente.

Teorema *Sean $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n)$ y $\mathbf{B} = (S_{k,j})_{k,j} \in \mathcal{M}_M(\mathcal{M}_n)$.*

(i) Si $M = N$, $\mathbf{A} \in \mathcal{M}_N^S(\mathcal{M}_n)$ y $\mathbf{B} \in \mathcal{M}_N^+(\mathcal{M}_n)$, entonces

$$\mathrm{tr}(\mathbf{A} \circledast \mathbf{B}) \leq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B}).$$

(ii) Si $\mathbf{A} \in \mathcal{M}_N^S(\mathcal{M}_n)$ *y* $\mathbf{B} \in \mathcal{M}_M^S(\mathcal{M}_n)$*, entonces*

$$\operatorname{tr}(\mathbf{A} \boxtimes \mathbf{B}) \leq \operatorname{tr}(\mathbf{A}) \cdot \operatorname{tr}(\mathbf{B}).$$

También damos una versión de este resultado para productos finitos, y proporcionamos una pequeña aplicación que consiste en analizar la traza de una versión de la exponencial de una matriz definida por medio del producto $\circledast$. Finalmente, también damos estimaciones superiores para trazas de productos de matrices que combinan tanto el producto $\circledast$ como el producto $\boxtimes$ en términos de trazas que sólo involucran a uno de los productos y que, por tanto, son más sencillas de calcular.

Teorema *Sean* $\mathbf{A}_i, \mathbf{B}_i \in \mathcal{M}_N^+(\mathcal{M}_n)$*, for* $1 \leq i \leq m$*. Entonces, tenemos*

(i) $\operatorname{tr}\Big((\mathbf{A}_1 \boxtimes \mathbf{A}_2 \boxtimes \cdots \boxtimes \mathbf{A}_m) \circledast (\mathbf{B}_1 \boxtimes \mathbf{B}_2 \boxtimes \cdots \boxtimes \mathbf{B}_m) \Big) \leq \prod_{i=1}^m \operatorname{tr}(\mathbf{A}_i \circledast \mathbf{B}_i)$.

(ii) $\operatorname{tr}\Big((\mathbf{A}_1 \circledast \mathbf{A}_2 \circledast \cdots \circledast \mathbf{A}_m) \boxtimes (\mathbf{B}_1 \circledast \mathbf{B}_2 \circledast \cdots \circledast \mathbf{B}_m) \Big) \leq \prod_{i=1}^m \operatorname{tr}(\mathbf{A}_i \boxtimes \mathbf{B}_i)$.

Parte de los contenidos de este capítulo se encuentran en el artículo:

> García-Bayona, I., Traces of Schur and Kronecker Products for Block Matrices, *Khayyam J. Math.*, 2019. DOI: 10.22034/kjm.2019.84207.

Resum

"Tota gran comesa ha de tindre un principi, però és en la continuïtat fins al final, fins que s'ha acabat totalment, on està la verdadera glòria."

—Sir Francis Drake.

Aquesta tesi es dedica a l'estudi de problemes en les àrees de mesurabilitat i anàlisi harmònic matricial en el context d'espais de matrius i operadors. Més concretament, l'objectiu és extendre conceptes i resultats d'aquestes teories del context escalar a un marc de treball més general, on les funcions prenen valors en espais d'operadors, o les matrius tenen com a entrades a altres matrius.

En primer lloc, es començarà amb una secció introductòria on tractarem aspectes relatius a notació, algunes definicions i ferramentes que apareixeràn al llarg de tot el treball, i donarem una idea general del tipus de resultats que podem esperar trobar en cada capítol.

El Capítol 1 tracta sobre qüestions de mesurabilitat per a funcions amb valors en operadors $f : \Omega \to \mathcal{L}(E_1, E_2)$, on (Ω, Σ) és un espai mesurable (és a dir, Σ és una σ-àlgebra sobre Ω) i $\mathcal{L}(E_1, E_2)$ denota l'espai d'operadors lineals i fitats entre dos espais de Banach E_1 i E_2. En $\mathcal{L}(E_1, E_2)$, s'utilitzaràn les tres topologies fonamentals: la topologia de la norma $\tau_{\|\cdot\|}$, la topologia forta d'operadors τ_{SOT} i la topologia dèbil d'operadors τ_{WOT}.

La notació $\mathcal{N}_{\|\cdot\|}$, $\mathcal{N}_{SOT}$ i $\mathcal{N}_{WOT}$ s'emprarà per a les bases de les corresponents topologies, donades per

$$B(T,\varepsilon) = \{S \in \mathcal{L}(E_1,E_2); \|S-T\| < \varepsilon\},$$

$$N(T;\mathbf{x},\varepsilon) = \{S \in \mathcal{L}(E_1,E_2) : \max_{1\le j\le n} \|(S-T)(x_j)\| < \varepsilon\}$$

i

$$N(T;\mathbf{x},\mathbf{y}^*,\varepsilon) = \{S \in \mathcal{L}(E_1,E_2) : \max_{1\le j\le n} |\langle (S-T)x_j, y_j^*\rangle| < \varepsilon\},$$

on $\varepsilon > 0$, $T \in \mathcal{L}(E_1,E_2)$, $\mathbf{x} = (x_1,x_2,\cdots,x_n) \in (E_1)^n$, $\mathbf{y}^* = (y_1^*,y_2^*,\cdots,y_n^*) \in (E_2^*)^n$ i $n \in \mathbb{N}$, respectivament.

És clar que $\mathcal{N}_{SOT(\mathbb{K},E_2)} = \mathcal{N}_{\|\cdot\|}$, i denotarem $\mathcal{N}_{weak} = \mathcal{N}_{WOT(\mathbb{K},E_2)}$. S'observa que quan el segon espai és el cos, es té $\mathcal{N}_{SOT(E_1,\mathbb{K})} = \mathcal{N}_{WOT(E_1,\mathbb{K})}$ i en aquest cas utilitzarem la notació $\mathcal{N}_{weak^*}$. En el cas en què E_2 siga un espai dual, $E_2 = F^*$, com $\mathcal{L}(E_1,E_2)$ és un espai dual, a més de $\mathcal{N}_{WOT}$ i $\mathcal{N}_{weak}$, també tindrem $\mathcal{N}_{weak^*}$, corresponent a la base donada per

$$N(T;\mathbf{x},\mathbf{y},\varepsilon) = \{S \in X : \max_{1\le j\le n} |\langle (S-T)x_j, y_j\rangle| < \varepsilon\},$$

on $\mathbf{x} = (x_1,\cdots,x_n) \in (E_1)^n$, $\mathbf{y} = (y_1,\cdots,y_n) \in (E_2)^n$ i $n \in \mathbb{N}$.

La noció de funció mesurable depén fortament de la formulació que adoptem, tant quan es tracta amb funcions que prenen valors en espais de Banach (veure [20, 23, 33]) com quan treballem amb espais d'operadors (veure [4, 23, 33, 37, 47]). La considerable varietat de nocions que podem trobar en la literatura, combinada amb el fet que habitualment les qüestions sobre measurabilitat són tractades amb la intenció de desenvolupar una teoria d'integració (i per tant no reben el tractament detallat que mereixen per dret propi) pot fer que una primera aproximació a la materia resulte un poc confusa. Per tant, en aquest primer capítol introduirem una terminologia, treballant únicament amb espais mesurables sense cap tipus de mesura subjacent, que creiem que pot ajudar a diferenciar les corresponents definicions de mesurabilitat i a desentranyar les relacions entre elles d'una manera sistemàtica.

Emprearem dos tipus de mesurabilitat diferents però, como veurem, relacionats. Un d'ells es defineix mitjançant les corresponents bases de cada topologia, mentre que l'altre utilitza la noció d'aproximabilitat. Siga (Ω, Σ) un espai mesurable, i (Y, τ) un espai vectorial topològic amb base $\beta \subseteq \tau$.

- Una funció $f : \Omega \to Y$ s'anomena **β-mesurable** quan $f^{-1}(A) \in \Sigma$ per a tot $A \in \beta$.
- Una funció $f : \Omega \to Y$ es diu que és **τ-aproximable** quan f és límit puntual de funcions finitament valuades. Dit d'una altra manera, quan $s_N \to f$ en la topologia τ, on $s_N = \sum_{k=1}^{N} y_k \chi_{A_k}$ amb $y_k \in Y$ i A_k conjunts disjunts tals que $\cup_{k=1}^{N} A_k = \Omega$.

A continuació recordem alguns resultats bàsics que connecten aquestes nocions, adaptats a la nostra terminologia. Probablement el més famós d'ells és el Teorema de mesurabilitat de Pettis.

Teorema *(veure [20, Capítol 2, Teorema 2]) Teorema de mesurabilitat de Pettis. Siga (Ω, Σ, μ) un espai de mesura finita complet i siga E un espai de Banach. Aleshores $f : \Omega \to E$ és $\|\cdot\|$-aproximable μ-q.p.t. (quasi per tot, és a dir, amb l'excepció d'un conjunt de mesura nul·la) si i només si f és $\mathcal{N}_{weak}$-mesurable i $f(\Omega)$ és essencialment separable, és a dir, $f(\Omega \setminus A)$ és separable per a algun $A \in \Sigma$ amb $\mu(A) = 0$.*

Un corol·lari de Dunford ens diu que per a $E = \mathcal{L}(E_1, E_2)$, $f : \Omega \to \mathcal{L}(E_1, E_2)$ satisfà que f_x és $\|\cdot\|$-aproximable μ-q.p.t per a $x \in E_1$ si i només si f és $\mathcal{N}_{WOT}$-mesurable i $f_x(\Omega)$ és essencialment separable per a tot $x \in E_1$, on $f_x(\omega) = f(\omega)(x)$.

Teorema *(veure [33, Teorema 3.5.5]) Teorema de mesurabilitat de Dunford. Siga (Ω, Σ, μ) un espai de mesura finita complet i siguen E_1, E_2 espais de Banach. Aleshores $f : \Omega \to \mathcal{L}(E_1, E_2)$ és $\|\cdot\|$-aproximable μ-q.p.t. si i només si f és $\mathcal{N}_{WOT}$-mesurable i $f(\Omega)$ és essencialment separable en $\mathcal{L}(E_1, E_2)$.*

Teorema *(veure [37, Teorema 2]) Teorema de mesurabilitat de Johnson. Siga (Ω, Σ, μ) un espai de mesura finita complet i siga H un espai de Hilbert separable. Aleshores*

$f : \Omega \to \mathcal{L}(H,H)$ satisfà que f_x és $\|\cdot\|$-aproximable μ-q.p.t. per a tot $x \in H$ si i només si f és $\mathcal{N}_{SOT}$-mesurable.

En la Secció 1.2 comencem analitzant la noció de mesurabilitat respecte de bases de diferents topologies. Es comença recordant un fet ben conegut en el context d'espais mètrics.

Proposició *Siga (Y,d) un espai mètric separable. Aleshores $f : \Omega \to Y$ és $\mathcal{N}_d$-mesurable si i només si f és τ_d-mesurable. En particular, si E és un espai de Banach, $f : \Omega \to E$ és $\mathcal{N}_{\|\cdot\|}$-mesurable i $f(\Omega)$ és separable aleshores f és $\tau_{\|\cdot\|}$-mesurable.*

Quan tractem amb funcions $f : \Omega \to \mathcal{L}(E_1,E_2)$, de vegades la següent notació pot resultar útil: per a $x \in E_1$ i $y^* \in E_2^*$, denotem $f_x(w) = f(w)(x)$ i $f_{x,y^*}(w) = \langle y^*, f(w)(x)\rangle$. La següent proposició dóna ús a l'esmentada notació i proporciona una caracterització d'algunes nocions de β-mesurabilitat.

Proposició *Siguen $X = \mathcal{L}(E_1,E_2)$ i $f : \Omega \to X$. Aleshores*

(i) f és $\mathcal{N}_{\|\cdot\|}$-mesurable $\iff$ $\|f(\cdot) - T\|$ és mesurable per a tot $T \in X$.

(ii) f és $\mathcal{N}_{SOT}$-mesurable $\iff$ f_x és $\mathcal{N}_{\|\cdot\|}$-mesurable per a tot $x \in E_1$.

(iii) f és $\mathcal{N}_{WOT}$-mesurable $\iff$ f_{x,y^*} és mesurable per a tot $x \in E_1$ i per a tot $y^* \in E_2^*$.

És clar que $\mathcal{N}_{WOT} \subset \mathcal{N}_{weak}$, i per tant la $\mathcal{N}_{weak}$-mesurabilitat implica la $\mathcal{N}_{WOT}$-mesurabilitat. Per altra banda, es té que $\tau_{weak} \subset \tau_{\|\cdot\|}$ en qualsevol espai de Banach i també $\tau_{WOT} \subset \tau_{SOT} \subset \tau_{\|\cdot\|}$ per a $X = \mathcal{L}(E_1,E_2)$. Per tant, funcions que són $\tau_{\|\cdot\|}$-mesurables són també $\mathcal{N}_{weak}$-mesurables i $\mathcal{N}_{SOT}$-mesurables. Sota certes condicions de separabilitat, es poden recuperar versions de mesurabilitat més fortes a partir d'altres més dèbils, com mostren les següents proposicions, que tanquen la Secció 1.2.

Proposició *Siga $f : \Omega \to X = \mathcal{L}(E_1,E_2)$, on E_1 és un espai de Banach separable. Si f és $\mathcal{N}_{SOT}$-mesurable, aleshores f és també $\mathcal{N}_{\|\cdot\|}$-mesurable.*

Proposició *Siga $f : \Omega \to \mathcal{L}(E_1,E_2)$ amb E_2 separable. Aleshores f és $\mathcal{N}_{WOT}$-mesurable si i només si f és $\mathcal{N}_{SOT}$-mesurable.*

L'estudi de les nocions de mesurabilitat en termes d'aproximació està cobert per la Secció 1.3. El concepte de funció simple i numerablement valorada és important i recordem la seua definició.

Definició *En un espai topològic de Hausdorff* (X, τ)*, diem que una funció* $f : \Omega \to X$ *és simple (respectivament comptablement valorada) si existeix un conjunt finit (respectivament successió)* $(x_n)_n \subset X$ *i una partició finita (respectivament una partició numerable) de conjunts* $(A_n)_n \subset \Sigma$ *disjunts dos a dos, tals que* $\Omega = \cup_k A_k$ *i* $f = \sum_n x_n \chi_{A_n}$*.*

La $\|\cdot\|$-, *weak*-, SOT- i WOT-aproximabilitat d'una funció $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ està determinada per l'existència d'una successió de funcions simples $s_n : \Omega \to X$ amb valors en operadors tal que

$$\|s_n(\omega) - f(\omega)\| \xrightarrow[n\to\infty]{} 0, \quad \forall \omega \in \Omega,$$

$$\langle s_n(\omega), T^* \rangle \xrightarrow[n\to\infty]{} \langle f(\omega), T^* \rangle, \quad \forall T^* \in X^*, \quad \forall \omega \in \Omega,$$

$$\lim_n \|s_n(\omega)(x) - f(\omega)(x)\| = 0, \quad \forall x \in E_1, \quad \forall \omega \in \Omega$$

o bé

$$\langle s_n(\omega)(x), y^* \rangle \xrightarrow[n\to\infty]{} \langle f(\omega)(x), y^* \rangle, \quad \forall x \in E_1, \quad \forall y^* \in E_2^*, \quad \forall \omega \in \Omega,$$

respectivament.

Les connexions entre l'aproximabilitat, la mesurabilitat respecte a bases i la separabilitat del rang en la topologia corresponent comencen a brillar en la resta de la secció. Per exemple, tenim el següent teorema.

Teorema *Siga* (Y, d) *un espai mètric i* $f : \Omega \to Y$*. Les següents afirmacions són equivalents:*

(i) f *és d-aproximable.*

(ii) f *és* τ_d*-mesurable i* $f(\Omega)$ *és d-separable.*

(iii) f *és* $\mathcal{N}_d$*-mesurable i* $f(\Omega)$ *és d-separable.*

Com a conseqüència del teorema previ i del Teorema de Banach-Alaoglu, obtindrem el següent resultat.

Proposició *Siga E un espai de Banach separable, $X = E^*$ i siga $f : \Omega \to X$ una funció fitada. Aleshores f és weak*-aproximable si i només si f és $\mathcal{N}_{weak^*}$-mesurable.*

És prou sorprenent el fet que, per a qualsevol espai de Banach, les nocions de $\|\cdot\|$-aproximabilitat i aproximabilitat dèbil coincidisquen.

Proposició *Siga X un espai de Banach i $f : \Omega \to X$. Aleshores, f és $\|\cdot\|$-aproximable si i només si f és weak-aproximable.*

Estudiarem alguns exemples per diferenciar certes nocions de mesurabilitat, amb la qual cosa tancarem la secció i passarem a la Secció 1.4. En aquesta, el primer objetiu serà donar una prova del Teorema de mesurabilitat de Dunford (i per tant també del de Pettis).

Teorema *Siga $f : \Omega \to X = \mathcal{L}(E_1, E_2)$. Les següents afirmacions són equivalents:*

(i) f és $\|\cdot\|$-aproximable.

(ii) f és $\tau_{\|\cdot\|}$-mesurable i $f(\Omega)$ és separable en X.

(iii) f és $\mathcal{N}_{\|\cdot\|}$-mesurable i $f(\Omega)$ és separable en X.

(iv) f és $\mathcal{N}_{SOT}$-mesurable i $f(\Omega)$ és separable en X.

(v) f és $\mathcal{N}_{WOT}$-mesurable i $f(\Omega)$ és separable en X.

La nostra principal contribució en aquesta secció, i en aquest capítol, ha sigut la versió del Teorema de mesurabilitat de Pettis per a la topologia SOT, en el cas en què l'espai E_1 és un espai de Banach separable.

Teorema *Siga $f : \Omega \to X = \mathcal{L}(E_1, E_2)$, on E_1 és separable. Les següents afirmacions són equivalents.*

(i) f és SOT-aproximable

(ii) f és WOT-aproximable.

(iii) f és $\mathcal{N}_{WOT}$-mesurable i $f(\Omega)$ és WOT-separable.

Per acabar el capítol, aplicarem aquest teorema per construir alguns exemples naturals de funcions amb valors en operadors que són SOT-aproximables.

Els resultats que apareixen en aquest capítol es troben publicats en el següent article:

> Blasco, O.; García-Bayona, I., Remarks on Measurability of Operator-valued Functions, *Mediterr. J. Math.* **13** (2016), 5147–5162. DOI: 10.1007/s00009-016-0798-1.

En el Capítol 2, passem a l'àrea de l'anàlisi harmònic matricial. El capítol comença amb una secció introductòria en la qual, una vegada recordats alguns aspectes i resultats clàssics del cas escalar, com el Teorema de Toeplitz o el Teorema de Bennett, comencem definint alguns dels elements més importants amb els quals treballarem al larg d'aquest capítol i alguns dels següents. En primer lloc, tenim l'espai $\mathcal{B}(\ell^2(H))$.

Definició *Donada una matriu* $\mathbf{A} = (T_{kj})$ *amb entrades* $T_{kj} \in \mathcal{B}(H)$ *i* $\mathbf{x} \in c_{00}(H)$, *denotem* $\mathbf{A}(\mathbf{x})$ *a la successió* $(\sum_{j=1}^{\infty} T_{kj}(x_j))_k$. *Direm que* $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *si l'aplicació* $\mathbf{x} \to \mathbf{A}(\mathbf{x})$ *extén a un operador lineal i fitat en* $\ell^2(H)$, *és a dir*

$$\Big(\sum_{k=1}^{\infty} \|\sum_{j=1}^{\infty} T_{kj}(x_j)\|^2\Big)^{1/2} \leq C\Big(\sum_{j=1}^{\infty} \|x_j\|^2\Big)^{1/2}.$$

La norma en aquest espai ve donada de la següent manera:

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{A}\mathbf{x}\|_{\ell^2(H)} \leq C\|\mathbf{x}\|_{\ell^2(H)}\}.$$

La principal operació que estudiarem serà una versió del clàssic producte de Schur (producte de matrius entrada a entrada) en el context de matrius les entrades de les quals són operadors, i el definim de la següent manera.

Definició *Siguen* $\mathbf{A} = (T_{kj})$ *i* $\mathbf{B} = (S_{kj})$ *matrius amb* $T_{kj}, S_{kj} \in \mathcal{B}(H)$. *Definim el seu producte de Schur com*

$$\mathbf{A} * \mathbf{B} = (T_{kj}S_{kj}),$$

on $T_{kj}S_{kj}$ és la composició dels operadors T_{kj} i S_{kj}.

Un concepte molt important que es començarà a estudiar en aquest capítol és el de multiplicador de Schur. Els multiplicadors per al nostre producte tenen una definició similar a la dels multiplicadors en el context escalar. No obstant, ja que el nostre producte no és commutatiu, és necessari definir multiplicadors a dreta i a esquerra.

Definició *Donada una matriu $\mathbf{A} = (T_{kj})$, es diu que $\mathbf{A}$ és un multiplicador de Schur a dreta (respectivament multiplicador de Schur a esquerra), i ho denotem per $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ (respectivament $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$), sempre que $\mathbf{B} * \mathbf{A} \in \mathcal{B}(\ell^2(H))$ (respectivament $\mathbf{A} * \mathbf{B} \in \mathcal{B}(\ell^2(H))$) per a qualsevol matriu $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. Les expressions de la norma en aquests espais són*

$$\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{B} * \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq C\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\}$$

i

$$\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{A} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \leq C\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\}.$$

La Secció 2.2 inclou algunes nocions bàsiques sobre successions i funcions amb valors vectorials que seràn utilitzades en la resta del capítol. En particular, s'estudiaràn espais com $\ell^2(\mathbb{N}, \mathcal{B}(H))$, $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ i $\ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$, i explorarem relacions entre ells, proporcionant exemples i contraexemples quan siga necessari.

El producte tensorial projectiu és una ferramenta que apareixerà en diverses ocasions al llarg de les proves. Més intens fins i tot és l'ús de certes idees de la teoria de mesures vectorials, i per aquesta raó hem inclòs la Secció 2.3. En ella, recordem tres identificacions entre operadors i mesures, que mencionem ací breument. La primera d'elles ens diu que $\mathfrak{M}(\mathbb{T}, E)$ pot identificar-se amb l'espai dels operadors dèbilment compactes $T_\mu : C(\mathbb{T}) \to E$ i que $\|T_\mu\| = \|\mu\|$ (veure [20, Capítol 6]). En el cas d'espais duals $E = F^*$, el Teorema de Singer (veure [53, 54, 32])) assegura que $M(\mathbb{T}, E) = C(\mathbb{T}, F)^*$. En altres paraules, que existeix un operador lineal i fitat $\Psi_\mu : C(\mathbb{T}, F) \to \mathbb{C}$ amb $\|\Psi_\mu\| = |\mu|$ tal que $\Psi_\mu(y\phi) =$

$T_\mu(\phi)(y)$, $\phi \in C(\mathbb{T})$, $y \in F$. A més, en el context d'operadors, encara hi ha una tercera opció que es pot considerar, utilitzant la topologia forta d'operadors, i és $\Phi_\mu : C(\mathbb{T}, X) \to Y^*$, definit per $\Phi_\mu(f)(y) = \Psi_\mu(f \otimes y)$, $f \in C(\mathbb{T}, X)$, $y \in Y$, on $f \otimes y(t) = f(t) \otimes y$.

Descriurem diferents tipus d'espais de mesures vectorials, i d'especial importància serà l'espai $M_{SOT}(\mathbb{T}, \mathcal{B}(H)))$, format per mesures $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ tals que $\mu_x \in M(\mathbb{T}, H)$ per a qualsevol $x \in H$. La norma en aquest espai s'expressa de la següent manera.

$$\|\mu\|_{SOT} = \sup\{|\mu_x| : x \in H, \|x\| = 1\}.$$

A més, l'anomenada "mesura adjunta" també jugarà un paper important.

Definició *Siga* $\mu : \mathfrak{B}(\mathbb{T}) \to \mathcal{L}(X, Y^*)$ *una mesura vectorial. Definim la "mesura adjunta"* $\mu^* : \mathfrak{B}(\mathbb{T}) \to \mathcal{L}(Y, X^*)$ *mitjançant la fórmula*

$$\mu^*(A)(y)(x) = \mu_x(A)(y), \quad A \in \mathfrak{B}(\mathbb{T}), x \in X, y \in Y.$$

Amb el concepte de mesura adjunta en ment, l'espai $M_{SOT}(\mathbb{T}, \mathcal{B}(H))$, molt relacionat amb els multiplicadorss de Schur, como es veurà en seccions posteriors, pot ser descrit mitjançant operadors de la següent forma.

Proposició *Siga* $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. *Aleshores* $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ *si i només si* $\Phi_{\mu^*} \in \mathcal{L}(C(\mathbb{T}, H), H)$. *A més,* $\|\mu\|_{SOT} = \|\Phi_{\mu^*}\|$.

En la Secció 2.4 presentem condicions suficients i necessàries perquè una matriu pertanga a $\mathcal{B}(\ell^2(H))$, i algunes involucren condicions sobre les files i columnes relacionades amb els espais vists en la Secció 2.2. Un dels resultats més importants de la Secció 2.4 és la versió del Teorema de Schur en el marc de les matrius amb entradas en operadors, és a dir, demostrarem que l'espai $\mathcal{B}(\ell^2(H))$ defineix una àlgebra de Banach amb el producte de Schur. No obstant això, perquè un producte de matrius caiga en l'espai $\mathcal{B}(\ell^2(H))$ no és necessari que ambdues matrius estiguen en aquest espai, per descomptat. De fet, en la subsecció 2.4.2, trobem el següent teorema, on comprovem que és suficient que una de les matrius estiga en $\mathcal{B}(\ell^2(H))$ i que l'altra matriu verifique certes condicions sobre les files relacionades amb espais de successions amb valors en operadors vists en la Secció 2.2.

Teorema *Siguen* $\mathbf{A} = (T_{k,j})$ *i* $\mathbf{B} = (S_{k,j})$ *matrius amb entrades en* $\mathcal{B}(H)$. *Si* $\mathbf{B} \in B(\ell^2(H))$ *i* $\mathbf{A} \in \ell^\infty(\mathbb{N}, \ell^2(\mathcal{B}(H))) \cup \ell^\infty\left(\mathbb{N}, \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))\right)$, *aleshores* $\mathbf{A} * \mathbf{B} \in B(\ell^2(H))$, *és a dir,* $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$.

La Secció 2.5 és l'última del capítol i conté alguns dels resultats més importants sobre matrius de Toeplitz (espai de matrius constants per diagonals, denotat per $\mathcal{T}$). El primer d'aquests resultats és la generalització del Teorema de Toeplitz, que proporciona una condició suficient i necessària en relació amb la funció associada a una matriu de Toeplitz perquè dita matriu pertanga a l'espai $\mathcal{B}(\ell^2(H))$. L'espai de mesures vectorials $V^\infty(\mathbb{T}, \mathcal{B}(H))$, definit en la Secció 2.3, fa la seua aparició ací substituint a $L^\infty(\mathbb{T})$ en aquesta versió generalitzada.

Teorema *Siga* $\mathbf{A} = (T_{kj}) \in \mathcal{T}$. *Aleshores,* $A \in \mathcal{B}(\ell^2(H))$ *si i només si existeix una mesura* $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ *tal que* $T_{kj} = \hat{\mu}(j-k)$ *per a tot* $k, j \in \mathbb{N}$. *A més,* $\|\mathbf{A}\| = \|\mu\|_\infty$.

La secció continua presentant algunes condicions suficients perquè una matriu siga un multiplicador de Schur en el nostre context, i termina amb un parell de resultats que tracten de generalitzar el Teorema de Bennett al nostre marc de treball.

Teorema *Si* $\mu \in M(\mathbb{T}, \mathcal{B}(H))$ *i* $\mathbf{A} = (T_{kj}) \in \mathcal{T}$ *amb* $T_{kj} = \hat{\mu}(j-k)$ *per a* $k, j \in \mathbb{N}$, *aleshores* $\mathbf{A} \in \mathcal{M}_l(\ell^2(H)) \cap \mathcal{M}_r(\ell^2(H))$. *A més,*

$$\max\{\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}, \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\} \leq |\mu|.$$

Teorema *Siga* $\mathbf{A} = (T_{kj}) \in \mathcal{T} \cap \mathcal{M}_r(\ell^2(H))$. *Aleshores, existeix* $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ *tal que* $T_{kj} = \hat{\mu}(j-k)$ *per a tot* $k, j \in \mathbb{N}$. *Més encara,*

$$\|\mu\|_{SOT} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}.$$

Els resultats d'aquest capítol es troben publicats en l'article següent:

Blasco, O.; García-Bayona, I., Schur Product with Operator-valued Entries, *Taiwanese J. Math.*, advance publication, 30 November 2018. DOI:10.11650/tjm/181110. https://projecteuclid.org/euclid.twjm/1543546839.

El Capítol 3 està dedicat principalment a l'estudi d'un tipus particular de matrius, anomenades "matrius contínues". Aquest espai es denotarà per $C(\ell^2(H))$, i és l'espai de matrius que poden ser aproximades en la norma d'operadors per matrius amb un nombre finit de diagonals no nul·les, o més precisament, "matrius polinomials". L'espai de les matrius contínues amb entrades escalars va ser introduït per Barza, Persson i Popa (veure [6]).

Direm que una matriu $\mathbf{A} = (T_{kj})$ amb entrades $T_{kj} \in \mathcal{B}(H)$ és una "matriu polinomial", de manera abreviada $\mathbf{A} \in \mathcal{P}(\ell^2(H))$, quan es satisfacen dues condicions: que $\sup_{k,j} \|T_{kj}\| < \infty$ i que existisquen $N, M \in \mathbb{N}$ tals que $\mathbf{A}$ puga ser escrita com a suma finita de diagonals, $\mathbf{A} = \sum_{l=-N}^{M} \mathbf{D_l}$.

La primera secció del capítol introdueix els conceptes necessaris i la notació que s'emprarà. Principalment es trata d'una continuació natural d'allò que es va tractar al Capítol 2. Un nou tipus de matrius que apareix per primera vegada és el de les matrius M_μ. Donada $\mu \in M(\mathbb{T})$, denotarem per $\mathbf{M}_\mu$ a la matriu de Toeplitz definida per

$$\mathbf{M}_\mu = (\hat{\mu}(j-k)Id)_{k,j} \in \mathcal{T}.$$

La següent fórmula, que és vàlida per exemple per a $\mu \in M(\mathbb{T})$ i $f \in L^1(\mathbb{T})$ (on $\mu * f(t) = \int_0^{2\pi} f(e^{i(t-s)})d\mu(s)$ és la convolució entre funcions i mesures en $\mathbb{T}$) suggereix la relació entre multiplicadors de Fourier i multiplicadors de Schur, i dóna una idea de la importància d'aquest nou tipus de matrius definit anteriorment.

$$\mathbf{M}_\mu * \mathbf{M}_f = \mathbf{M}_f * \mathbf{M}_\mu = \mathbf{M}_{\mu * f}.$$

En el procés d'exploració de les connexions entre l'anàlisi de Fourier i l'anàlisi matricial (veure [44]) la següent funció amb valors en matrius tindrà un paper important.

Definició *Siga* $\mathbf{A} = (T_{kj})$ *amb* $T_{kj} \in \mathcal{B}(H)$ *per a* $k, j \in \mathbb{N}$. *Definim*

$$f_{\mathbf{A}}(t) = \mathbf{M}_t * \mathbf{A} = (e^{i(j-k)t} T_{kj}), \quad t \in [0, 2\pi].$$

En el cas de matrius triangulars superiors (denotades per $\mathcal{U}$*) treballarem amb*

$$F_{\mathbf{A}}(z) = (z^{(j-k)} T_{kj}), \quad |z| < 1.$$

Després de la secció preliminar, la Secció 3.2 presenta exemples particulars de matrius amb entrades en operadors, i alguns procediments per construir-les. Destaquem el següent, que permet incloure el cas escalar en el context d'operadors, generant senzills exemples de matrius en el nostre marc de treball.

Exemple *Siguen* $A = (a_{k,j}) \in \mathcal{B}(\ell^2)$ *i* $T \in \mathcal{B}(H)$. *Aleshores*

$$\mathbf{A} = (a_{k,j} T) \in \mathcal{B}(\ell^2(H)) \;\; i \;\; \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|A\|_{\mathcal{B}(\ell^2)} \|T\|_{\mathcal{B}(H)}.$$

A més, definirem una versió matricial de l'àlgebra de Wiener. La Secció 3.3 tracta en la seua totalitat sobre multiplicadors de Schur. Donarem condicions necessàries perquè una matriu siga un multiplicador de Schur, i presentarem la versió de multiplicadors de l'exemple anterior per construir de manera senzilla exemples de multiplicadors amb entrades en operadors. Utilitzant nuclis de sumabilitat, definim les matrius $M_n(\mathbf{A})$ com $M_n(\mathbf{A}) = \mathbf{M}_{K_n} * \mathbf{A}$, i demostrem la següent proposició, que mostra la seua importància, ja que permet determinar quan una matriu defineix un operador o un multiplicador.

Proposició *Siguen* $\mathbf{A}$ *una matriu amb entrades en* $\mathcal{B}(H)$ *i* $\{k_n\}$ *un nucli de sumabilitat, i denotem* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$. *Aleshores,*

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H)) \;\Leftrightarrow\; \sup_n \|M_n(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} < \infty$.

(ii) $\mathbf{A} \in \mathcal{M}_r(\ell^2(H)) \;\Leftrightarrow\; \sup_n \|M_n(\mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} < \infty$.

(iii) $\mathbf{A} \in \mathcal{M}_l(\ell^2(H)) \;\Leftrightarrow\; \sup_n \|M_n(\mathbf{A})\|_{\mathcal{M}_l(\ell^2(H))} < \infty$.

La Secció 3.4 comença fent ús de la funció $f_{\mathbf{A}}$ definida dalt per mostrar la connexió entre aquest nou espai de matrius i l'espai de les funcions contínues. Un primer resultat senzill però útil mostra que aquesta funció és una isometria entre els espais corresponents. La secció avança presentant diversos resultats entre els quals volem destacar un parell d'ells per la seua importància i utilitat. El primer d'ells és una caracterització de les matrius de $C(\ell^2(H))$ que ens mostra diferents formes en las quals es pot veure aquest espai i la relació amb les funcions contínues.

Teorema *Siga* $\mathbf{A} = (T_{k,j})_{k,j}$ *una matriu amb entrades en* $\mathcal{B}(H)$*, complint la condició* $\sup_{k,j}\|T_{k,j}\| < \infty$*. Les següents afirmacions són equivalents:*

1) $\mathbf{A} \in C(\ell^2(H))$.

2) $\lim_{n\to\infty} M_n(\mathbf{A}) = \mathbf{A}$ *en* $\mathcal{B}(\ell^2(H))$ *on* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$ *i* $\{k_n\} \subseteq L^1(\mathbb{T})$ *és un nucli de sumabilitat.*

3) $\lim_{n\to\infty} \sigma_n(\mathbf{A}) = \mathbf{A}$ *en* $\mathcal{B}(\ell^2(H))$.

4) $t \to f_{\mathbf{A}}(t)$ *és una funció contínua amb valors en* $\mathcal{B}(\ell^2(H))$.

El segon resultat mostra que, quan treballem amb multiplicadors de Schur, és suficient si considerem aquells que envien $C(\ell^2(H))$ a si mateix. Més precisament, tenim:

Teorema $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ *(respectivament* $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$*) si i només si* $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_l$ *(respectivament* $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_r$*).*

Passem, doncs, a la Subsecció 3.4.1, centrada en matrius de Toeplitz. D'aquesta secció, destaquem també dos resultats. El primer d'ells presenta altra caracterització que deixa clara la total connexió entre l'espai $C(\ell^2(H))$ i l'espai de les funcions contínues.

Teorema *Siga* $(T_l)_{l\in\mathbb{Z}}$ *una successió d'operadors en* $\mathcal{B}(H)$ *i siga* $\mathbf{A} = (T_{j-k})_{k,j}$*. Aleshores,* $\mathbf{A} \in C(\ell^2(H))_{\mathcal{T}}$ *si i només si existeix* $g_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(H))$ *tal que* $\widehat{g_{\mathbf{A}}}(l) = T_l$*. Altrament,* $\|g_{\mathbf{A}}\|_{C(\mathbb{T},\mathcal{B}(H))} = \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}$.

El segon resultat, ja vist en l'anterior capítol, és la caracterització dels multiplicadors de Toeplitz en termes de mesures SOT. No obstant això, en aquesta ocasió, la prova alternativa que proporcionem utilitza tècniques i resultats d'aquest capítol, sense dependre de mesures vectorials.

La subsecció final, 3.4.2, introdueix una versió matricial de l'àlgebra del disc, per a matrius triangulars superiors. Assumint que una matriu $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{U}$ satisfà $\sup_{k,j}\|T_{k,j}\| < \infty$, és clar que

$$F_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} \mathbf{D_l} z^l \in \mathcal{H}(\mathbb{D}, \mathcal{B}(\ell^2(H)))$$

és una funció holomorfa ben definida. Provem el següent resultat.

Teorema *Siga* $\mathbf{A} = (T_{kj}) \in \mathcal{U}$ *satisfent* $\sup_{k,j}\|T_{k,j}\| < \infty$.

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *si i només si* $F_{\mathbf{A}} \in H^\infty(\mathbb{D}, \mathcal{B}(\ell^2(H)))$. *A més,* $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|F_{\mathbf{A}}\|_{H^\infty(\mathbb{D},\mathcal{B}(\ell^2(H)))}$.

(ii) $\mathbf{A} \in C(\ell^2(H))$ *si i només si* $F_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(\ell^2(H)))$.

Els resultats d'aquest capítol es troben recollits en el següent article:

> Blasco, O.; García-Bayona, I., New spaces of matrices with operator entries, *Quaest. Math*, 2019. DOI: 10.2989/16073606.2019.1605416.

L'enfocament del Capítol 4 està en sintonia amb l'emprat al capítol previ, però en aquesta ocasió l'èmfasi sobre els multiplicadors és major. Considerarem la classe dels multiplicadors de Schur que poden aproximar-se en la norma de multiplicadors per matrius polinomials. També, com és habitual, donarem un tractament especial al cas de matrius de Toeplitz i triangulars superiors.

La primera secció del capítol és de caràcter preliminar, i fa èmfasi en alguns dels conceptes que seràn necessaris al llarg d'aquest. També, es presenta la nova classe de matrius que serà objecte d'estudi al capítol: l'espai de les "matrius integrables", denotades per $\mathcal{L}^1(\ell^2(H))$.

Definició *Definim* $\mathcal{L}^1_l(\ell^2(H))$ *(respectivament* $\mathcal{L}^1_r(\ell^2(H))$*) com la clausura de* $\mathcal{P}(\ell^2(H))$ *en* $\mathcal{M}_l(\ell^2(H))$ *(respectivament* $\mathcal{M}_r(\ell^2(H))$*). Utilitzarem la notació* $\mathcal{L}^1(\ell^2(H)) = \mathcal{L}^1_l(\ell^2(H)) \cap \mathcal{L}^1_r(\ell^2(H))$.

Quan vam estudiar les matrius contínues, vam veure que aquest nom era raonable per a aquesta classe de matrius, degut a les propietats que les relacionen amb l'espai de funcions contínues. El significat del nom "matrius integrables" serà també àmpliament justificat en el capítol pels resultats que presentarem. La Secció 4.2 comença estudiant el nou espai $\mathcal{L}^1(\ell^2(H))$ i proporcionant alguns exemples de matrius en ell. A banda d'això, s'obtindrà una formulació equivalent utilitzant el producte de Schur amb matrius de Toeplitz donades per nuclis de sumabilitat en la línia de l'obtinguda per a l'espai $C(\ell^2(H))$.

Teorema *Siga* $\mathbf{A} = (T_{k,j})_{k,j}$ *una matriu amb entrades en* $\mathcal{B}(H)$ *satisfent* $\sup_{k,j}\|T_{k,j}\| < \infty$. *Les següents afirmacions són equivalents:*

1) $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$.

2) $\lim_{n\to\infty} M_n(\mathbf{A}) = \mathbf{A}$ *en* $\mathcal{M}_r(\ell^2(H))$ *on* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$ *i* $\{k_n\} \subseteq L^1(\mathbb{T})$ *és un nucli de sumabilitat.*

3) $\lim_{n\to\infty} \sigma_n(\mathbf{A}) = \mathbf{A}$ *en* $\mathcal{M}_r(\ell^2(H))$.

4) $\lim_{r\to 1} P_r(\mathbf{A}) = \mathbf{A}$ *en* $\mathcal{M}_r(\ell^2(H))$.

Estudiarem també matrius columna i matrius diagonals en $\mathcal{L}^1_l(\ell^2(H))$, comprovant, en particular, que una versió del lema de Riemann-Lebesgue es verifica en el nostre context.

Proposició *(Lema de Riemann-Lebesgue) Si* $A = \sum_l \mathbf{D_l} \in \mathcal{L}^1_r(\ell^2(H))$, *aleshores*

$$\|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} \xrightarrow[|l|\to\infty]{} 0.$$

La Secció 4.3 comença analitzant la connexió entre l'espai $\mathcal{L}^1(\ell^2(H))$ i les funcions integrables mitjançant la funció $f_{\mathbf{A}}$ vista com a funció amb valors en multiplicadors. Les matrius en $\mathcal{L}^1_l(\ell^2(H))$ es poden caracteritzar de la següent manera.

Teorema *Siga* $\mathbf{A}$ *una matriu amb entrades en* $\mathcal{B}(H)$*. Aleshores* $\mathbf{A} \in \mathcal{L}^1_l(\ell^2(H))$ *si i només si la funció* $t \to f_{\mathbf{A}}(t)$ *és una funció contínua amb valors en* $\mathcal{M}_l(\ell^2(H))$*.*

En la Subsecció 4.3.1 prestem una atenció especial al cas de matrius de Toeplitz el qual, com és habitual, demostra que és el pont més directe que connecta el món de les matrius amb el món de les funcions/mesures. Utilizant diversos resultats intermedis, obtenim la caracterització de l'espai $\mathcal{L}^1_r(\ell^2(H)) \cap \mathcal{T}$ de la següent forma.

Teorema

$$\mathcal{L}^1_r(\ell^2(H)) \cap \mathcal{T} = \tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H)),$$

on l'espai $\tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H))$ és la clausura dels polinomis en la norma $\|\cdot\|_{L^1_{SOT}}$ donada per

$$\|P\|_{L^1_{SOT}} = \sup_{\|x\|=1} \int_0^{2\pi} \left\| \sum_l T_l(x) e^{ilt} \right\| \frac{dt}{2\pi},$$

amb $P \in P(\mathbb{T}, \mathcal{B}(H))$, $P(t) = \sum_l T_l e^{ilt}$, $(T_l)_{l\in\mathbb{Z}} \in c_{00}(\mathcal{B}(H))$.

El capítol es tanca amb la Subsecció 4.3.2, que completa els resultats vists al final del Capítol 3 relacionats amb funcions holomorfes amb valors en operadors i matrius triangulars. Per exemple, tenim el següent teorema.

Teorema *Siga* $\mathbf{A} = (T_{j-k}) \in \mathcal{U} \cap \mathcal{T}$ *amb* $\sup_{l\geq 0} \|T_l\| < \infty$*, i considerem* $G_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} T_l z^l, |z| < 1$.

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *si i només si* $G_{\mathbf{A}} \in H^\infty(\mathbb{D}, \mathcal{B}(H))$*.*

(ii) $\mathbf{A} \in C(\ell^2(H))$ *si i només si* $G_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(H))$*.*

(iii) Si $G_{\mathbf{A}} \in H^1(\mathbb{D}, \mathcal{B}(H))$ *aleshores* $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$*.*

(iv) Si $G_{\mathbf{A}} \in H^1(\mathbb{T}, \mathcal{B}(H))$ *aleshores* $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$*.*

Els resultats del capítol es troben recollits en el següent article:

> Blasco, O.; García-Bayona, I., A class of Schur multipliers of matrices with operator entries, *Mediterr. J. Math*, to appear.

El Capítol 5 és el capítol final de la tesi. En els anteriors tres capítols s'ha estudiat un producte de Schur basat en la composició d'operadors. Aquest capítol introdueix altra versió de producte de tipus Schur per a matrius amb entrades en operadors, denotat per $\circledast$, on l'operació entre les entrades de la matriu és el producte de Schur mateix. També amb la mateixa idea, un producte de tipus Kronecker, $\boxtimes$, serà definit en aquest context.

La Secció 5.2 explora algunes propietats que aquest producte de tipus Schur satisfà. El resultat més destacat, provat en diversos passos, mostra que els operadors lineals i fitats, amb l'esmentat producte, formen una estructura d'àlgebra de Banach commutativa. En el procés de provar aquest teorema, ens adonarem d'una relació que hi ha entre les matrius escalars i les matrius amb entrades en operadors que ens permetrà obtindre algunes aplicacions, atés que hi ha una manera de calcular la norma d'operador o multiplicador de matrius amb entrades en operadors en termes de matrius escalars.

La primera aplicació proporciona un mètode per obtindre multiplicadors per al nou producte (un espai denotat per $\mathcal{M}^{\circledast}$) en termes de multiplicadors per al producte clàssic de Schur, i viceversa.

Teorema *(i) Siga $A = (a_{i,j})_{i,j}$ una matriu de $\mathcal{M}(\ell^2)$. Aleshores, donat $n \geq 1$, la matriu $\mathbf{A}^{n)}$ formada prenent blocs de mida $n \times n$ en A és una matriu amb entrades en operadors que defineix un element de $\mathcal{M}^{\circledast}\left(\mathcal{B}\left(\ell^2\left(\ell_n^2(\mathbb{C})\right)\right)\right)$. Notem que en el cas en què A és Toeplitz, $\mathbf{A}^{n)}$ també ho és.*

(ii) Siga $\mathbf{A}^{n)}$ una matriu les entrades de la qual són matrius de mida $n \times n$ amb $\mathbf{A}^{n)}$ a l'espai $\mathcal{M}^{\circledast}\left(\mathcal{B}\left(\ell^2\left(\ell_n^2(\mathbb{C})\right)\right)\right)$. Aleshores, la matriu A amb entrades escalars obtinguda alliberant les entrades de $\mathbf{A}^{n)}$, defineix un element de $\mathcal{M}(\ell^2)$. A més, si $\mathbf{A}^{n)}$ és Toeplitz, A no necessàriament ho és.

L'última aplicació de la secció és un mètode per construir una quantitat numerable d'elements pertanyents a diferents espais de mesures vectorials a partir d'un únic element en $L^\infty(\mathbb{T})$.

Teorema *Siga* $f(t) := \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{ikt} \in L^\infty(\mathbb{T})$*. Aleshores, donat* $N \in \mathbb{N}$*, tenim que la distribució*

$$f_N(t) \sim \sum_{k=-\infty}^{\infty} T_k^{N)} e^{ikt}$$

pertany a $V^\infty(\mathbb{T}, \mathcal{B}(\ell_N^2(\mathbb{C})))$*, on* $T_k^{N)}$ *és una matriu de Toeplitz donada per la successió* $(\widehat{f}(Nk+j))_{j=-N+1}^{j=N-1}$.

La Secció 5.3 té com a objecte d'estudi a les matrius bloc finites, i per comoditat s'emprarà la notació $\mathcal{M}_N(\mathcal{M}_n) := \mathcal{M}_{N\times N}(\mathcal{M}_{n\times n}(\mathbb{R}))$. Introduirem el nou producte de tipus Kronecker $\boxtimes$ per a matrius bloc esmentat anteriorment, també basat en el producte de Schur clàssic. El propòsit d'aquesta secció és estudiar traces de matrius bloc en conjunció amb aquests dos nous productes. Recordem que l'operador traça per a matrius bloc, $\mathrm{tr} : \mathcal{M}_N(\mathcal{M}_n) \to \mathbb{R}$, actua como segueix: donada $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n)$, aleshores

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{N} \mathrm{tr}(T_{i,i}) = \sum_{i=1}^{N}\sum_{l=1}^{n} T_{i,i}(l,l),$$

on la traça que apareix després de la primera igualtat és la traça usual per a matrius amb entrades escalars.

Estudiarem algunes igualtats i desigualtats en relació amb aquest operador i els productes $\circledast$ i $\boxtimes$. Comprovarem que la traça no és submultiplicativa per a cap d'ells. No obstant això, introduirem dos espais en el context dels quals açò pot canviar: els espais $\mathcal{M}_N^S(\mathcal{M}_n)$ i $\mathcal{M}_N^+(\mathcal{M}_n)$.

Definició *Donats* $N, n \in \mathbb{N}$*, definim els següents subconjunts de* $\mathcal{M}_N(\mathcal{M}_n)$*:*

$$\mathcal{M}_N^S(\mathcal{M}_n) := \{(T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n) \ / \ \sum_{k=1}^{N} T_{k,k}(l,l) \geq 0, \ \forall\, 1 \leq l \leq n\},$$

$$\mathcal{M}_N^+(\mathcal{M}_n) := \{(T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n) \ / \ T_{k,k}(l,l) \geq 0, \ \forall\, 1 \leq k \leq N, \ \forall\, 1 \leq l \leq n\}.$$

En efecte, es té el següent resultat.

Teorema *Siguen* $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n)$ *i* $\mathbf{B} = (S_{k,j})_{k,j} \in \mathcal{M}_M(\mathcal{M}_n)$.

(i) Si $M = N$, $\mathbf{A} \in \mathcal{M}_N^S(\mathcal{M}_n)$ *i* $B \in \mathcal{M}_N^+(\mathcal{M}_n)$, *aleshores*

$$\mathrm{tr}(\mathbf{A} \circledast \mathbf{B}) \leq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B}).$$

(ii) Si $\mathbf{A} \in \mathcal{M}_N^S(\mathcal{M}_n)$ *i* $\mathbf{B} \in \mathcal{M}_M^S(\mathcal{M}_n)$, *aleshores*

$$\mathrm{tr}(\mathbf{A} \boxtimes \mathbf{B}) \leq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B}).$$

També donem una versió d'aquest resultat per a productes finits, i proporcionem una xicoteta aplicació que consisteix a analitzar la traça d'una versió de l'exponencial d'una matriu definida mitjançant el producte $\circledast$. Finalment, també donem estimacions superiors per a traces de productes de matrius que combinen tant el producte $\circledast$ com el producte $\boxtimes$ en termes de traces que únicament involucren a un dels productes i que, per tant, són més senzilles de calcular.

Teorema *Siguen* $\mathbf{A}_i, \mathbf{B}_i \in \mathcal{M}_N^+(\mathcal{M}_n)$, *per a* $1 \leq i \leq m$. *Aleshores, tenim*

(i) $\mathrm{tr}\Big((\mathbf{A}_1 \boxtimes \mathbf{A}_2 \boxtimes \cdots \boxtimes \mathbf{A}_m) \circledast (\mathbf{B}_1 \boxtimes \mathbf{B}_2 \boxtimes \cdots \boxtimes \mathbf{B}_m) \Big) \leq \prod_{i=1}^m \mathrm{tr}(\mathbf{A}_i \circledast \mathbf{B}_i)$.

(ii) $\mathrm{tr}\Big((\mathbf{A}_1 \circledast \mathbf{A}_2 \circledast \cdots \circledast \mathbf{A}_m) \boxtimes (\mathbf{B}_1 \circledast \mathbf{B}_2 \circledast \cdots \circledast \mathbf{B}_m) \Big) \leq \prod_{i=1}^m \mathrm{tr}(\mathbf{A}_i \boxtimes \mathbf{B}_i)$.

Part dels continguts d'aquest capítol es troben en l'article:

> García-Bayona, I., Traces of Schur and Kronecker Products for Block Matrices, *Khayyam J. Math.*, 2019. DOI: 10.22034/kjm.2019.84207.

Abstract

"There must be a beginning of any great matter, but the continuing unto the end until it be thoroughly finished yields the true glory."

—Sir Francis Drake.

This thesis is devoted to the study of problems of the areas of measurability and matricial harmonic analysis in the context of spaces of matrices and operators. More precisely, the purpose is to extend notions and results of these theories from the classical scalar setting to a more general framework where the functions take values in spaces of operators, or the matrices have operator entries.

First, we start with an introductory section to the topics where we introduce some notation, basic definitions, and some tools that will appear throughout the dissertation, and give a general idea of the kind of results that one can expect to find in each chapter.

Chapter 1 deals with questions of measurability of operator-valued functions $f : \Omega \to \mathcal{L}(E_1, E_2)$, where (Ω, Σ) is a measurable space (i.e. Σ is a σ-algebra over Ω) and $\mathcal{L}(E_1, E_2)$ denotes the space of bounded and linear operators between two Banach spaces E_1 and E_2. In $\mathcal{L}(E_1, E_2)$, the three fundamental topologies shall be dealt with, namely the norm topology $\tau_{\|\cdot\|}$, the strong operator topology τ_{SOT} and the weak operator topology τ_{WOT}. The notation $\mathcal{N}_{\|\cdot\|}$, $\mathcal{N}_{SOT}$ and $\mathcal{N}_{WOT}$ will be used for the bases of the corresponding

topologies given by

$$B(T,\varepsilon)=\{S\in\mathcal{L}(E_1,E_2);\|S-T\|<\varepsilon\},$$

$$N(T;\mathbf{x},\varepsilon)=\{S\in\mathcal{L}(E_1,E_2):\max_{1\leq j\leq n}\|(S-T)(x_j)\|<\varepsilon\}$$

and

$$N(T;\mathbf{x},\mathbf{y}^*,\varepsilon)=\{S\in\mathcal{L}(E_1,E_2):\max_{1\leq j\leq n}|\langle(S-T)x_j,y_j^*\rangle|<\varepsilon\},$$

where $\varepsilon>0$, $T\in\mathcal{L}(E_1,E_2)$, $\mathbf{x}=(x_1,x_2,\cdots,x_n)\in(E_1)^n$, $\mathbf{y}^*=(y_1^*,y_2^*,\cdots,y_n^*)\in(E_2^*)^n$ and $n\in\mathbb{N}$, respectively.

It is clear that $\mathcal{N}_{SOT(\mathbb{K},E_2)}=\mathcal{N}_{\|\cdot\|}$, and we shall denote $\mathcal{N}_{weak}=\mathcal{N}_{WOT(\mathbb{K},E_2)}$. Observe that when the second space is the field, one has $\mathcal{N}_{SOT(E_1,\mathbb{K})}=\mathcal{N}_{WOT(E_1,\mathbb{K})}$ and in this case the notation $\mathcal{N}_{weak^*}$ shall be put to use. In the event of E_2 being a dual space, $E_2=F^*$, since $\mathcal{L}(E_1,E_2)$ is a dual space itself, besides $\mathcal{N}_{WOT}$ and $\mathcal{N}_{weak}$, we will also have $\mathcal{N}_{weak^*}$ corresponding to the basis given by

$$N(T;\mathbf{x},\mathbf{y},\varepsilon)=\{S\in X:\max_{1\leq j\leq n}|\langle(S-T)x_j,y_j\rangle|<\varepsilon\}.$$

where $\mathbf{x}=(x_1,\cdots,x_n)\in(E_1)^n$, $\mathbf{y}=(y_1,\cdots,y_n)\in(E_2)^n$ and $n\in\mathbb{N}$.

The notion of measurable function strongly relies on the formulation we use, both in the case of functions with values in a Banach space (see [20, 23, 33]) or in spaces of operators (see [4, 23, 33, 37, 47]). The considerable variety of notions that can be found in the literature, together with the fact that usually measurability questions are treated just with the intention of developing an integration theory (and thus not getting the detailed treatment they deserve in their own right) can make a first approach to the subject somewhat confusing. Therefore, in this chapter we introduce some terminology, working only with measurable spaces without an underlying measure on them, that we believe it allows to differentiate the corresponding definitions of measurability and unravel the relations among them sistematically.

We shall be dealing with two different, but related, concepts of measurability. One is defined by using bases of the corresponding toplogies, and the other one utilizes the notion of approximability. Let (Ω, Σ) be a measurable space, and (Y, τ) a topological vector space with a basis $\beta \subseteq \tau$.

- A function $f : \Omega \to Y$ is said to be β**-measurable** whenever $f^{-1}(A) \in \Sigma$ for any $A \in \beta$.

- A function $f : \Omega \to Y$ is called τ**-approximable** whenever f is a pointwise limit of finitely valued functions. That is to say, whenever $s_N \to f$ in the τ-topology where $s_N = \sum_{k=1}^{N} y_k \chi_{A_k}$ with $y_k \in Y$ and A_k are pairwise disjoint sets such that $\cup_{k=1}^{N} A_k = \Omega$.

Here we recall some basic results connecting these notions in our terminology. Perhaps the most famous one is Pettis's measurability Theorem.

Theorem *(see [20, Chapter 2, Theorem 2]) Pettis's measurability Theorem. Let (Ω, Σ, μ) be a finite complete measure space and let E be a Banach space. Then $f : \Omega \to E$ is $\|\cdot\|$-approximable μ-a.e. if and only if f is $\mathcal{N}_{weak}$-measurable and $f(\Omega)$ is essentially separable, i.e $f(\Omega \setminus A)$ is separable for some $A \in \Sigma$ with $\mu(A) = 0$.*

An easy corollary by Dunford tells us that for $E = \mathcal{L}(E_1, E_2)$, $f : \Omega \to \mathcal{L}(E_1, E_2)$ satisfies that f_x is $\|\cdot\|$-approximable μ-a.e for any $x \in E_1$ if and only if f is $\mathcal{N}_{WOT}$-measurable and $f_x(\Omega)$ is essentially separable for any $x \in E_1$, where $f_x(\omega) = f(\omega)(x)$.

Theorem *(see [33, Theorem 3.5.5]) Dunford's measurability Theorem. Let (Ω, Σ, μ) be a finite complete measure space and let E_1, E_2 be Banach spaces. Then $f : \Omega \to \mathcal{L}(E_1, E_2)$ is $\|\cdot\|$-approximable μ-a.e. if and only if f is $\mathcal{N}_{WOT}$-measurable and $f(\Omega)$ is essentially separable in $\mathcal{L}(E_1, E_2)$.*

Theorem *(see [37, Theorem 2]) Johnson's measurability Theorem. Let (Ω, Σ, μ) be a finite complete measure space and let H be a separable Hilbert space. Then $f : \Omega \to \mathcal{L}(H, H)$ satisfies that f_x is $\|\cdot\|$-approximable μ-a.e. for any $x \in H$ if and only if f is $\mathcal{N}_{SOT}$-measurable.*

In Section 1.2 we start analyzing the notion of measurability with respect to bases of different topologies. We begin recalling a well known fact in the context of metric spaces.

Proposition *Let (Y, d) be a separable metric space. Then $f : \Omega \to Y$ is $\mathcal{N}_d$-measurable if and only if f is τ_d-measurable. In particular if E is a Banach space, $f : \Omega \to E$ is $\mathcal{N}_{\|\cdot\|}$-measurable and $f(\Omega)$ is separable then f is $\tau_{\|\cdot\|}$-measurable.*

Dealing with functions $f : \Omega \to \mathcal{L}(E_1, E_2)$, sometimes the following notation proves to be very useful: for $x \in E_1$ and $y^* \in E_2^*$, we denote $f_x(w) = f(w)(x)$ and $f_{x,y^*}(w) = \langle y^*, f(w)(x)\rangle$. The next proposition puts this notation to use and gives a characterization of some notions of β-measurability.

Proposition *Let $X = \mathcal{L}(E_1, E_2)$ and $f : \Omega \to X$. Then*

(i) f is $\mathcal{N}_{\|\cdot\|}$-measurable $\Longleftrightarrow$ $\|f(\cdot) - T\|$ is measurable for any $T \in X$.

(ii) f is $\mathcal{N}_{SOT}$-measurable $\Longleftrightarrow$ f_x is $\mathcal{N}_{\|\cdot\|}$-measurable for any $x \in E_1$.

(iii) f is $\mathcal{N}_{WOT}$-measurable $\Longleftrightarrow$ f_{x,y^*} is measurable for any $x \in E_1$ and $y^* \in E_2^*$.

It is clear that $\mathcal{N}_{WOT} \subset \mathcal{N}_{weak}$, therefore $\mathcal{N}_{weak}$-measurable implies $\mathcal{N}_{WOT}$-measurable. On the other hand $\tau_{weak} \subset \tau_{\|\cdot\|}$ for any Banach space and also $\tau_{WOT} \subset \tau_{SOT} \subset \tau_{\|\cdot\|}$ for $X = \mathcal{L}(E_1, E_2)$. Therefore, $\tau_{\|\cdot\|}$-measurable functions are also $\mathcal{N}_{weak}$-measurable and $\mathcal{N}_{SOT}$-measurable. Under certain conditions of separability, one can recover stronger versions of measurability from weaker ones, as the two propositions that close Section 1.2 show.

Proposition *Let $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ where E_1 is separable. If f is $\mathcal{N}_{SOT}$-measurable then f is also $\mathcal{N}_{\|\cdot\|}$-measurable.*

Proposition *Let $f : \Omega \to \mathcal{L}(E_1, E_2)$ with E_2 separable. Then f is $\mathcal{N}_{WOT}$-measurable if and only if f is $\mathcal{N}_{SOT}$-measurable.*

The study of the notions of measurability in terms of approximation is covered by Section 1.3. The concept of simple and countably valued function is important and we recall its definition.

Definition *In a Hausdorff topological space* (X, τ), *we say that a function* $f : \Omega \to X$ *is simple (respectively countably valued) if there exist a finite set (respectively sequence)* $(x_n)_n \subset X$ *and a finite partition (respectively countable partition) of pairwise disjoint sets* $(A_n)_n \subset \Sigma$ *such that* $\Omega = \cup_k A_k$ *and* $f = \sum_n x_n \chi_{A_n}$.

The $\|\cdot\|$-, *weak*-, SOT- and WOT-approximability of a function $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ is determined by the existence of a sequence of simple functions $s_n : \Omega \to X$ with values in operators such that

$$\|s_n(\omega) - f(\omega)\| \xrightarrow[n\to\infty]{} 0, \quad \forall \omega \in \Omega,$$

$$\langle s_n(\omega), T^* \rangle \xrightarrow[n\to\infty]{} \langle f(\omega), T^* \rangle, \quad \forall T^* \in X^*, \quad \forall \omega \in \Omega,$$

$$\lim_n \|s_n(\omega)(x) - f(\omega)(x)\| = 0, \quad \forall x \in E_1, \quad \forall \omega \in \Omega$$

or

$$\langle s_n(\omega)(x), y^* \rangle \xrightarrow[n\to\infty]{} \langle f(\omega)(x), y^* \rangle, \quad \forall x \in E_1, \quad \forall y^* \in E_2^*, \quad \forall \omega \in \Omega$$

respectively.

The connections between the approximability, the measurability with respect to bases and the separability of the range in the corresponding topologies start to shine in the rest of the section. For example, we have the following theorem.

Theorem *Let* (Y, d) *be a metric space and* $f : \Omega \to Y$. *The following are equivalent:*

(i) f *is d-approximable.*

(ii) f *is* τ_d*-measurable and* $f(\Omega)$ *is d-separable.*

(iii)f *is* $\mathcal{N}_d$*-measurable and* $f(\Omega)$ *is d-separable.*

As a consequence of the previous theorem and Banach-Alaoglu Theorem, we shall obtain the following result.

Proposition *Let* E *be a separable Banach space,* $X = E^*$ *and let* $f : \Omega \to X$ *be a bounded function. Then* f *is weak***-approximable if and only if* f *is* $\mathcal{N}_{weak^*}$*-measurable.*

It is rather surprising that, for any Banach space, the notions of $\|\cdot\|$-approximability and weak-approximability coincide.

Proposition *Let X be a Banach space and $f:\Omega\to X$. Then f is $\|\cdot\|$-approximable if and only if f is weak-approximable.*

Some examples to differentiate certain notions of measurability will be given, closing the section and leading to Section 1.4. There, our first goal will be to give a proof of Dunford's (and hence Pettis's) measurability Theorem.

Theorem *Let $f:\Omega\to X=\mathcal{L}(E_1,E_2)$. The following statements are equivalent:*

(i) f is $\|\cdot\|$-approximable.

(ii) f is $\tau_{\|\cdot\|}$-measurable and $f(\Omega)$ is separable in X.

(iii) f is $\mathcal{N}_{\|\cdot\|}$-measurable and $f(\Omega)$ is separable in X.

(iv) f is $\mathcal{N}_{SOT}$-measurable and $f(\Omega)$ is separable in X.

(v) f is $\mathcal{N}_{WOT}$-measurable and $f(\Omega)$ is separable in X.

Our main contribution in this section and in the chapter is the version of Pettis's measurability Theorem for the SOT-topology in the case where E_1 is a separable Banach space.

Theorem *Let $f:\Omega\to X=\mathcal{L}(E_1,E_2)$ where E_1 is separable. The following are equivalent.*

(i) f is SOT-approximable

(ii) f is WOT-approximable.

(iii) f is $\mathcal{N}_{WOT}$-measurable and $f(\Omega)$ is WOT-separable.

To finish the chapter, we apply this theorem to produce several examples of natural operator-valued functions that are SOT-approximable.

The results that appear in this chapter are published in the following paper:

Blasco, O.; García-Bayona, I., Remarks on Measurability of Operator-valued Functions, *Mediterr. J. Math.* **13** (2016), 5147–5162. DOI: 10.1007/s00009-016-0798-1.

In Chapter 2, we switch to the area of matricial harmonic analysis. This chapter starts with an introductory section in which, after recalling some facts and classic results of the scalar case, such as Toeplitz's theorem or Bennett's theorem, we start defining some of the most important elements on which we shall be working throughout this chapter and some of the next ones too. First of all, the space $\mathcal{B}(\ell^2(H))$.

Definition *Given a matrix $\mathbf{A} = (T_{kj})$ with entries $T_{kj} \in \mathcal{B}(H)$ and $\mathbf{x} \in c_{00}(H)$ we write $\mathbf{A}(\mathbf{x})$ for the sequence $(\sum_{j=1}^{\infty} T_{kj}(x_j))_k$. We say that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ if the map $\mathbf{x} \to \mathbf{A}(\mathbf{x})$ extends to a bounded linear operator in $\ell^2(H)$, that is*

$$\Big(\sum_{k=1}^{\infty} \|\sum_{j=1}^{\infty} T_{kj}(x_j)\|^2\Big)^{1/2} \leq C\Big(\sum_{j=1}^{\infty} \|x_j\|^2\Big)^{1/2}.$$

The norm in this space is given as follows:

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{A}\mathbf{x}\|_{\ell^2(H)} \leq C\|\mathbf{x}\|_{\ell^2(H)}\}.$$

The main operation we shall be studying is a version of the classical Schur product (entry-wise product of matrices) in the context of matrices with operator entries, and is defined as follows.

Definition *Let $\mathbf{A} = (T_{kj})$ and $\mathbf{B} = (S_{kj})$ be matrices with $T_{kj}, S_{kj} \in \mathcal{B}(H)$. We define their Schur product as*

$$\mathbf{A} * \mathbf{B} = (T_{kj}S_{kj}),$$

where $T_{kj}S_{kj}$ stands for the composition of the operators T_{kj} and S_{kj}.

A very important notion that will be studied is the notion of Schur multiplier. The multipliers for our product have a similar definition to that of the ones in the scalar setting, but notice that since our product is not commutative, it is necessary to define the right multipliers and the left multipliers.

Definition *Given a matrix* $\mathbf{A} = (T_{kj})$*,* $\mathbf{A}$ *is said to be a right Schur multiplier (respectively left Schur multiplier), to be denoted by* $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ *(respectively* $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ *), whenever* $\mathbf{B} * \mathbf{A} \in \mathcal{B}(\ell^2(H))$ *(respectively* $\mathbf{A} * \mathbf{B} \in \mathcal{B}(\ell^2(H))$ *) for any* $\mathbf{B} \in \mathcal{B}(\ell^2(H))$*. The expressions of the norm in these spaces are*

$$\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{B} * \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq C\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\}$$

and

$$\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{A} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \leq C\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\}.$$

Section 2.2 includes some basic notions on vector-valued sequences and functions that will be used in the rest of the chapter. In particular, spaces like $\ell^2(\mathbb{N}, \mathcal{B}(H))$, $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ and $\ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$ will be studied and relations between them shall be explored providing examples and counterexamples when required.

The projective tensor product is a tool that will be used several times throughout the proofs. Even more intense is the use of certain ideas from vector measure theory, and this is why Section 2.3 exists. This section recalls three identifications between operators and measures that we briefly mention here. The first one tells us that $\mathfrak{M}(\mathbb{T}, E)$ can be identified with the space of weakly compact linear operators $T_\mu : C(\mathbb{T}) \to E$ and that $\|T_\mu\| = \|\mu\|$ (see [20, Chapter 6]). In the case of dual spaces $E = F^*$, Singer's Theorem (see [53, 54, 32])) ensures that $M(\mathbb{T}, E) = C(\mathbb{T}, F)^*$. In other words, there exists a bounded linear map $\Psi_\mu : C(\mathbb{T}, F) \to \mathbb{C}$ with $\|\Psi_\mu\| = |\mu|$ such that $\Psi_\mu(y\phi) = T_\mu(\phi)(y)$, $\phi \in C(\mathbb{T})$, $y \in F$. Also, in the context of operators, there is still a third possibility to consider, by using the strong operator topology, namely $\Phi_\mu : C(\mathbb{T}, X) \to Y^*$ defined by $\Phi_\mu(f)(y) = \Psi_\mu(f \otimes y)$, $f \in C(\mathbb{T}, X)$, $y \in Y$, where $f \otimes y(t) = f(t) \otimes y$.

Different types of spaces of vector measures will be described, and it is specially important the space $M_{SOT}(\mathbb{T}, \mathcal{B}(H)))$, that is those measures $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ such that

$\mu_x \in M(\mathbb{T}, H)$ for any $x \in H$. The norm in this space is written as

$$\|\mu\|_{SOT} = \sup\{|\mu_x| : x \in H, \|x\| = 1\}.$$

Also, the "adjoint measure" will play an important role.

Definition *Let $\mu : \mathfrak{B}(\mathbb{T}) \to \mathcal{L}(X, Y^*)$ be a vector measure. We define "the adjoint measure" $\mu^* : \mathfrak{B}(\mathbb{T}) \to \mathcal{L}(Y, X^*)$ by the formula*

$$\mu^*(A)(y)(x) = \mu_x(A)(y), \quad A \in \mathfrak{B}(\mathbb{T}), x \in X, y \in Y.$$

If one considers this concept of adjoint measure, the space $M_{SOT}(\mathbb{T}, \mathcal{B}(H))$, which is very related with Schur multipliers as we shall see in later sections, can be described via operators as follows.

Proposition *Let $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. Then $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ if and only if $\Phi_{\mu^*} \in \mathcal{L}(C(\mathbb{T}, H), H)$. Moreover $\|\mu\|_{SOT} = \|\Phi_{\mu^*}\|$.*

In Section 2.4 we present necessary and sufficient conditions for a matrix to belong to $\mathcal{B}(\ell^2(H))$, and some of them involve conditions on the rows and the columns related with spaces seen at Section 2.2. One of the most important results of Section 2.4 is the version in our framework of matrices with operator entries of Schur's Theorem, that is, we prove that the space $\mathcal{B}(\ell^2(H))$ defines a Banach algebra when equipped with the Schur product. Nevertheless, for a product of matrices to end up in the space $\mathcal{B}(\ell^2(H))$ it is not necessary that both matrices belong to that space, of course. Indeed, in Subsection 2.4.2, the following theorem is presented, where we see that it is enough that one matrix belongs to $\mathcal{B}(\ell^2(H))$ and the other matrix satisfies certain conditions on its rows related with spaces of operator-valued sequences seen at Section 2.2.

Theorem *Let $\mathbf{A} = (T_{k,j})$ and $\mathbf{B} = (S_{k,j})$ be matrices with entries in $\mathcal{B}(H)$. If $\mathbf{B} \in B(\ell^2(H))$ and $\mathbf{A} \in \ell^\infty(\mathbb{N}, \ell^2(\mathcal{B}(H))) \cup \ell^\infty\left(\mathbb{N}, \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))\right)$, then $\mathbf{A} * \mathbf{B} \in B(\ell^2(H))$, that is, $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$.*

Section 2.5 is the final section of this chapter and contains the most important results on Toeplitz matrices (the space of matrices that are constant in their diagonals, and denoted by $\mathcal{T}$). The first one of these results is the generalized version of Toeplitz's Theorem, giving a sufficient and necessary condition regarding the associated function of a Toeplitz matrix for that matrix to be in the space $\mathcal{B}(\ell^2(H))$. The space of vector measures $V^\infty(\mathbb{T}, \mathcal{B}(H))$, defined in Section 2.3, makes an appearance as the substitute of $L^\infty(\mathbb{T})$ in this generalized version.

Theorem *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{T}$. *Then* $A \in \mathcal{B}(\ell^2(H))$ *if and only if there exists* $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ *such that* $T_{kj} = \hat{\mu}(j-k)$ *for all* $k, j \in \mathbb{N}$. *Moreover,* $\|\mathbf{A}\| = \|\mu\|_\infty$.

The section continues presenting some sufficient conditions for a matrix to be a Schur multiplier in our setting, and ends presenting a couple of results that try to generalize Bennett's Theorem to our framework.

Theorem *If* $\mu \in M(\mathbb{T}, \mathcal{B}(H))$ *and* $\mathbf{A} = (T_{kj}) \in \mathcal{T}$ *with* $T_{kj} = \hat{\mu}(j-k)$ *for* $k, j \in \mathbb{N}$ *then* $\mathbf{A} \in \mathcal{M}_l(\ell^2(H)) \cap \mathcal{M}_r(\ell^2(H))$. *Moreover,*

$$\max\{\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}, \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\} \leq |\mu|.$$

Theorem *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{T} \cap \mathcal{M}_r(\ell^2(H))$. *Then there exists* $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ *such that* $T_{kj} = \hat{\mu}(j-k)$ *for all* $k, j \in \mathbb{N}$. *Moreover,*

$$\|\mu\|_{SOT} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}.$$

The results of this chapter are published in the following paper:

> Blasco, O.; García-Bayona, I., Schur Product with Operator-valued Entries, *Taiwanese J. Math.*, advance publication, 30 November 2018. DOI:10.11650/tjm/181110. https://projecteuclid.org/euclid.twjm/1543546839.

Chapter 3 is mainly devoted to the study of a particular type of matrices, called "continuous matrices". This space shall be denoted by $C(\ell^2(H))$, and is the space of matrices that can be approached in the operator norm by matrices with a finite number of non-zero diagonals, or more precisely, "polynomial matrices". The space of continuous matrices with scalar entries was introduced by Barza, Persson and Popa (see [6]).

We say that a matrix $\mathbf{A} = (T_{kj})$ with entries $T_{kj} \in \mathcal{B}(H)$ is a "polynomial matrix", in short $\mathbf{A} \in \mathcal{P}(\ell^2(H))$, whenever two conditions are satisfied: $\sup_{k,j} \|T_{kj}\| < \infty$ and there exist $N, M \in \mathbb{N}$ such that $\mathbf{A}$ can be written as a finite sum of diagonal matrices $\mathbf{A} = \sum_{l=-N}^{M} \mathbf{D_l}$.

The first section of the chapter introduces the necessary concepts and the notation to be used. Most of it is a natural continuation of what we treated in Chapter 2. Some new matrices that appear for the first time are the matrices M_μ. Given $\mu \in M(\mathbb{T})$ we shall denote by $\mathbf{M}_\mu$ the Toeplitz matrix given by

$$\mathbf{M}_\mu = (\hat{\mu}(j-k)Id)_{k,j} \in \mathcal{T}.$$

The following formula, which holds for example for any $\mu \in M(\mathbb{T})$ and $f \in L^1(\mathbb{T})$ (where $\mu * f(t) = \int_0^{2\pi} f(e^{i(t-s)})d\mu(s)$ is the convolution between functions and measures in $\mathbb{T}$) suggests the relation between Fourier multipliers and Schur multipliers, and gives a first idea of the importance of this new type of matrices we defined above.

$$\mathbf{M}_\mu * \mathbf{M}_f = \mathbf{M}_f * \mathbf{M}_\mu = \mathbf{M}_{\mu * f}.$$

When exploring the connections between Fourier analysis and matricial analysis (see [44]) the following matrix-valued functions will be relevant.

Definition *Let $\mathbf{A} = (T_{kj})$ with $T_{kj} \in \mathcal{B}(H)$ for $k, j \in \mathbb{N}$. We define*

$$f_{\mathbf{A}}(t) = \mathbf{M}_t * \mathbf{A} = (e^{i(j-k)t}T_{kj}), \quad t \in [0, 2\pi].$$

In the case of upper triangular matrices (denoted $\mathcal{U}$) we will work with

$$F_{\mathbf{A}}(z) = (z^{(j-k)}T_{kj}), \quad |z| < 1.$$

After the preliminary section, Section 3.2 presents particular examples of matrices with operator-entries, and some ways to construct them. We highlight the following one, which allows to embed the scalar case in the operator setting, generating easy examples of matrices in our framework.

Example *Let $A = (a_{k,j}) \in \mathcal{B}(\ell^2)$ and $T \in \mathcal{B}(H)$. Then*

$$\mathbf{A} = (a_{k,j}T) \in \mathcal{B}(\ell^2(H)) \ \ \textit{and} \ \ \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|A\|_{\mathcal{B}(\ell^2)}\|T\|_{\mathcal{B}(H)}.$$

Also, a matricial version of the Wiener algebra is defined. Section 3.3 is all about the study of Schur multipliers. We give some necessary conditions for a matrix to be a multiplier, and give a multiplier version of the previous example to easily construct examples of multipliers with operator entries. By using summability kernels, we define the matrices $M_n(\mathbf{A})$ as $M_n(\mathbf{A}) = \mathbf{M}_{K_n} * \mathbf{A}$, and we prove the following proposition that shows us their importance, since they give a criteria to determine when a matrix is a bounded operator or a multiplier.

Proposition *Let $\mathbf{A}$ be a matrix with entries in $\mathcal{B}(H)$ and $\{k_n\}$ a summability kernel, and denote $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$. Then:*

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H)) \ \Leftrightarrow \ \sup_n \|M_n(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} < \infty$.

(ii) $\mathbf{A} \in \mathcal{M}_r(\ell^2(H)) \ \Leftrightarrow \ \sup_n \|M_n(\mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} < \infty$.

(iii) $\mathbf{A} \in \mathcal{M}_l(\ell^2(H)) \ \Leftrightarrow \ \sup_n \|M_n(\mathbf{A})\|_{\mathcal{M}_l(\ell^2(H))} < \infty$.

Section 3.4 starts making use of the function $f_{\mathbf{A}}$ defined above to show the connection between our new space of matrices and the space of continuous functions. A first easy but useful result shows that this function is an isometry between the corresponding spaces. The section advances presenting several results among which we would like to highlight two of them for their importance and utility. The first one is a characterization of the matrices in $C(\ell^2(H))$ that shows us different ways in which this space can be seen and the relation with continuous functions.

Theorem *Let $\mathbf{A} = (T_{k,j})_{k,j}$ be a matrix with entries in $\mathcal{B}(H)$, satisfying $\sup_{k,j} \|T_{k,j}\| < \infty$. The following statements are equivalent:*

1) $\mathbf{A} \in C(\ell^2(H))$.

2) $\lim_{n\to\infty} M_n(\mathbf{A}) = \mathbf{A}$ in $\mathcal{B}(\ell^2(H))$ where $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$ and $\{k_n\} \subseteq L^1(\mathbb{T})$ is a summability kernel.

3) $\lim_{n\to\infty} \sigma_n(\mathbf{A}) = \mathbf{A}$ in $\mathcal{B}(\ell^2(H))$.

4) $t \to f_{\mathbf{A}}(t)$ is a $\mathcal{B}(\ell^2(H))$-valued continuous function.

The second one shows us that, when working with Schur multipliers, it is enough to consider those that map the space $C(\ell^2(H))$ to itself. More precisely, we have:

Theorem $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ *(respectively $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$) if and only if* $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_l$ *(respectively $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_r$).*

Then we move on to Subsection 3.4.1, whose main topic is the class of Toeplitz matrices. We remark two results from this section. In one of them, we present another characterization that makes clear the total connection between $C(\ell^2(H))$ and the space of continuous functions.

Theorem *Let $(T_l)_{l\in\mathbb{Z}}$ be a sequence of operators in $\mathcal{B}(H)$ and let $\mathbf{A} = (T_{j-k})_{k,j}$. Then, $\mathbf{A} \in C(\ell^2(H))_{\mathcal{T}}$ if and only if there exists $g_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(H))$ such that $\widehat{g_{\mathbf{A}}}(l) = T_l$. Moreover, $\|g_{\mathbf{A}}\|_{C(\mathbb{T},\mathcal{B}(H))} = \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}$.*

The second result, already seen in the previous chapter, is the characterization of Toeplitz multipliers in terms of SOT-measures. In contrast, this time the alternative proof provided is achieved by using the techniques developed in this chapter and without relying on vector measures.

The final subsection, 3.4.2, presents a matricial version of the disc algebra for upper triangular matrices. Assuming that a matrix $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{U}$ satisfies $\sup_{k,j} \|T_{k,j}\| < \infty$, it is clear that

$$F_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} \mathbf{D_l} z^l \in \mathcal{H}(\mathbb{D}, \mathcal{B}(\ell^2(H)))$$

is a well defined holomorphic function. We prove the following result.

Theorem *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{U}$ *satisfying* $\sup_{k,j} \|T_{k,j}\| < \infty$.

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *if and only if* $F_{\mathbf{A}} \in H^\infty(\mathbb{D}, \mathcal{B}(\ell^2(H)))$. *Furthermore,* $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|F_{\mathbf{A}}\|_{H^\infty(\mathbb{D},\mathcal{B}(\ell^2(H)))}$.

(ii) $\mathbf{A} \in C(\ell^2(H))$ *if and only if* $F_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(\ell^2(H)))$.

The content of this chapter is included in the following paper:

> Blasco, O.; García-Bayona, I., New spaces of matrices with operator entries, *Quaest. Math*, 2019. DOI: 10.2989/16073606.2019.1605416.

The approach of Chapter 4 is in tune with the one taken in the previous chapter, but this time the emphasis on multipliers is stronger. We shall consider the class of Schur multipliers that can be approached in the multiplier norm by polynomial matrices. Also, as usual, a special treatment to the case of Toeplitz and upper triangular matrices will be given.

The first section of the chapter is of preliminary type and we stress on some of the concepts that will be necessary throughout the chapter. Also, the new class of matrices that shall be the object of study in this chapter is presented: the space of "integrable matrices", also denoted $\mathcal{L}^1(\ell^2(H))$.

Definition *We define* $\mathcal{L}^1_l(\ell^2(H))$ *(respectively* $\mathcal{L}^1_r(\ell^2(H))$*) as the closure of* $\mathcal{P}(\ell^2(H))$ *in* $\mathcal{M}_l(\ell^2(H))$ *(respectively* $\mathcal{M}_r(\ell^2(H))$*). We use the notation* $\mathcal{L}^1(\ell^2(H)) = \mathcal{L}^1_l(\ell^2(H)) \cap \mathcal{L}^1_r(\ell^2(H))$.

When studying continuous matrices we saw that this was a reasonable label for such class of matrices, due to the properties that related them with the space of continous functions. The meaning of the name "integrable matrices" will be also amply justified in this chapter by the results we shall present. Section 4.2 starts studying the new space $\mathcal{L}^1(\ell^2(H))$ and providing some examples of matrices in it. Also, we obtain an equivalent formulation using Schur product with Toeplitz matrices given by summability kernels in the same line as the one obtained for the space $C(\ell^2(H))$.

Theorem *Let $\mathbf{A} = (T_{k,j})_{k,j}$ be a matrix whose entries are in $\mathcal{B}(H)$ and satisfying that $\sup_{k,j}\|T_{k,j}\| < \infty$. The following are equivalent:*

1) $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$.

2) $\lim_{n\to\infty} M_n(\mathbf{A}) = \mathbf{A}$ *in* $\mathcal{M}_r(\ell^2(H))$ *where* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$ *and* $\{k_n\} \subseteq L^1(\mathbb{T})$ *is a summability kernel.*

3) $\lim_{n\to\infty} \sigma_n(\mathbf{A}) = \mathbf{A}$ *in* $\mathcal{M}_r(\ell^2(H))$.

4) $\lim_{r\to 1} P_r(\mathbf{A}) = \mathbf{A}$ *in* $\mathcal{M}_r(\ell^2(H))$.

Columns and diagonal matrices in $\mathcal{L}^1(\ell^2(H))$ are also studied. In particular, a version of Riemann-Lebesgue lemma holds in our framework.

Proposition *(Riemann-Lebesgue lemma) If $A = \sum_l \mathbf{D_l} \in \mathcal{L}^1_r(\ell^2(H))$, then*

$$\|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} \xrightarrow[|l|\to\infty]{} 0.$$

Section 4.3 starts analyzing the connection between the space $\mathcal{L}^1(\ell^2(H))$ and the integrable functions by means of the function $f_{\mathbf{A}}$ as a multiplier-valued function. Matrices in $\mathcal{L}^1_l(\ell^2(H))$ can be characterized in the following way.

Theorem *Let $\mathbf{A}$ be a matrix with entries in $\mathcal{B}(H)$. Then $\mathbf{A} \in \mathcal{L}^1_l(\ell^2(H))$ iff the associated function $t \to f_{\mathbf{A}}(t)$ is a $\mathcal{M}_l(\ell^2(H))$-valued continuous function.*

In Subsection 4.3.1 we pay attention to the special case of Toeplitz matrices which, as usual, reveals to be the most direct bridge connecting the world of matrices and the world of functions/measures. Using different intermediate results, we get to characterize the space $\mathcal{L}_r^1(\ell^2(H)) \cap \mathcal{T}$ as follows.

Theorem

$$\mathcal{L}_r^1(\ell^2(H)) \cap \mathcal{T} = \tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H)),$$

where the space $\tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H))$ is the closure of polynomials under the norm $\|\cdot\|_{L^1_{SOT}}$ given by

$$\|P\|_{L^1_{SOT}} = \sup_{\|x\|=1} \int_0^{2\pi} \left\| \sum_l T_l(x) e^{ilt} \right\| \frac{dt}{2\pi},$$

with $P \in P(\mathbb{T}, \mathcal{B}(H))$, $P(t) = \sum_l T_l e^{ilt}$, $(T_l)_{l \in \mathbb{Z}} \in c_{00}(\mathcal{B}(H))$.

The chapter is closed with Subsection 4.3.2, which completes the results seen at the end of Chapter 3 regarding operator-valued holomorphic functions and upper triangular matrices. For example, we show that

Theorem *Let* $\mathbf{A} = (T_{j-k}) \in \mathcal{U} \cap \mathcal{T}$ *with* $\sup_{l \geq 0} \|T_l\| < \infty$*, and consider* $G_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} T_l z^l, |z| < 1$.

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *if and only if* $G_{\mathbf{A}} \in H^\infty(\mathbb{D}, \mathcal{B}(H))$.

(ii) $\mathbf{A} \in C(\ell^2(H))$ *if and only if* $G_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(H))$.

(iii) If $G_{\mathbf{A}} \in H^1(\mathbb{D}, \mathcal{B}(H))$ *then* $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$.

(iv) If $G_{\mathbf{A}} \in H^1(\mathbb{T}, \mathcal{B}(H))$ *then* $\mathbf{A} \in \mathcal{L}_r^1(\ell^2(H))$.

The resuls of this chapter are contained in the following paper:

> Blasco, O.; García-Bayona, I., A class of Schur multipliers of matrices with operator entries, *Mediterr. J. Math*, to appear.

Chapter 5 is the final chapter of the thesis. In the previous three chapters we have been studying a Schur product based on the composition of operators. This chapter introduces another version of a Schur-type product for matrices with operator entries, denoted $\circledast$, where the operation between the entries of the matrix is the Schur product itself. Also with the same idea, a Kronecker-type product, ⊠, will be defined in this framework.

Section 5.2 explores some properties satisfied by this new Schur product. The most important result, proved in various steps, shows that the bounded linear operators endowed with such product are a commutative Banach algebra. In the process of proving this theorem, we shall realize about a relation between scalar matrices and matrices with operator entries that will allow us to obtain some applications since there is a way to compute the operator and multiplier norms of matrices with operator entries in terms of scalar matrices.

The first application shows a way to obtain multipliers for the new product (a space we denote by $\mathcal{M}^{\circledast}$) in terms of multipliers for the classical Schur product, and vice versa.

Theorem *(i) Let $A = (a_{i,j})_{i,j}$ be a matrix from $\mathcal{M}(\ell^2)$. Then, given $n \geq 1$, the matrix $\mathbf{A}^{n)}$ formed by taking $n \times n$ size blocks in A is a matrix with operator entries that defines an element of $\mathcal{M}^{\circledast}\left(\mathcal{B}\left(\ell^2\left(\ell^2_n(\mathbb{C})\right)\right)\right)$. Observe that in the event that A is Toeplitz, $\mathbf{A}^{n)}$ is too.*

(ii) Let $\mathbf{A}^{n)}$ be a matrix whose entries are $n \times n$ matrices, with $\mathbf{A}^{n)}$ in the space $\mathcal{M}^{\circledast}\left(\mathcal{B}\left(\ell^2\left(\ell^2_n(\mathbb{C})\right)\right)\right)$. Then, the matrix A with scalar entries, obtained by freeing the entries of $\mathbf{A}^{n)}$, defines an element of $\mathcal{M}\left(\ell^2\right)$. Also, if $\mathbf{A}^{n)}$ was Toeplitz, A needs not be.

The last application of the section is a method to construct a countable amount of elements belonging to different vector measure spaces from a single element of $L^\infty(\mathbb{T})$.

Theorem *Consider $f(t) := \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{ikt} \in L^\infty(\mathbb{T})$. Then, given $N \in \mathbb{N}$, we have that the distribution*

$$f_N(t) \sim \sum_{k=-\infty}^{\infty} T_k^{N)} e^{ikt}$$

belongs to $V^\infty\left(\mathbb{T}, \mathcal{B}(\ell^2_N(\mathbb{C}))\right)$, where $T_k^{N)}$ is a Toeplitz matrix given by the sequence $(\widehat{f}(Nk+j))_{j=-N+1}^{j=N-1}$.

Section 5.3 has finite block matrices as the object of study, and for convenience we denote $\mathcal{M}_N(\mathcal{M}_n) := \mathcal{M}_{N\times N}(\mathcal{M}_{n\times n}(\mathbb{R}))$. We introduce the new Kronecker-type product $\boxtimes$ for block matrices mentioned above, also based on the classical Schur product. The purpose of the section is to study traces of block matrices in conjunction with these two new products. We recall that the trace operator for block matrices $\mathrm{tr} : \mathcal{M}_N(\mathcal{M}_n) \to \mathbb{R}$ acts as follows: given $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n)$, then

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{N} \mathrm{tr}(T_{i,i}) = \sum_{i=1}^{N}\sum_{l=1}^{n} T_{i,i}(l,l),$$

where the trace after the first equality is the usual trace for matrices with scalar entries.

We study some equalities and inequalities regarding this operator and the products $\circledast$ and $\boxtimes$. We check that the trace is not sub-multiplicative for any of them. Nevertheless, we introduce two spaces in the context of which that can change: the spaces $\mathcal{M}_N^S(\mathcal{M}_n)$ and $\mathcal{M}_N^+(\mathcal{M}_n)$.

Definition *Given $N, n \in \mathbb{N}$, we define the following subsets of $\mathcal{M}_N(\mathcal{M}_n)$:*

$$\mathcal{M}_N^S(\mathcal{M}_n) := \{(T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n) \ / \ \sum_{k=1}^{N} T_{k,k}(l,l) \geq 0, \ \forall\, 1 \leq l \leq n\},$$

$$\mathcal{M}_N^+(\mathcal{M}_n) := \{(T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n) \ / \ T_{k,k}(l,l) \geq 0, \ \forall\, 1 \leq k \leq N, \ \forall\, 1 \leq l \leq n\}.$$

Indeed, we have the following result.

Theorem *Let $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n)$ and $\mathbf{B} = (S_{k,j})_{k,j} \in \mathcal{M}_M(\mathcal{M}_n)$.*

(i) If $M = N$, $\mathbf{A} \in \mathcal{M}_N^S(\mathcal{M}_n)$ and $\mathbf{B} \in \mathcal{M}_N^+(\mathcal{M}_n)$, then

$$\mathrm{tr}(\mathbf{A} \circledast \mathbf{B}) \leq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B}).$$

(ii) If $\mathbf{A} \in \mathcal{M}_N^S(\mathcal{M}_n)$ and $\mathbf{B} \in \mathcal{M}_M^S(\mathcal{M}_n)$, then

$$\mathrm{tr}(\mathbf{A} \boxtimes \mathbf{B}) \leq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B}).$$

We also give a version of this result for finite products, and provide a small application consisting of analyzing the trace of a version of the exponential of a matrix defined by means of the product $\circledast$. Finally, we also give some upper estimates for traces of products that combine both the product $\circledast$ and the product $\boxtimes$ in terms of traces only involving one of these products that are, therefore, easier to compute.

Theorem *Let $\mathbf{A}_i, \mathbf{B}_i \in \mathcal{M}_N^+(\mathcal{M}_n)$, for $1 \leq i \leq m$. Then, we have*

(i) $\operatorname{tr}\Big((\mathbf{A}_1 \boxtimes \mathbf{A}_2 \boxtimes \cdots \boxtimes \mathbf{A}_m) \circledast (\mathbf{B}_1 \boxtimes \mathbf{B}_2 \boxtimes \cdots \boxtimes \mathbf{B}_m) \Big) \leq \prod_{i=1}^m \operatorname{tr}(\mathbf{A}_i \circledast \mathbf{B}_i)$.

(ii) $\operatorname{tr}\Big((\mathbf{A}_1 \circledast \mathbf{A}_2 \circledast \cdots \circledast \mathbf{A}_m) \boxtimes (\mathbf{B}_1 \circledast \mathbf{B}_2 \circledast \cdots \circledast \mathbf{B}_m) \Big) \leq \prod_{i=1}^m \operatorname{tr}(\mathbf{A}_i \boxtimes \mathbf{B}_i)$.

Part of the contents of this chapter are included in the following paper:

García-Bayona, I., Traces of Schur and Kronecker Products for Block Matrices, *Khayyam J. Math.*, 2019. DOI: 10.22034/kjm.2019.84207.

Contents

Introduction

"Life breaks free. Life expands to new territories. Painfully, perhaps even dangerously. But life finds a way."

—Ian Malcolm, *Jurassic Park.*

This dissertation, which extends into five chapters, will mainly deal with three areas of mathematical analysis: the measurability theory, the harmonic analysis and the matricial analysis. Our study is interested in an operator-valued approach. More precisely, we want to extract the concepts, elements and some of the classical results of the aforementioned theories from the scalar setting, and then try to see how they can be extended or generalized to a vectorial context.

In Chapter 1, the one devoted to measurability, the vectorial approach that we take consists of considering functions taking values in $\mathcal{L}(E_1, E_2)$, that is the space of bounded and linear operators between E_1 and E_2, which are considered to be Banach spaces (with some eventual hypothesis of separability on them for certain results). In this operator-valued setting, as expected, the three main topologies (namely the norm topology, the strong operator topology and the weak operator topology) shall play an important role.

The number of different notions of measurability that one can find in the literature is considerable (see [4, 20, 23, 33, 37, 47]) and the measurable spaces are usually covered there with an underlying measure with the intention of developing an integration theory after. In this chapter we shall give importance to measurability itself, and introduce some terminology that hopefully allows to differentiate all the possible notions of measurability that appear in the operator-valued setting, and sheds some light on the relations between them.

The notions of measurability will be organized in two types: measurability with respect to a basis of the corresponding topology (see Section 1.2) and measurability in terms of approximability by a sequence of countably-valued functions (see Section 1.3). We highlight Theorem 1.4.3 as possibly the most important result of the chapter that relates these notions. It is a version of the well-known Pettis's measurability Theorem for the strong operator topology.

The subject of the next chapters is the matricial harmonic analysis. We started this line of research inspired by the works of Persson and Popa (see [44]). This time, the operator-valued philosophy leads us to consider matrices whose entries belong to $\mathcal{B}(H)$, the space of bounded and linear operators from a Hilbert space H to itself.

A very important device that we utilize in this thesis is the Hadamard product. If $A = (a_{k,j})_{k,j}$ and $B = (b_{k,j})_{k,j}$ are matrices of the same size (possibly infinite), their Hadamard product is the element-wise product

$$A * B = (a_{k,j} \cdot b_{k,j})_{k,j}.$$

The term Hadamard product was coined by J. von Neumann, and it was P. R. Halmos in [30] who introduced it in the literature. Since it was Schur who provided the initial studies on this product (see [51]), it is widely known as the Schur product. For an historical discussion, we refer the reader to [55]. Although it is not the purpose of the thesis to explore the applications of this product outside of the matricial harmonic analysis, we point out that the Schur product has implications in areas such as Banach spaces theory (see [7] and [38]), complex function theory (see [46] and [52]), operator theory (see [3] and [43]) and statistics (see [40] and [55]).

An important concept that will be used extensively in the thesis is the concept of Schur multiplier. If X, Y are Banach spaces of matrices, we define the space of multipliers between X and Y as

$$\mathcal{M}(X,Y) = \{M \ : \ A * M \in Y \ \text{ for every } A \in X\},$$

with the norm $\|M\| = \sup_{\|A\|_X=1} \|A * M\|_Y$. When $X = Y = \mathcal{B}(\ell^2)$ (where ℓ^2 is the space of 2-power summable sequences), we shall denote it simply by $\mathcal{M}(\ell^2)$, and then a matrix $M \in \mathcal{M}(\ell^2)$ is called a Schur multiplier.

A particular and important type of matrices is the class of Toeplitz matrices. These are matrices that are constant in their diagonals, i.e., there exists a sequence $(\alpha_l)_l$ such that $A = (\alpha_{j-k})_{k,j}$. For these matrices, one could consider an identification with functions given by $A \leftrightarrow f_A$, where f_A is a function such that $\hat{f}_A(l) = \alpha_l$. We highlight two classical results that compare properties of certain Toeplitz matrices with properties of their corresponding functions via the aforementioned identification, beautifully connecting the worlds of functions and matrices.

Theorem *(Toeplitz [57]) Let $A = (\alpha_{kj})$ be a Toeplitz matrix. Then $A \in \mathcal{B}(\ell^2)$ if and only if there exists $f \in L^\infty(\mathbb{T})$ such that $\alpha_{kj} = \hat{f}(j-k)$ for all $k, j \in \mathbb{N}$. Moreover $\|A\| = \|f\|_{L^\infty(\mathbb{T})}$.*

Theorem *(Bennett [7]) Let $A = (\alpha_{kj})$ be a Toeplitz matrix. Then $A \in \mathcal{M}(\ell^2)$ if and only if there exists $\mu \in M(\mathbb{T})$ such that $\alpha_{kj} = \hat{\mu}(j-k)$ for all $k, j \in \mathbb{N}$. Moreover $\|A\| = \|\mu\|_{M(\mathbb{T})}$.*

In Chapter 2, we work with a version of the Schur product for matrices with operator entries based on the composition of operators. With this philosophy, we introduce versions of the spaces $\mathcal{B}(\ell^2)$ and $\mathcal{M}(\ell^2)$, and provide results that extend some classical ones from the scalar setting, like the two above (see Section 2.5). Of course, since the composition of operators is not commutative, the notion of multipliers will have right and left versions in the framework of operator entries with this product. The lack of commutativity also forces to take different approaches in the proof of the results.

In Chapter 3 we explore a particular type of matrices that reveals to be very important regarding the study of Schur multipliers, that is, the space of "continuous matrices".

Barza, Persson and Popa (see [6]) introduced the space $C(\ell^2)$ as those matrices in $\mathcal{B}(\ell^2)$ such that $\sigma_n(A)$ converges to A in $\mathcal{B}(\ell^2)$. In this chapter we use a different approach and introduce such a class of matrices with entries in the space $\mathcal{B}(H)$, to be called "continuous matrices", as those that are limit in the operator norm of "polynomial matrices", that are defined below.

Definition *Given a matrix* $\mathbf{A} = (T_{kj})$ *with entries* $T_{kj} \in \mathcal{B}(H)$ *we say that* $\mathbf{A}$ *is a "polynomial matrix", or in short,* $\mathbf{A} \in \mathcal{P}(\ell^2(H))$*, whenever* $\sup_{k,j} \|T_{kj}\| < \infty$ *and also there exist* $N, M \in \mathbb{N}$ *such that* $\mathbf{A}$ *can be written as a finite sum of diagonal matrices* $\mathbf{A} = \sum_{l=-N}^{M} \mathbf{D_l}$.

We will use the Schur product with Toeplitz matrices generated by summability kernels in order to describe this class of "continuous matrices". Also, we shall see that in the Toeplitz case there is a natural identification of this class with the space of continuous functions with values in $\mathcal{B}(H)$. Finally, the chapter also considers upper triangular matrices to present matricial versions with operator entries of some classical spaces of holomorphic functions. Furthermore, the chapter contains alternative proofs for some of the results seen in Chapter 2, but this time without relying on vector measures as the first proofs did. Also, a characterization of Toeplitz Schur multipliers acting on Toeplitz matrices in terms of SOT-measures is achieved.

Moving on to Chapter 4, there the reader will find that another important type of matrices is studied: the "integrable matrices". This class of matrices is composed by those matrices that are limit of polynomial matrices in the multiplier norm. They will be classified via summability kernels again to show the connection with integrable functions, and the cases of Toeplitz and upper triangular matrices will also be dealt with. Some results in the chapter (like Lemma 4.3.4 or Proposition 4.3.6) can be derived from others seen in Chapter 2, but here we present different proofs that again will not utilize vector measures.

Given $P \in P(\mathbb{T}, \mathcal{B}(H))$, say $P(t) = \sum_l T_l e^{ilt}$ for some $(T_l)_{l\in\mathbb{Z}} \in c_{00}(\mathcal{B}(H))$, we denote

$$\|P\|_{L^1_{SOT}} = \sup_{\|x\|=1} \int_0^{2\pi} \left\| \sum_l T_l(x) e^{ilt} \right\| \frac{dt}{2\pi}.$$

Towards the end of the chapter, we provide some sufficient and necessary conditions for a Toeplitz matrix to be integrable. Furthermore, in Theorem 4.3.8 we obtain a characterization of the Toeplitz integrable matrices in terms of elements of $\tilde{L}^1_{SOT}$, which is the space of the closure of polynomials in the L^1_{SOT} norm defined above. Finally, upper triangular matrices are considered, and we complete some results regarding matrix versions of spaces of holomorphic functions started in Chapter 3.

The thesis is closed with Chapter 5, which takes a rather algebraic tone. We present new alternative versions of Schur and Kronecker products for block matrices and explore some of their properties. These definitions are also very connected with the classical Schur and Kronecker products and recover them as particular cases. One of the main results of the chapter is Schur's Theorem for the space of block matrices with the new Schur product. In other words, we prove that the space of block matrices endowed with such product defines a Banach algebra. The ideas in the chapter allow us to present a couple of interesting applications, namely we show a way to obtain multipliers for the new product from multipliers for the classical one and vice versa, and explain a method to construct a countable amount of elements belonging to different vector measure spaces, from a single element of $L^\infty(\mathbb{T})$.

Finally, in the context of finite block matrices, the chapter investigates how the trace operator acts on matrices that are a result of multiplying matrices with each of the new products, or combinations of both. Some equalities and inequalities regarding the trace operator are presented, extending results of Das, Vashisht, Taskara and Gumus (see [17] and [56]) to block matrices. We find spaces of matrices in which the trace is submultiplicative for each product, exhibit some examples and applications, and finally we present trace inequalities combining both products.

Chapter 1

Measurability for operator-valued functions

"In man, the things which are not measurable are more important than those which are measurable."

—Alexis Carrel

1.1 Preliminaries

In this chapter, (Ω, Σ) stands for a measurable space (that is, Σ is a σ-algebra over the set Ω). We remind here that a σ-algebra over a set Ω is a family Σ of subsets of Ω satisfying the following properties:

1) $\emptyset \in \Sigma$.

2) $A \in \Sigma \implies \Omega \setminus A \in \Sigma$.

3) $(A_n)_{n\geq 1} \subset \Sigma \implies \bigcup_{n=1}^{\infty} A_n \in \Sigma$.

The elements of a Σ algebra are called "Σ-measurable" sets, or simply "measurable sets" when there is no ambiguity. Note that property 2 gives that $\Omega \in \Sigma$, and a combined use of property 3 and De Morgan's laws implies that countable intersections of measurable sets also belong to Σ.

E, E_1 and E_2 will always denote Banach spaces over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). B_E will stand for the unit ball, E^* for the dual space of E and $\mathcal{L}(E_1, E_2)$ for the space of bounded and linear operators between E_1 and E_2. The three fundamental topologies in $\mathcal{L}(E_1, E_2)$ will be used in what follows, namely the norm topology, $\tau_{\|\cdot\|}$; the strong operator topology, τ_{SOT}; and the weak operator topology, τ_{WOT}. The last two topologies are often refered to as the SOT and WOT topologies, respectively.

Given a sequence of operators $(T_n) \in \mathcal{L}(E_1, E_2)$, we write $T_n \to T$ (or $T_n \xrightarrow{\|\cdot\|} T$) for the convergence in norm, $T_n \xrightarrow{SOT} T$ for the convergence in the strong operator topology, i.e. $\|T_n x - Tx\| \to 0, \quad \forall x \in E_1$; and $T_n \xrightarrow{WOT} T$ for the convergence in weak operator topology, i.e. $\langle T_n x - Tx, y^* \rangle \to 0, \quad \forall x \in E_1$ and $y^* \in E_2^*$.

In the case $E_2 = \mathbb{K}$, the convergence in norm coincides with the weak convergence in E_2. In this case, $\mathcal{L}(E_1, E_2) = E_1^*$ and therefore the SOT convergence and the WOT convergence are simply weak* convergence in X^*. The same happens with $E_2 = \mathbb{K}^d$, with $\mathcal{L}(X, Y) = \prod_{k=1}^d \mathcal{L}(E_1, \mathbb{K})$. On the other hand, when $E_1 = \mathbb{K}$, then $\mathcal{L}(E_1, E_2) = E_2$ ($\mathcal{L}(E_1, E_2) = \prod_{k=1}^d E_2$ in the case $E_1 = \mathbb{K}^d$), we have that the SOT convergence reduces to the norm convergence, whilst the WOT convergence is nothing but the weak convergence of (T_n).

Cauchy-Schwarz inequality reveals the relation between these three types of convergence of sequences of operators,

$$T_n \xrightarrow{\|\cdot\|} T \implies T_n \xrightarrow{SOT} T \implies T_n \xrightarrow{WOT} T.$$

Remark 1.1.1 *The converse implications are not true in general. Consider the sequence space ℓ^2, and operators $T_n, S_n : \ell^2 \to \ell^2$ defined by $T_n x = (0, \overset{n}{\cdots}, 0, a_{n+1}, a_{n+2}, \cdots)$ and $S_n(x) = (0, \overset{n}{\cdots}, 0, a_1, a_2, \cdots)$, for $x = (a_1, a_2, \cdots) \in \ell^2$. It is easy to check that (T_n) converges in the strong operator topology, but not in the norm toplogy, while (S_n) converges in the weak operator topology, but does not converge in the SOT topology.*

We write $\mathcal{N}_{\|\cdot\|}$, $\mathcal{N}_{SOT}$ and $\mathcal{N}_{WOT}$ (sometimes $\mathcal{N}_{SOT(E_1,E_2)}$ and $\mathcal{N}_{WOT(E_1,E_2)}$ shall be used to avoid misunderstandings) for the bases of the corresponding topologies given by

$$B(T,\varepsilon) = \{S \in \mathcal{L}(E_1,E_2); \|S-T\| < \varepsilon\},$$

$$N(T;\mathbf{x},\varepsilon) = \{S \in \mathcal{L}(E_1,E_2) : \max_{1\leq j\leq n} \|(S-T)(x_j)\| < \varepsilon\}$$

and

$$N(T;\mathbf{x},\mathbf{y}^*,\varepsilon) = \{S \in \mathcal{L}(E_1,E_2) : \max_{1\leq j\leq n} |\langle (S-T)x_j, y_j^*\rangle| < \varepsilon\},$$

where $\varepsilon > 0$, $T \in \mathcal{L}(E_1,E_2)$, $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in (E_1)^n$, $\mathbf{y}^* = (y_1^*, y_2^*, \cdots, y_n^*) \in (E_2^*)^n$ and $n \in \mathbb{N}$, respectively.

Observe that $\mathcal{N}_{SOT(\mathbb{K},E_2)} = \mathcal{N}_{\|\cdot\|}$. We shall use the notation $\mathcal{N}_{weak} = \mathcal{N}_{WOT(\mathbb{K},E_2)}$ for the corresponding basis. Although $\mathcal{L}(E_1,E_2) = \mathcal{L}(\mathbb{K},X)$ where $X = \mathcal{L}(E_1,E_2)$, the notations $\mathcal{N}_{WOT}$ and $\mathcal{N}_{weak}$ distinguish between the two topologies. Notice also that $\mathcal{N}_{SOT(E_1,\mathbb{K})} = \mathcal{N}_{WOT(E_1,\mathbb{K})}$ and the notation $\mathcal{N}_{weak^*}$ shall be used in this case. Therefore, in the case where E_2 is a dual space, $E_2 = F^*$, since $\mathcal{L}(E_1,E_2)$ is a dual space itself, besides $\mathcal{N}_{WOT}$ and $\mathcal{N}_{weak}$, we will have $\mathcal{N}_{weak^*}$ which corresponds to the basis given by

$$N(T;\mathbf{x},\mathbf{y},\varepsilon) = \{S \in X : \max_{1\leq j\leq n} |\langle (S-T)x_j, y_j\rangle| < \varepsilon\},$$

where $\mathbf{x} = (x_1, \cdots, x_n) \in (E_1)^n$, $\mathbf{y} = (y_1, \cdots, y_n) \in (E_2)^n$ and $n \in \mathbb{N}$.

In the literature, many definitions and approaches regarding measurability have been followed. It is our goal in this chapter to consider possible notions of measurability with respect to these topologies, and to try to clarify the different perspectives about this subject, and the relations between the different notions. Let us first recall the basic definition of measurability.

Definition 1.1.2 *Let (Ω_1,Σ_1) and (Ω_2,Σ_2) be measurable spaces. A function $f : \Omega_1 \to \Omega_2$ is called (Σ_1,Σ_2)-measurable, whenever $f^{-1}(A) \in \Sigma_1$ for any $A \in \Sigma_2$.*

When (Ω_2, τ) is a topological space, we say that f is measurable (or sometimes called Σ_1-measurable) when referring to Σ_2 as the Borel σ-algebra $\sigma(\tau)$, which is clearly equivalent to $f^{-1}(A) \in \Sigma_1$ for any open set $A \in \tau$. A basic result of measure theory is the following one.

Proposition 1.1.3 *Consider the measurable spaces* (Ω_1, Σ_1) *and* $(\mathbb{K}, \sigma(\tau))$ *(where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*). Let* $f : \Omega_1 \to \mathbb{K}$*. The following facts are equivalent:*

(a) f *is measurable.*

(b) $f^{-1}(B(\alpha, \varepsilon)) \in \Sigma_1$ *for any* $\alpha \in \mathbb{K}$ *and* $\varepsilon > 0$ *where* $B(\alpha, \varepsilon) = \{\beta \in \mathbb{K}; |\alpha - \beta| < \varepsilon\}$.

(c) There exists a sequence of simple functions (s_n) *such that* $f(w) = \lim_n s_n(w)$ *for all* $w \in \Omega$.

In fact it is not difficult to see (a) is equivalent to (b) whenever (Ω_2, d) is a separable metric space.

The above statements are, of course, not equivalent for general topological Banach spaces (Y, τ). The concept of "measurable" function in the framework of functions with values in Banach spaces (or more generally on spaces of operators) depends strongly on the formulation that we take as definition.

The variety of notions can be overwhelming at first. In the case that $\Omega_2 = E$ is a Banach space, endowed with the norm or weak topology, concepts such as strongly measurable, weakly measurable (also called scalarly measurable) or weak*-measurable have been used in the literature (for instance, the reader is referred to [20, 23, 33]). If we consider the context of operators, $\Omega_2 = \mathcal{L}(E_1, E_2)$ for two Banach spaces E_1 and E_2 equipped with either the norm topology, the SOT topology or the WOT topology, notions such as uniformly measurable, strong operator measurable or weak operator measurable (see [4, 23, 33, 37, 47]) appear in the literature.

Throughout this chapter we present a terminology that tries to be clear enough to allow to differentiate all the corresponding notions of measurability and the existing connections between them. Also, we point out that we shall be focusing only on measurable spaces (Ω, Σ), without any kind of underlying measure μ on them. We believe a pure measurability study is interesting in itself, since in the literature measurability machinery is often relegated to an instrument utilized with the intention of developing an integration theory.

In the next definition, we present two different groups of notions of measurability, one related with bases and the other with approximability.

Definition 1.1.4 *Let (Ω, Σ) be a measurable space, and (Y, τ) a topological vector space with a basis $\beta \subseteq \tau$.*

- *A function $f : \Omega \to Y$ is said to be β-**measurable** whenever $f^{-1}(A) \in \Sigma$ for any $A \in \beta$.*
- *A function $f : \Omega \to Y$ is called τ-**approximable** whenever f is a pointwise limit of finitely valued functions. In other words, whenever $s_N \to f$ in the τ-topology where $s_N = \sum_{k=1}^{N} y_k \chi_{A_k}$ with $y_k \in Y$ and A_k are pairwise disjoint sets such that $\cup_{k=1}^{N} A_k = \Omega$.*

Perhaps the simplest examples of τ-approximable functions are the countably valued functions. Also, note that τ-approximable functions are always τ-measurable. Observe that a first difference between both notions is that τ-approximability implies the τ-separability of the range, $f(\Omega)$, while τ-measurability does not (it is enough to consider $(\Omega, \Sigma) = (Y, \sigma(\tau))$ and f as the identity map on a non-separable topological space).

In summary, throughout the rest of the exposition, the terms $\| \cdot \|$-, *weak*-, *weak**-, *SOT*- or *WOT*-approximable will mean the existence of a sequence of simple functions such that $s_n(w)$ converges to $f(w)$ for any $w \in \Omega$, in the norm, weak, weak*, *SOT* or *WOT* topologies. We shall use the terms $\mathcal{N}_{\|\cdot\|}$-, $\mathcal{N}_{weak}$-, $\mathcal{N}_{weak^*}$-, $\mathcal{N}_{SOT}$- and $\mathcal{N}_{WOT}$-measurable for the β-measurability with respect to the corresponding standard basis in each of the mentioned topologies.

Having already established the concepts and their notation, let us collect now three basic results connecting them adapted to our terminology.

Theorem A: (see [20, Chapter 2, Theorem 2]) *Pettis's measurability Theorem.* Let (Ω, Σ, μ) be a finite complete measure space and let E be a Banach space. Then $f : \Omega \to E$ is $\|\cdot\|$-approximable μ-a.e. if and only if f is $\mathcal{N}_{weak}$-measurable and $f(\Omega)$ is essentially separable, i.e. $f(\Omega \setminus A)$ is separable for some $A \in \Sigma$ with $\mu(A) = 0$.

Dunford proved a corollary stating that for $E = \mathcal{L}(E_1, E_2)$, $f : \Omega \to \mathcal{L}(E_1, E_2)$ satisfies that f_x is $\|\cdot\|$-approximable μ-a.e for any $x \in E_1$ if and only if f is $\mathcal{N}_{WOT}$-measurable and $f_x(\Omega)$ is essentially separable for any $x \in E_1$, where $f_x(\omega) = f(\omega)(x)$.

Theorem B: (see [33, Theorem 3.5.5]) *Dunford's measurability Theorem.* Let (Ω, Σ, μ) be a finite complete measure space and let E_1, E_2 be Banach spaces. Then $f : \Omega \to \mathcal{L}(E_1, E_2)$ is $\|\cdot\|$-approximable μ-a.e. if and only if f is $\mathcal{N}_{WOT}$-measurable and the range $f(\Omega)$ is essentially separable in $\mathcal{L}(E_1, E_2)$.

Theorem C: (see [37, Theorem 2]) *Johnson's measurability Theorem.* Let (Ω, Σ, μ) be a finite complete measure space and let H be a separable Hilbert space. Then $f : \Omega \to \mathcal{L}(H, H)$ satisfies that f_x is $\|\cdot\|$-approximable μ-a.e. for any $x \in H$ if and only if f is $\mathcal{N}_{SOT}$-measurable.

Some problems regarding operator-valued functions have appeared recently (for example, see [1, 8, 5, 15, 13, 26]), where the strong operator topology plays an important role. We believe that a better understanding of the different types of measurability and their connections could be beneficial to address them. Our main result in this chapter, which can be found in section 1.4, establishes the following version of Pettis's measurability Theorem for the SOT-topology under the assumption of separability for the space E_1.

Theorem 1.1.5 *Let $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ where E_1 is separable. Then f is SOT-approximable $\iff$ f is WOT-approximable $\iff$ f is $\mathcal{N}_{WOT}$-measurable and $f(\Omega)$ is WOT-separable.*

Notice that in the case $E_1 = \mathbb{K}$, the classic Pettis's measurability Theorem is recovered.

Let us compare this last result with the previous ones to see that they offer different information. Observe that the WOT-separability of $f(\Omega)$ used in Theorem 1.1.5 gives the SOT-approximability of f, while in Theorem B the $\|\cdot\|$-separability of $f(\Omega)$ gives the $\|\cdot\|$-approximability of f. We shall prove in Proposition 1.2.8 that under the assumption of the separability of E_2, then $\mathcal{N}_{WOT}$-measurability and $\mathcal{N}_{SOT}$-measurability coincide. Therefore, if $E_1 = E_2 = H$ where H is a separable Hilbert space, the assumption $f(\Omega)$ is WOT-separable together with $\mathcal{N}_{SOT}$-measurability of f implies the SOT-approximability of f, which is stronger than the condition f_x is $\|\cdot\|$-approximable for any $x \in E_1$ that was obtained in Theorem C where there was no separability assumption on $f(\Omega)$.

Theorem 1.1.5 allows us to verify that several "natural" operator-valued functions which are not $\|\cdot\|$-approximable can still be SOT-approximable. For instance, the "translation" and "dilation" functions $f(t) = \tau_t$ and $g(\delta) = D_\delta$ defined by

$$\tau_t(\phi)(s) = \phi(s+t), \quad D_\delta(\phi)(s) = \delta^{1/p}\phi(\delta s)$$

are SOT-approximable but not $\|\cdot\|$-approximable considered as operator-valued functions from $\mathbb{R}$ and $\mathbb{R}^+$ into $\mathcal{L}(L^p(\mathbb{R}), L^p(\mathbb{R}))$ and $1 \leq p < \infty$ respectively.

It is known and easy to check that $f(\mathbb{R})$ and $g(\mathbb{R}^+)$ are not $\|\cdot\|$-separable. Hence f and g can't be $\|\cdot\|$-approximable. To prove that they are SOT-approximable, invoking Theorem 1.1.5 it is enough to show that f and g are $\mathcal{N}_{WOT}$-measurable, and that $f(\mathbb{R})$ and $g(\mathbb{R}^+)$ are WOT-separable. Notice that for $\phi \in L^p(\mathbb{R})$ and $\psi \in L^{p'}(\mathbb{R})$ we have that

$$t \to \langle \tau_t(\phi), \psi\rangle = \int_{\mathbb{R}} \phi(t+s)\psi(s)ds$$

and

$$\delta \to \langle D_\delta(\phi), \psi\rangle = \delta^{-1/p}\int_{\mathbb{R}} \phi(\delta s)\psi(s)ds$$

are continuous functions, and therefore measurable. Regarding the ranges, $\{f(q) : q \in \mathbb{Q}\}$ and $\{g(q) : q \in \mathbb{Q}, q > 0\}$ are WOT-dense sets in $f(\mathbb{R})$ and $g(\mathbb{R}^+)$ respectively.

Another application is given by the following example. Let $\Omega = [0,1]$ and $\Sigma = \mathcal{B}$ the Borel σ-algebra. Then the function $f : [0,1] \to \mathcal{L}(C([0,1]\times[0,1]), C([0,1]))$ given by

$$f(t)(\phi) = \phi_t, \quad \phi_t(s) = \phi(t,s), \quad \phi \in C([0,1]\times[0,1])$$

is SOT-approximable but not $\|\cdot\|$-approximable.

To check that f is not $\|\cdot\|$-approximable, just take $t \neq t'$ and select a function $\psi \in C([0,1])$ such that $\|\psi\|_\infty = 1$, $\psi(0) = 0$ and $\psi(t-t') = 1$. For $\phi(t,s) = \psi(t-s)$ we have

$$\|f(t) - f(t')\| \geq \|\phi_t - \phi_{t'}\| \geq 1$$

which shows that $f([0,1])$ is not separable, which prevents f from being $\|\cdot\|$-approximable.

To see that that f is SOT-approximable we can apply Theorem 1.1.5 and show the $\mathcal{N}_{WOT}$-measurability of f, together with the WOT-separability of its range, $f([0,1])$. This follows easily since for each $\phi \in C([0,1]\times[0,1])$ and $\mu \in M([0,1]) = E_2^*$ we have

$$t \to \langle f(t)(\phi), \mu\rangle = \int_0^1 \phi(t,s)d\mu(s),$$

which is continuous (and therefore Borel measurable). Also, observe that $\{f(q) : q \in \mathbb{Q}\cap[0,1]\}$ is SOT-dense (in particular WOT-dense) in $f([0,1])$, because for any $\phi \in C([0,1]\times[0,1]) = C([0,1], C[0,1])$ and $0 \leq t \leq 1$ we have $\|\phi_t - \phi_{q_n}\| \to 0$ for any sequence $(q_n) \subset \mathbb{Q}\cap[0,1]$ converging to t.

This chapter, besides this preliminary part, has three more sections. In the next one we shall focus on the study of the concept of β-measurability for the standard bases in the SOT- and WOT-topologies. In the following section, our attention will move to the analysis of the concept of τ-approximability showing that d-separability and d-approximability coincide for metric spaces and also the striking (but no so hard to prove) fact that $\|\cdot\|$-approximability coincides with *weak*-approximability in any Banach space. In the last section of the chapter we give the proof of Theorem 1.1.5, and apply this theorem to produce some examples of SOT-approximable functions that illustrate the theory.

1.2 Measurability with respect to bases

Definition 1.2.1 *Let (Y, τ) be a topological vector space and $\beta \subseteq \tau$ be a basis of the topology. A function $f : \Omega \to Y$ is called β-measurable whenever $f^{-1}(A) \in \Sigma$ for any $A \in \beta$.*

Let us take a look at this well known fact for metric spaces.

Proposition 1.2.2 *Let (Y, d) be a separable metric space. Then $f : \Omega \to Y$ is $\mathcal{N}_d$-measurable if and only if f is τ_d-measurable.*

In particular if E is a Banach space, $f : \Omega \to E$ is $\mathcal{N}_{\|\cdot\|}$-measurable and $f(\Omega)$ is separable then f is $\tau_{\|\cdot\|}$-measurable.

Proof: It is obvious that if f is τ_d-measurable, then f is also $\mathcal{N}_d$-measurable.

Assume now that f is $\mathcal{N}_d$-measurable. To see that f is τ_d-measurable it will be enough to prove that the σ-algebra generated by $\mathcal{N}_d$ and τ_d coincide, that is $\sigma(\mathcal{N}_d) = \sigma(\tau_d)$, whenever Y is d-separable. In order to do that, let $\mathcal{A} = (y_n)_{n=1}^{\infty}$ be a dense set in Y and an open set $G \in \tau_d$. For each point $y \in G$, there exist $\varepsilon > 0$ and y_k such that $y \in B(y_k, \varepsilon)$. Taking ε_k such that $y \in B(y_k, \varepsilon_k) \subseteq G$, we can conclude that $G = \cup_k B(y_k, \varepsilon_k)$ and therefore any open set G is in $\sigma(\mathcal{N}_d)$. This shows that $f^{-1}(G) = \cup_k f^{-1}(B(y_k, \varepsilon_k))$, which belongs to Σ by hypothesis, and the proof is complete. ∎

We present now some particular cases of the notion of measurability with respect to bases seen at Definition 1.1.4, corresponding to bases of different topologies.

Definition 1.2.3 *Let $f : \Omega \to \mathcal{L}(E_1, E_2)$. Then f is* $\mathcal{N}_{\|\cdot\|}$**-measurable** *whenever*

$$f^{-1}(B(T, \varepsilon)) \in \Sigma \quad \forall T \in \mathcal{L}(E_1, E_2), \forall \varepsilon > 0. \tag{1.1}$$

*f is $\mathcal{N}_{SOT}$-**measurable** whenever*

$$f^{-1}(N(T;\mathbf{x},\varepsilon)) \in \Sigma \quad \forall T \in \mathcal{L}(E_1,E_2), \forall \mathbf{x} \in (E_1)^n, \forall \varepsilon > 0. \tag{1.2}$$

*f is $\mathcal{N}_{WOT}$-**measurable** whenever*

$$f^{-1}(N(T;\mathbf{x},\mathbf{y}^*,\varepsilon)) \in \Sigma, \quad \forall T \in \mathcal{L}(E_1,E_2), \forall \mathbf{x} \in (E_1)^n, \forall \mathbf{y}^* \in (E_2^*)^n, \forall \varepsilon > 0. \tag{1.3}$$

The notation $\tau_{\|\cdot\|}$-, τ_{SOT} and τ_{WOT} will be reserved to indicate that $f^{-1}(G) \in \Sigma$ for any open set G in the norm, SOT or WOT topologies, respectively. Should it be necessary, we shall use $\mathcal{N}_{SOT}(E_1,E_2)$ or $\mathcal{N}_{WOT}(E_1,E_2)$ in certain contexts to avoid misunderstandings.

Given a Banach space X (respectively a dual space E^*) and $f : \Omega \to X$ (respectively $f : \Omega \to E^*$) the term *weak*-measurable (respectively *weak**-measurable) is used to indicate that $w \to \langle x^*, f(w)\rangle$ (respectively $w \to \langle f(w), x\rangle$) is a measurable function for all $x^* \in X^*$ (respectively $x \in E$). The next result will prove that this corresponds to $\mathcal{N}_{weak}$-measurable when $X = \mathcal{L}(\mathbb{K}, X)$ and $\mathcal{N}_{weak^*}$-measurable when $X = \mathcal{L}(E,\mathbb{K})$.

When working with functions $f : \Omega \to \mathcal{L}(E_1,E_2)$, the following notation comes in handy and will be put to use in what follows. For $x \in E_1$ and $y^* \in E_2^*$, we denote:

$$f_x(w) = f(w)(x), \quad f_{x,y^*}(w) = \langle y^*, f(w)(x)\rangle.$$

Proposition 1.2.4 *Let $X = \mathcal{L}(E_1,E_2)$ and $f : \Omega \to X$. Then*

(i) f is $\mathcal{N}_{\|\cdot\|}$-measurable $\Longleftrightarrow$ $\|f(\cdot) - T\|$ is measurable for any $T \in X$.

(ii) f is $\mathcal{N}_{SOT}$-measurable $\Longleftrightarrow$ f_x is $\mathcal{N}_{\|\cdot\|}$-measurable for any $x \in E_1$.

(iii) f is $\mathcal{N}_{WOT}$-measurable $\Longleftrightarrow$ f_{x,y^*} is measurable for any $x \in E_1$ and $y^* \in E_2^*$.

Proof: (i) It is an immediate consequence of the definitions.

(ii) Let us assume first that f_x is $\mathcal{N}_{\|\cdot\|}$-measurable for any $x \in E_1$. Since $N(T;\mathbf{x},\varepsilon) = \cap_{i=1}^n N(T;x_i,\varepsilon)$ where $\mathbf{x} = (x_1,\cdots,x_n) \in (E_1)^n$, it suffices to see that $f^{-1}(N(T;x_i,\varepsilon)) \in \Sigma$ for $i = 1,\cdots,n$. This clearly follows using that $f(\cdot)(x)$ is measurable, since this gives that

$$\{\omega \in \Omega : \|f(\omega)(x) - y\| \leq \varepsilon\} \in \Sigma \text{ for any } \varepsilon > 0, y \in E_2.$$

Now, assume that f is $\mathcal{N}_{SOT}$-measurable and let $x \in E_1$. We have to see that $\{\omega \in \Omega : \|f_x(\omega) - y\| < \varepsilon\} \in \Sigma$ for any $\varepsilon > 0$, $y \in E_2$. Select x^* satisfying that $\langle x^*, x\rangle = \|x\|$ and define $T := \frac{x^*}{\|x\|} \otimes y$, that is $T(x_0) = \langle \frac{x^*}{\|x\|}, x_0\rangle y$ for $x_0 \in E_1$. Since $T(x) = y$, with this choice we have

$$\{\omega \in \Omega : \|f_x(\omega) - y\| < \varepsilon\} = \{\omega \in \Omega : f(\omega) \in N(T; x, \varepsilon)\} \in \Sigma.$$

(iii) Let us start with f being $\mathcal{N}_{WOT}$-measurable. Given $x \in E_1$, $y^* \in E_2^*$, $\alpha \in \mathbb{K}$ and $\varepsilon > 0$ we need to verify that $\{\omega \in \Omega : |\langle y^*, f(\omega)(x)\rangle - \alpha| < \varepsilon\} \in \Sigma$. We may assume that $x \neq 0$ and $y^* \neq 0$. Now, select x_0^* and y_0 such that $\langle x_0^*, x\rangle = \|x\|$, and with $\langle y^*, y_0\rangle = 1$. Define $T := \frac{x_0^*}{\|x\|} \otimes y_0\alpha$. With this choice, observe that $\langle y^*, T(x)\rangle = \alpha$. Therefore

$$\{\omega \in \Omega : |\langle y^*, f(\omega)(x)\rangle - \alpha| < \varepsilon\} = \{\omega \in \Omega : f(\omega) \in N(T; x, y^*, \varepsilon)\} \in \Sigma.$$

The converse follows the same line as (ii). ∎

Let us remark that for $X = \mathcal{L}(E_1, E_2)$ and $f : \Omega \to X = \mathcal{L}(\mathbb{K}, X)$ the $\mathcal{N}_{weak}$-measurability means that $\langle f(\cdot), T^*\rangle$ is measurable for any $T^* \in X^*$, while the $\mathcal{N}_{WOT}$-measurability means $\langle y^*, f(\cdot)(x)\rangle$ is measurable for any $x \in E_1$ and $y^* \in E_2^*$. If we take into account that for each $x \in E_1$ and $y^* \in E_2^*$ the map $T \to \langle Tx, y^*\rangle$ belongs to X^*, one gets $\mathcal{N}_{WOT} \subset \mathcal{N}_{weak}$. Therefore $\mathcal{N}_{weak}$-measurable implies $\mathcal{N}_{WOT}$-measurable. On the other hand, $\tau_{weak} \subset \tau_{\|\cdot\|}$ for any Banach space and also $\tau_{WOT} \subset \tau_{SOT} \subset \tau_{\|\cdot\|}$ for $X = \mathcal{L}(E_1, E_2)$. In particular, then, $\tau_{\|\cdot\|}$-measurable functions are also $\mathcal{N}_{weak}$-measurable and $\mathcal{N}_{SOT}$-measurable.

Since $\mathcal{N}_{WOT}$-measurability for $E_2 = \mathbb{K}$ corresponds to $\mathcal{N}_{weak^*}$-measurability, we can make use of some classical examples for vector-valued functions (see, for example, [20, page 43]) to prove that the above inclusions are strict.

Example 1.2.5 *Let $\Omega = [0,1]$, $\Sigma = \mathcal{B}$ the Borel σ-algebra, $E_1 = \ell^1$, $E_2 = \mathbb{K}$ and $f : [0,1] \to \ell^\infty = \mathcal{L}(\ell^1, \mathbb{K})$ given by $f(t) = (r_n(t))_{n\in\mathbb{N}}$ where r_n stand for the Rademacher functions. Then f is $\mathcal{N}_{WOT}$-measurable ($\mathcal{N}_{weak^*}$-measurable) but not $\mathcal{N}_{weak}$-measurable.*

By the Riesz theorem, for Hilbert spaces we have that $H = H^* = \mathcal{L}(\mathbb{K}, H) = \mathcal{L}(H, \mathbb{K})$. In this setting, the notions of $\mathcal{N}_{weak}$-, $\mathcal{N}_{WOT}$-, $\mathcal{N}_{SOT}$- and $\mathcal{N}_{weak^*}$- measurability coincide. Now we present an example of a $\mathcal{N}_{SOT}$-measurable function which is not $\mathcal{N}_{\|\cdot\|}$-measurable (and therefore not $\tau_{\|\cdot\|}$-measurable either).

Example 1.2.6 *Let $\Omega = [0,1]$, $\Sigma = \mathcal{B}$ the Borel σ-algebra and let A be a non-Borel set. Take $E_1 = \ell^2([0,1])$, $E_2 = \mathbb{K}$ and $f : [0,1] \to \mathcal{L}(\ell^2([0,1]), \mathbb{K}) = \ell^2([0,1])$ given by*

$$t \to e_t \chi_A(t)$$

where $(e_t)_t$ stands for the canonical basis of $\ell^2([0,1])$. Then f is $\mathcal{N}_{SOT}$-measurable but not $\mathcal{N}_{\|\cdot\|}$-measurable.

Proof: By Proposition 1.2.4, we need to check that $f(\cdot)(x)$ is Borel measurable for any $x \in \ell^2([0,1])$. For each $x \in \ell^2([0,1])$ we can find $(t_n) \in [0,1]$ such that $x = \sum_{n\in\mathbb{N}} \alpha_n e_{t_n}$ with $\sum_n |\alpha_n|^2 < \infty$. Therefore $t \to f(t)(x) = \sum_{n:t_n\in A} \alpha_n \langle e_{t_n}, e_t\rangle$ is countably valued, therefore measurable.

Finally, since the set $\{t \in [0,1] : \|f(t)\| < 1/2\} = [0,1] \setminus A$ is not Borel measurable by hypothesis, then f can't be $\mathcal{N}_{\|\cdot\|}$-measurable. ∎

When E_1 or E_2 happens to be separable, there are some notions of measurability that coincide. We know that $\mathcal{N}_{\|\cdot\|}$-measurability implies $\mathcal{N}_{SOT}$-measurability. Let us now show that when E_1 is separable, both notions are the same.

Proposition 1.2.7 *Let $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ where E_1 is separable. If f is $\mathcal{N}_{SOT}$-measurable then f is also $\mathcal{N}_{\|\cdot\|}$-measurable.*

Proof: Let (x_n) be a dense set in the unit ball of E_1. Due to Proposition 1.2.4 and hypothesis, each map $\|f_{x_n}(\cdot) - Tx_n\|$ is measurable for each $n \in \mathbb{N}$ and $T \in X$. Since

$$\|f(w) - T\| = \sup_n \|f_{x_n}(w) - Tx_n\|,$$

taking into acount the fact that the countable supremum of measurable functions is measurable, we obtain that f is $\mathcal{N}_{\|\cdot\|}$-measurable. ∎

We also know that $\mathcal{N}_{SOT}$-measurability implies $\mathcal{N}_{WOT}$-measurability. Let us prove that under the separability condition on E_2, $\mathcal{N}_{WOT}$- and $\mathcal{N}_{SOT}$-measurability coincide.

Proposition 1.2.8 *Let $f : \Omega \to \mathcal{L}(E_1, E_2)$ with E_2 separable. Then f is $\mathcal{N}_{WOT}$-measurable if and only if f is $\mathcal{N}_{SOT}$-measurable.*

Proof: Assume that f is $\mathcal{N}_{WOT}$-measurable. Now, select $(y_n) \subset E_2$ and $(y_n^*) \subset E_2^*$ such that (y_n) is dense in E_2 and $\|y_n\| = \langle y_n, y_n^* \rangle$ for each $n \in \mathbb{N}$. Hence, for each $x \in E_1$ and $T \in \mathcal{L}(E_1, E_2)$, we have

$$\|f(w)x - Tx\| = \sup_n |\langle f(w)x, y_n^* \rangle - \langle Tx, y_n^* \rangle|,$$

which is a countable supremum of measurable functions, since f_{x,y_n^*} is measurable for each n. Therefore f is $\mathcal{N}_{SOT}$-measurable.

Assume now that f is $\mathcal{N}_{SOT}$-measurable. Applying Proposition 1.2.4 (parts (ii) and (iii)) and also Proposition 1.2.2, we obtain that f_x is $\tau_{\|\cdot\|}$-measurable for any $x \in E_1$. Hence f is $\mathcal{N}_{WOT}$-measurable. ∎

1.3 SOT- and WOT-approximable functions

In this section, we will explore the remaining type of definition of measurability, which involves the existence of an approximating sequence of simple functions. We recall the definition of simple and countably valued function.

Definition 1.3.1 *Let* (X, τ) *be a Hausdorff topological space. A function* $f : \Omega \to X$ *is said to be simple (respectively countably valued) if there exist a finite set (respectively sequence)* $(x_n)_n \subset X$ *and a finite partition (respectively countable partition) of pairwise disjoint sets* $(A_n)_n \subset \Sigma$ *such that* $\Omega = \cup_k A_k$ *and* $f = \sum_n x_n \chi_{A_n}$.

Definition 1.3.2 *We say that a function* $f : \Omega \to X$ *is* τ*-approximable if there exists a sequence of* X*-valued simple functions* $s_n : \Omega \to X$ *such that*

$$\lim_n s_n(\omega) = f(\omega), \quad \forall \omega \in \Omega.$$

Specifically, a function $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ *is said to be* $\|\cdot\|$*-, weak-,* SOT*-,* WOT*-approximable if there exists a sequence of operator-valued simple functions* $s_n : \Omega \to X$ *satisfying*

$$\|s_n(\omega) - f(\omega)\| \xrightarrow[n\to\infty]{} 0, \quad \forall \omega \in \Omega,$$

$$\langle s_n(\omega), T^* \rangle \xrightarrow[n\to\infty]{} \langle f(\omega), T^* \rangle, \quad \forall T^* \in X^*, \quad \forall \omega \in \Omega,$$

$$\lim_n \|s_n(\omega)(x) - f(\omega)(x)\| = 0, \quad \forall x \in E_1, \quad \forall \omega \in \Omega$$

or

$$\langle s_n(\omega)(x), y^* \rangle \xrightarrow[n\to\infty]{} \langle f(\omega)(x), y^* \rangle, \quad \forall x \in E_1, \quad \forall y^* \in E_2^*, \quad \forall \omega \in \Omega$$

respectively.

As usual, when E_1 or E_2 are equal to $\mathbb{K}$, some of the notions coincide. More precisely, when $E_2 = \mathbb{K}$, the SOT-approximability and the WOT-approximability are the same concept and will just be called $weak^*$-approximability. If $E_1 = \mathbb{K}$, the SOT-approximability is just the $\|\cdot\|$-approximability, whilst the WOT-approximability is the weak-approximability.

Proposition 1.3.3 *Let* (X, τ) *be a Haussdorff topological space. If* $f : \Omega \to X$ *is* τ*-approximable then* f *is* τ*-measurable and there exists a countable set* $\mathcal{A} \subset X$ *such that* $f(\Omega) \subset \overline{\mathcal{A}}^{\tau}$.

Proof: Let us assume that $f = \lim_n s_n$ pointwise, with $s_n = \sum_{k=1}^{m_n} x_{n,k}\chi_{A_{n,k}}$ for some $x_{n,k} \in X$ and $A_{n,k} \in \Sigma$. If $G \in \tau$ then $s_n^{-1}(G) \in \Sigma$ since s_n is τ-measurable. Also, observe that

$$\{w : f(w) \in G\} = \limsup\{w : s_n(w) \in G\} \in \Sigma.$$

Finally, notice that the set $\mathcal{A} = \cup_n\{s_n(w) : w \in \Omega\}$ is countable and $f(\Omega) \subset \overline{\mathcal{A}}^{\tau}$. ∎

In the proof of the following corollary, the notion of "hereditary separability" will appear. We include here its definition.

Definition 1.3.4 *Let (X, τ) be a topological space. We say that X is hereditarily separable if every $S \subseteq X$ is separable in the relative topology.*

Corollary 1.3.5 *Let E_1 and E_2 be separable Banach spaces and $f : \Omega \to \mathcal{L}(E_1, E_2)$. If f is SOT-approximable (respect. WOT-approximable) then $f(\Omega)$ is SOT-separable (respect. WOT-separable).*

Proof: By using Proposition 1.3.3, this will follow clearly if we show that $\mathcal{L}(E_1, E_2)$ is hereditarily separable for SOT- and WOT-topologies. Using that the weak operator topology is weaker than the strong operator topology, we just need to prove it for the SOT-topology.

Using the known fact (see [48, 42]) that the space of continuous functions from a separable metric space into another separable metric space is hereditarily separable with the pointwise topology and taking into account that $\mathcal{L}(E_1, E_2) \subset C(E_1, E_2)$ considered with the pointwise topology, we obtain the result. ∎

Although the following result for metric spaces is part of the folklore, it will be useful later and we include the proof here for completeness.

Lemma 1.3.6 *Let (Y, d) be a metric space and $\Sigma = \mathcal{B}(Y)$ the Borel σ-algebra. Then Y is d-separable if and only if $id : Y \to Y$ is d-approximable.*

Proof: Assume first that Y is d-separable. Then, we can choose $\{y_n : n \in \mathbb{N}\}$ a countable dense subset of Y. For each $n \in \mathbb{N}$, and $1 \leq k \leq n$, we denote by $B_{k,n}$ the set $y \in Y$ satisfying

$$d(y, y_k) < d(y, y_m) \;\; 1 \leq m < k, \quad d(y, y_k) \leq d(y, y_m) \;\; k \leq m \leq n.$$

With this construction, clearly the sets $B_{k,n}$ are pairwise disjoint Borel sets satisfying that $Y = \cup_{k=1}^{n} B_{k,n}$ for each $n \in \mathbb{N}$.

Now, let us define $\phi_n : Y \to Y$ by

$$\phi_n = \sum_{k=1}^{n} y_k \, \chi_{B_{k,n}}. \tag{1.4}$$

In particular $\phi_n(y) = y_{k(n)}$ for some $1 \leq k(n) \leq n$ and satisfies

$$d(y, \phi_n(y)) \leq \min_{1 \leq k \leq n} d(y, y_k).$$

By density, notice that for each $y \in Y$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(y, y_{n_0}) < \varepsilon$. Therefore $d(y, \phi_n(y)) \leq d(y, y_{n_0}) < \varepsilon$ for any $n \geq n_0$. This gives that $id = \lim_n \phi_n$ where ϕ_n are $\mathcal{B}(Y)$-simple functions.

Conversely, if $id = \lim_n s_n$ for some sequence $s_n : Y \to Y$ of simple functions, then $Y = \overline{\mathcal{A}}$ where $\mathcal{A} = \{s_n(y) : n \in \mathbb{N}, y \in Y\}$ is a countable set, and the proof is complete. ∎

Theorem 1.3.7 *Let (Y, d) be a metric space and $f : \Omega \to Y$. The following are equivalent:*

(i) f is d-approximable.

(ii) f is τ_d-measurable and $f(\Omega)$ is d-separable.

(iii) f is $\mathcal{N}_d$-measurable and $f(\Omega)$ is d-separable.

Proof: (i) $\Longrightarrow$ (ii) Applying Proposition 1.3.3, we obtain that f is τ_d-measurable and also that $f(\Omega) \subseteq \overline{\mathcal{A}}$ where $\mathcal{A} = \{s_n(w) : n \in \mathbb{N}, w \in \Omega\}$ is a countable set. Now, taking into account that metric spaces are hereditarily separable, and that $\overline{\mathcal{A}}$ is d-separable, we get that $f(\Omega)$ is also d-separable.

(ii) $\Longrightarrow$ (iii) This implication is obvious.

(iii) $\Longrightarrow$ (i) As in Lemma 1.3.6, we construct $\phi_n = \sum_{k=1}^{n} y_k \ \chi_{B_{k,n}}$ with $\phi_n(y) \to y$ for any $y \in f(\Omega)$. Since $B_{k,n} \in \sigma(\mathcal{N}_d)$ and f is $\mathcal{N}_d$-measurable by hypothesis, we obtain that $A_{n,k} = f^{-1}(B_{n,k}) \in \Sigma$ and we have that $s_n(w) \to f(w)$ for each $w \in \Omega$, where $s_n(w) = \phi_n(f(w)) = \sum_{k=1}^{n} y_k \chi_{A_{n,k}}$. Therefore, f is d-approximable. ∎

A combined use of Theorem 1.3.7 and Proposition 1.2.4 provides the following corollary, which can be viewed as a version of Johnson's result (see Theorem C in the introduction).

Corollary 1.3.8 *A function $f : \Omega \to \mathcal{L}(E_1, E_2)$ satisfies that f_x is $\|\cdot\|$-approximable for any $x \in E_1$ if and only if f is $\mathcal{N}_{SOT}$-measurable and $f_x(\Omega)$ is $\|\cdot\|$-separable for any $x \in E_1$.*

The relation between the convergence of the main topologies we are studying quickly reveals that $\|\cdot\|$-approximable implies *SOT*- and *weak*-approximable and also *SOT*-approximable implies *WOT*-approximable. It is a natural question to ask about the converse implications, which will not be true in all the cases. Examples of this shall be provided in the sequel.

We point out that $\|\cdot\|$-approximable corresponds to the classical definition of strongly measurable that can be found in the literature. Let us see now a rather surprising, yet easy to prove fact: the notion of $\|\cdot\|$-approximability is exactly the same as the notion of *weak*-approximability in any Banach space.

Proposition 1.3.9 *Let X be a Banach space and $f : \Omega \to X$. Then f is $\|\cdot\|$-approximable if and only if f is weak-approximable.*

Proof: We already know that $\|\cdot\|$-approximability implies *weak*-approximability since the convergence in norm implies the *weak*-convergence.

Let us assume that f is *weak*-approximable. Then, we can find a sequence of simple functions (s_n) that converges pointwise to f with respect to the weak topology. Then, $f(\Omega)$ is contained in the weak closure of $U = \cup_n s_n(\Omega)$, and, therefore, $f(\Omega)$ is contained in $V = \overline{co(U)}^{\|\cdot\|}$ by Mazur's Lemma. As U is separable, it turns out that V is a separable metric space (hence heriditarily separable), so its subset $f(\Omega)$ is also separable. Now, from Proposition 1.3.3, we obtain that f is weakly-measurable, and finally Pettis's measurability theorem gives that f is $\|\cdot\|$-approximable. ∎

The following example shows that for $E_2 = \mathbb{K}$, the notion of *weak**-approximability differs from the $\|\cdot\|$-approximability.

Example 1.3.10 *Let $\Omega = [0,1]$ with the Borel σ-algebra, $E_1 = \ell^1$ and $E_2 = \mathbb{K}$. Let $f(t) = (e^{int})_{n\in\mathbb{N}}$. Then f is weak*-approximable but not $\|\cdot\|$-approximable.*

Proof: First of all, it is well known and easy to check that f has non-separable range, so it can't be $\|\cdot\|$-approximable.

Let $f_n(t) = \sum_{k=1}^{n} e^{ikt} e_k$ where e_k stands for the sequence $(e_k)_{j\in\mathbb{N}} = \delta_{kj}$. Since f_n are continuous functions and $f_n \xrightarrow{w^*} f$ then we have that f is *weak**-approximable. ∎

Proposition 1.3.11 *Let E be a separable Banach space, $X = E^*$ and let $f : \Omega \to X$ be a bounded function. Then f is weak*-approximable if and only if f is $\mathcal{N}_{weak^*}$-measurable.*

Proof: The direct implication is immediate from Theorem 1.3.7.

Assume that f is $\mathcal{N}_{weak^*}$-measurable, and let us check that f is *weak**-approximable. Without loss of generality we can assume that $f(\Omega) \subseteq B_X$. We know that the *weak**-topology on the unit ball of B_X is metrizable (the reader is referred to [23, page 426]) with the metric given by the distance $d(x^*, y^*) = \sum_{n=1}^{\infty} \frac{|\langle x^* - y^*, x_n\rangle|}{2^n(1+|\langle x^* - y^*, x_n\rangle|)}$ for $x^*, y^* \in B_X$ for a given sequence (x_n) dense in E. Also, it is a well known fact that B_X is *weak**-compact set (see [33, page 37]). All in all, B_X is a metrizable separable space with the *weak**-topology. Therefore $f(\Omega)$ is also separable in this topology. Since f is $\mathcal{N}_d$-measurable, invoking Theorem 1.3.7 we finish the proof. ∎

Before moving on to the next section of the chapter, let us present a couple more examples. Of course, $\|\cdot\|$-approximability implies SOT-approximability. The converse is false as one sees when taking $E_2 = \mathbb{K}$. Indeed, there are examples of $weak^*$-approximable (SOT-approximable) functions that are not $weak$-approximable (equivalently $\|\cdot\|$-approximable). Here is one of them.

Example 1.3.12 *Let E be a separable Banach space such that E^* is not separable. Let $\Omega = B_{E^*}$ with the $weak^*$-topology and Σ the Borel σ-algebra. Then $id : \Omega \to \mathcal{L}(E, \mathbb{K})$ is $weak^*$-approximable but it is not $weak$-approximable.*

Proof: Since Ω is $weak^*$-separable, Lemma 1.3.6 gives that $id : \Omega \to E^*$ is $weak^*$-approximable. By the same result, Ω not being $\|\cdot\|$-separable implies that $id : \Omega \to E^*$ is not $\|\cdot\|$-approximable, which by Proposition 1.3.9 gives that it is not $weak$-approximable. ∎

Let us wrap up this section with an example of a function that is not WOT-approximable.

Example 1.3.13 *Let $\Omega = [0,1]$ and $\Sigma = \mathcal{B}$ the Borel σ-algebra. Let $E_1 = E_2 = \ell^\infty$. Define $f(t) \in \mathcal{L}(E_1, E_2)$ as*

$$f(t)((\alpha_n)) = (e^{int}\alpha_n), \quad (\alpha_n) \in \ell^\infty.$$

Then f is not WOT-approximable.

Proof: Assume that f is WOT-approximable and let us find a contradiction. The WOT-approximability of f would mean that there exists a countable set $\mathcal{A}$ with $\{f(t) : t \in [0,1]\} \subseteq \overline{\mathcal{A}}^{WOT}$. As a consequence, selecting the element of ℓ^∞ given by $\mathbf{1} = (\alpha_n)$ with $\alpha_n = 1$ for all n we would have that $\{f(t)(\mathbf{1}) : t \in [0,1]\} \subseteq \overline{\mathcal{A}_1}^{weak}$ for a countable set $\mathcal{A}_1$, and by Mazur's theorem $\{f(t)(\mathbf{1}) : t \in [0,1]\} \subseteq \overline{\mathcal{A}_1}^{weak} \subseteq \overline{co(\mathcal{A}_1)}^{weak} = \overline{co(\mathcal{A}_1)}^{\|\cdot\|}$. But the set $\{f(t)(\mathbf{1}) : t \in [0,1]\} = \{(e^{int})_n : t \in [0,1]\}$ is clearly not separable in norm. Hence, f is not WOT-approximable. ∎

1.4 On Pettis's measurability Theorem

Let us present now a proof of Dunford's measurability theorem (and hence Pettis's measurability too).

Theorem 1.4.1 *Let $f : \Omega \to X = \mathcal{L}(E_1, E_2)$. The following statements are equivalent:*

(i) f is $\|\cdot\|$-approximable.

(ii) f is $\tau_{\|\cdot\|}$-measurable and $f(\Omega)$ is separable in X.

(iii) f is $\mathcal{N}_{\|\cdot\|}$-measurable and $f(\Omega)$ is separable in X.

(iv) f is $\mathcal{N}_{SOT}$-measurable and $f(\Omega)$ is separable in X.

(v) f is $\mathcal{N}_{WOT}$-measurable and $f(\Omega)$ is separable in X.

Proof: First, observe that at this point, due to Theorem 1.3.7, the only implications that actually remain to be shown are that if f is either $\mathcal{N}_{SOT}$-measurable or $\mathcal{N}_{WOT}$-measurable together with $f(\Omega)$ separable, then f is $\mathcal{N}_{\|\cdot\|}$-measurable.

In order to do it, select (T_n) a dense set in $f(\Omega)$, and take $x_n \in E_1$ and $y_n^* \in E_2^*$ satisfying the conditions $\|T_n\| < \|T_n(x_n)\| + 1/n$ and $\|T_n(x_n)\| = |\langle T_n x_n, y_n^*\rangle|$. It is easy to see that for each $w \in \Omega$ and $T \in \mathcal{L}(E_1, E_2)$, one has

$$\|f(w) - T\| = \sup_n \|f(w)x_n - Tx_n\| = \sup_n |\langle (f(w) - T)x_n, y_n^*\rangle|.$$

Therefore, using that a countable supremum of measurable functions is measurable, then either $\mathcal{N}_{SOT}$- or $\mathcal{N}_{WOT}$-measurability implies $\mathcal{N}_{\|\cdot\|}$-measurability. ∎

It is natural to ask ourselves if any $\mathcal{N}_{WOT}$-measurable function with SOT-separable range is also SOT-approximable. Theorem 1.4.3 will give some information in that regard. But before that, we present the following example with $E_2 = \mathbb{K}$.

Example 1.4.2 *Let $\Omega = [0,1]$, $E_1 = \ell^2([0,1])$, $E_2 = \mathbb{K}$ and $f : [0,1] \to \mathcal{L}(\ell^2([0,1]), \mathbb{K})$ given by $f(t) = e_t$ the corresponding element in the canonical basis. f is $\mathcal{N}_{weak^*}$-measurable, but not weak*-approximable.*

Proof: For each $x = (\alpha_t)_t \in \ell^2([0,1])$, one clearly obtains that $t \to \alpha_t = f(t)(x)$ is measurable.

Let us see that f is not *weak**-approximable. If that was the case, we would have that $f([0,1]) \subset \overline{\mathcal{A}}^{weak^*}$ for certain countable set $\mathcal{A}$. Then for any $t \in [0,1]$ there exists $g \in \mathcal{A}$ such that $g \in N(e_t; e_t, 1/2)$. In particular

$$|g(t) - 1| < 1/2. \tag{1.5}$$

On the other hand if the countable set $\mathcal{A}$ is $\mathcal{A} = (g_m)_{m\in\mathbb{N}}$, and $g_m = \sum_{t\in F_m} a_t e_t$ for a given countable set F_m, it suffices to select $t \notin \cup_m F_m$ to obtain a contradiction with (1.5). Hence, Proposition 1.3.3 rules out f from being *weak**-approximable. ∎

Now, we present the main result of the chapter: a version of Pettis's measurability theorem for the SOT-topology, under the assumption that E_1 is separable.

Theorem 1.4.3 *Let $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ and assume E_1 is separable. The following statements are equivalent:*

(i) f is SOT-approximable.

(ii) f is WOT-approximable.

(iii) f is $\mathcal{N}_{WOT}$-measurable and $f(\Omega)$ is WOT-separable.

Proof: (i) $\Longrightarrow$ (ii) This implication requires no explanation.

(ii) $\Longrightarrow$ (iii) Invoking Proposition 1.3.3 one obtains that f is τ_{WOT}-measurable (and, as a consequence, $\mathcal{N}_{WOT}$-measurable).

Let us see that $f(\Omega)$ is WOT-separable. By the approximability hypothesis, we can find a sequence of simple functions (s_n) such that $s_n(w) \to f(w)$ in the WOT-topology. Also, select (x_m) a dense set in E_1. Consider the countable sets

$$\mathcal{A} = \cup_n s_n(\Omega), \quad \mathcal{A}_{E_2} = \cup_{n,m}\{s_n(w)(x_m) : w \in \Omega\}$$

and define $\tilde{E}_2 = \overline{span\mathcal{A}_{E_2}}$ (observe that $\tilde{E}_2$ is a separable Banach subspace of E_2). Since

$\tilde{E}_2 = \overline{span\mathcal{A}_{E_2}}^{weak}$ then $f : \Omega \to \mathcal{L}(E_1, \tilde{E}_2)$ and $f(\Omega) \subset \overline{\mathcal{A}}^{WOT}$. Now, since both E_1 and $\tilde{E}_2$ are separable, we can use Corollary 1.3.5 to get the WOT-separability of $f(\Omega)$.

(iii) $\Longrightarrow$ (i) Since $\overline{co(f(\Omega))}^{WOT} = \overline{co(f(\Omega))}^{SOT}$ (see [23, Corollary VI.1.5]) we can find a countable set $\{T_n\}$ such that $f(\Omega) \subset \overline{span\{T_n\}}^{SOT}$. The separability of the space E_1 allows us to select (x_m) dense in B_{E_1}. Consider now the separable subspace of E_2 given by $\tilde{E}_2 = \overline{span\{T_n(x_k) : n, k \in \mathbb{N}\}}$. Also notice that Hahn-Banach's theorem gives that $\mathcal{N}_{WOT(E_1,E_2)}$-measurability implies $\mathcal{N}_{WOT(E_1,\tilde{E}_2)}$-measurability.

Since $\tilde{E}_2$ is separable, Proposition 1.2.8 implies that f is $\mathcal{N}_{SOT(E_1,\tilde{E}_2)}$-measurable. In particular, since E_1 is separable too, we can use Proposition 1.2.7 to get that $\{w : \|f(w)\| \leq m\} \in \Sigma$ for any $m \in \mathbb{N}$. Taking into account that $f = \lim_m f\chi_{\{\|f\| \leq m\}}$, and the fact that pointwise limit of τ-approximable functions is also τ-approximable, it will be enough to prove the result for bounded functions. Hence we may assume that $K_0 = \sup_{w\in\Omega} \|f(w)\| < \infty$.

Denote $N_k = N(0; x_1, \cdots, x_k, \frac{1}{k})$ and, for each $n, k \in \mathbb{N}$, define the set

$$A_{k,n} = \{w \in \Omega : f(w) - T_n \in N_k\}.$$

Since f is $\mathcal{N}_{SOT}$-measurable, $A_{k,n} \in \Sigma$. Now consider $B_{k,1} = A_{k,1}$ and

$$B_{k,n} = A_{k,n} \setminus (\cup_{1\leq j<n} B_{k,j}),$$

which are pairwise disjoint sets in Σ with $\Omega = \cup_n A_{k,n} = \cup_n B_{k,n}$ for any $k \in \mathbb{N}$. Define now the countably valued function

$$f_k = \sum_n T_n \chi_{B_{k,n}}.$$

Let us see that f is SOT-approximable by checking that $f(w) = SOT - \lim_k f_k(w)$, that is to say that for each $w \in \Omega$, $x \in E_1$ and $\varepsilon > 0$ there exists $k_0 = k(x, \varepsilon)$ such that $f(w) - f_k(w) \in N(0; x, \varepsilon)$ for any $k \geq k_0$.

By density, we can select $j \in \mathbb{N}$ such that $\|x - x_j\| < \frac{\varepsilon}{4K_0}$. Take also $k \geq \max\{j, \frac{2}{\varepsilon}\}$ and $n \in \mathbb{N}$ satisfying $w \in B_{k,n}$. Therefore

$$\begin{aligned}
\|f(w)(x) - f_k(w)(x)\| &\leq \|f(w)(x_j) - f_k(w)(x_j)\| + \frac{\varepsilon(\|f(w)\| + \|f_k(w)\|)}{4K_0} \\
&\leq \|f(w)(x_j) - T_n(x_j)\| + \varepsilon/2 \\
&\leq \frac{1}{k} + \varepsilon/2 < \varepsilon.
\end{aligned}$$

Finally we use the fact that countably valued functions are SOT-limits of simple functions, to obtain that f is $SOT(E_1, \tilde{E}_2)$-approximable, and this completes the proof. ∎

1.5 Some examples

In this final section, we utilize the previous theorem to construct some examples of SOT-approximable functions.

Proposition 1.5.1 *Let $\Omega = [0,1]$, $\Sigma = \mathcal{B}$ the Borel σ-algebra, $E_1 = L^1([0,1])$ and $E_2 = C([0,1])$. For each $K : [0,1] \times [0,1] \to \mathbb{R}^+$ measurable and bounded we define $f_K : [0,1] \to \mathcal{L}(E_1, E_2)$ as*

$$f_K(t)(\phi)(s) = \int_0^s K(t,u)\phi(u)du, \quad \phi \in L^1([0,1]).$$

Then f_K is SOT-approximable.

Furthermore, if $\tilde{K} : [0,1] \to L^\infty([0,1])$ where $\tilde{K}(t)(u) = K(t,u)$ is assumed to be $\|\cdot\|$-approximable then f_K is also $\|\cdot\|$-approximable.

Proof: Observe that $f_K(t)$ is well defined. Indeed, for each t and $\phi \in L^1([0,1])$ we have that $K(t,\cdot)\phi(\cdot)$ belongs to $L^1([0,1])$. Therefore, $\int_0^s K(t,u)\phi(u)du$ is continuous. Invoking Theorem 1.4.3 we have to prove that f_K is WOT-approximable.

From the scalar-valued measurability of K, we know that K is pointwise limit of simple functions, that is, $K(s,t) = \lim_n K_n(s,t)$ where

$$K_n = \sum_{k=1}^{m_n} \alpha_{k,n} \chi_{A_{n,k} \times B_{n,k}}, \quad \alpha_{k,n} \in \mathbb{R}^+, A_{n,k}, B_{n,k} \in \Sigma.$$

Define the simple functions

$$f_n(t) = \sum_{k=1}^{m_n} \alpha_{k,n} \Phi_{n,k} \chi_{A_{n,k}}(t)$$

where $\Phi_{n,k}(\phi)(s) = \int_{[0,s]\cap B_{n,k}} \phi(u)du \in \mathcal{L}(E_1, E_2)$. Then, for each $\phi \in L^1([0,1])$ and each measure $\mu \in M([0,1]) = E_2^*$ we have that

$$t \to \langle f(t)(\phi), \mu \rangle = \lim_n \langle f_n(t)(\phi), \mu \rangle.$$

Assume that $\tilde{K} = \lim_n \tilde{K}_n$ in the $\|\cdot\|$-topology where $\tilde{K}_n = \sum_{k=1}^{m_n} \psi_{k,n} \chi_{A_{n,k}}$ for some $\psi_{k,n} \in L^\infty$ and $A_{n,k} \in \Sigma$. Denote

$$f_n(t)(\phi)(s) = \sum_{k=1}^{\infty} \left(\int_0^s \psi_{k,n}(u)\phi(u)du \right) \chi_{A_{n,k}}(t), \quad \phi \in L^1([0,1]).$$

Now we obtain the result using that

$$\|f_K(t) - f_n(t)\| \le \sup_{\|\phi\|_1 = 1} \int_0^1 |K(t,u)\phi(u) - K_n(t,u)\phi(u)| du \le \|\tilde{K}(t) - \tilde{K}_n(t)\|.$$

∎

In the next example, given $x^* \in E_1^*$ and $y \in E_2$ we use the notation $x^* \otimes y$ for the operator in $\mathcal{L}(E_1, E_2)$ given by $x \to \langle x^*, x \rangle y$. We recall that the dual of the projective tensor product $E_1 \hat{\otimes} F$ can be identified with $\mathcal{L}(E_1, E_2) = (E_1 \hat{\otimes} F)^*$ where $E_2 = F^*$ (the reader is referred to [20]) .

By using these special sequences of elementary operators, we shall construct now functions with values in the space of operators between general Banach spaces.

Proposition 1.5.2 *Let E_1 and $E_2 = F^*$ be Banach spaces. Let $\phi_n : [0,1] \to \mathbb{C}$ be a sequence of measurable functions satisfying the condition*

$$M = \sup_n |\phi_n(t)| < \infty. \tag{1.6}$$

Let $(y_n^*) \in B_{E_2}$ and $(x_n^*) \in B_{E_1^*}$ be such that

$$\sum_n |\langle x_n^*, x\rangle \langle y_n^*, y\rangle| < \infty, \quad x \in E_1, \quad y \in F. \tag{1.7}$$

Define $f : [0,1] \to \mathcal{L}(E_1, E_2)$ as

$$f(t) = \sum_n \phi_n(t) x_n^* \otimes y_n^*. \tag{1.8}$$

(i) f is weak*-approximable.

(ii) f is SOT-approximable whenever $\sum_n |\langle x_n^*, x\rangle| < \infty$ for all $x \in E_1$.

(iii) f is $\|\cdot\|$-approximable whenever $\sum_n \|x_n^*\|\|y_n^*\| < \infty$.

Proof: Hypothesis ensure that $f(t)$ is well defined, because the series

$$\langle f(t)(x), y\rangle = \sum_n \phi_n(t)\langle x_n^*, x\rangle\langle y_n^*, y\rangle$$

is an absolutely convergent series for each $t \in [0,1]$, $x \in E_1$ and $y \in F$.

Consider $f_N(t) = \sum_{n=1}^N \phi_n(t) x_n^* \otimes y_n^*$. Clearly f_N can be approximated by simple functions in the norm topology. Also, observe that $f_N(t) \to f(t)$ in the *weak**-topology since for each $\varepsilon > 0$ and $x \in E_1$ and $y \in F$ there exists N such that $\sum_{n=N}^\infty |\langle x_n^*, x\rangle||\langle y_n^*, y\rangle| <$

ε/M. This gives that

$$|\langle f_N(t)(x) - f(t)(x), y\rangle| \leq M \sum_{n=N+1}^{\infty} |\langle x_n^*, x\rangle||\langle y_n^*, y\rangle| < \varepsilon.$$

Since we can get

$$\|f_N(t)(x) - f(t)(x)\| \leq M \sum_{n=N+1}^{\infty} |\langle x_n^*, x\rangle|\|y_n^*\|$$

and

$$\|f_N(t) - f(t)\| \leq M \sum_{n=N+1}^{\infty} \|x_n^*\|\|y_n^*\|$$

from the assumptions in (ii) and (iii) respectively, we get that $f_N(t) \to f(t)$ in the SOT and $\|\cdot\|$ topologies in each case. ∎

Corollary 1.5.3 *Let $1 < p < \infty$ and (e_n) the canonical basis of ℓ^p and define $f(t) = \sum_n (e_n \otimes e_n)e^{int} \in \mathcal{L}(\ell^p, \ell^p)$. Then f is SOT-approximable but not $\|\cdot\|$-approximable.*

Proof: By using Theorem 1.4.3, it is enough to prove that f is WOT-approximable (or $weak^*$-approximable, which is the same in this case since ℓ^p is a reflexive space). Observe that conditions (1.6) and (1.7) hold in our case, and then the result follows directly applying Proposition 1.5.2.

The range is not separable though, since $\|f(t)-f(s)\| = \sup_n |e^{int}-e^{ins}| \geq 1$ whenever $t \neq s$. Therefore f is not $\|\cdot\|$-approximable. ∎

Chapter 2

Schur product for matrices with operator entries

"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

—Emil Artin.

2.1 Preliminaries

In this preliminary section we shall present the notation, definitions and basic concepts that will be necessary to develop this chapter, which is the first one of a set of three chapters that will mainly be focused on the topic of matricial harmonic analysis for matrices with operator entries.

Throughout the chapter, X, Y and E will be complex Banach spaces, and H will denote a separable complex Hilbert space with ortonormal basis (e_n). The notation X^* will be used for the dual space. As usual, $\mathcal{L}(X, Y)$ stands for the space of bounded linear operators, and in the case where $X = Y$, we shall denote $\mathcal{B}(X) = \mathcal{L}(X, X)$.

More standard notation that will be used: $\ell^2(E)$, $C(\mathbb{T}, E)$, $L^p(\mathbb{T}, E)$ or $M(\mathbb{T}, E)$ will stand for the space of sequences $\mathbf{z} = (z_n)$ in E such that $\|\mathbf{z}\|_{\ell^2(E)} = (\sum_{n=1}^{\infty} \|z_n\|^2)^{1/2} < \infty$, the space of E-valued continuous functions, the space of strongly measurable functions from $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ into E such that $\|f\|_{L^p(\mathbb{T},E)} = (\int_0^{2\pi} \|f(e^{it})\|^p \frac{dt}{2\pi})^{1/p} < \infty$ for $1 \leq p \leq \infty$ (with the usual modification for $p = \infty$) and the space of regular vector-valued measures of bounded variation, respectively. As usual, for $E = \mathbb{C}$ we shall write ℓ^2, $C(\mathbb{T})$, $L^p(\mathbb{T})$ and $M(\mathbb{T})$.

If we take two matrices $A = (\alpha_{k,j})_{k,j}$, $B = (\beta_{k,j})_{k,j}$ of the same size, with entries in the complex or real field, their Hadamard product is just their element-wise product, that is:

$$A * B = (\alpha_{k,j} \cdot \beta_{k,j})_{k,j}.$$

It was Schur who provided the initial studies about its properties, that is why it is widely known also as the "Schur product". Horn, in 1990, gave a profound insight on this product (the reader is referred to [34]).

This operation endows the space $\mathcal{B}(\ell^2)$ with a structure of Banach algebra, a fact that was originally proved by J. Schur. To see a proof of this result, we refer the reader to [7, Proposition 2.1] or [44, Theorem 2.20].

Theorem 2.1.1 *(Schur, [51]) If $A = (\alpha_{kj}) \in \mathcal{B}(\ell^2)$ and $B = (\beta_{kj}) \in \mathcal{B}(\ell^2)$ then $A * B \in \mathcal{B}(\ell^2)$. Moreover*

$$\|A * B\|_{\mathcal{B}(\ell^2)} \leq \|A\|_{\mathcal{B}(\ell^2)} \|B\|_{\mathcal{B}(\ell^2)}.$$

A recurring definition in this area is the notion of Schur multiplier.

Definition 2.1.2 *We say that a matrix $A = (\alpha_{kj})$ is a Schur multiplier whenever $A * B \in \mathcal{B}(\ell^2)$ for any $B \in \mathcal{B}(\ell^2)$.*

The space of multipliers from $\mathcal{B}(\ell^2)$ to $\mathcal{B}(\ell^2)$ will be denoted by $\mathcal{M}(\ell^2)$. For the study of Schur multipliers, we refer the reader to [7, 44].

There is a particular type of matrices that will prove to be very important specially when it comes to the identification between spaces of matrices and spaces of functions. They are called "Toeplitz matrices", and are defined as follows.

Definition 2.1.3 *Let A be a matrix $A = (\alpha_{k,j})_{k,j}$. A is called a Toeplitz matrix if there exists a sequence of complex numbers $(\alpha_n)_{n \in \mathbb{Z}}$ such that $\alpha_{k,j} = \alpha_{j-k}$. This kind of matrix,* *constant on its diagonals, looks as follows:*

$$A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \\ \alpha_{-1} & \ddots & \ddots & \ddots & \ddots \\ \alpha_{-2} & \ddots & \ddots & \ddots & \ddots \\ \alpha_{-3} & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The set of Toeplitz matrices will be denoted by $\mathcal{T}$.

The study of Toeplitz matrices defining bounded operators or Schur multipliers goes back to work of Toeplitz in [57]. We refer the reader to [2, 7, 44] for recent proofs of the following classic results regarding Toeplitz matrices.

The first one is a theorem of Toeplitz that characterizes bounded Toeplitz matrices in terms of the defining sequence of the matrix.

Theorem 2.1.4 *(Toeplitz [57]) Let $A = (\alpha_{kj})$ be a Toeplitz matrix. Then $A \in \mathcal{B}(\ell^2)$ if and only if there exists $f \in L^\infty(\mathbb{T})$ such that $\alpha_{kj} = \hat{f}(j-k)$ for all $k, j \in \mathbb{N}$. Moreover $\|A\| = \|f\|_{L^\infty(\mathbb{T})}$.*

A similar theorem, by Bennett, characterizes the Toeplitz Schur multipliers in terms of their corresponding sequence.

Theorem 2.1.5 *(Bennett [7]) Let $A = (\alpha_{kj})$ be a Toeplitz matrix. Then $A \in \mathcal{M}(\ell^2)$ if and only if there exists $\mu \in M(\mathbb{T})$ such that $\alpha_{kj} = \hat{\mu}(j-k)$ for all $k, j \in \mathbb{N}$. Moreover $\|A\| = \|\mu\|_{M(\mathbb{T})}$.*

One of the goals in this chapter will be to formulate the analogues of the theorems above in the framework of matrices $\mathbf{A} = (T_{kj})$ with entries $T_{kj} \in \mathcal{B}(H)$. This objective will lead us to consider operator-valued measures. Therefore we shall need different notions and spaces from the theory of vector-valued measures and in this regard, we refer the reader to classical books like [21, 20], or to [8] for some new results in connection with Fourier analysis.

In the sequel, we utilize the standard notation for the scalar product in H, $\langle \cdot, \cdot \rangle$, and $\ll \cdot, \cdot \gg$ will be utilized to denote the scalar product in $\ell^2(H)$, where

$$\ll \mathbf{x}, \mathbf{y} \gg = \sum_{j=1}^{\infty} \langle x_j, y_j \rangle.$$

The notation $x\mathbf{e_j} = (0, \cdots, 0, x, 0, \cdots)$ will be used for the element in $\ell^2(H)$ in which $x \in H$ is located in the j-coordinate for $j \in \mathbb{N}$. Also, $c_{00}(H) = span\{x\mathbf{e_j} : x \in H, j \in \mathbb{N}\}$ is the space of sequences with values in H, with a finite number of non-zero terms.

Let us give now the definition of the space $\mathcal{B}(\ell^2(H))$, which will be the generalized version of the space $\mathcal{B}(\ell^2)$ in the framework of matrices with operator entries.

Definition 2.1.6 *Given a matrix* $\mathbf{A} = (T_{kj})$ *with entries* $T_{kj} \in \mathcal{B}(H)$ *and* $\mathbf{x} \in c_{00}(H)$ *we write* $\mathbf{A}(\mathbf{x})$ *for the sequence* $(\sum_{j=1}^{\infty} T_{kj}(x_j))_k$*. We say that* $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *if the map* $\mathbf{x} \to \mathbf{A}(\mathbf{x})$ *extends to a bounded linear operator in* $\ell^2(H)$*, that is*

$$\Big(\sum_{k=1}^{\infty} \| \sum_{j=1}^{\infty} T_{kj}(x_j) \|^2 \Big)^{1/2} \leq C \Big(\sum_{j=1}^{\infty} \|x_j\|^2 \Big)^{1/2}.$$

The norm in this space shall be

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{A}\mathbf{x}\|_{\ell^2(H)} \leq C \|\mathbf{x}\|_{\ell^2(H)}\}.$$

This is how $\mathbf{Ax}$ looks like in matricial notation:

$$\mathbf{Ax} = \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} & \cdots & \cdots \\ T_{2,1} & T_{2,2} & T_{2,3} & \cdots & \cdots \\ T_{3,1} & T_{3,2} & T_{3,3} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_j T_{1,j}(x_j) \\ \sum_j T_{2,j}(x_j) \\ \sum_j T_{3,j}(x_j) \\ \vdots \\ \vdots \end{pmatrix}.$$

The notion of Schur product that we will use in our context will be the following one. A first version of this definition of Schur product for block matrices appeared first in print in a paper by Horn, Mathias and Nakamura (see [35]).

Definition 2.1.7 *Given two matrices $\mathbf{A} = (T_{kj})$ and $\mathbf{B} = (S_{kj})$ with entries $T_{kj}, S_{kj} \in \mathcal{B}(H)$ we define their Schur product as*

$$\mathbf{A} * \mathbf{B} = (T_{kj}S_{kj}),$$

where $T_{kj}S_{kj}$ stands for the composition of the operators T_{kj} and S_{kj}.

In a natural way, we can define the notion of multiplier for this product. Since (contrary to the scalar-valued case) this product is not commutative, it is necessary to define the right multipliers and the left multipliers.

Definition 2.1.8 *Given a matrix $\mathbf{A} = (T_{kj})$, we say that $\mathbf{A}$ is a right Schur multiplier (respectively left Schur multiplier), to be denoted by $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ (respectively $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$), whenever $\mathbf{B} * \mathbf{A} \in \mathcal{B}(\ell^2(H))$ (respectively $\mathbf{A} * \mathbf{B} \in \mathcal{B}(\ell^2(H))$) for any $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. We shall write*

$$\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{B} * \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq C\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\}$$

and

$$\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{A} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \leq C\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\}.$$

Remark 2.1.9 *Denoting by $\mathbf{A}^*$ the adjoint matrix given by $S_{kj} = T^*_{jk}$ for all $k, j \in \mathbb{N}$, one easily sees that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ if and only if $\mathbf{A}^* \in \mathcal{B}(\ell^2(H))$ with $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|\mathbf{A}^*\|_{\mathcal{B}(\ell^2(H))}$ and also that $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ if and only if $\mathbf{A}^* \in \mathcal{M}_r(\ell^2(H))$ and $\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} = \|\mathbf{A}^*\|_{\mathcal{M}_r(\ell^2(H))}$.*

If X and Y are Banach spaces, we use the notation $X\hat{\otimes}Y$ for the projective tensor product, which shall be used frequently throughout this chapter. We refer the reader to [20, Chap.8], [49, Chap.2] or [18] for all possible results needed.

It is known that the dual of the projective tensor product of two spaces can be identified with a space of bounded and linear operators in the following way:

$$(X\hat{\otimes}Y)^* = \mathcal{L}(X, Y^*).$$

This identification will be key in some of the proofs, and to avoid misunderstandings, for each $T \in \mathcal{L}(X, Y^*)$, we shall write $\mathcal{J}T$ when T is considered as an element in $(X\hat{\otimes}Y)^*$. In other words, we write $\mathcal{J} : \mathcal{L}(X, Y^*) \to (X\hat{\otimes}Y)^*$ for the isometry given by $\mathcal{J}T(x \otimes y) = T(x)(y)$ for any $T \in \mathcal{L}(X, Y^*)$, $x \in X$ and $y \in Y$. In addition, given $x^* \in X^*$ and $y^* \in Y^*$, we write $\widetilde{x^* \otimes y^*}$ for the operator in $\mathcal{L}(X, Y^*)$ given by $\widetilde{x^* \otimes y^*}(z) = x^*(z)y^*$ for each $z \in X$.

For the most part, we shall restrict ourselves to the case $\mathcal{L}(X, Y^*) = \mathcal{B}(H)$, in other words, the case in which $X = Y^* = H$. Using the Riesz theorem, we can identify $Y = Y^* = H$. Hence, for $T, S \in \mathcal{B}(H)$ and $x, y \in H$, we shall use the following formulae

$$\langle T(x), y\rangle = \mathcal{J}T(x\otimes y), \tag{2.1}$$

$$(\widetilde{x \otimes y})(z) = \langle z, x\rangle y, \quad z \in H, \tag{2.2}$$

$$T(\widetilde{x \otimes y}) = (x \otimes \widetilde{(Ty)}), \quad (\widetilde{x \otimes y})T = (\widetilde{T^*x) \otimes y}, \tag{2.3}$$

$$\mathcal{J}(TS)(x \otimes y) = \mathcal{J}T(Sx \otimes y) = \mathcal{J}S(x \otimes T^*y). \tag{2.4}$$

The chapter includes (besides this preliminary part) four sections. The first one contains definitions and results regarding basic notions on vector-valued sequences and functions that will be useful in the sequel. Next section is focused on regular operator-valued measures, that will play a key role in the proofs to come. In the following section, we present necessary and sufficient conditions for a matrix $\mathbf{A}$ to belong to the space $\mathcal{B}(\ell^2(H))$, and we shall prove a version in our framework of Schur's theorem, namely we show that the Schur product endows $\mathcal{B}(\ell^2(H))$ with a Banach algebra structure in the context of matrices with operator entries. The last section is devoted to the study of Toeplitz matrices $\mathbf{A}$ with entries in $\mathcal{B}(H)$. One of the key results presented is the one that characterizes $\mathcal{T}\cap\mathcal{B}(\ell^2(H))$ as those matrices where $T_{kj}=\hat{\mu}(j-k)$ for certain regular operator-valued vector measure μ belonging to $V^\infty(\mathbb{T},\mathcal{B}(H))$ (see Definition 2.3.10 below). The other main result that we prove in this final section is an analogue of Bennett's Theorem: we shall show that $M(\mathbb{T},\mathcal{B}(H))\subseteq\mathcal{M}_r(\ell^2(H))\subseteq M_{SOT}(\mathbb{T},\mathcal{B}(H))$ where $M(\mathbb{T},\mathcal{B}(H))$ stands for the space of regular operator-valued measures with bounded variation and $M_{SOT}(\mathbb{T},\mathcal{B}(H))$ is defined, using the strong operator topology, as the space of those vector measures μ such that $\mu_x\in M(\mathbb{T},H)$ where $\mu_x(A)=\mu(A)(x)$ for any $x\in H$.

2.2 Operator-valued sequences and functions

Let us now define a variety of spaces of sequences of operators that shall play a role in the following sections, and see some relations between them.

Write $\ell^2_{weak}(\mathbb{N},\mathcal{B}(H))$ and $\ell^2_{weak}(\mathbb{N}^2,\mathcal{B}(H))$ for the space of sequences $\mathbf{T}=(T_n)\subset\mathcal{B}(H)$ and matrices $\mathbf{A}=(T_{kj})\subset\mathcal{B}(H)$ such that

$$\|\mathbf{T}\|_{\ell^2_{weak}(\mathbb{N},\mathcal{B}(H))}=\sup_{\|x\|=1,\|y\|=1}\left(\sum_{n=1}^{\infty}|\langle T_n(x),y\rangle|^2\right)^{1/2}<\infty$$

and

$$\|\mathbf{A}\|_{\ell^2_{weak}(\mathbb{N}^2,\mathcal{B}(H))}=\sup_{\|x\|=1,\|y\|=1}\left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|\langle T_{kj}(x),y\rangle|^2\right)^{1/2}<\infty.$$

The reader is invited to check that these spaces actually coincide with the ones appearing using notation in [19]. Clearly, $\ell^2(E)\subsetneq\ell^2_{weak}(E)$. In the case $\mathcal{B}(H)$ we can actually introduce certain spaces between $\ell^2(E)$ and $\ell^2_{weak}(E)$ by means of the strong operator topology.

Definition 2.2.1 *Given a sequence* $\mathbf{T}=(T_n)$ *and a matrix* $\mathbf{A}=(T_{kj})$ *with entries in*

$\mathcal{B}(H)$, we write

$$\|\mathbf{T}\|_{\ell^2_{SOT}(\mathbb{N},\mathcal{B}(H))} = \sup_{\|x\|=1} \left(\sum_{n=1}^{\infty} \|T_n(x)\|^2 \right)^{1/2} \tag{2.5}$$

and

$$\|\mathbf{A}\|_{\ell^2_{SOT}(\mathbb{N}^2,\mathcal{B}(H))} = \sup_{\|x\|=1} \left(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \|T_{kj}(x)\|^2 \right)^{1/2}. \tag{2.6}$$

We say that a sequence of operators $\mathbf{T} = (T_n)$ belongs to $\ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$ whenever $\|\mathbf{T}\|_{\ell^2_{SOT}(\mathbb{N},\mathcal{B}(H))} < \infty$. A matrix $\mathbf{A} = (T_{kj})$ is said to belong to $\ell^2_{SOT}(\mathbb{N}^2, \mathcal{B}(H))$ when $\|\mathbf{A}\|_{\ell^2_{SOT}(\mathbb{N}^2,\mathcal{B}(H))} < \infty$.

Remark 2.2.2 *It is easy to check that*

$$\ell^2(\mathbb{N}^2, \mathcal{B}(H)) \subsetneq \ell^2(\mathbb{N}, \ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H)) \subsetneq \ell^2_{SOT}(\mathbb{N}^2, \mathcal{B}(H)).$$

As usual, we shall use the notation $\varphi_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$, and given a complex Banach space E, we write $\mathcal{P}(\mathbb{T}, E) = span\{e\varphi_j : j \in \mathbb{Z}, e \in E\}$ for the E-valued trigonometric polynomials, $\mathcal{P}_a(\mathbb{T}, E) = span\{e\varphi_j : j \in \mathbb{N}, e \in E\}$ for the E-valued analytic polynomials.

It is well-known that $\mathcal{P}(\mathbb{T}, E)$ is a dense subset of $C(\mathbb{T}, E)$ and $L^p(\mathbb{T}, E)$ for $1 \leq p < \infty$. Also, we shall denote $H^2_0(\mathbb{T}, E) = \{f \in L^2(\mathbb{T}, E) : \hat{f}(k) = 0, k \leq 0\}$, where $\hat{f}(k) = \int_0^{2\pi} f(t)\overline{\varphi_k(t)}\frac{dt}{2\pi}$ for $k \in \mathbb{Z}$. Recall that $H^2_0(\mathbb{T}, E)$ coincides with the closure of $\mathcal{P}_a(\mathbb{T}, E)$ with the norm in $L^2(\mathbb{T}, E)$. Similarly $H^2_0(\mathbb{T}^2, E) = \{f \in L^2(\mathbb{T}^2, E) : \hat{f}(k, j) = 0, \quad k, j \leq 0\}$, where $\hat{f}(k, j) = \int_0^{2\pi}\int_0^{2\pi} f(t, s)\overline{\varphi_k(t)\varphi_j(s)}\frac{dt}{2\pi}\frac{ds}{2\pi}$ for $k, j \in \mathbb{Z}$.

It is now time to introduce some new spaces that shall be needed later on.

Definition 2.2.3 *Let $\mathbf{T} = (T_n) \subset \mathcal{B}(H)$ and $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$. We say that $\mathbf{T} \in \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ whenever*

$$\|\mathbf{T}\|_{\tilde{H}^2(\mathbb{T},\mathcal{B}(H))} = \sup_N \left(\int_0^{2\pi} \|\sum_{j=1}^{N} T_j\varphi_j(t)\|^2 \frac{dt}{2\pi} \right)^{1/2} < \infty.$$

We say that $\mathbf{A} \in \tilde{H}^2(\mathbb{T}^2, \mathcal{B}(H))$ *whenever*

$$\|\mathbf{A}\|_{\tilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))} = \sup_{N,M} \left(\int_0^{2\pi} \int_0^{2\pi} \| \sum_{j=1}^{N} \sum_{k=1}^{M} T_{kj} \varphi_j(t) \varphi_k(s) \|^2 \frac{dt}{2\pi} \frac{ds}{2\pi} \right)^{1/2} < \infty.$$

Remark 2.2.4 *Observe that* $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H)) \not\subseteq H_0^2(\mathbb{T}, \mathcal{B}(H))$.

In order to check it, let us consider $T_j = \widetilde{e_j \otimes e_j}$. *Then for any* $t \in [0, 2\pi)$ *and* $N \in \mathbb{N}$,

$$\left\| \sum_{j=1}^{N} (\widetilde{e_j \otimes e_j}) \varphi_j(t) \right\|_{\mathcal{B}(H)} = \sup_{\|x\|=1} \left\| \sum_{j=1}^{N} \langle x, e_j \rangle \varphi_j(t) e_j \right\|_H = 1.$$

Hence we have $\mathbf{T} = (e_j \otimes e_j)_j \in \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$.

However, since $\|T_j\| = 1$ *for all* j, *we have* $\lim_{j\to\infty} \|T_j\| = 1 \neq 0$, *which implies, by the Riemann-Lebesgue Lemma, that* $\mathbf{T} \notin L^1(\mathbb{T}, \mathcal{B}(H))$. *Therefore* $\mathbf{T} \notin H_0^2(\mathbb{T}, \mathcal{B}(H))$, *as desired.*

Remark 2.2.5

$$\ell^2(\mathbb{N}, \mathcal{B}(H)) = \{(T_k)_k \subseteq \mathcal{B}(H) \ / \ \sup_{\|x_k\|=1} (\sum_k \|T_k(x_k)\|^2)^{\frac{1}{2}} < \infty\}.$$

Clearly, $\sup_{\|x_k\|=1} \sum_k \|T_k(x_k)\|^2 \leq \sum_k \|T_k\|^2$. *To check the other inequality, note that given* $\varepsilon > 0$, *by using the definition of supremum we can find for each* j *some unitary* x_j *such that* $\|T_j\|^2 < \|T_j(x_j)\|^2 + \frac{\varepsilon}{2^j}$,*and therefore*

$$\sum_j \|T_j\|^2 < \sum_j \|T_j(x_j)\|^2 + \varepsilon \sum_j \frac{1}{2^j} = \sum_j \|T_j(x_j)\|^2 + \varepsilon.$$

$$\Longrightarrow \sum_j \|T_j\|^2 < \sup_{\|x_j\|=1} \sum_j \|T_j(x_j)\|^2.$$

Proposition 2.2.6 *(i)* $\ell^2(\mathbb{N}, \mathcal{B}(H)) \subsetneq \ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$.

(ii) $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H)) \subsetneq \ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$ *and* $\tilde{H}^2(\mathbb{T}^2, \mathcal{B}(H)) \subsetneq \ell^2_{SOT}(\mathbb{N}^2, \mathcal{B}(H))$.

Proof: (i) To check the first content relation, just take into account that

$$\sum_k \|T_k(h)\|^2 \leq \sum_k \|T_k\|^2 \|h\|^2$$

To verify that the content is strict, consider $S_k : \ell^2 \to \ell^2$ with $S_k(x) = \langle e_k, x\rangle e_k$. This way, we have that $(S_k) \in \ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$, since given $x \in \ell^2$ in the unit ball

$$\sum_{k=0}^{\infty} \|S_k(x)\|^2 = \sum_{k=0}^{\infty} \|\langle e_k, x\rangle e_k\|^2 = \sum_{k=0}^{\infty} |x_k|^2 \leq 1 < \infty$$

On the other hand

$$\sum_{k=0}^{\infty} \|S_k\|^2 = \sum_{k=0}^{\infty} 1^2 = \infty,$$

and therefore $(S_k) \notin \ell^2(\mathbb{N}, \mathcal{B}(H))$.

(ii) Both inclusions are consequence Plancherel's theorem (which holds for Hilbert-valued functions). It suffices to see that there exists $\mathbf{T} \in \ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H)) \setminus \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ because choosing matrices with a single row we obtain also a counterexample for the other inclusion.

Consider $T_k : \ell^2 \to \ell^2$ with $T_k = \widetilde{e_k \otimes e_1}$, that is $T_k(x) = (x_k, 0, \cdots)$. Observe that

$$\sum_{k=1}^{\infty} \|T_k(x)\|^2 = \sum_{k=1}^{\infty} \|(x_k, 0, \cdots)\|^2 = \sum_{k=1}^{\infty} |x_k|^2 = \|x\| < \infty,$$

which gives that $(T_k) \in \ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$. Now, let us see that it doesn't belong to $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$. We have to study $\sup_N \int \|\sum_{k=1}^N T_k e^{ikt}\|^2 dt$.

$$\begin{aligned}\|\sum_{k=1}^{N} T_k e^{ikt}\| &= \sup_{\|x\|=1} \|\sum_{k=1}^{N} T_k(x) e^{ikt}\| = \sup_{\|x\|=1} \|\sum_{k=1}^{N} (x_k e^{ikt}, 0, \cdots)\| = \sup_{\|x\|=1} \|(\sum_{k=1}^{N} x_k e^{ikt}, 0, \cdots)\| \\ &= \sup_{\|x\|=1} |\sum_{k=1}^{N} x_k e^{ikt}| = \sup_{\|x\|=1} |\langle x, (e^{-ikt})_{k=1}^N\rangle| = \|(e^{-ikt})_{k=1}^N\|_2 = \sqrt{N},\end{aligned}$$

so we have that

$$\sup_N \int \|\sum_{k=1}^N T_k e^{ikt}\|^2 dt = \sup_N \int N dt = \infty.$$

∎

Proposition 2.2.7 *$\tilde{H}^2(\mathbb{T}, \mathcal{B}(H)) \nsubseteq \ell^2(\mathbb{N}, \mathcal{B}(H))$ and $\ell^2(\mathbb{N}, \mathcal{B}(H)) \nsubseteq \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$.*

Proof: This result can be deduced from Kwapien's Theorem, as we see now. However, we shall also provide examples of sequences of operators for each case.

Given a Banach space X, Kwapien's Theorem states that the following are equivalent:

1) $\exists C > 0$ such that $(\int \|\sum_{k=1}^N x_k e^{int}\|^2 dt)^{\frac{1}{2}} \leq C(\sum_{k=1}^N \|x_k\|^2)^{\frac{1}{2}}$.

2) $\exists C > 0$ such that $(\sum_{k=1}^N \|x_k\|^2)^{\frac{1}{2}} \leq C(\int \|\sum_{k=1}^N x_k e^{int}\|^2 dt)^{\frac{1}{2}}$.

3) X is isomorphic to a Hilbert space.

Therefore, by taking supremums on N, from 1 (resp. 2), we can deduce that if $(T_k)_k$ is in $\ell^2(\mathbb{N}, \mathcal{B}(H))$ (resp. $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$) then it can't be in $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ (resp. $\ell^2(\mathbb{N}, \mathcal{B}(H))$) because if that was the case, applying 1 (resp. 2), the space X would be isomorphic to a Hilbert space.

Remark 2.2.4 provided an example of a sequence of operators that belongs to $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ but not to $\ell^2(\mathbb{N}, \mathcal{B}(H))$. Finally, let us construct an example that will be in $\ell^2(\mathbb{N}, \mathcal{B}(H))$ but not in $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$.

Let us work in $(C([-\pi, \pi]), \|\cdot\|_\infty)$. Consider $\varphi_k(s) = \alpha_k e^{iks}$, with $\alpha_k \geq 0$ selected in such a way that $\sum_k \alpha_k^2 < \infty$ (for example, $\alpha_k = \frac{1}{k}$). We have,

$$\sum_{k=1}^N \|\varphi_k\|_\infty^2 = \sum_{k=1}^N \alpha_k^2 < \infty,$$

On the other hand,

$$\int \|\sum_{k=1}^N \varphi_k e^{ikt}\|_\infty^2 dt = \int \sup_s \left|\sum_{k=1}^N \alpha_k e^{ik(t+s)}\right|^2 dt = \sup_s \left|\sum_{k=1}^N \alpha_k e^{iks}\right|^2 \overset{\alpha_k \geq 0}{=} \left|\sum_{k=1}^N \alpha_k\right|^2,$$

and taking supremums with respect to N,

$$\sup_N \int \|\sum_{k=1}^{N} \varphi_k e^{ikt}\|_\infty^2 dt = \left|\sum_{k=1}^{\infty} \alpha_k\right|^2 \overset{\alpha_k=\frac{1}{k}}{=} \infty.$$

To translate this to the context of operators and get the desired example, it is enough to include the continuous functions in a space of operators, by means of multiplication, as follows:

$$\begin{aligned} G_\varphi : C(\mathbb{T}) &\longrightarrow \mathcal{L}\left(L^2(\mathbb{T}), L^2(\mathbb{T})\right) \\ \varphi &\longrightarrow \qquad G_\varphi(f) = \varphi.f \end{aligned}$$

Observe that the norm of the operator G_φ equals to the norm of the function φ, and therefore (G_{φ_k}) satisfies the desired properties. ∎

2.3 On regular vector measures

Let us recall some facts regarding vector measures that can be found in [20, 21]. Let us consider the measure space $(\mathbb{T}, \mathfrak{B}(\mathbb{T}), m)$ where $\mathfrak{B}(\mathbb{T})$ stands for the Borel sets over $\mathbb{T}$ and m for the Lebesgue measure on $\mathbb{T}$.

Definition 2.3.1 *Given a vector measure $\mu : \mathfrak{B}(\mathbb{T}) \to E$ and $B \in \mathfrak{B}(\mathbb{T})$, we define the variation and the semivariation of μ of the set B, to be denoted $|\mu|(B)$ and $\|\mu\|(B)$ respectively, as the quantities*

$$|\mu|(B) = \sup\{\sum_{A\in\pi} \|\mu(A)\|, A \in \mathfrak{B}(\mathbb{T}), \pi \textit{ finite partition of } B\}$$

and

$$\|\mu\|(B) = \sup\{|\langle e^*, \mu\rangle|(B) : e^* \in E^*, \|e^*\| = 1\},$$

respectively, where $\langle e^*, \mu\rangle(A) = e^*(\mu(A))$ for all $A \in \mathfrak{B}(\mathbb{T})$.

Of course $|\mu|(\cdot)$ becomes a positive measure on $\mathfrak{B}(\mathbb{T})$, whereas $\|\mu\|(\cdot)$ is only subadditive in general. We shall simplify the notation by using $|\mu| = |\mu|(\mathbb{T})$ and $\|\mu\| = \|\mu\|(\mathbb{T})$. In the case of dual spaces $E = F^*$ it is easy to see that $\|\mu\| = \sup\{|\langle \mu, f\rangle| : f \in F, \|f\| = 1\}$ where $\langle \mu, f\rangle(A) = \mu(A)(f)$.

In the following, we shall be considering regular vector measures, that is to say vector measures $\mu : \mathfrak{B}(\mathbb{T}) \to E$ such that for each $\varepsilon > 0$ and $B \in \mathfrak{B}(\mathbb{T})$ there exists a compact set K, an open set O such that $K \subset B \subset O$ with $\|\mu\|(O \setminus K) < \varepsilon$. We will use $\mathfrak{M}(\mathbb{T}, E)$ and $M(\mathbb{T}, E)$ to denote the space of regular Borel measures with values in E endowed with the norm given by the semi-variation and the space of regular Borel measures with values in E endowed with the norm given by the variation, respectively. Of course $M(\mathbb{T}, E) \subsetneq \mathfrak{M}(\mathbb{T}, E)$ when E is infinite dimensional.

It is a well known fact that the space $\mathfrak{M}(\mathbb{T}, E)$ can be identified with the space of weakly compact linear operators $T_\mu : C(\mathbb{T}) \to E$ and that $\|T_\mu\| = \|\mu\|$ (see [20, Chapter 6]). Hence, it is natural to define the k-Fourier coefficient of a measure m as the image of the continuous function e^{-ikt} by the associated operator T_μ. That is, for each $\mu \in \mathfrak{M}(\mathbb{T}, E)$ and $k \in \mathbb{Z}$ we can define (see [8]) the k-Fourier coefficient by

$$\hat{\mu}(k) = T_\mu(\varphi_{-k}). \tag{2.7}$$

The reader should note that one approach to describe measures in $M(\mathbb{T}, E)$ is the one that uses absolutely summing operators (see [19]), and the variation can be described as the norm in such space (see [20]). We shall not follow this procedure. Since we deal with either $E = \mathcal{B}(H)$ or $E = H$ we have at our disposal Singer's theorem (see, for example, [53, 54, 32]). This theorem, in the case of dual spaces $E = F^*$ asserts that $M(\mathbb{T}, E) = C(\mathbb{T}, F)^*$. In other words, there exists a bounded linear map $\Psi_\mu : C(\mathbb{T}, F) \to \mathbb{C}$ with $\|\Psi_\mu\| = |\mu|$ such that

$$\Psi_\mu(y\phi) = T_\mu(\phi)(y), \quad \phi \in C(\mathbb{T}), y \in F. \tag{2.8}$$

In particular, for $k \in \mathbb{Z}$ one has $\hat{\mu}(-k)(y) = \Psi_\mu(y\varphi_k)$ for each $y \in F$.

As mentioned above since $M(\mathbb{T}, \mathcal{L}(X, Y^*)) = C(\mathbb{T}, X\hat{\otimes}Y)^*$, for each $\mu \in M(\mathbb{T}, \mathcal{L}(X, Y^*))$ we can associate two operators T_μ and Ψ_μ. Of course the connection between them is given by the formula

$$T_\mu(\phi)(x)(y) = \Psi_\mu((x \otimes y)\phi), \quad \phi \in C(\mathbb{T}), x \in X, y \in Y. \tag{2.9}$$

In this context of operators, there is still one more possibility to be considered, by using the strong operator topology, namely $\Phi_\mu : C(\mathbb{T}, X) \to Y^*$ defined by

$$\Phi_\mu(f)(y) = \Psi_\mu(f \otimes y), \quad f \in C(\mathbb{T}, X), y \in Y, \tag{2.10}$$

where $f \otimes y(t) = f(t) \otimes y$.

In summary, given $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{L}(X, Y^*))$, we have three approaches to identify it with a linear operator defined on the corresponding spaces of polynomials, namely: $T_\mu : \mathcal{P}(\mathbb{T}) \to \mathcal{L}(X, Y^*)$, $\Psi_\mu : \mathcal{P}(\mathbb{T}, X\hat{\otimes}Y) \to \mathbb{C}$ and $\Phi_\mu : \mathcal{P}(\mathbb{T}, X) \to Y^*$, and these are defined by the following formulae

$$T_\mu\left(\sum_{j=-M}^{N} \alpha_j \varphi_j\right) = \sum_{j=-M}^{N} \alpha_j \hat{\mu}(-j), \quad N, M \in \mathbb{N}, \alpha_j \in \mathbb{C}, \tag{2.11}$$

$$\Psi_\mu\left(\sum_{j=-M}^{N}\left(\sum_{n=1}^{n_j} x_{jn}\right) \otimes \left(\sum_{m=1}^{m_j} y_{jm}\right)\varphi_j\right) = \sum_{j=-M}^{N}\left(\sum_{n=1}^{n_j}\sum_{m=1}^{m_j} \hat{\mu}(-j)(x_{jn})(y_{jm})\right), \tag{2.12}$$

$$\Phi_\mu\left(\sum_{j=-M}^{N} x_j \varphi_j\right) = \sum_{j=-M}^{N} \hat{\mu}(-j)(x_j), \quad N, M \in \mathbb{N}, x_j \in X. \tag{2.13}$$

When restricting to the case $X = Y^* = H$ we obtain the following connections between them.

$$\mathcal{J}T_\mu(\psi)(x \otimes y) = \Psi_\mu((x \otimes y)\psi) = \langle \Phi_\mu(x\psi), y \rangle, \quad \psi \in \mathcal{P}(\mathbb{T}), x, y \in H. \tag{2.14}$$

Definition 2.3.2 *Given $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{L}(X, Y^*))$ and $x \in X$, we define μ_x as the Y^*-valued measure given by*

$$\mu_x(A) = \mu(A)(x), \quad A \in \mathfrak{B}(\mathbb{T}).$$

It is elementary to see that μ_x is a regular measure because one can associate to it the weakly compact operator $T_{\mu_x} = \delta_x \circ T_\mu : C(\mathbb{T}) \to Y^*$ where δ_x stands for the operator $\delta_x : \mathcal{L}(X, Y^*) \to Y^*$ given by $\delta_x(T) = T(x)$ for $T \in \mathcal{L}(X, Y^*)$.

If $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$, $k \in \mathbb{Z}$ and $x, y \in H$ then $\mu_x \in \mathfrak{M}(\mathbb{T}, H)$,

$$\langle \mu_x(A), y \rangle = \mathcal{J}\mu(A)(x \otimes y), \quad A \in \mathfrak{B}(\mathbb{T}) \tag{2.15}$$

and

$$\langle \hat{\mu}(k)(x), y \rangle = \langle \hat{\mu}_x(k), y \rangle = \mathcal{J}\hat{\mu}(k)(x \otimes y). \tag{2.16}$$

In the case $E = \mathcal{B}(H)$, a new space of measures appears. It is the space of SOT-measures, which has great importance when analyzing multipliers, as we shall see throughout the chapter.

Definition 2.3.3 *Let $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. We say that $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ if $\mu_x \in M(\mathbb{T}, H)$ for any $x \in H$. We write the norm in this space as*

$$\|\mu\|_{SOT} = \sup\{|\mu_x| : x \in H, \|x\| = 1\}.$$

Proposition 2.3.4 *$M(\mathbb{T}, \mathcal{B}(H)) \subsetneq M_{SOT}(\mathbb{T}, \mathcal{B}(H)) \subsetneq \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$.*

Proof: The inclusions between these spaces follow from the inequalities

$$|\langle \mu(A)(x), y \rangle| \leq \|\mu(A)(x)\|\|y\| \leq \|\mu(A)\|\|x\|\|y\|$$

which lead to

$$|\langle \mu_x, y \rangle| \leq |\mu_x|\|y\| \leq |\mu|\|x\|\|y\|$$

and the corresponding embeddings with norm 1 trivially follow.

Let us see now that these relations of content are strict. Let $H = \ell^2$. We shall find measures $\mu_1 \in M_{SOT}(\mathbb{T}, \mathcal{B}(H)) \setminus M(\mathbb{T}, \mathcal{B}(H))$ and $\mu_2 \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H)) \setminus M_{SOT}(\mathbb{T}, \mathcal{B}(H))$. Both measures will be constructed by means of a similar procedure. Let $y_0 \in H$ with $\|y_0\| = 1$ and select a Hilbert-valued regular measure ν with $|\nu| = \infty$ (for instance, take an example of a Pettis integrable, but not Bochner integrable function $f : \mathbb{T} \to H$ given by $t \to (f_n(t))_n$ and $\nu(A) = (\int_A f_n(t) \frac{dt}{2\pi})_n$ for $A \in \mathfrak{B}(\mathbb{T})$). Denote by $T_\nu : C(\mathbb{T}) \to H$ the bounded (and hence weakly compact) operator that is associated to the measure ν, with $\|T_\nu\| = \|\nu\|$.

Now we define μ_1 and μ_2 as follows:

$$\mu_1(A)(x) = \langle x, \nu(A) \rangle y_0, \quad A \in \mathfrak{B}(\mathbb{T})$$

and

$$\mu_2(A)(x) = \langle x, y_0 \rangle \nu(A), \quad A \in \mathfrak{B}(\mathbb{T}).$$

In other words, if $J_y : H \to \mathcal{B}(H)$ and $I_y : H \to \mathcal{B}(H)$ are the operators

$$J_y(x)(z) = \langle z, x \rangle y, \quad I_y(x)(z) = \langle x, y \rangle z, \quad x, y, z \in H,$$

then we have that $T_{\mu_1} = J_{y_0} T_\nu$ and $T_{\mu_2} = I_{y_0} T_\nu$ are compositions of a weakly compact operator with a continuous operator, therefore they are weakly compact. This gives that $\mu_1, \mu_2 \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$.

Note that $|(\mu_1)_x| = |\langle x, \nu \rangle|$ and $|(\mu_2)_x| = |\langle x, y_0 \rangle||\nu|, \quad x \in H$. Hence, we have that

$$\|\mu_1\|_{SOT} = \|\nu\|, \quad \|\mu_2\|_{SOT} = |\nu|.$$

Finally, observe that $\|\mu_1(A)\|_{\mathcal{B}(H)} = \|\nu(A)\|_H$, and therefore $|\mu_1| = |\nu|$, which gives the results we were looking for. ∎

Another special measure that can be considered when the range is a space of bounded and linear operators is the adjoint measure.

Definition 2.3.5 *Let $\mu : \mathfrak{B}(\mathbb{T}) \to \mathcal{L}(X, Y^*)$ be a vector measure. We define "the adjoint measure" $\mu^* : \mathfrak{B}(\mathbb{T}) \to \mathcal{L}(Y, X^*)$ by the formula*

$$\mu^*(A)(y)(x) = \mu_x(A)(y), \quad A \in \mathfrak{B}(\mathbb{T}), x \in X, y \in Y. \tag{2.17}$$

Observe that in the case $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ with the identification $Y^* = H$, one clearly has that

$$\langle x, \mu^*(A)(y)\rangle = \langle \mu(A)(x), y\rangle, \quad A \in \mathfrak{B}(\mathbb{T}), x, y \in H. \tag{2.18}$$

Remark 2.3.6 *μ^* belongs to $\mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ (respect. $M(\mathbb{T}, \mathcal{B}(H))$) if and only if the original measure μ belongs to $\mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ (respect. $M(\mathbb{T}, \mathcal{B}(H))$). Furthermore, $\|\mu\| = \|\mu^*\|$ (respect.$|\mu| = |\mu^*|$).*

These assertions follow using that $T_{\mu^*}(\phi) = (T_\mu(\phi))^*$ for any $\phi \in C(\mathbb{T})$ and $\|\mu(A)\| = \|\mu^*(A)\|$ for any $A \in \mathfrak{B}(\mathbb{T})$.

The following result shows how to describe the norm in the space $M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ by using the adjoint measure.

Proposition 2.3.7 *Let $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. Then $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ if and only if $\Phi_{\mu^*} \in \mathcal{L}(C(\mathbb{T}, H), H)$. Moreover $\|\mu\|_{SOT} = \|\Phi_{\mu^*}\|$.*

Proof: By definition, we know that $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ if and only if the operator $S_\mu(x) = \mu_x$ is well defined and belongs to $\mathcal{L}(H, M(\mathbb{T}, H))$. Moreover, $\|\mu\|_{SOT} = \|S_\mu\|$. Therefore the result will follow if we show that S_μ is the adjoint of Φ_{μ^*}. Recall that, identifying $H = H^*$, we have $\mu^* \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. Hence $\Phi_{\mu^*} : \mathcal{P}(\mathbb{T}, H) \to H$ is generated by linearity using

$$\Phi_{\mu^*}(x\varphi_k) = \widehat{\mu^*}(-k)(x) = \hat{\mu}(-k)^*(x), \quad x \in H, k \in \mathbb{Z}.$$

Therefore, if $k \in \mathbb{Z}$, $x, y \in H$, since $M(\mathbb{T}, H) = (C(\mathbb{T}, H))^*$, we have

$$S_\mu(y)(x\varphi_k) = \Psi_{\mu_y}(x\varphi_k) = \langle \widehat{\mu_y}(-k), x\rangle = \langle \hat{\mu}(-k)(y), x\rangle = \langle y, \Phi_{\mu^*}(x\varphi_k)\rangle.$$

By linearity we can extend to $\langle y, \Phi_{\mu^*}(x\phi)\rangle = S_\mu(y)(x\phi)$ for any polynomial ϕ and the density of $\mathcal{P}(\mathbb{T}, H)$ in $C(\mathbb{T}, H)$ gives the result and completes the proof. ∎

Definition 2.3.8 *Let $1 \leq p < \infty$. Given a linear and bounded operator $T : L^p(\mathbb{T}) \to X$, we can identify this operator with a regular measure μ_T with values in X as follows: $\mu_T(A) := T(\chi_A)$, $A \in \mathfrak{B}(\mathbb{T})$.*

Observe that if $T_C : C(\mathbb{T}) \to X$ denotes the restriction of T to $C(\mathbb{T})$, we can associate a measure μ_{T_C} with values in X^{**} such that $\mu_{T_C}(A)(x^*) = T_C^*(x^*)(A)$, $x^* \in X^*$, $A \in \mathfrak{B}(\mathbb{T})$. Since the inclusion map $i : C(\mathbb{T}) \hookrightarrow L^p(\mathbb{T})$ is weakly compact, we get that $T_C = T \circ i$ is a weakly compact operator and hence that associated measure is regular and actually takes values in X (see [20, Chapter 6]). Also, it coincides with the one in Definition 2.3.8, since for every $x^* \in X^*$ and $A \in \mathfrak{B}(\mathbb{T})$, we have

$$\langle \mu_T(A), x^*\rangle = \langle T(\chi_A), x^*\rangle = \langle \chi_A, T^*(x^*)\rangle = \langle \chi_A, i^* \circ T^*(x^*)\rangle = \langle \chi_A, T_C^*(x^*)\rangle = T_C^*(x^*)(A).$$

When the domain of the operator is not the space of continuous functions as in Proposition 2.3.7, but the space of integrable ones instead, we have this correspondence.

Remark 2.3.9 *$\Psi : L^1(\mathbb{T}, X) \to Y$ is continuous if and only if the associated measure, μ_Ψ belongs to $V^\infty(\mathbb{T}, \mathcal{L}(X, Y))$.*

To prove it, one has to take into account the following three facts: $L^1(\mathbb{T}, X) = L^1(\mathbb{T})\hat{\otimes}X$, $\mathcal{L}(L^1(\mathbb{T}), E) = V^\infty(\mathbb{T}, E)$ (see Definition 2.3.10) and that given three Banach spaces E_1, E_2, E_3, one has:

$$\mathcal{L}(E_1\hat{\otimes}E_2, E_3) = \mathcal{L}(E_1, \mathcal{L}(E_2, E_3)) \text{ (see [20])},$$

so combining everything, one has

$$\mathcal{L}(L^1(\mathbb{T}, X), Y) = \mathcal{L}(L^1(\mathbb{T})\hat{\otimes}X, Y) = \mathcal{L}(L^1(\mathbb{T}), \mathcal{L}(X, Y)) = V^\infty(\mathbb{T}, \mathcal{L}(X, Y)).$$

Let us now present some subspaces of regular measures. Some of them will play an important role in what follows, specially the space $V^\infty(\mathbb{T}, E)$.

Definition 2.3.10 *Let us write $V^\infty(\mathbb{T}, E)$ for the subspace of those measures $\mu \in \mathfrak{M}(\mathbb{T}, E)$ such that there exists $C > 0$ with*

$$\|\mu(A)\| \leq Cm(A), \quad A \in \mathfrak{B}(\mathbb{T}). \tag{2.19}$$

We define

$$\|\mu\|_\infty = \sup\left\{\frac{\|\mu(A)\|}{m(A)} : m(A) > 0\right\}.$$

It is clear that any $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ also belongs to $M(\mathbb{T}, \mathcal{B}(H))$ and it is absolutely continuous with respect to m. More generally, one can define the measures of finite p-variation.

Definition 2.3.11 *We say that a vector measure $\mu : \Sigma \to E$ has finite p-variation $(1 \leq p < \infty)$ if it satisfies*

$$|\mu|_p := \sup_\Pi \left\{ \left(\sum_{A\in\Pi} \frac{\|\mu(A)\|^p}{m(A)^{p-1}} \right)^{\frac{1}{p}} \right\} = \sup_\Pi \left\{ \| \sum_{A\in\Pi} \frac{\mu(A)}{m(A)} \chi_A \|_{L^p(m,X)} \right\} < \infty$$

where Π is a partition of the total set Ω in subsets of Σ, and it is assumed that $m(A) > 0$. The space of measures with finite p-variation is denoted by $V^p(\mathbb{T}, E)$.

It is a known fact that $V^p(\mathbb{T}, X^*)$ can be identified with the space $L^{p'}(\mathbb{T}, X)^*$. Also, we point out that in the case of a measure with values in operators, one can define the spaces of measures of finite SOT_p-variation and finite WOT_p-variation as those measures

$\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{L}(X, Y))$ satisfying

$$|\nu|_{SOT_p}(\Omega) := \sup_{\|x\|=1} \{|\nu_x|_p(\Omega)\} < \infty,$$

and

$$|\nu|_{WOT_p}(\Omega) := \sup_{\|x\|=1, \|y^*\|=1} \{|\nu_{x,y^*}|_p(\Omega)\} < \infty,$$

respectively.

It is easy to check that if $p > 1$, a measure with finite p-variation is absolutely continuous with respect to m. Therefore, if Y has the Radon-Nikodym property, then $|\nu_x|_p(\Omega)$ in the previous definition would be just the L^p norm of the p-integrable density function associated to ν_x (due to Radon-Nikodym Theorem). Of course ν_{x,y^*} always has a density function associated since it is a scalar measure.

Remark 2.3.12 *Observe that in the case $p = \infty$, if we consider the space of measures with finite SOT_∞-variation, or those with finite WOT_∞-variation, the resulting spaces coincide with the space V^∞. We check this fact for the SOT case:*

$$|\mu|_{SOT_\infty} = \sup_{\|x\|=1} \sup_{m(A)>0} \frac{\|\mu(A)(x)\|}{m(A)} = \sup_{m(A)>0} \sup_{\|x\|=1} \frac{\|\mu(A)(x)\|}{m(A)} = \sup_{m(A)>0} \frac{\|\mu(A)\|}{m(A)} = |\mu|_\infty.$$

Now, we prove a version of Proposition 2.3.7, where the domain of the operator is the space of 2-power integrable functions. We present it in a more general context (without the restriction $X = Y^* = H$), and also a different proof is given. This time, the identification is made with the space of measures with finite SOT_2-variation (that we can denote by V^2_{SOT}).

Corollary 2.3.13 *An operator $\Psi : L^2(\mathbb{T}, X) \to Y^*$ is continuous if and only if the operator $T_\Psi : L^2(\mathbb{T}) \to \mathcal{L}(X, Y^*)$ (defined by $T_\Psi(\varphi)(x) = \Psi(\varphi x)$) satisfies that its associated measure μ_Ψ has its adjoint measure in the space $V^2_{SOT}(\mathbb{T}, \mathcal{L}(Y^{**}, X^*))$, that is,*

$\sup_{\|y^{**}\|=1} |(\mu_\Psi^*)_{y^{**}}|_2 < \infty$. *Moreover,*

$$\|\Psi\| = \sup_{\|y^{**}\|=1} |(\mu_\Psi^*)_{y^{**}}|_2 < \infty.$$

Proof: First of all, starting from a continuous operator $\Psi : L^2(\mathbb{T}, X) \to Y^*$, we define another operator $T : L^2(\mathbb{T}) \longmapsto \mathcal{L}(X, Y^*)$ as follows:

$$\begin{array}{rcl} T : L^2(\mathbb{T}) & \longmapsto & \mathcal{L}(X, Y^*) \\ \varphi & \longmapsto & T(\varphi)(x) = \Psi(\varphi x). \end{array}$$

We can denote its associated measure by μ_Ψ. The question now is which variation corresponds to μ_Ψ to obtain the norm of Ψ. We recall the following equivalence of continuity for operators.

$$T : E \to F \text{ is continuous} \quad \Leftrightarrow \quad T_{y^*} : E \to \mathbb{C} \text{ is continous } \forall y^* \in Y^*$$

where $T_{y^*}(x) = \langle T(x), y^* \rangle$, and $\|T\| = \sup_{\|y^*\|=1} \|T_{y^*}\|_{E^*}$.

Applying this to our general case, we have that $\Psi : L^2(\mathbb{T}, X) \to Y^*$ is continuous if and only if $\Psi_{y^{**}} \in (L^2(\mathbb{T}, X))^*$ $\forall y^{**} \in Y^{**}$ and $\sup_{\|y^{**}\|=1} \|\Psi_{y^{**}}\| < \infty$. Taking into account that $(L^2(\mathbb{T}, X))^* = V^2(\mathbb{T}, X^*)$, this leads to obtain that the continuity of $\Psi : L^2(\mathbb{T}, X) \to Y^*$ is equivalent to the fact that $\nu_{y^{**}} \in V^2(\mathbb{T}, X^*)$ $\forall y^{**} \in Y^{**}$ with $\sup_{\|y^{**}\|=1} |\nu_{y^{**}}|_2 < \infty$.

Let $\nu_{y^{**}}$ the measure associated to $\Psi_{y^{**}}$. Let us find out the relation with μ_Ψ. In order to do that, consider now the adjoint measure of μ_Ψ, $\mu_\Psi^* \in \mathcal{M}(\mathbb{T}, \mathcal{L}(Y^{**}, X^*))$, and consider also the measures obtained by its composition with elements of Y^{**}, $(\mu_\Psi^*)_{y^{**}} \in \mathcal{M}(\mathbb{T}, X^*)$. We shall see that, in fact, $(\mu_\Psi^*)_{y^{**}} = \nu_{y^{**}}$.

First, observe thaty $\nu_{y^{**}}$ acts as follows:

$$\begin{array}{rcl} L^2(\mathbb{T}) & \longmapsto & X^* \\ \varphi & \longmapsto & \Psi_{y^{**}}(\cdot\, \varphi) \end{array}$$

$$x \to \Psi_{y^{**}}(x\varphi) = y^{**}(\Psi(x\varphi)).$$

On the other hand, $(\mu_\Psi^*)_{y^{**}}$ acts in the following way:

$$\begin{array}{ccl} L^2(\mathbb{T}) & \longmapsto & X^* \\ \varphi & \longmapsto & (T_{\mu_\Psi(\varphi)})^*(y^{**}) \end{array}$$

$$x \to (T_{\mu_\Psi(\varphi)})^*(y^{**})(x) \overset{\text{Adjoint operator}}{=} y^{**}((T_{\mu_\Psi(\varphi)}(x)).$$

And looking back at the beginning of the proof, we see that $y^{**}(T_{\mu_\Psi(\varphi)}(x)) = y^{**}(\Psi(x\varphi))$, and therefore $(\mu_\Psi^*)_{y^{**}}$ and $\nu_{y^{**}}$ coincide. Therefore, we have proved that the continuity of $\Psi : L^2(\mathbb{T}, X) \to Y^*$ implies that $(\mu_\Psi^*)_{y^{**}} \in V^2(\mathbb{T}, X^*)$, $\forall y^{**} \in Y^{**}$ and with $\sup_{\|y^{**}\|=1} |(\mu_\Psi^*)_{y^{**}}|_2 < \infty$.

Let us prove the converse. Assume by hypothesis that we have $T_\mu : L^2(\mathbb{T}) \to \mathcal{L}(X, Y^*)$ such that $T_{\mu^*} : L^2(\mathbb{T}) \to \mathcal{L}(Y^{**}, X^*)$ has its associated measure with finite SOT_2-variation. First, we define

$$\begin{array}{rcl} \Psi_\mu : L^2(\mathbb{T}, X) & \longmapsto & Y^* \\ \varphi x & \longmapsto & \Psi_\mu(\varphi x) = T_\mu(\varphi)(x) \end{array}$$

only for functions of the form φx, where $\varphi \in L^2(\mathbb{T}), x \in X$.

It is clear that by linearity, a polynomial $\sum_{n=1}^M \varphi_n x_n$, (with $\varphi_n \in L^2(\mathbb{T})$, $x_n \in X$), is mapped by Ψ_μ to $\sum_{n=1}^M T_\mu(\varphi_n)(x_n)$. Let us check that

$$\|\sum_{n=1}^M T_\mu(\varphi_n)(x_n)\|_{Y^*} \leq C\|\sum_{n=1}^M \varphi_n x_n\|_{L^2(\mathbb{T},X)},$$

and then, by using the density of the polynomials in $L^2(\mathbb{T}, X)$, we will obtain the continuity of Ψ_μ.

$$\begin{aligned}
\|\sum_{n=1}^{M} T_\mu(\varphi_n)(x_n)\|_{Y^*} &= \sup_{\|y^{**}\|=1} \left| y^{**}\left(\sum_{n=1}^{M} T_\mu(\varphi_n)(x_n)\right)\right| = \sup_{\|y^{**}\|=1}\left|\sum_{n=1}^{M} y^{**}\left(T_\mu(\varphi_n)\right)(x_n)\right| \\
&= \sup_{\|y^{**}\|=1}\left|\sum_{n=1}^{M} \left(T_\mu(\varphi_n)\right)^*(y^{**})(x_n)\right| = \sup_{\|y^{**}\|=1}\left|\sum_{n=1}^{M} T_{\mu^*}(\varphi_n)(y^{**})(x_n)\right| \\
&\overset{\substack{\Phi \text{ denotes the} \\ \text{associated functional}}}{=} \sup_{\|y^{**}\|=1}\left|\sum_{n=1}^{M} \Phi_{\mu,y^{**}}(\varphi_n x_n)\right| = \sup_{\|y^{**}\|=1}\left|\Phi_{\mu,y^{**}}(\sum_{n=1}^{M}\varphi_n x_n)\right| \\
&\leq \sup_{\|y^{**}\|=1} \|\Phi_{\mu,y^{**}}\|_{(L^2(\mathbb{T},X))^*}\|\sum_{n=1}^{M}\varphi_n x_n\|_{L^2(\mathbb{T},X)}.
\end{aligned}$$

Using the SOT hypothesis, we have that for each y^{**},

$$\begin{array}{rcl} L^2(\mathbb{T}) & \longmapsto & X^* \\ \varphi & \longmapsto & T_{\mu^*}(\varphi)(y^{**}) \end{array}$$

is a measure with finite 2-variation. Therefore, it has a functional in $(L^2(\mathbb{T},X))^*$ associated, that we denoted above by $\Phi_{\mu,y^{**}}$, whose norm will be such variation. Therefore, we get that $\sup_{\|y^{**}\|=1}\|\Phi_{\mu,y^{**}}\|_{(L^2(\mathbb{T},X))^*}$ is finite, and this concludes the proof. ∎

Let us recall two possible descriptions of $V^\infty(\mathbb{T},E)$. One option is to look at $V^\infty(\mathbb{T},E) = \mathcal{L}(L^1(\mathbb{T}),E)$ (see [21, page 261]), that is to say that T_μ has a bounded extension to $L^1(\mathbb{T})$. With this point of view, a measure $\mu \in \mathfrak{M}(\mathbb{T},E)$ belongs to $V^\infty(\mathbb{T},E)$ if and only if

$$\|T_\mu(\psi)\| \leq C\|\psi\|_{L^1(\mathbb{T})}, \quad \psi \in C(\mathbb{T}). \tag{2.20}$$

Moreover $\|T_\mu\|_{L^1(\mathbb{T})\to E} = \|\mu\|_\infty$.

When the range is a dual space, $E = F^*$, we have another possibility of identification, as we mentioned before: $V^\infty(\mathbb{T},E) = L^1(\mathbb{T},F)^*$. In other words, $V^\infty(\mathbb{T},E)$ can be identified with the dual of the space of Bochner integrable functions. In this case a measure

$\mu \in V^\infty(\mathbb{T}, E)$ if and only if Ψ_μ has a bounded extension to $L^1(\mathbb{T}, F)^*$, that is

$$\|\Psi_\mu(p)\| \leq C\|p\|_{L^1(\mathbb{T},F)}, \quad p \in \mathcal{P}(\mathbb{T}, F). \tag{2.21}$$

Moreover $\|\Psi_\mu\|_{L^1(\mathbb{T},F)^*} = \|\mu\|_\infty$.

Although, as it was pointed out before, measures in $V^\infty(\mathbb{T}, \mathcal{B}(H))$ are absolutely continuous with respect to m, the reader should be aware that they might not have a Radon-Nikodym derivative in $L^1(\mathbb{T}, E)$ (see [20, Chap. 3]).

We shall give now an example of this situation in the case $E = \mathcal{B}(H)$.

Proposition 2.3.14 *Let $H = \ell^2$ and $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ such that $T_\mu \in \mathcal{L}(C(\mathbb{T}), \mathcal{B}(H))$ is defined as*

$$T_\mu(\phi) = \sum_{n=1}^\infty \hat{\phi}(n)\widetilde{e_n \otimes e_n}.$$

Then $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ with $\|\mu\|_\infty = 1$,

$$\hat{\mu}(k) = \begin{cases} \widetilde{e_k \otimes e_k} & k \geq 1 \\ 0, & k \leq 0 \end{cases}$$

but it does not have a Radon-Nikodym derivative in $L^1(\mathbb{T}, \mathcal{B}(H))$.

Proof: Let us show first that T_μ defines a continuous operator from $L^1(\mathbb{T})$ to $\mathcal{B}(H)$ with norm 1 (in other words, that $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ with $\|\mu\|_\infty = 1$). In such a case, using that the inclusion $C(\mathbb{T}) \to L^1(\mathbb{T})$ is weakly compact, one automatically gets that $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. Moreover, for $x = (\alpha_n) \in H$ and $y = (\beta_n) \in H$, we have

$$\begin{aligned} |\langle T_\mu(\phi)(x), y\rangle| &= \left|\sum_{n=1}^\infty \hat{\phi}(n)\alpha_n\beta_n\right| \\ &\leq \sup_{n\geq 1} |\hat{\phi}(n)|\|x\|\|y\| \\ &\leq \|\phi\|_{L^1(\mathbb{T})}\|x\|\|y\|. \end{aligned}$$

which means that $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ and $\|\mu\|_\infty \leq 1$. Finally, observe that $T_\mu(\varphi_j) = \widetilde{e_j \otimes e_j}$

and $\|\widetilde{e_j \otimes e_j}\|_{\mathcal{B}(H)} = 1$, which gives the equality of norms.

The assertion regarding the Fourier coefficients is immediate. It only remains to show that μ does not have a Bochner integrable Radon-Nikodym derivative. If that was the case, we would have $\hat{\mu}(k) = \hat{f}(k)$ for some $f \in L^1(\mathbb{T}, \mathcal{B}(H))$ which would imply that $\|\hat{f}(k)\| \to 0$ as $k \to \infty$. But we know that $\|\hat{\mu}(k)\| = 1$ for $k \geq 1$. Therefore μ can't have a Bochner integrable Radon-Nikodym derivative, and the proof is complete. ∎

We shall end this section with a known result that characterizes measures in $M(\mathbb{T}, F^*)$. It will be useful later on.

Lemma 2.3.15 *Let $E = F^*$ be a dual Banach space and $\mu \in \mathfrak{M}(\mathbb{T}, E)$. For each $0 < r < 1$, define*

$$P_r * \mu(t) = \sum_{k \in \mathbb{Z}} \hat{\mu}(k) r^{|k|} \varphi_k(t), \quad t \in [0, 2\pi). \tag{2.22}$$

Then

(i) $P_r * \mu \in C(\mathbb{T}, E)$ with $\|P_r * \mu\|_{C(\mathbb{T},E)} \leq \|\mu\| \frac{1+r}{1-r}$ for any $0 < r < 1$.

(ii) $\mu \in M(\mathbb{T}, E) \iff \sup_{0<r<1} \|P_r * \mu\|_{L^1(\mathbb{T},E)} < \infty$. Furthermore,

$$|\mu| = \sup_{0<r<1} \|P_r * \mu\|_{L^1(\mathbb{T},E)}.$$

Proof: (i) Notice that

$$\sum_{k \in \mathbb{Z}} |\hat{\mu}(k)| r^{|k|} \|\varphi_k\|_{C(\mathbb{T})} \leq \|T_\mu\| \left(1 + 2 \sum_{k=1}^{\infty} r^k \right) = \|\mu\| \frac{1+r}{1-r}.$$

Hence, the series in (2.22) is absolutely convergent in $C(\mathbb{T}, E)$, and (i) is obtained.

(ii) Assume that $\mu \in M(\mathbb{T}, E)$. In particular, one has that $|\mu| \in M(\mathbb{T})$ and

$$\int_0^{2\pi} \|P_r * \mu(t)\| \frac{dt}{2\pi} \leq \int_0^{2\pi} P_r * |\mu|(t) \frac{dt}{2\pi}.$$

Now, using the scalar-valued result, we obtain

$$\sup_{0<r<1} \|P_r * \mu\|_{L^1(\mathbb{T},E)} \leq \sup_{0<r<1} \|P_r * |\mu|\|_{L^1(\mathbb{T})} \leq \sup_{0<r<1} |\mu| \|P_r\|_{L^1(\mathbb{T})} = |\mu|.$$

Let us prove the converse. Assume that $\sup_{0<r<1} \|P_r * \mu\|_{L^1(\mathbb{T},E)} < \infty$. Since $L^1(\mathbb{T},E) \subseteq M(\mathbb{T},E) = C(\mathbb{T},F)^*$, applying the Banach-Alaoglu theorem allows us to find a sequence $r_n \to 1$ and a measure $\nu \in M(\mathbb{T},E)$ such that $P_{r_n} * \mu \to \nu$ in the w^*-topology. Now, selecting particular functions in $C(\mathbb{T},F)$ given by $y\varphi_{-k}$ for all $y \in F$ and $k \in \mathbb{Z}$, it follows that $\hat{\nu}(k) = \hat{\mu}(k)$. Therefore, $\mu = \nu$ and $\mu \in M(\mathbb{T},E)$. Observe that

$$|\mu| = \sup\{|\Psi_\mu(p)| : p \in \mathcal{P}(\mathbb{T},F), \|p\|_{C(\mathbb{T},F)} = 1\}.$$

If we take $p = \sum_{k=-M}^{N} y_k \varphi_k$, we have $P_r * p = \sum_{k=-M}^{N} y_k r^{|k|} \varphi_k$ and

$$\Psi_\mu(P_r * p) = \sum_{k=-M}^{N} \hat{\mu}(k)(y_k) r^{|k|} = \int_0^{2\pi} P_r * \mu(t)(p(t)) \frac{dt}{2\pi}.$$

Since $p = \lim_{r\to 1} P_r * p$ is in $C(\mathbb{T},F)$, we have

$$\begin{aligned} |\Psi_\mu(p)| &= \lim_{r\to 1} |\Psi_\mu(P_r * p)| \\ &\leq \sup_{0<r<1} \left| \int_0^{2\pi} P_r * \mu(t)(p(t)) \frac{dt}{2\pi} \right| \\ &\leq \sup_{0<r<1} \|P_r * \mu\|_{L^1(\mathbb{T},E)} \|p\|_{C(\mathbb{T},F)}. \end{aligned}$$

This gives the inequality $|\mu| \leq \sup_{0<r<1} \|P_r * \mu\|_{L^1(\mathbb{T},E)}$ and concludes the proof. ∎

2.4 Results on matrices with operator entries

In this section, we go back to the notation $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$ for the matrices with operator entries, and with $\mathbf{R_k}$ and $\mathbf{C_j}$ we will be referring to the k-row and the j-column matrix respectively, that is $\mathbf{R_k} = (T_{kj})_{j=1}^{\infty}$, $\mathbf{C_j} = (T_{kj})_{k=1}^{\infty}$. We shall use the notation

$A_{N,M}$ for

$$\mathbf{A}_{N,M}(s,t) = \sum_{k=1}^{M}\sum_{j=1}^{N} T_{kj}\overline{\varphi_j(s)}\varphi_k(t), \quad 0 \le t, s < 2\pi, \quad N, M \in \mathbb{N}. \tag{2.23}$$

Also, to each $\mathbf{x} = (x_j) \in \ell^2(H)$ we can associate the function $h_{\mathbf{x}}$ given by

$$h_{\mathbf{x}}(t) = \sum_{j=1}^{\infty} x_j \varphi_j(t), \quad t \in [0, 2\pi). \tag{2.24}$$

Remark 2.4.1 *Notice that* $\mathbf{A} \in \tilde{H}^2(\mathbb{T}^2, \mathcal{B}(H))$ *if and only if*

$$\sup_{N,M} \|\mathbf{A}_{N,M}\|_{L^2(\mathbb{T}^2,\mathcal{B}(H))} < \infty.$$

Also, observe that by Plancherel, $\mathbf{x} \in \ell^2(H)$ *if and only if* $h_{\mathbf{x}} \in H_0^2(\mathbb{T}, H)$*. Moreover*

$$\|\mathbf{x}\|_{\ell^2(H)} = \|h_{\mathbf{x}}\|_{H^2(\mathbb{T},H)}.$$

Proposition 2.4.2 *Let* $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$ *be a matrix with operator entries.*

(i) If $\mathbf{A} \in \ell^2_{SOT}(\mathbb{N}^2, \mathcal{B}(H))$ *then* $\mathbf{R_k}, \mathbf{C_j} \in \ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$ *for all* $k, j \in \mathbb{N}$*.*

(ii) If $\mathbf{A} \in \tilde{H}^2(\mathbb{T}^2, \mathcal{B}(H))$ *then* $\mathbf{C_j}, \mathbf{R_k} \in \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ *for all* $j, k \in \mathbb{N}$*.*

Proof: (i) By looking at Definition 2.6, it follows immediately.

(ii) Let $k' \in \mathbb{N}$, $M \in \mathbb{N}$ and $t \in [0, 2\pi)$. For $N \ge k'$, the orthogonality of the exponentials gives that

$$\sum_{j=1}^{N} T_{k'j}\varphi_j(t) = \int_0^{2\pi} \Big(\sum_{k=1}^{N}\sum_{j=1}^{M} T_{kj}\varphi_j(t)\varphi_k(s)\Big)\overline{\varphi_{k'}(s)}\frac{ds}{2\pi}.$$

This implies

$$\int_0^{2\pi} \left\|\sum_{j=1}^{N} T_{k'j}\varphi_j(t)\right\|^2 \frac{dt}{2\pi} \le \int_0^{2\pi}\int_0^{2\pi} \left\|\sum_{k=1}^{N}\sum_{j=1}^{M} T_{kj}\varphi_j(t)\varphi_k(s)\right\|^2 \frac{ds}{2\pi}\frac{dt}{2\pi}.$$

Hence $\|\mathbf{R_{k'}}\|_{\tilde{H}^2(\mathbb{T},\mathcal{B}(H))} \leq \|\mathbf{A}\|_{\tilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))}$. The same argument can be used to prove that $\|\mathbf{C_j}\|_{\tilde{H}^2(\mathbb{T},\mathcal{B}(H))} \leq \|\mathbf{A}\|_{\tilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))}$. ∎

2.4.1 Boundedness conditions and Schur's Theorem

In this subsection, different necessary and sufficient conditions for a matrix to be in $\mathcal{B}(\ell^2(H))$ will be given. Towards the end, we shall present the proof of Schur's theorem for matrices with operator entries.

First, let us characterize the elements of $\mathcal{B}(\ell^2(H))$ in terms of bilinear maps.

Definition 2.4.3 *Let* $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$. *Define* $B_{\mathbf{A}} : \mathcal{P}_a(\mathbb{T}, H) \times \mathcal{P}_a(\mathbb{T}, H) \to \mathbb{C}$ *as the map*

$$(h_{\mathbf{x}}, h_{\mathbf{y}}) \to \int_0^{2\pi}\int_0^{2\pi} \mathcal{J}\mathbf{A}_{N,M}(s,t)(h_{\mathbf{x}}(s) \otimes h_{\mathbf{y}}(t))\frac{ds}{2\pi}\frac{dt}{2\pi}, \tag{2.25}$$

where $h_{\mathbf{x}} = \sum_{j=1}^N x_j\varphi_j$ *and* $h_{\mathbf{y}} = \sum_{k=1}^M y_k\varphi_k$ *for* $x_j, y_k \in H$.

Proposition 2.4.4 *If* $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$ *then*

$$\ll \mathbf{A}(\mathbf{x}), \mathbf{y} \gg = B_{\mathbf{A}}(h_{\mathbf{x}}, h_{\mathbf{y}}), \quad \mathbf{x}, \mathbf{y} \in c_{00}(H). \tag{2.26}$$

In particular, $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *if and only if* $B_{\mathbf{A}}$ *extends to a bounded bilinear map on* $H_0^2(\mathbb{T}, H) \times H_0^2(\mathbb{T}, H)$. *Moreover,* $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|B_{\mathbf{A}}\|$.

Proof: Notice that that for $h_{\mathbf{x}} = \sum_{j=1}^N x_j\varphi_j$ and $h_{\mathbf{y}} = \sum_{k=1}^M y_k\varphi_k$ we have $y_k = \int_0^{2\pi} h_{\mathbf{y}}(t)\overline{\varphi_k(t)}\frac{dt}{2\pi}$ and $x_j = \int_0^{2\pi} h_{\mathbf{x}}(t)\overline{\varphi_j(s)}\frac{ds}{2\pi}$. Hence, we can write

$$\begin{aligned}
\sum_{k=1}^M \left\langle \sum_{j=1}^N T_{kj}x_j, y_k \right\rangle &= \int_0^{2\pi} \left\langle \sum_{k=1}^M \left(\sum_{j=1}^N T_{kj}x_j\right)\varphi_k(t), h_{\mathbf{y}}(t)\right\rangle \frac{dt}{2\pi} \\
&= \int_0^{2\pi} \left\langle \sum_{k=1}^M \left(\sum_{j=1}^N T_{kj}\varphi_k(t)\right)(x_j), h_{\mathbf{y}}(t)\right\rangle \frac{dt}{2\pi} \\
&= \int_0^{2\pi} \left\langle \int_0^{2\pi} \mathbf{A}_{N,M}(s,t)(h_{\mathbf{x}}(s))\frac{ds}{2\pi}, h_{\mathbf{y}}(t)\right\rangle \frac{dt}{2\pi}
\end{aligned}$$

$$= \int_0^{2\pi}\int_0^{2\pi} \mathcal{J}\mathbf{A}_{N,M}(s,t)(h_{\mathbf{x}}(s)\otimes h_{\mathbf{y}}(t))\frac{ds}{2\pi}\frac{dt}{2\pi}.$$

By taking supremums appropriately, the equality of norms follows. ∎

From Proposition 2.4.4 one can produce some sufficient conditions for $\mathbf{A}$ to belong to $\mathcal{B}(\ell^2(H))$.

Corollary 2.4.5 *If* $\mathbf{A} \in \tilde{H}^2(\mathbb{T}^2, \mathcal{B}(H)) \cup \ell^2(\mathbb{N}^2, \mathcal{B}(H))$ *then* $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *and* $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq \min\{\|\mathbf{A}\|_{\tilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))}, \|\mathbf{A}\|_{\ell^2(\mathbb{N}^2,\mathcal{B}(H))}\}$.

Proof: Let us assume first that $\mathbf{A} \in \ell^2(\mathbb{N}^2, \mathcal{B}(H))$. Then

$$| \ll \mathbf{A}(\mathbf{x}), \mathbf{y} \gg | \leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|T_{kj}\|\|x_j\|\|y_k\|$$

And therefore, a use of Cauchy-Schwarz's inequality in $\ell^2(\mathbb{N}^2)$ gives that

$$\begin{aligned} | \ll \mathbf{A}(\mathbf{x}), \mathbf{y} \gg | &\leq \|\mathbf{A}\|_{\ell^2(\mathbb{N}^2,\mathcal{B}(H))}\|(\|x_j\|\|y_k\|)\|_{\ell^2(\mathbb{N}^2)} \\ &= \|\mathbf{A}\|_{\ell^2(\mathbb{N}^2,\mathcal{B}(H))}\|\mathbf{x}\|\|\mathbf{y}\|. \end{aligned}$$

Now, suppose $\mathbf{A} \in \tilde{H}^2(\mathbb{T}^2, \mathcal{B}(H))$ and apply Cauchy-Schwarz in $L^2(\mathbb{T}^2)$

$$\begin{aligned} &\left|\int_0^{2\pi}\int_0^{2\pi} \mathcal{J}\mathbf{A}_{N,M}(s,t)(h_{\mathbf{x}}(s)\otimes h_{\mathbf{y}}(t))\frac{ds}{2\pi}\frac{dt}{2\pi}\right| \\ \leq\ & \|\mathbf{A}_{N,M}\|_{H_0^2(\mathbb{T}^2,\mathcal{B}(H))}\|h_{\mathbf{x}}\|_{H_0^2(\mathbb{T},H)}\|h_{\mathbf{y}}\|_{H_0^2(\mathbb{T},H)}. \end{aligned}$$

The result follows from Proposition 2.4.4 and Remark 2.4.1. ∎

In the following proposition, we give a sufficient condition better than $\mathbf{A} \in \ell^2(\mathbb{N}^2, \mathcal{B}(H))$, where the strong operator topology comes into play.

Proposition 2.4.6 *Let* $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$ *such that* $\mathbf{C_j} \in \ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$ *for all* $j \in \mathbb{N}$ *or* $\mathbf{R}_k^* \in \ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))$ *for all* $k \in \mathbb{N}$ *and satisfy*

$$\min\{\|(\mathbf{C_j})\|_{\ell^2(\mathbb{N},\ell^2_{SOT}(\mathbb{N},\mathcal{B}(H)))}, \|(\mathbf{R}_k^*)\|_{\ell^2(\mathbb{N},\ell^2_{SOT}(\mathbb{N},\mathcal{B}(H)))}\} = M < \infty.$$

Then $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *and* $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq M$.

Proof: Let $\mathbf{x}, \mathbf{y} \in \ell^2(H)$, we have

$$\begin{aligned} |\ll \mathbf{A}(\mathbf{x}), \mathbf{y} \gg| &\leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|y_k\| \left\| T_{kj}\left(\frac{x_j}{\|x_j\|}\right)\right\| \|x_j\| \\ &\leq \left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left\| T_{kj}\left(\frac{x_j}{\|x_j\|}\right)\right\|^2\right)^{1/2} \left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|y_k\|^2\|x_j\|^2\right)^{1/2} \\ &\leq \|\mathbf{x}\|_{\ell^2(H)}\|\mathbf{y}\|_{\ell^2(H)} \left(\sum_{j=1}^{\infty} \|\mathbf{C_j}\|^2_{\ell^2_{SOT}(\mathbb{N},\mathcal{B}(H))}\right)^{1/2}. \end{aligned}$$

Similar argument works with $\mathbf{R_k^*}$, which completes the proof. ∎

In the following we present some necessary conditions for $\mathbf{A}$ to belong to $\mathcal{B}(\ell^2(H))$. A first easy observation is that since $\ll \mathbf{A}(x\mathbf{e_j}), y\mathbf{e_k} \gg = \langle T_{kj}(x), y\rangle$, we have that if $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ then $\mathbf{A} \in \ell^\infty(\mathbb{N}^2, \mathcal{B}(H))$ and $\sup_{k,j} \|T_{kj}\| \leq \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}$.

Lemma 2.4.7 *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{B}(\ell^2(H))$. *Then*

$$(\mathbf{C_j})_j, (\mathbf{R}_k^*)_k \in \ell^\infty(\mathbb{N}, \ell^2_{SOT}(\mathbb{N}, \mathcal{B}(H))).$$

Proof: Since for each $\mathbf{y} \in \ell^2(H)$, $x, y \in H$ and $k, j \in \mathbb{N}$ we have

$$\ll \mathbf{A}(x\mathbf{e_j}), \mathbf{y} \gg = \ll \mathbf{C}_j(x), \mathbf{y} \gg$$

and

$$\ll \mathbf{A}(\mathbf{x}), y\mathbf{e_k} \gg = \ll \mathbf{x}, \mathbf{R}_k^*(y) \gg,$$

we clearly get that

$$\|\mathbf{C}_j\|_{\ell^2_{SOT}(\mathbb{N},\mathcal{B}(H))} = \sup_{\|x\|=1}\sup_{\|\mathbf{y}\|_{\ell^2(H)}=1} |\ll \mathbf{A}(x\mathbf{e_j}), \mathbf{y} \gg| \leq \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}.$$

A similar argument works to show that $\|\mathbf{R}_k^*\|_{\ell^2_{SOT}(\mathbb{N},\mathcal{B}(H))} \leq \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}$. ∎

The following proposition gives another necessary condition for boundedness that will be the main tool in the proof of the generealization of Schur's theorem.

Proposition 2.4.8 *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{B}(\ell^2(H))$. *Then*

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|T_{kj}x_j\|^2 \leq \|\mathbf{A}\|^2_{\mathcal{B}(\ell^2(H))} \sum_{j=1}^{\infty} \|x_j\|^2. \tag{2.27}$$

Proof: Let $\mathbf{x} \in \ell^2(H)$ and assume that $\sum_{j=1}^{\infty} \|x_j\|^2 = 1$.We use the notation $F_{\mathbf{x}} : [0, 2\pi] \to \ell^2(H)$ for the continuous function given by $F_{\mathbf{x}}(s) = \Big(x_j\varphi_j(s)\Big)$. It is clear that $\|\mathbf{x}\| = \|F_{\mathbf{x}}\|_{C(\mathbb{T},\ell^2(H))}$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|T_{kj}x_j\|^2 &= \sum_{k=1}^{\infty} \int_0^{2\pi} \left\| \sum_{j=1}^{\infty} T_{kj}x_j\varphi_j(s) \right\|^2 \frac{ds}{2\pi} \\
&= \int_0^{2\pi} \sum_{k=1}^{\infty} \left\| \sum_{j=1}^{\infty} T_{kj}x_j\varphi_j(s) \right\|^2 \frac{ds}{2\pi} \\
&= \int_0^{2\pi} \|\mathbf{A}(F_{\mathbf{x}}(s))\|^2 \frac{ds}{2\pi} \\
&\leq \|\mathbf{A}\|^2_{\mathcal{B}(\ell^2(H))} \int_0^{2\pi} \|F_{\mathbf{x}}(s)\|^2 \frac{ds}{2\pi} = \|\mathbf{A}\|^2_{\mathcal{B}(\ell^2(H))}.
\end{aligned}$$

This completes the proof. ∎

Now we can prove the extension of Schur's theorem to matrices with operator entries.

Theorem 2.4.9 *If* $\mathbf{A} = (T_{kj})$ *and* $\mathbf{B} = (S_{kj})$. *If* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\ell^2(H))$ *then* $\mathbf{A} * \mathbf{B} \in \mathcal{B}(\ell^2(H))$. *Moreover*

$$\|\mathbf{A} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \leq \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}.$$

Proof: It is enough to show that, if $\mathbf{x}, \mathbf{y} \in c_{00}(H)$, then

$$|\ll \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \gg| \leq \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\|\mathbf{x}\|\|\mathbf{y}\|. \tag{2.28}$$

Observe that as a consequence of Cauchy-Schwarz inequality, we have

$$\begin{aligned}
| \ll \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \gg | &= \left| \sum_{k=1}^{\infty} \left\langle \sum_{j=1}^{\infty} T_{kj} S_{kj}(x_j), y_k \right\rangle \right| \\
&= \left| \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle S_{kj}(x_j), T_{kj}^*(y_k) \rangle \right| \\
&\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \|T_{kj}^*(y_k)\| \|S_{kj}(x_j)\| \\
&\leq \left(\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \|T_{kj}^*(y_k)\|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \|S_{kj}(x_j)\|^2 \right)^{1/2}.
\end{aligned}$$

This estimate, combined with Proposition 2.4.8 (applied to $\mathbf{B}$ and $\mathbf{A}^*$) gives (2.28), since $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|\mathbf{A}^*\|_{\mathcal{B}(\ell^2(H))}$. The proof is completed. ∎

Let us make some remarks on the norm of certain submatrices. Given $S \subset \mathbb{N} \times \mathbb{N}$ and $\mathbf{A} = (T_{kj})$, we write $P_S\mathbf{A} = (T_{kj}\chi_S)$, that is the matrix with entries T_{kj} if $(k, j) \in S$ and 0 otherwise. We have already been working with some examples of these matrices. For instance, matrices with a single row, column or diagonal correspond to the sets $S = \{k\} \times \mathbb{N}$, $S = \mathbb{N} \times \{j\}$ and $D_l = \{(k, k + l) : k \in \mathbb{N}\}$ for $l \in \mathbb{Z}$, respectively. Moreover, the case of finite or upper (or lower) triangular matrices coincides with $P_S\mathbf{A}$ for $S = [1, N] \times [1, M] = \{(k, j) : 1 \leq k \leq N, 1 \leq j \leq M\}$ or $S = \Delta = \{(k, j) : j \geq k\}$ (or $S = \{(k, j) : j \leq k\}$), respectively.

It is a well known fact that the mapping $\mathbf{A} \to P_S\mathbf{A}$ is not continuous in $\mathcal{B}(H)$ for all sets S (for instance, we refer the reader to [44, Chap.2, Thm.2.19], where it is proved that $S = \Delta$, the triangle projection, is unbounded). However, there are cases where this actually holds true. It is clear that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ if and only if $\|\mathbf{A}\| = \sup_{N,M} \|P_{[1,N]\times[1,M]}\mathbf{A}\| < \infty$. This easily follows taking into account that to obtain the norm of $\mathbf{A}$ is is enough to consider vectors in c_{00}, and also noticing that

$$\ll P_{[1,N]\times[1,M]}\mathbf{A}(\mathbf{x}), \mathbf{y} \gg = \ll \mathbf{A}(P_N\mathbf{x}), P_M\mathbf{y} \gg,$$

where $P_N\mathbf{x}$ stands for the projection on the N-first coordinates of $\mathbf{x}$.

Let us compute the norm of some basic submatrices.

Corollary 2.4.10 *Let* $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$. *Then*

(i) $\|P_{\mathbb{N}\times\{j\}}\mathbf{A}\| = \|\mathbf{C_j}\|_{\ell^2_{SOT}(\mathbb{N},\mathcal{B}(H))}$ *for each* $j \in \mathbb{N}$.

(ii) $\|P_{\{k\}\times\mathbb{N}}\mathbf{A}\| = \|\mathbf{R}_k^*\|_{\ell^2_{SOT}(\mathbb{N},\mathcal{B}(H))}$ *for each* $k \in \mathbb{N}$.

(iii) $\|P_{D_l}\mathbf{A}\| = \sup_k \|T_{k,k+l}\|$ *for each* $l \in \mathbb{Z}$ *(where* $T_{k,k+l} = 0$ *whenever* $k + l \leq 0$*).*

Proof: (i) and (ii) are a trivial consequence of Lemma 2.4.7.

To see (iii), note that $(P_{D_l}\mathbf{A}(\mathbf{x}))_k = (T_{k,k+l}x_{k+l})_k$. Hence, we have that $\|P_{D_l}\mathbf{A}(\mathbf{x})\| \leq (\sup_k \|T_{k,k+l}\|)\|\mathbf{x}\|$. Since the other inequality always holds, the proof is complete. ∎

2.4.2 Some results on multipliers

By considering a matrix $\mathbf{B}$ as the sequence of its rows, $\mathbf{B} = (R_k^{\mathbf{B}})_k$, we define now two intersection subspaces of $\mathcal{B}(\ell^2(H))$.

Definition 2.4.11

$$B_{f,\ell^2}(\ell^2(H)) := \left\{\mathbf{B} \in \mathcal{B}(\ell^2(H)) \ / \ \mathbf{B} = (R_k^{\mathbf{B}})_k \in \ell^\infty\left(\mathbb{N}, \ell^2(\mathcal{B}(H))\right)\right\},$$

$$B_{f,\tilde{H}^2}(\ell^2(H)) := \left\{\mathbf{B} \in \mathcal{B}(\ell^2(H)) \ / \ \mathbf{B} = (R_k^{\mathbf{B}})_k \in \ell^\infty\left(\mathbb{N}, \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))\right)\right\}.$$

Observe that it is meaningful to define these spaces in the sense that one can find matrices in $\mathcal{B}(\ell^2(H))$ that don't fulfill the additional conditions that these spaces require. Let us see an example of this situation.

Example 2.4.12 *The following matrix* $\mathbf{A}$ *is in the space* $\mathcal{B}(\ell^2(H))$, *but neither its rows nor the rows of the adjoint matrix are in* $\ell^2(\mathbb{N}, \mathcal{B}(H)) \cup \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$.

$$\mathbf{A} = \begin{pmatrix} e_1 \otimes e_1 & e_1 \otimes e_2 & e_1 \otimes e_3 & \cdots & \cdots \\ e_2 \otimes e_1 & e_2 \otimes e_2 & e_2 \otimes e_3 & \cdots & \cdots \\ e_3 \otimes e_1 & e_3 \otimes e_2 & e_3 \otimes e_3 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Clearly, the rows are not in $\ell^2(\mathbb{N}, \mathcal{B}(H))$*, since each entry of the matrix has operator norm equal to* 1*. To check that the* k*-row is not in* $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ *either, simply observe that*

$$\|\sum_{j=1}^{N} e_k \otimes e_j(\mathbf{x}) e^{ijt}\|^2 = \|(x_k e^{it}, x_k e^{2it}, x_k e^{3it}, \cdots, x_k e^{iNt})\|^2 = N x_k^2.$$

For the adjoint matrix, one just has to take into account that $(e_i \otimes e_j)^* = e_j \otimes e_i$ *and apply the same argument.*

Finally, the matrix $\mathbf{A}$ *is in* $\mathcal{B}(\ell^2(H))$ *(actually, it is an isometry). Indeed,*

$$\|\mathbf{A}\mathbf{x}\|_{\ell^2(H)}^2 = \sum_j \|\sum_i e_j \otimes e_i(x_i)\|^2 = \sum_j \|(x_1^j, x_2^j, x_3^j, \cdots)\|^2 = \|\mathbf{x}\|_{\ell^2(H)}^2.$$

In these spaces, we could consider the norm as the minimum between the norm in $\mathcal{B}(\ell^2(H))$ and the norm given by the other space. Observe that both are algebras for the Schur product, since if two sequences of operators (T_i), (S_i) are in $\ell^2(\mathbb{N}, \mathcal{B}(H))$ (respectively $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$), the sequence that results from taking the composition of the operators, $(T_i \circ S_i)$, is in $\ell^2(\mathbb{N}, \mathcal{B}(H))$ (respectively $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$). This fact is completely immediate in the case $\ell^2(\mathbb{N}, \mathcal{B}(H))$. Regarding the case $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$, we include the proof here.

Lemma 2.4.13 $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ *is an algebra with the convolution. In other words, if* $F(t) = \sum_j T_j e^{2\pi ijt}$*,* $T_j \in \mathcal{B}(H)$*, and* $G(t) = \sum_j S_j e^{2\pi ijt}$*,* $S_j \in \mathcal{B}(H)$ *with* $(T_j)_j, (S_j)_j \in \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$*, then* $(T_j \circ S_j) \in \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$*.*

Proof:

$$\begin{aligned}
\int \|\sum_{j=1}^{N}(T_j \circ S_j)e^{2\pi ijt}\|^2 dt &= \int \| \int (\sum_j^N T_j e^{2\pi isj})(\sum_j^N S_j e^{2\pi(t-s)j}) ds\|^2 dt \\
&\leq \int (\int \|\sum_j^N T_j e^{2\pi isj}\| \|\sum_j^N S_j e^{2\pi(t-s)j}\| ds)^2 dt \\
&\leq \int (\int \|\sum_j^N T_j e^{2\pi ijs}\|^2 ds)(\int \|\sum_j^N S_j e^{2\pi i(t-s)j}\|^2 ds) dt \\
&= (\int \|\sum_j^N T_j e^{2\pi ijs}\|^2 ds) \cdot \int (\int \|\sum_j^N S_j e^{2\pi i(t-s)j}\|^2 ds) dt.
\end{aligned}$$

Taking supremums with respect to N and using the hypothesis, we get the result. ∎

The following theorem gives an interesting sufficient condition on the rows of a matrix for it to become a Schur multiplier.

Theorem 2.4.14 *Let* $\mathbf{A} = (T_{k,j})$ *and* $\mathbf{B} = (S_{k,j})$ *be matrices with entries in* $\mathcal{B}(H)$. *If* $\mathbf{B} \in \mathcal{B}(\ell^2(H))$ *and* $\mathbf{A} \in \ell^\infty(\mathbb{N}, \ell^2(\mathcal{B}(H))) \cup \ell^\infty\Big(\mathbb{N}, \tilde{H}^2(\mathbb{T}, \mathcal{B}(H))\Big)$, *then* $\mathbf{A} * \mathbf{B} \in \mathcal{B}(\ell^2(H))$, *that is,* $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$.

Proof: To simplify the proof, let us consider vectors x with finite support (since they form a dense subset in the unit ball). We are also considering that the matrices have a finite number of rows N, which is not a problem since the norm of a full matrix can be computed as the supremum of the norms of the row-truncated matrices.

We use the notation $\mathcal{L}_k^{\mathbf{A}} = \sum_j T_{k,j} e^{2\pi ijt}$ and $\mathcal{L}_k^{\mathbf{B}} = \sum_j S_{k,j} e^{2\pi ijt}$, and the convolutions that appear are defined as usual, but taking the composition as the operation involved. For $x \in \ell^2(H)$, denote by $h(t)$ the sum $\sum_{j=1}^\infty x_j e^{2\pi ijt}$. First, let us assume that $\mathbf{A} \in \ell^\infty(\mathbb{N}, \ell^2(\mathcal{B}(H)))$. We have

$$\|(\mathbf{A} * \mathbf{B})x\|_2^2 = \sup_N \sum_{k=1}^N \| \int_0^1 (\mathcal{L}_k^{\mathbf{A}} * \mathcal{L}_k^{\mathbf{B}})(t) h(-t) dt\|^2$$

$$\begin{aligned}
&= \sup_N \sum_{k=1}^{N} \| \int_0^1 \mathcal{L}_k^{\mathbf{A}}(s) \left(\int_0^1 \mathcal{L}_k^{\mathbf{B}}(t-s)h(-t)dt \right) ds \|^2 \\
&= \sup_N \sum_{k=1}^{N} \| \int_0^1 \mathcal{L}_k^{\mathbf{A}}(s) \left((\mathcal{L}_k^{\mathbf{B}} * h)(-s) \right) ds \|^2 \\
&= \sup_N \sum_{k=1}^{N} \| \int_0^1 \left(\sum_{l}^{M} T_{k,l} e^{ils} \right) \left(\sum_j S_{k,j}(x_j) e^{-ijs} \right) ds \|^2 \\
&= \sup_N \sum_{k=1}^{N} \| \sum_j T_{k,j} S_{k,j}(x_j) \|^2 \leq \sup_N \sum_{k=1}^{N} \left(\sum_j \|T_{k,j}\| \|S_{k,j}(x_j)\| \right)^2 \\
&\overset{\text{Cauchy-Schwarz}}{\leq} \sup_N \sum_{k=1}^{N} \sum_j \|T_{k,j}\|^2 \sum_j \|S_{k,j}(x_j)\|^2 \\
&\leq \sup_N \left(\sup_k \sum_j \|T_{k,j}\|^2 \right) \sum_{k=1}^{N} \sum_j \|S_{k,j}(x_j)\|^2 \\
&\overset{\text{Prop. 2.4.8}}{\leq} \|\mathbf{A}\|^2_{\ell^\infty(\mathbb{N}, \ell^2(\mathcal{B}(H)))} \|\mathbf{B}\|_2^2 \|x\|^2.
\end{aligned}$$

And taking supremums, this leads to

$$\|(\mathbf{A} * \mathbf{B})\|_{\mathcal{B}(\ell^2(H))} \leq \|\mathbf{A}\|_{\ell^\infty(\mathbb{N}, \ell^2(\mathcal{B}(H)))} \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}.$$

This time, suppose that $A \in \ell^\infty \left(\mathbb{N}, \tilde{H}^2(\mathbb{T}, \mathcal{B}(H)) \right)$. We proceed in a similar manner.

$$\|(\mathbf{A} * \mathbf{B})x\|_2^2 = \sup_N \sum_{k=1}^{N} \| \int_0^1 (\mathcal{L}_k^{\mathbf{A}} * \mathcal{L}_k^{\mathbf{B}})(t) h(-t) dt \|^2$$

$$\begin{aligned}
&= \sup_N \sum_{k=1}^{N} \| \int_0^1 \mathcal{L}_k^{\mathbf{A}}(s) \left(\int_0^1 \mathcal{L}_k^{\mathbf{B}}(t-s)h(-t)dt \right) ds \|^2 \\
&= \sup_N \sum_{k=1}^{N} \| \int_0^1 \mathcal{L}_k^{\mathbf{A}}(s) \left((\mathcal{L}_k^{\mathbf{B}} * h)(-s) \right) ds \|^2 \\
&= \sup_N \sum_{k=1}^{N} \| \int_0^1 \left(\sum_{l}^{M} T_{k,l} e^{ils} \right) \left(\sum_j S_{k,j}(x_j) e^{-ijs} \right) ds \|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad \sup_N \sum_{k=1}^{N} \left(\int_0^1 \|\sum_{l}^{M} T_{k,l} e^{ils}\| \|\sum_j S_{k,j}(x_j) e^{-ijs}\| ds \right)^2 \\
&\leq \quad \sup_N \sum_{k=1}^{N} \left(\int_0^1 \|\sum_{l}^{M} T_{k,l} e^{ils}\|^2 ds \right) \left(\int_0^1 \|\sum_j S_{k,j}(x_j) e^{-ijs}\|^2 ds \right) \\
&\leq \quad \|\mathbf{A}\|^2_{\ell^\infty\left(\mathbb{N},\tilde{H}^2(\mathbb{T},\mathcal{L}(H))\right)} \sup_N \sum_{k=1}^{N} \left(\int_0^1 \|\sum_j S_{k,j}(x_j) e^{-ijs}\|^2 ds \right) \\
&= \quad \|\mathbf{A}\|^2_{\ell^\infty\left(\mathbb{N},\tilde{H}^2(\mathbb{T},\mathcal{L}(H))\right)} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \|S_{k,j}(x_j)\|^2 \\
&\overset{\text{Prop. 2.4.8}}{\leq} \quad \|\mathbf{A}\|^2_{\ell^\infty\left(\mathbb{N},\tilde{H}^2(\mathbb{T},\mathcal{B}(H))\right)} \|\mathbf{B}\|_2^2 \|x\|^2.
\end{aligned}
$$

And taking supremums yields that

$$
\|(\mathbf{A} * \mathbf{B})\|_{\mathcal{B}(\ell^2(H))} \leq \|\mathbf{A}\|_{\ell^\infty\left(\mathbb{N},\tilde{H}^2(\mathbb{T},\mathcal{B}(H)))\right)} \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}. \quad \blacksquare
$$

Corollary 2.4.15 *If* $\mathbf{A} \in \ell^\infty(\mathbb{N}, \ell^2(\mathcal{B}(H)))$ *and* $\mathbf{B} \in \mathcal{B}(\ell^2(H))$ *then* $\mathbf{A} * \mathbf{B} \in B_{f,\ell^2}(\ell^2(H))$. *In other words,*

$$
\ell^\infty\left(\mathbb{N}, \ell^2(\mathcal{B}(H))\right) \subset \mathcal{M}_l(\mathcal{B}(\ell^2(H)), B_{f,\ell^2}(\ell^2(H)))
$$

Proof: The result follows using Theorem 2.4.14 and taking into account that a sequence of operators in $\ell^\infty(\mathbb{N}, \mathcal{B}(H))$ composed by a sequence in $\ell^2(\mathbb{N}, \mathcal{B}(H))$, stays in $\ell^2(\mathbb{N}, \mathcal{B}(H))$. $\blacksquare$

Remark 2.4.16 *Observe that the analogous inclusion for* $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ *is not true, that is*

$$
\ell^\infty\left(\mathbb{N}, \tilde{H}^2(\mathbb{T}, \mathcal{L}(H)))\right) \not\subseteq \mathcal{M}_l(\mathcal{B}(\ell^2(H)), B_{f,\tilde{H}^2}(\ell^2(H))),
$$

and the reason is the following. Although Theorem 2.4.14 ensures that whenever $\mathbf{A} \in \ell^\infty\left(\mathbb{N}, \tilde{H}^2(\mathbb{T}, \mathcal{B}(H)))\right)$ *and* $\mathbf{B} \in \mathcal{B}(\ell^2(H))$ *then* $\mathbf{A} * \mathbf{B} \in \mathcal{B}(\ell^2(H))$, *the rows of* $\mathbf{A} * \mathbf{B}$ *need not be in* $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$. *A simple example to see it, with row matrices: take* $\mathbf{A}$ *as* *the matrix such that* $\mathbf{R}_1^{\mathbf{A}} = (e_j \otimes e_j)_j$ *and* $\mathbf{R}_j^{\mathbf{A}} = 0\ \forall j > 1$*; and select* $\mathbf{B}$ *as the matrix satisfying* $\mathbf{R}_1^{\mathbf{B}} = (e_1 \otimes e_j)_j$ *and* $\mathbf{R}_j^{\mathbf{B}} = 0\ \forall j > 1$.

It turns out that the sequence $(e_j \otimes e_j)_j$ *is in* $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ *(see Remark 2.2.4) and the row matrix* $(e_1 \otimes e_j)_j$ *defines a bounded operator (since it is just the first row of the matrix seen at Example 2.4.12). Moreover,* $(e_j \otimes e_j) \circ (e_1 \otimes e_j) = e_1 \otimes e_j$*, so it suffices to see that* $(e_1 \otimes e_j)_j$ *is not in en* $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$*. Indeed, it is not even in the space* $\tilde{H}^\infty(\mathbb{T}, \mathcal{B}(H))$*, since*

$$\sup_{\|x\|=1} \|\sum_{k=1}^{N} e_1 \otimes e_k(x) e^{ikt}\| = \sup_{\|x\|=1} \|(x_1 e^{it}, x_1 e^{2it}, \cdots, x_1 e^{iNt}, 0, 0, \cdots)\| = \sup_{\|x\|=1} |x_1| \sqrt{N} = \sqrt{N}$$

$$\Rightarrow \sup_N \sup_{\|x\|=1} \|\sum_{k=1}^{N} e_1 \otimes e_k(x) e^{ikt}\| = \sup_N \sqrt{N} = \infty.$$

However, the following inclusion is true

$$\ell^\infty\left(\mathbb{N}, \tilde{H}^2(\mathbb{T}, \mathcal{L}(H))\right) \subset \mathcal{M}_l(B_{f,\tilde{H}^2}(\ell^2(H)), B_{f,\tilde{H}^2}(\ell^2(H))),$$

because, on the one hand, Theorem 2.4.14 ensures again the boundedness of the matrix $\mathbf{A} * \mathbf{B}$*; and on the other hand, this time since both* $\mathbf{A}$ *and* $\mathbf{B}$ *have their rows in the space* $\tilde{H}^2(\mathbb{T}, \mathcal{B}(H))$*, the matrix* $\mathbf{A} * \mathbf{B}$ *also does, due to Lemma 2.4.13.*

2.5 Generalized Toeplitz's and Bennett's Theorems

The main results that will be presented in this section are the analogues for matrices with operator entries of the theorems of Toeplitz and Bennett that appeared at the introductory section.

2.5.1 Toeplitz's Theorem

Theorem 2.5.1 *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{T}$. *Then* $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *if and only if there exists* $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ *such that* $T_{kj} = \hat{\mu}(j-k)$ *for all* $k, j \in \mathbb{N}$. *Moreover,* $\|\mathbf{A}\| = \|\mu\|_\infty$.

Proof: Start assuming that $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ and $T_{kj} = \hat{\mu}(j-k)$ for all $k, j \in \mathbb{N}$. Then, for $\mathbf{x}, \mathbf{y} \in c_{00}(H)$, we have

$$\begin{aligned}
\ll \mathbf{A}(\mathbf{x}), \mathbf{y} \gg &= \sum_{k=1}^{M}\sum_{j=1}^{N}\langle T_{kj}(x_j), y_k\rangle \\
&= \sum_{k=1}^{M}\sum_{j=1}^{N}\langle T_\mu(\overline{\varphi_j}\varphi_k)(x_j), y_k\rangle \\
&= \sum_{k=1}^{M}\sum_{j=1}^{N}\Psi_\mu(\overline{\varphi_j}x_j \otimes \overline{\varphi_k}y_k) \\
&= \Psi_\mu\left(\sum_{k=1}^{M}\sum_{j=1}^{N}\overline{\varphi_j}x_j \otimes \overline{\varphi_k}y_k\right) \\
&= \Psi_\mu\left(\left(\sum_{j=1}^{N}\overline{\varphi_j}x_j\right) \otimes \left(\sum_{k=1}^{M}\overline{\varphi_k}y_k\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
| \ll \mathbf{A}(\mathbf{x}), \mathbf{y} \gg | &\leq \|\Psi_\mu\|_{L^1(\mathbb{T}, H\hat{\otimes}H)^*}\int_0^{2\pi}\|h_{\mathbf{x}}(-t)\otimes h_{\mathbf{y}}(-t))\|_{H\hat{\otimes}H}\frac{dt}{2\pi} \\
&= \|\mu\|_\infty \int_0^{2\pi}\|h_{\mathbf{x}}(-t)\|\|h_{\mathbf{y}}(-t)\|\frac{dt}{2\pi} \\
&\leq \|\mu\|_\infty\left(\int_0^{2\pi}\|h_{\mathbf{x}}(-t)\|^2\frac{dt}{2\pi}\right)^{1/2}\left(\int_0^{2\pi}\|h_{\mathbf{y}}(-t))\|^2\frac{dt}{2\pi}\right)^{1/2} \\
&\leq \|\mu\|_\infty\|\mathbf{x}\|_{\ell^2(H)}\|\mathbf{y}\|_{\ell^2(H)}.
\end{aligned}$$

Therefore, $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ and $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq \|\mu\|_\infty$.

Conversely, assume that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ with $T_{kj} = T_{j-k}$ for certain sequence $\mathbf{T} =$

$(T_n)_{n\in\mathbb{Z}}$ of operators in $\mathcal{B}(H)$. We define

$$T\left(\sum_{n=-M}^{N}\alpha_n\varphi_n\right) = \alpha_0 T_{1,1} + \sum_{n=1}^{M}\alpha_{-n}T_{n+1,1} + \sum_{n=1}^{N}\alpha_n T_{1,n+1}. \tag{2.29}$$

We shall prove that $T \in \mathcal{L}(L^1(\mathbb{T}), \mathcal{B}(H))$. Since $L^1(\mathbb{T}) = \overline{span\{\varphi_k : k \in \mathbb{Z}\}}^{\|\cdot\|_1}$, it is enough to show that

$$\left\| T(\sum_{n=-M}^{N}\alpha_n\varphi_n)\right\| \le \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}\int_0^{2\pi}\left|\sum_{n=-M}^{N}\alpha_n\varphi_n(t)\right|\frac{dt}{2\pi}. \tag{2.30}$$

Let $x, y \in H$ and observe that

$$\left\langle T\left(\sum_{n=-M}^{N}\alpha_n\varphi_n\right)(x), y\right\rangle = \sum_{n=-M}^{N}\alpha_n\beta_n(x,y),$$

where $\beta_n(x,y) = \langle T_n(x), y\rangle$. Now, taking into account that $A_{x,y} = (\langle T_{kj}(x), y\rangle)$ is a Toeplitz matrix and defines a bounded operator $A_{x,y} \in \mathcal{B}(\ell^2)$ with $\|A_{x,y}\| \le \|\mathbf{A}\|\|x\|\|y\|$ we obtain, invoking Theorem 2.1.4, that

$$\psi_{x,y} = \sum_{n\in\mathbb{Z}}\beta_n(x,y)\varphi_n \in L^\infty(\mathbb{T})$$

with $\|\psi_{x,y}\|_{L^\infty(\mathbb{T})} \le \|\mathbf{A}\|\|x\|\|y\|$. Finally, we have

$$\begin{aligned}\left|\left\langle T\left(\sum_{n=-M}^{N}\alpha_n\varphi_n\right)(x), y\right\rangle\right| &= \left|\int_0^{2\pi}\left(\sum_{n=-M}^{N}\alpha_n\varphi_n(t)\right)\overline{\psi_{x,y}(t)}\frac{dt}{2\pi}\right| \\ &\le \left\|\sum_{n=-M}^{N}\alpha_n\varphi_n(t)\right\|_{L^1(\mathbb{T})}\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}\|x\|\|y\|.\end{aligned}$$

This shows (2.30) which gives $\|T\|_{L^1(\mathbb{T})\to\mathcal{B}(H)} \le \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}$. Finally, from the embedding $C(\mathbb{T}) \to L^1(\mathbb{T})$ we have that there exists $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ such that $T_\mu = T$ and $\|\mu\|_\infty \le \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}$. This completes the proof. ∎

2.5.2 Some sufficient conditions for multipliers

Before moving on to the proof of our version of Bennett's theorem, we shall present a couple of results that provide sufficient conditions for a matrix to be a multiplier in $\mathcal{M}(\ell^2(H))$, related with some spaces of measures defined in 2.3.11.

Theorem 2.5.2 *Let* $\mathbf{A}$ *be a Toeplitz matrix with entries in* $\mathcal{B}(H)$*, and assume its associated distribution,* $\mu_{\mathbf{A}}$*, belongs to* $M(\mathbb{T}, \mathcal{B}(H))$ *and satisfies that the associated operator* $\mu_{\mathbf{A}} \sim \Phi_{\mu_{\mathbf{A}}} : C(\mathbb{T}, H \otimes H) \to \mathbb{C}$ *extends to* $L^2(\mathbb{T}, H \otimes H)$*. Then*

$$\|\mathbf{A} * \mathbf{B}\| \leq K_{\mu_{\mathbf{A}}} \|\mathbf{B}\|$$

where $K_{\mu_{\mathbf{A}}}$ *is a positive constant, and* $\mathbf{B}$ *is any matrix of* $\mathcal{B}(\ell^2(H))$*. That is,* $\mathbf{A}$ *is a Schur multiplier.*

Proof: Let $h = (h_j)_j \in \ell^2(H)$. We denote $\mathbf{B} = (S_{k,j})_{k,j}$, and $\mathbf{A} = (T_l)_l$, where $\hat{\mu_A}(l) = T_l$. Those f_k below are obtained by duality, and we denote $y_t = \sum_k f_k e^{ikt}$.

$$\begin{aligned}
\|(\mathbf{A} * \mathbf{B})(h)\| &= \left\| \left(\sum_j T_{k,j}(S_{k,j}(h_j)) \right)_k \right\|_2 = |\sum_k \sum_j \langle T_{k,j}(S_{k,j}(h_j)), f_k \rangle| \\
&= |\sum_k \sum_{l=1-k} \langle T_l(S_{k,l+k}(h_{l+k})), f_k \rangle| = |\sum_k \langle \sum_{l=1-k} T_l(S_{k,l+k}(h_{l+k})), f_k \rangle| \\
&= \int |\langle \sum_k \left(\sum_{l=1-k} T_l(S_{k,l+k}(h_{l+k})) \right) e^{ikt}, y_t \rangle| dt \\
&= \int |\sum_l \langle T_l \left(\sum_k S_{k,l+k}(h_{l+k}) e^{ikt} \right), y_t \rangle| dt \\
&= \int |\Phi_{\mu_{\mathbf{A}}} \left(\sum_l \left[(\sum_k S_{k,l+k}(h_{l+k}) e^{ikt}) \otimes y_t \right] e^{ils} \right) | dt \\
&\leq \|\Phi_{\mu_{\mathbf{A}}}\| \int (\sum_l \| \sum_k S_{k,l+k}(h_{l+k}) e^{ikt} \otimes y_t \|_2^2)^{\frac{1}{2}} dt \\
&= \|\Phi_{\mu_{\mathbf{A}}}\| \int (\sum_l \| \sum_k S_{k,l+k}(h_{l+k}) e^{ikt} \|_2^2 \|y_t\|_2^2)^{\frac{1}{2}} dt
\end{aligned}$$

$$
\begin{aligned}
&= \|\Phi_{\mu_{\mathbf{A}}}\| \int \|y_t\|_2 (\sum_l \|\sum_k S_{k,l+k}(h_{l+k}) e^{ikt}\|_2^2)^{\frac{1}{2}} dt \\
&\overset{\text{Cauchy-Schwarz}}{\leq} \|\Phi_{\mu_{\mathbf{A}}}\| \left(\int \sum_l \|\sum_k S_{k,l+k}(h_{l+k}) e^{ikt}\|_2^2 dt \right)^{\frac{1}{2}} \\
&= \|\Phi_{\mu_{\mathbf{A}}}\| \left(\sum_l \int \|\sum_k S_{k,l+k}(h_{l+k}) e^{ikt}\|_2^2 dt \right)^{\frac{1}{2}} \\
&= \|\Phi_{\mu_{\mathbf{A}}}\| \left(\sum_l \sum_k \|S_{k,l+k}(h_{l+k})\|_2^2 \right)^{\frac{1}{2}} \\
&\overset{\text{Prop. 2.4.8}}{\leq} \|\Phi_{\mu_{\mathbf{A}}}\| \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \|h\|_{\ell^2(H)} = |\mu_{\mathbf{A}}|_2 \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \|h\|_{\ell^2(H)}.
\end{aligned}
$$

Finally, we take supremums in norm 1 with respect to h, and we obtain the result. Notice that the constant $K_{\mu_{\mathbf{A}}}$ is the 2-variation of the measure. ∎

Theorem 2.5.3 *Let* $\mathbf{A}$ *be a Toeplitz matrix with entries in* $\mathcal{B}(H)$*, and assume that its associated distribution,* $\mu_{\mathbf{A}}$*, belongs to* $\mathcal{M}(\mathbb{T}, \mathcal{B}(H))$ *and satisfies that the associated operator* $\mu_{\mathbf{A}} \sim \Psi_{\mu_{\mathbf{A}}} : C(\mathbb{T}, H) \to H$ *extends to* $L^2(\mathbb{T}, H)$*. Then*

$$\|\mathbf{A} * \mathbf{B}\| \leq K_{\mu_{\mathbf{A}}} \|\mathbf{B}\|,$$

where $K_{\mu_{\mathbf{A}}}$ *is a positive constant, and* $\mathbf{B}$ *is any matrix of* $\mathcal{B}(\ell^2(H))$*. That is,* $\mathbf{A}$ *is a Schur multiplier.*

Proof: Let $h = (h_j)_j \in \ell^2(H)$. Take any $\mathbf{B} = (S_{k,j})_{k,j} \in \mathcal{B}(\ell^2(H))$, and write $\mathbf{A} = (T_l)_l$, where $\hat{\mu_{\mathbf{A}}}(l) = T_l$. Again, those f_k appearing below are obtained by duality, and we denote $y_t = \sum_k f_k e^{ikt}$.

$$
\begin{aligned}
\|(\mathbf{A} * \mathbf{B})(h)\| &= \left\| \left(\sum_j T_{k,j}(S_{k,j}(h_j)) \right)_k \right\|_2 = |\sum_k \sum_j \langle T_{k,j}(S_{k,j}(h_j)), f_k \rangle| \\
&= |\sum_k \sum_{l=1-k} \langle T_l(S_{k,l+k}(h_{l+k})), f_k \rangle|
\end{aligned}
$$

$$
\begin{aligned}
&= && |\sum_k \langle \sum_{l=1-k} T_l(S_{k,l+k}(h_{l+k})), f_k\rangle| \\
&= && \int |\langle \sum_k \left(\sum_{l=1-k} T_l(S_{k,l+k}(h_{l+k}))\right) e^{ikt}, y_t\rangle| dt \\
&= && \int |\langle \sum_l T_l \left(\sum_k S_{k,l+k}(h_{l+k}) e^{ikt}\right), y_t\rangle| dt \\
&= && \int |\langle \Psi_{\mu_{\mathbf{A}}} \left(\sum_l \left(\sum_k S_{k,l+k}(h_{l+k}) e^{ikt}\right) e^{ils}\right), y_t\rangle| dt \\
&\leq && \|\Psi_{\mu_{\mathbf{A}}}\| \int \|y_t\|_2 \left\| \sum_l \left(\sum_k S_{k,l+k}(h_{l+k}) e^{ikt}\right) e^{ils}\right\|_2 dt \\
&= && \|\Psi_{\mu_{\mathbf{A}}}\| \int \|y_t\|_2 \left(\sum_l \|\sum_k S_{k,l+k}(h_{l+k}) e^{ikt}\|_2^2 \right)^{\frac{1}{2}} dt \\
&\overset{\text{Cauchy-Schwarz}}{\leq} && \|\Psi_{\mu_{\mathbf{A}}}\| \left(\int \sum_l \|\sum_k S_{k,l+k}(h_{l+k}) e^{ikt}\|_2^2 dt \right)^{\frac{1}{2}} \\
&= && \|\Psi_{\mu_{\mathbf{A}}}\| \left(\sum_l \int \|\sum_k S_{k,l+k}(h_{l+k}) e^{ikt}\|_2^2 dt \right)^{\frac{1}{2}} \\
&= && \|\Psi_{\mu_{\mathbf{A}}}\| \left(\sum_l \sum_k \|S_{k,l+k}(h_{l+k})\|_2^2 \right)^{\frac{1}{2}} \\
&\overset{\text{Prop. 2.4.8}}{\leq} && \|\Psi_{\mu_{\mathbf{A}}}\| \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \|h\|_{\ell^2(H)} \\
&= && |\mu_{\mathbf{A}}|_{2-SOT} \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \|h\|_{\ell^2(H)}.
\end{aligned}
$$

To conclude, we take supremums in norm 1 with respect to h, and we obtain the result, where $K_{\mu_{\mathbf{A}}}$ is the SOT_2- variation of the measure. ∎

2.5.3 Bennett's Theorem

The following lemmas shall be necessary to deal with the proof of Bennett's theorem on Schur multipliers with operator entries.

Lemma 2.5.4 *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{M}_l(\ell^2(H)) \cup \mathcal{M}_r(\ell^2(H))$ *and* $x_0, y_0 \in H$ *with* $\|x_0\| =$

$\|y_0\| = 1$. *Denote by* $A_{x_0,y_0} = (\gamma_{kj})$ *the matrix with entries*

$$\gamma_{kj} = \langle T_{kj}(x_0), y_0\rangle, \quad k, j \in \mathbb{N}.$$

Then A_{x_0,y_0} *is a Schur multiplier with scalar entries and*

$$\|A_{x_0,y_0}\|_{\mathcal{M}(\ell^2)} \leq \min\{\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}, \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\}.$$

Proof: Let $z_0 \in H$ and $\|z_0\| = 1$ and consider the bounded operators $\pi_{z_0} : \ell^2(H) \to \ell^2$ and $i_{z_0} : \ell^2 \to \ell^2(H)$ given by

$$\pi_{z_0}((x_j)) = (\langle x_j, z_0\rangle)_j, \quad i_{z_0}((\alpha_k)) = (\alpha_k z_0)_k.$$

Now, given $B = (\beta_{kj}) \in \mathcal{B}(\ell^2)$ with $\|B\| = 1$, we set $\mathbf{B} = i_{z_0} B\, \pi_{z_0}$.

Hence $\mathbf{B} \in \mathcal{B}(\ell^2(H))$ since it is a composition of bounded linear operators. Moreover, observe that $\|\mathbf{B}\| = \|B\|$ because $\|i_{z_0}\| = \|\pi_{z_0}\| = 1$ and $B((\alpha_j))z_0 = \mathbf{B}((\alpha_j z_0))$ for any $(\alpha_j) \in \ell^2$.

Write $\mathbf{B} = (S_{kj})$ and observe that $S_{kj} = \beta_{kj}\widetilde{z_0 \otimes z_0}$. Indeed,

$$\langle S_{kj}(x), y\rangle = \ll \mathbf{B}(x\mathbf{e_j}), y\mathbf{e_k} \gg = \ll ((\langle x, z_0\rangle \beta_{kj} z_0)_k, y\mathbf{e_k} \gg = \beta_{kj}\langle x, z_0\rangle\langle z_0, y\rangle.$$

Recall that $T(\widetilde{x \otimes y}) = \widetilde{x \otimes T(y)}$ and $(\widetilde{x \otimes y})T = \widetilde{T^* x \otimes y}$ for any $T \in \mathcal{B}(H)$ and $x, y \in H$. In particular, we obtain

$$\langle (T_{kj}S_{kj})(x_0), y_0\rangle = \beta_{kj}\langle T_{kj}(z_0), y_0\rangle\langle x_0, z_0\rangle$$

and

$$\langle (S_{kj}T_{kj})(x_0), y_0\rangle = \beta_{kj}\langle T_{kj}(x_0), z_0\rangle\langle z_0, y_0\rangle.$$

Therefore, choosing $z_0 = x_0$ and $\mathbf{C} = \mathbf{A} * \mathbf{B}$ one has $C_{x_0,y_0} = A_{x_0,y_0} * B$, and using that

$\|C_{x_0,y_0}\| \leq \|\mathbf{C}\|$ we have

$$\|A_{x_0,y_0} * B\|_{\mathcal{B}(\ell^2)} \leq \|\mathbf{A} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \leq \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}.$$

In the same way, choosing $z_0 = y_0$ and $\mathbf{C} = \mathbf{B} * \mathbf{A}$ we obtain

$$\|B * A_{x_0,y_0}\|_{\mathcal{B}(\ell^2)} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))},$$

and this completes the proof. ∎

Lemma 2.5.5 *Let* $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$, $\mathbf{A} = (T_{kj}) \in \mathcal{T}$ *with* $T_{kj} = \hat{\mu}(j-k)$ *for* $k, j \in \mathbb{N}$, $\mathbf{B} = (S_{kj}) \subset \mathcal{B}(H)$ *and* $\mathbf{x}, \mathbf{y} \in c_{00}(H)$. *Then*

$$\ll \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \gg = \Psi_\mu \left(\int_0^{2\pi} \int_0^{2\pi} \mathbf{B}_{N,M}(s - \cdot, t - \cdot)(h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{ds}{2\pi} \frac{dt}{2\pi} \right). \quad (2.31)$$

Proof: Let $\mathbf{x}, \mathbf{y} \in c_{00}(H)$, and denote $h_{\mathbf{x}} = \sum_{j=1}^N x_j \varphi_j$ and $h_{\mathbf{y}} = \sum_{k=1}^M y_k \varphi_k$. Recall that $x_j = \int_0^{2\pi} h_{\mathbf{x}}(s) \overline{\varphi_j(s)} \frac{ds}{2\pi}$ and $y_k = \int_0^{2\pi} h_{\mathbf{y}}(t) \overline{\varphi_k(t)} \frac{dt}{2\pi}$. Then

$$\begin{aligned}
& \ll \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \gg \\
= & \sum_{k=1}^M \sum_{j=1}^N \langle \hat{\mu}(j-k) S_{kj}(x_j), y_k \rangle \\
= & \int_0^{2\pi} \left\langle \sum_{k=1}^M \left(\sum_{j=1}^N \hat{\mu}(j-k) S_{kj}(x_j) \right) \varphi_k(t), h_{\mathbf{y}}(t) \right\rangle \frac{dt}{2\pi} \\
= & \int_0^{2\pi} \left\langle \sum_{l=-M}^N \hat{\mu}(l) \left(\sum_{j-k=l} S_{kj}(x_j) \varphi_k(t) \right), h_{\mathbf{y}}(t) \right\rangle \frac{dt}{2\pi} \\
= & \int_0^{2\pi} \int_0^{2\pi} \left\langle \sum_{l=-M}^N \hat{\mu}(l) \left(\left(\sum_{j-k=l} S_{kj} \overline{\varphi_j(s)} \varphi_k(t) (h_{\mathbf{x}}(s)) \right) \right), h_{\mathbf{y}}(t) \right\rangle \frac{ds}{2\pi} \frac{dt}{2\pi} \\
= & \int_0^{2\pi} \int_0^{2\pi} \sum_{l=-M}^N \mathcal{J}\mu(l) \left(\left(\sum_{j-k=l} S_{kj} \overline{\varphi_j(s)} \varphi_k(t) \right) (h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \right) \frac{ds}{2\pi} \frac{dt}{2\pi}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=-M}^{N} \mathcal{J}\mu(l) \left(\int_0^{2\pi} \int_0^{2\pi} \left(\sum_{j-k=l} S_{kj} \overline{\varphi_j(s)} \varphi_k(t) \right) (h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{ds}{2\pi} \frac{dt}{2\pi} \right) \\
&= \Psi_\mu \left(\sum_{l=-M}^{N} \left(\int_0^{2\pi} \int_0^{2\pi} \left(\sum_{j-k=l} S_{kj} \overline{\varphi_j(s)} \varphi_k(t) \right) (h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{ds}{2\pi} \frac{dt}{2\pi} \right) \varphi_l \right) \\
&= \Psi_\mu \left(\int_0^{2\pi} \int_0^{2\pi} \left(\sum_{k=1}^{M} \sum_{j=l}^{N} S_{kj} \overline{\varphi_j(s)} \varphi_k(t) \varphi_j \varphi_{-k} \right) (h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{ds}{2\pi} \frac{dt}{2\pi} \right) \\
&= \Psi_\mu \left(\int_0^{2\pi} \int_0^{2\pi} \mathbf{B}_{N,M}(s-\cdot, t-\cdot)(h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{dt}{2\pi} \frac{ds}{2\pi} \right).
\end{aligned}$$

The proof is completed. ∎

Theorem 2.5.6 *If $\mu \in M(\mathbb{T}, \mathcal{B}(H))$ and $\mathbf{A} = (T_{kj}) \in \mathcal{T}$ with $T_{kj} = \hat{\mu}(j-k)$ for $k, j \in \mathbb{N}$ then $\mathbf{A} \in \mathcal{M}_l(\ell^2(H)) \cap \mathcal{M}_r(\ell^2(H))$. Moreover,*

$$\max\{\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}, \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\} \leq |\mu|.$$

Proof: Since $\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} = \|\mathbf{A}^*\|_{\mathcal{M}_l(\ell^2(H))}$ and $|\mu| = |\mu^*|$ it is enough to focus on the case of left Schur multipliers. Let $\mathbf{x}, \mathbf{y} \in c_{00}(H)$ and $\mathbf{B} = (S_{kj}) \subset \mathcal{B}(H)$ such that $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. Now, define

$$G(u) = \int_0^{2\pi} \int_0^{2\pi} \mathbf{B}_{N,M}(s-u, t-u)(h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{dt}{2\pi} \frac{ds}{2\pi}.$$

Since $(\lambda x) \otimes y = x \otimes \bar{\lambda} y$, we can rewrite

$$G(u) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} S_{kj}(x_j \varphi_j(u)) \otimes y_k \varphi_k(u).$$

In particular, Cauchy-Schwarz inequality gives that

$$\|G(u)\|_{H \hat{\otimes} H} \leq \sum_{k=1}^{\infty} \left\| \sum_{j=1}^{\infty} S_{kj}(x_j \varphi_j(u)) \right\| \|y_k \varphi_k(u)\|$$

$$\begin{aligned} &\leq \left(\sum_{k=1}^{\infty} \left\| \sum_{j=1}^{\infty} S_{kj}(x_j \varphi_j(u)) \right\|^2 \right)^{1/2} \|\mathbf{y}\| \\ &\leq \|\mathbf{B}\| \|\mathbf{x}\| \|\mathbf{y}\|. \end{aligned}$$

Using Lemma 2.5.5, we have

$$\begin{aligned} | \ll \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \gg | &\leq \|\Psi_\mu\|_{C(\mathbb{T}, H\hat{\otimes}H)^*} \sup_{0 \leq u < 2\pi} \|G(u)\|_{H\hat{\otimes}H} \\ &= |\mu| \|\mathbf{B}\| \|\mathbf{x}\| \|\mathbf{y}\|. \end{aligned}$$

and this concludes the proof. ∎

Lemma 2.5.7 *Let $\mu, \nu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$, $\mathbf{A} = (T_{kj}) \in \mathcal{T}$ with $T_{kj} = \hat{\mu}(j-k)$, $\mathbf{B} = (S_{kj}) \in \mathcal{T}$ with $S_{kj} = \hat{\nu}(j-k)$ for $k, j \in \mathbb{N}$ and $\mathbf{x}, \mathbf{y} \in c_{00}(H)$. Then*

$$\ll \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \gg = \Psi_\mu \left(\sum_{k=1}^{M} \left(\sum_{j=1}^{N} \hat{\nu}(j-k)(x_j)\overline{\varphi_j} \right) \otimes y_k \overline{\varphi_k} \right). \tag{2.32}$$

Proof: Let us denote $h_{\mathbf{x}} = \sum_{k=1}^{M} y_k \varphi_k$ and $h_{\mathbf{y}} = \sum_{j=1}^{N} x_j \varphi_j$. Then

$$\begin{aligned} \ll \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \gg &= \sum_{k=1}^{M} \sum_{j=1}^{N} \langle \hat{\mu}(j-k)\hat{\nu}(j-k)(x_j), y_k \rangle \\ &= \sum_{l=-M}^{N} \sum_{k=1}^{M} \langle \hat{\mu}(l)\hat{\nu}(l)(x_{k+l}), y_k \rangle \\ &= \sum_{l=-M}^{N} \sum_{k=1}^{M} \mathcal{J}\hat{\mu}(l) \Big(\nu(l)(x_{k+l}) \otimes y_k \Big) \\ &= \sum_{l=-M}^{N} \mathcal{J}\hat{\mu}(l) \left(\sum_{k=1}^{M} \hat{\nu}(l)(x_{k+l}) \otimes y_k \right) \\ &= \Psi_\mu \left(\sum_{l=-M}^{N} \left(\sum_{k=1}^{M} \hat{\nu}(l)(x_{k+l}) \otimes y_k \right) \varphi_{-l} \right) \\ &= \Psi_\mu \left(\sum_{k=1}^{M} \left(\sum_{j=1}^{N} \hat{\nu}(j-k)(x_j)\overline{\varphi_j} \right) \otimes y_k \overline{\varphi_k} \right). \end{aligned}$$

The proof is complete. ∎

Corollary 2.5.8 *Let* $\mathbf{A} = (S_{kj}) \in \mathcal{T}$ *such that* $S_{kj} = \hat{\nu}(j-k)$ *for some* $\nu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. *For each* $\mathbf{x}, \mathbf{y} \in c_{00}(H)$ *we denote*

$$F_{\mathbf{x},\mathbf{y},\mathbf{A}}(t) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \hat{\nu}(j-k)(x_j)\overline{\varphi_j}(t) \right) \otimes y_k \overline{\varphi_k}(t).$$

If $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ *then*

$$\|F_{\mathbf{x},\mathbf{y},\mathbf{A}}\|_{L^1(\mathbb{T}, H\hat{\otimes}H)} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{x}\|_{\ell^2(H)} \|\mathbf{y}\|_{\ell^2(H)}.$$

Proof: If $\mathbf{A}$ is a multiplier in $\mathcal{M}_r(\ell^2(H))$ then $\mathbf{B} * \mathbf{A} \in \mathcal{B}(\ell^2(H))$ for any matrix $\mathbf{B} \in \mathcal{B}(\ell^2(H)) \cap \mathcal{T}$. In particular (and here we are using Theorem 2.5.1) for any $\mathbf{B} = (T_{kj})$ with $T_{kj} = \hat{\mu}(j-k)$ for some $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ with $\|\mu\|_\infty = \|\mathbf{B}\|$. Taking into account that $L^1(\mathbb{T}, H\hat{\otimes}H) \subseteq (V^\infty(\mathbb{T}, B(H)))^*$ isometrically, we can invoke Lemma 2.5.7 to obtain

$$\begin{aligned} \|F_{\mathbf{x},\mathbf{y},\mathbf{A}}\|_{L^1(\mathbb{T}, H\hat{\otimes}H)} &= \sup\{|\Psi_\mu(F_{\mathbf{x},\mathbf{y},\mathbf{A}})| : \|\mu\|_\infty = 1\} \\ &= \sup\{| \ll \mathbf{B} * \mathbf{A}(\mathbf{x}), \mathbf{y} \gg | : \|\mathbf{B}\| = 1\} \\ &\leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{x}\|_{\ell^2(H)} \|\mathbf{y}\|_{\ell^2(H)}. \end{aligned}$$

The proof is complete ∎.

Theorem 2.5.9 *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{T} \cap \mathcal{M}_r(\ell^2(H))$. *Then there exists* $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ *such that* $T_{kj} = \hat{\mu}(j-k)$ *for all* $k, j \in \mathbb{N}$. *Moreover,*

$$\|\mu\|_{SOT} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}.$$

Proof: Let $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$. As seen previously, for each $x_0, y_0 \in H$ we can consider the Toeplitz matrix with scalar entries $A_{x_0,y_0} = (\langle T_{kj}(x_0), y_0 \rangle)$. A use of Lemma 2.5.4, yields that $A_{x_0,y_0} \in \mathcal{M}(\ell^2)$ and $\|A_{x_0,y_0}\|_{\mathcal{M}(\ell^2)} \leq \|\mathbf{A}\|_{\mathcal{M}(\ell^2(H))}$. This guarantees, invoking Bennett's theorem (see Theorem 2.1.5), that there exists $\eta_{x_0,y_0} \in M(\mathbb{T})$ such that $\langle T_{kj}(x_0), y_0\rangle = \widehat{\eta_{x_0,y_0}}(j-k)$ for all $j,k \in \mathbb{N}$ and $|\eta_{x_0,y_0}| = \|A_{x_0,y_0}\|_{\mathcal{M}_r(\ell^2)}$.

Now, define $\mu(A) \in \mathcal{B}(H)$ given by

$$\langle \mu(A)(x), y\rangle = \eta_{x,y}(A), \quad x, y \in H.$$

It can be easily seen that such μ (which is finitely additive) is actually a countably additive vector measure, and also regular, by adapting the proofs of theorems [23, IV.10.1] and [50], respectively. Let us prove that $\mu \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ and $\|\mu\|_{SOT} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}$.

The first thing we need to show is that $\mu(A) \in \mathcal{B}(H)$ for any $A \in \mathfrak{B}(\mathbb{T})$. This follows using that

$$\widehat{\eta_{\lambda x+\beta x',y}}(l) = \lambda\widehat{\eta_{x,y}}(l) + \beta\widehat{\eta_{x',y}}(l), \quad l \in \mathbb{Z}$$

for any $\lambda, \beta \in \mathbb{C}$ and $x, x', y \in H$. This guarantees that $\eta_{\lambda x+\beta x',y} = \lambda\eta_{x,y} + \beta\eta_{x',y}$ and therefore $\mu(A) : H \to H$ is a linear map. The continuity follows from the estimate $|\eta_{x,y}| \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\|x\|\|y\|$.

If we observe that

$$\langle T_\mu(\phi)(x), y\rangle = T_{\eta_{x,y}}(\phi)$$

for each $\phi \in C(\mathbb{T})$, where $T_{\eta_{x,y}} \in \mathcal{L}(C(\mathbb{T}), \mathbb{C})$ is the operator associated to $\eta_{x,y} \in M(\mathbb{T})$, we clearly get that $T_{kj} = \hat{\mu}(j-k)$ for all $j,k \in \mathbb{N}$.

Select $y_k = y\beta_k$ for some $\beta_k \in \mathbb{C}$ and $\|y\| = 1$. Using Corollary 2.5.8 we obtain that

$$\begin{aligned} &\int_0^{2\pi} \left\| \left(\sum_{k=1}^{M}\sum_{j=1}^{N} \hat{\mu}(j-k)(x_j)\beta_k\bar{\varphi}_j(t)\varphi_k(t) \right) \otimes y \right\|_{H\hat{\otimes}H} \frac{dt}{2\pi} \\ = &\int_0^{2\pi} \left\| \sum_{l=-M}^{N} \hat{\mu}(l) \left(\sum_{k=1}^{M} x_{k+l}\beta_k \right) \varphi_{-l}(t) \right\| \frac{dt}{2\pi} \\ \leq &\ \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\|\mathbf{x}\|_{\ell^2(H)} \left(\sum_{k=1}^{M} |\beta_k|^2 \right)^{1/2}. \end{aligned}$$

Taking $x_j = x\alpha_j$ such that $\|x\| = 1$, gives

$$\int_0^{2\pi} \left\| \sum_{l=-M}^{N} \hat{\mu}(l)(x) \left(\sum_{j-k=l} \alpha_j \bar{\varphi}_j(t) \beta_k \varphi_k(t) \right) \right\| \frac{dt}{2\pi}$$
$$\leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \left(\sum_{j=1}^{N} |\alpha_j|^2 \right)^{1/2} \left(\sum_{k=1}^{M} |\beta_k|^2 \right)^{1/2}.$$

Let us use now the notation

$$\gamma(s) = \sum_{l=-M}^{N} \left(\sum_{j-k=l} \beta_k \alpha_j \right) \varphi_l(s).$$

Now recall that $\hat{\mu}(l)(x) = \hat{\mu}_x(l)$ and

$$\sum_{l=-M}^{N} \hat{\mu}_x(l) \left(\sum_{j-k=l} \alpha_j \bar{\varphi}_j(t) \beta_k \varphi_k(t) \right) = \int_0^{2\pi} \left(\sum_{l=-M}^{N} \hat{\mu}_x(l) \varphi_l(s) \right) \gamma(-t-s) \frac{ds}{2\pi}.$$

Hence, if $\alpha = \sum_{j=1}^{\infty} \alpha_j \varphi_j$ and $\beta = \sum_{k=1}^{\infty} \beta_k \varphi_k$ are elements of $L^2(\mathbb{T})$, we have that $\gamma(t) = \alpha(t)\beta(-t)$ and

$$\int_0^{2\pi} \|\mu_x * \gamma(-t)\| \frac{dt}{2\pi} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \|\alpha\|_{L^2(\mathbb{T})} \|\beta\|_{L^2(\mathbb{T})}. \tag{2.33}$$

Due to Lemma 2.3.15, to show that $\mu_x \in M(\mathbb{T}, H)$ it is enough to prove that

$$\sup_{0<r<1} \|\mu_x * P_r\|_{L^1(\mathbb{T},H)} < \infty. \tag{2.34}$$

Choosing $\beta(t) = \alpha(t) = \frac{\sqrt{1-r^2}}{|1-re^{it}|}$ we obtain that $\gamma(t) = P_r(t)$. Therefore (2.33) leads to (2.34) and the estimate $\|\mu_x\|_{M(\mathbb{T},H)} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}$ follows. This finishes the proof. ∎

Chapter 3

Continuous matrices

"The world is continuous, but the mind is discrete."

—David Mumford.

3.1 Preliminaries

This chapter is a natural continuation of Chapter 2, where we started the study of matrices with operator entries. All the notations used in that chapter will be kept here, and many of the concepts and ideas explained that appeared there will be relevant to develop the results in the present one.

We present a type of matrices that will play an important role in what follows. Given $\mu \in M(\mathbb{T})$ we shall denote by $\mathbf{M}_\mu$ the Toeplitz matrix given by

$$\mathbf{M}_\mu = (\hat{\mu}(j-k)Id)_{k,j} \in \mathcal{T}.$$

The particular cases $\mu = \delta_{-t}$ or $d\mu = f dt$ with $f \in L^1(\mathbb{T})$ will be denoted by $\mathbf{M}_t$ and $\mathbf{M}_f$ respectively, that is $\mathbf{M}_t = (e^{i(j-k)t} Id)$ and $\mathbf{M}_f = (\hat{f}(j-k)Id)$. In particular invoking (2.1.4) and (2.1.5), for $H = \mathbb{C}$, we obtain that $\mathbf{M}_f \in \mathcal{B}(\ell^2)$ whenever $f \in L^\infty(\mathbb{T})$ and that $\mathbf{M}_\mu \in \mathcal{M}(\ell^2)$ for any $\mu \in M(\mathbb{T})$.

The following formula provides a first sign of the connections between Fourier multipliers and Schur multipliers:

$$\mathbf{M}_\mu * \mathbf{M}_f = \mathbf{M}_f * \mathbf{M}_\mu = \mathbf{M}_{\mu * f} \tag{3.1}$$

for any $\mu \in M(\mathbb{T})$ and $f \in L^1(\mathbb{T})$ where $\mu * f(t) = \int_0^{2\pi} f(e^{i(t-s)})d\mu(s)$ is the convolution between functions and measures in $\mathbb{T}$.

To further explore the connection between Fourier analysis and matricial analysis (see [44]) we shall introduce now the following matrix-valued functions and operators.

Definitions 3.1.1 *Let $\mathbf{A} = (T_{kj})$ with $T_{kj} \in \mathcal{B}(H)$ for $k, j \in \mathbb{N}$. Define*

$$f_{\mathbf{A}}(t) = \mathbf{M}_t * \mathbf{A} = (e^{i(j-k)t}T_{kj}), \quad t \in [0, 2\pi].$$

In the case of Toeplitz matrices $\mathbf{A} \in \mathcal{T}$, if $T_{kj} = T_{j-k}$ and denoting by $\varphi_l(t) = e^{-ilt}$ for $l \in \mathbb{Z}$, we define $\Phi_{\mathbf{A}} : P(\mathbb{T}) \to \mathcal{B}(H)$ and $\Psi_{\mathbf{A}} : P(\mathbb{T}, H) \to H$ given by

$$\Phi_{\mathbf{A}}(\sum_l \alpha_l \varphi_l) = \sum_l \alpha_l T_l, \quad \sum_l \alpha_l \varphi_l \in P(\mathbb{T})$$

and

$$\Psi_{\mathbf{A}}(\sum_l x_l \varphi_l) = \sum_l T_l(x_l), \quad \sum_l x_l \varphi_l \in P(\mathbb{T}, H).$$

For upper triangular matrices $\mathbf{A} \in \mathcal{U}$ we define

$$F_{\mathbf{A}}(z) = (z^{(j-k)}T_{kj}), \quad |z| < 1.$$

and for $\mathbf{A} \in \mathcal{U} \cap \mathcal{T}$ we define

$$\tilde{F}_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} T_l z^l, \quad |z| < 1.$$

In particular if $z = re^{it}$ gives that

$$F_{\mathbf{A}}(z) = \mathbf{M}_{P_r} * \mathbf{M}_t * \mathbf{A}$$

where P_r stands for the Poisson kernel. We shall use the notations

$$\sigma_n(\mathbf{A}) = \mathbf{M}_{K_n} * \mathbf{A}, \quad P_r(\mathbf{A}) = \mathbf{M}_{P_r} * \mathbf{A}$$

where K_n stands for the Féjer kernel.

In a paper of Barza, Persson and Popa (see [6]), the space $C(\ell^2)$ was introduced as those matrices in $\mathcal{B}(\ell^2)$ such that $\sigma_n(A)$ converges to A in $\mathcal{B}(\ell^2)$. We shall use here a different approach and introduce such a class of matrices, to be called "continuous matrices", with entries in the space $\mathcal{B}(H)$. It will be seen that these matrices play an important role in the study of Schur multipliers.

Here, we present the definition of a "polynomial matrix".

Definition 3.1.2 *Given a matrix $\mathbf{A} = (T_{kj})$ with entries $T_{kj} \in \mathcal{B}(H)$ we say that $\mathbf{A}$ is a "polynomial matrix", or in short, $\mathbf{A} \in \mathcal{P}(\ell^2(H))$, whenever there exist $N, M \in \mathbb{N}$ such that $\mathbf{A}$ can be written as a finite sum of diagonal matrices $\mathbf{A} = \sum_{l=-N}^{M} \mathbf{D_l}$, and*

$$\sup_{k,j} \|T_{kj}\| < \infty. \tag{3.2}$$

Observe that (3.2) ensures that $\mathcal{P}(\ell^2(H)) \subset \mathcal{B}(\ell^2(H))$.

Definition 3.1.3 *We define the space $C(\ell^2(H))$ as the closure of $\mathcal{P}(\ell^2(H))$ in $\mathcal{B}(\ell^2(H))$.*

This chapter is divided into several sections. In Section 3.2 we deal with matrices in $\mathcal{B}(\ell^2(H))$. The easy but very useful observation that $\mathcal{B}(\ell^2)\hat{\otimes}\mathcal{B}(H) \subseteq \mathcal{B}(\ell^2(H))$ is made (see Example 3.2.3). Also, the class $\mathcal{A}(\ell^2(H))$ is introduced as the matricial analogue of the Wiener algebra, to produce easy examples of continuous matrices.

Section 3.3 is devoted to the study of Schur multipliers. For instance, we show that

$\mathcal{M}(\ell^2)\hat{\otimes}\mathcal{B}(H) \subseteq \mathcal{M}(\ell^2(H))$ (see Proposition 3.3.2), and also that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ if and only if $\sup_n \|\mathcal{M}_{k_n} * \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} < \infty$ (see Proposition 3.3.5).

It is in Section 3.4 where the main results of the chapter are contained. We shall show (see Theorem 3.4.4) that $\mathbf{A} \in C(\ell^2(H))$ if and only if $f_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(\ell^2(H)))$ and equivalently $P_r(\mathbf{A}) \to \mathbf{A}$ in $\mathcal{B}(\ell^2(H))$ as $r \to 1$ or $\sigma_n(\mathbf{A}) \to \mathbf{A}$ in $\mathcal{B}(\ell^2(H))$ as $n \to \infty$. This class of matrices will be used to describe Schur multipliers, by showing (see Theorem 3.4.8) that $\mathcal{M}_l(\ell^2(H)) = (C(\ell^2(H)), C(\ell^2(H)))_l$.

Section 3.4 includes two subsections. The first one is focused on Toeplitz matrices. Our main contribution there is the description of "continuous" Toeplitz matrices $\mathbf{A} = (T_{j-k})$ as functions $g_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(H))$ with Fourier coefficients $\widehat{g_{\mathbf{A}}}(l) = T_l$ for $l \in \mathbb{Z}$ (see proposition 3.4.10) and the characterization of Toeplitz Schur multipliers acting on $\mathcal{B}(\ell^2(H)) \cap \mathcal{T}$ as those matrices $\mathbf{A}$ such that $\Psi_{\mathbf{A}}$ extends to a bounded operator from $C(\mathbb{T}, H)$ into H (Theorems 3.4.11 and 3.4.12). This completes and offers an alternative approach, without using vector measures, to some results that we saw in Chapter 2.

The final section deals with upper triangular matrices. Clearly $F_{\mathbf{A}}$ defines a $\mathcal{B}(\ell^2(H))$-valued holomorphic function. We show that $H^\infty(\mathbb{D}, \mathcal{B}(\ell^2(H)))$ and the disc algebra $A(\mathbb{D}, \mathcal{B}(\ell^2(H)))$ correspond to matrices in $\mathcal{B}(\ell^2(H))$ and $C(\ell^2(H))$ respectively. Also results for $\mathbf{A} \in \mathcal{U} \cap \mathcal{T}$ in terms of $\tilde{F}_{\mathbf{A}}$ are presented.

3.2 Examples of matrices with operator entries

The goal of this section is to present some examples that will come in handy when checking some analogues to classical results in our operator-valued setting. Given $x, y \in H$ we write $x \otimes y \in \mathcal{B}(H)$ for the operator given by $(x \otimes y)(z) = \langle z, x\rangle y$ for $z \in H$.

Example 3.2.1 *Take* $\mathbf{x} = (x_j), \mathbf{y} = (y_j) \in \ell^2(H)$ *and let* $\mathbf{x} \otimes \mathbf{y} = (x_j \otimes y_k)$. *Then* $\mathbf{x} \otimes \mathbf{y} \in \mathcal{B}(\ell^2(H))$ *and we have*

$$(\mathbf{x} \otimes \mathbf{y})(\mathbf{z}) = \ll \mathbf{z}, \mathbf{x} \gg \mathbf{y}, \quad \mathbf{z} \in \ell^2(H).$$

Moreover $\|\mathbf{x} \otimes \mathbf{y}\|_{\mathcal{B}(\ell^2(H))} = \|\mathbf{x}\|_{\ell^2(H)} \|\mathbf{y}\|_{\ell^2(H)}$.

The following example shows how to compute the norm of diagonal, row and column-type matrices.

Example 3.2.2 *Let* $\mathbf{A} = (T_{kj})$ *and let* $l \in \mathbb{Z}$ *and* $k, j \in \mathbb{N}$. *Then*

(i) $\mathbf{D_l} \in \mathcal{B}(\ell^2(H))$ *iff* $\sup_k \|T_{k,k+l}\| = \|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} < \infty$.

(ii) $\mathbf{C_j} \in \mathcal{B}(\ell^2(H))$ *iff* $\sup_{\|x\|=1} (\sum_{k=1}^{\infty} \|T_{kj}(x)\|^2)^{1/2} = \|\mathbf{C_j}\|_{\mathcal{B}(\ell^2(H))} < \infty$.

(iii) $\mathbf{R_k} \in \mathcal{B}(\ell^2(H))$ *iff* $\sup_{\|y\|=1} (\sum_{j=1}^{\infty} \|T^*_{kj}(y)\|^2)^{1/2} = \|\mathbf{R_k}\|_{\mathcal{B}(\ell^2(H))} < \infty$.

Proof: (i) and (ii) are straightforward.

(iii) follows by using the duality $(\ell^2(H))^* = \ell^2(H)$, since

$$\begin{aligned}
\sup_{\|\mathbf{x}\|=1} \|\sum_j T_{k,j}(x_j)\| &= \sup_{\substack{\|\mathbf{x}\|=1 \\ \|y\|=1}} |\sum_j \langle T_{k,j}(x_j), y\rangle| \\
&= \sup_{\substack{\|\mathbf{x}\|=1 \\ \|y\|=1}} |\sum_j \langle x_j, T^*_{k,j}(y)\rangle| \\
&= \sup_{\|y\|=1} (\sum_j \|T^*_{k,j}(y)\|^2)^{1/2}. \qquad \blacksquare
\end{aligned}$$

The next example illustrates how the scalar setting can be embedded in the operator setting via a Kronecker-type operation. It is an easy but useful way to generate examples of matrices with operator entries.

Example 3.2.3 *Let* $A = (a_{k,j}) \in \mathcal{B}(\ell^2)$ *and* $T \in \mathcal{B}(H)$. *Then*

$$\mathbf{A} = (a_{k,j}T) \in \mathcal{B}(\ell^2(H)) \quad \textit{and} \quad \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|A\|_{\mathcal{B}(\ell^2)} \|T\|_{\mathcal{B}(H)}.$$

Proof: Initially, let us assume that $\mathbf{A}$ is a rectangular matrix, and take $\mathbf{x}, \mathbf{y} \in \ell^2(H)$. Let $(v_l)_{l=1}^{\infty}$ be an orthonormal basis in H and use the notation $x^l(j)$ for the coordinate with respect to such a basis, i.e. $x_j = \sum_{l=1}^{\infty} x^l(j) v_l$.

$$\begin{aligned}
|\langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle| &= |\sum_{k,j=1}^{N,M} a_{k,j}\langle T(x_j), y_k\rangle| \\
&= |\sum_{k,j=1}^{N,M} a_{k,j} \sum_l T(x_j)^l y^l(k)| = |\sum_l (\sum_{k,j} a_{k,j} T(x_j)^l y^l(k))| \\
&\leq \sum_l \left|\sum_{k,j=1}^{N,M} a_{k,j} T(x_j)^l y^l(k)\right| \leq \sum_l \|A\|_{\mathcal{B}(\ell^2)} (\sum_{j=1}^{M} |T(x_j)^l|^2)^{\frac{1}{2}} (\sum_{k=1}^{N} |y^l(k)|^2)^{\frac{1}{2}} \\
&\leq \|A\|_{\mathcal{B}(\ell^2)} (\sum_l (\sum_{j=1}^{M} |T(x_j)^l|^2))^{\frac{1}{2}} (\sum_l (\sum_{k=1}^{N} |y^l(k)|^2))^{\frac{1}{2}} \\
&= \|A\|_{\mathcal{B}(\ell^2)} (\sum_{j=1}^{M} \|T(x_j)\|^2))^{\frac{1}{2}} (\sum_{k=1}^{N} \|y_k\|^2)^{\frac{1}{2}} \\
&\leq \|A\|_{\mathcal{B}(\ell^2)} \cdot \|T\| \cdot \|\mathbf{x}\| \cdot \|\mathbf{y}\|.
\end{aligned}$$

Now, for the general case, recall that the norm of a matrix $\mathbf{A}$ can be computed as the supremum of the norms of its truncated rectangular matrices $\|\mathbf{A}_{N,M}\|_{\mathcal{B}(\ell^2(H))}$. So this gives us the first of the inequalities, since

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \sup_{N,M} \|\mathbf{A}_{N,M}\|_{\mathcal{B}(\ell^2(H))} \leq \|T\| \sup_{N,M} \|A_{N,M}\|_{B(\ell^2)} = \|T\| \|A\|_{B(\ell^2)}.$$

Now, by taking into account that all vectors of the form $(\alpha_j x)_j$ where $x \in H$ with $\|x\| = 1$ and $(\alpha_j)_j$ is in the unit ball of ℓ^2, are in the unit ball of $\ell^2(H)$, we can obtain the other inequality. Indeed,

$$\begin{aligned}
\|\mathbf{A}\|^2_{\mathcal{B}(\ell^2(H))} &\geq \sup_{\substack{\|(\alpha_j)_j\|_2=1 \\ \|x\|=1}} \sum_k \|\sum_j a_{k,j}\alpha_j T(x)\|^2 = \sup_{\substack{\|(\alpha_j)_j\|_2=1 \\ \|x\|=1}} \|Tx\|^2 \sum_k \left|\sum_j a_{k,j}\alpha_j\right|^2 = \\
&= \|T\|^2 \cdot \|A\|^2_{\mathcal{B}(\ell^2)}
\end{aligned}$$

∎

Let us define a class of matrices which generalizes the Wiener algebra and which is contained in $C(\ell^2(H))$.

Definition 3.2.4 *We say that a matrix* $\mathbf{A} = (T_{kj})$ *belongs to the space* $\mathcal{A}(\ell^2(H))$ *whenever*

$$\|\mathbf{A}\|_{\mathcal{A}(\ell^2(H))} = \sum_{l\in\mathbb{Z}} \sup_j \|T_{j+l,j}\| < \infty.$$

Notice that, whenever we take supremums through the l-th diagonal, j will be a natural number in the interval $[\max\{1, 1-l\}, \infty)$.

Remark 3.2.5 *If* $\mathbf{A} = (T_{j-k}) \in \mathcal{T} \cap \mathcal{A}(\ell^2(H))$ *then* $(T_l) \in \ell^1(\mathbb{Z}, B(H))$. *Therefore, the function* $t \to \sum_{l\in\mathbb{Z}} T_l e^{ilt}$ *belongs to the Wiener algebra* $A(\mathbb{T}, B(H))$.

Remark 3.2.6 *In [41], for the case* $H = \mathbb{C}$ *it was mentioned another possible extension which coincides with* $A(\mathbb{T})$ *for Toeplitz matrices, but which was not even contained in the space of bounded operators.*

Proposition 3.2.7 $\mathcal{A}(\ell^2(H)) \subsetneq C(\ell^2(H))$.

Proof: Notice that $\|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} = \sup_j \|T_{j+l,j}\|$ and therefore if $\mathbf{A} \in \mathcal{A}(\ell^2(H))$ then $\sum_{l\in\mathbb{Z}} \mathbf{D_l}$ is absolutely convergent in $\mathcal{B}(\ell^2(H))$. Hence $\mathbf{A} \in C(\ell^2(H))$ and

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq \|\mathbf{A}\|_{\mathcal{A}(\ell^2(H))}.$$

It is clear that $\mathbf{M}_f \in \mathcal{A}(\ell^2(H))$ if and only if $f \in A(\mathbb{T})$. To get a matrix in $C(\ell^2(H)) \setminus \mathcal{A}(\ell^2(H))$ it suffices to take $f \in C(\mathbb{T}) \setminus A(\mathbb{T})$ and consider $\mathbf{A} = \mathbf{M}_f$. Due to Example 3.2.3 and Theorem 2.1.4 one has that

$$\|\mathbf{M}_{f-\sigma_n(f)}\|_{\mathcal{B}(\ell^2(H))} = \|f - \sigma_n(f)\|_{L^\infty(\mathbb{T})}$$

where $\sigma_n(f) = K_n * f$ is the convolution with the Féjer kernel K_n. This shows that $\mathbf{M}_f \in C(\ell^2(H))$. ∎

3.3 Schur multipliers for matrices with operator entries

Recall (see Proposition 2.4.8) that if $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ then

$$\sup_{\|\mathbf{x}\|=1} \sum_{k,j} \|T_{kj}(x_j)\|^2 \leq \|\mathbf{A}\|^2_{\mathcal{B}(\ell^2(H))}.$$

Let us present now a necessary condition for Schur multipliers.

Proposition 3.3.1 *If* $\mathbf{A} = (T_{kj}) \in \mathcal{M}_r(\ell^2(H))$ *then*

$$\sup_{k,j} \|T_{k,j}\| \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}.$$

Proof: Let $\mathbf{x}, \mathbf{y} \in \ell^2(H)$, since $(\mathbf{x} \otimes \mathbf{y}) = (x_j \otimes y_k) \in \mathcal{B}(\ell^2(H))$ then $(\mathbf{x} \otimes \mathbf{y}) * \mathbf{A} \in \mathcal{B}(\ell^2(H))$. So, we have that

$$\|(\mathbf{x} \otimes \mathbf{y}) * \mathbf{A}(z)\|_{\ell^2(H)} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{x}\|_{\ell^2(H)} \|\mathbf{y}\|_{\ell^2(H)} \|\mathbf{z}\|_{\ell^2(H)}.$$

Using now that $(x_j \otimes y_k)T_{kj} = T^*_{kj} x_j \otimes y_k$ we obtain that

$$\left\| \left(\Big(\sum_{j=1}^{\infty} \langle T_{k,j}(z_j), x_j \rangle \Big) y_k \right) \right\|_{\ell^2(H)} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{x}\|_{\ell^2(H)} \|\mathbf{y}\|_{\ell^2(H)} \|\mathbf{z}\|_{\ell^2(H)}.$$

This, in particular, yields

$$\sup_k \left| \sum_{j=1}^{\infty} \langle T_{kj}(z_j), x_j \rangle \right| \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{x}\|_{\ell^2(H)} \|\mathbf{z}\|_{\ell^2(H)}.$$

And applying duality, one has

$$\sup_{\|\mathbf{z}\|_{\ell^2(H)}} \sup_k \left(\sum_{j=1}^{\infty} \|T_{k,j}(z_j)\|^2 \right)^{\frac{1}{2}} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}.$$

Finally, observing the fact that $\sup_{\|\mathbf{z}\|_{\ell^2(H)}} \sup_k \left(\sum_{j=1}^{\infty} \|T_{k,j}(z_j)\|^2\right)^{\frac{1}{2}} = \sup_{k,j} \|T_{k,j}\|$, we conclude the proof. ∎

The following proposition is the analogue of Example 3.2.3 in the multiplier setting.

Proposition 3.3.2 *Let $A = (\alpha_{kj}) \in \mathcal{M}(\ell^2)$ and $T \in \mathcal{B}(H)$. Then $\mathbf{A} = (a_{k,j}T) \in \mathcal{M}(\ell^2(H))$ and*

$$\|\mathbf{A}\|_{M_l(\ell^2(H))} = \|\mathbf{A}\|_{M_r(\ell^2(H))} = \|A\|_{\mathcal{M}(\ell^2)} \|T\|_{\mathcal{B}(H)}$$

Proof: Let us prove it for left multipliers. Let $\mathbf{B} = (B_{k,j})_{k,j} \in \mathcal{B}(\ell^2(H))$, and take $\mathbf{x}, \mathbf{y} \in \ell^2(H)$.

$$\begin{aligned} | \ll \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \gg | &= |\sum_{k,j} a_{k,j} \langle T \circ B_{k,j}(x_j), y_k\rangle| = \\ &= |\sum_{k,j} a_{k,j} \langle T \circ B_{k,j}(\frac{x_j}{\|x_j\|}), \frac{y_k}{\|y_k\|}\rangle \|x_j\| \|y_k\|| \end{aligned}$$

Since A is a Schur multiplier, it verifies

$$|\sum_{k,j} a_{k,j} b_{k,j} \alpha_j \beta_k| \leq \|A\|_{\mathcal{M}(\ell^2)} \|(\alpha_j)\|_{\ell^2} \|(\beta_k)\|_{\ell^2} \|B\|_{\mathcal{B}(\ell^2)}, \quad \forall (b_{k,j}) \in \mathcal{B}(\ell^2), (\alpha_j), (\beta_k) \in \ell^2.$$

Let us show that $\left(\left\langle T \circ B_{k,j}(\frac{x_j}{\|x_j\|}), \frac{y_k}{\|y_k\|}\right\rangle\right) \in \mathcal{B}(\ell^2)$ to apply the previous condition and to obtain the first inequality. Indeed, using the boundedness of $\mathbf{B}$, we have for $\sum_j |\alpha_j|^2 = \sum_k |\beta_k|^2 = 1$

$$\begin{aligned} \left|\sum_{k,j} \left\langle T \circ B_{k,j}(\frac{x_j}{\|x_j\|}), \frac{y_k}{\|y_k\|}\right\rangle \alpha_j \beta_k\right| &= \left|\sum_{k,j} \left\langle T \circ B_{k,j}\left(\alpha_j \frac{x_j}{\|x_j\|}\right), \beta_k \frac{y_k}{\|y_k\|}\right\rangle\right| \\ &\leq \left(\sum_k \left\|\sum_j T \circ B_{k,j}\left(\alpha_j \frac{x_j}{\|x_j\|}\right)\right\|^2\right)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_k \left\| T \sum_j B_{k,j} \left(\alpha_j \frac{x_j}{\|x_j\|}\right)\right\|^2\right)^{\frac{1}{2}} \leq \\
&\leq \|T\| \left(\sum_k \left\| \sum_j B_{k,j} \left(\alpha_j \frac{x_j}{\|x_j\|}\right)\right\|^2\right)^{\frac{1}{2}} \\
&\leq \|T\| \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}.
\end{aligned}
$$

Therefore,

$$
| \ll \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \gg | \leq \|\mathbf{A}\|_{\mathcal{M}(\ell^2)} \|T\| \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \|\mathbf{x}\|_{\ell^2(H)} \|\mathbf{y}\|_{\ell^2(H)},
$$

and taking supremums, we get the desired inequality

$$
\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} \leq \|T\| \cdot \|A\|_{\mathcal{M}(\ell^2)}.
$$

To check that $\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} \geq \|T\| \cdot \|A\|_{\mathcal{M}(\ell^2)}$, we shall select a particular set of matrices from $\mathcal{B}(\ell^2(H))$ and apply Example 3.2.3.

$$
\begin{aligned}
\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} &= \sup_{\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}=1} \|(a_{k,j}T)_{k,j} * \mathbf{B}\|_{B(\ell^2(H))} \geq \\
&\geq \sup_{\|(b_{k,j}Id)_{k,j}\|_{\mathcal{B}(\ell^2(H))}=1} \|(a_{k,j}T)_{k,j} * (b_{k,j}Id)_{k,j}\|_{\mathcal{B}(\ell^2(H))} = \\
&= \sup_{\|(b_{k,j}Id)_{k,j}\|_{\mathcal{B}(\ell^2(H))}=1} \|(a_{k,j}b_{k,j}T)_{k,j}\|_{\mathcal{B}(\ell^2(H))} = \\
&= \sup_{\|(b_{k,j})_{k,j}\|_{\mathcal{B}(\ell^2)}=1} \|T\| \cdot \|(a_{k,j}b_{k,j})_{k,j}\|_{B(\ell^2)} = \\
&= \|T\| \cdot \|A\|_{\mathcal{M}(\ell^2)}
\end{aligned}
$$

The case $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ follows the same argument. ∎

The purpose of the rest of the section is to give a characterization of matrices in $\mathcal{B}(\ell^2(H))$ and $\mathcal{M}(\ell^2(H))$ by means of Schur products. First, we recall the definition of a summability kernel.

A sequence $(k_n)_n \subset L^1(\mathbb{T})$ is called a "summability kernel" (also denoted a "bounded approximation of the identity") if the following properties are satisfied:

1) $\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)dt = 1$.

2) $\sup_{n\in\mathbb{N}} \frac{1}{2\pi}\int_{-\pi}^{\pi} |k_n(t)|dt = C < \infty$.

3) $\frac{1}{2\pi}\int_{\delta\le|t|\le\pi} k_n(t)dt \underset{n\to\infty}{\longrightarrow} 0, \ \forall 0<\delta<\pi$.

Classical examples to be used in the sequel are the Féjer kernel

$$K_n(t) = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikt}$$

and the Poisson kernel

$$P_r(t) = \sum_{k\in\mathbb{Z}} r^{|k|}e^{ikt}.$$

As we pointed out in the introduction, for $\mathbf{A} = (T_{k,j})$, we shall use the notation

$$\sigma_n(\mathbf{A}) = \mathbf{M}_{K_n} * \mathbf{A}$$

and

$$P_r(\mathbf{A}) = \mathbf{M}_{P_r} * \mathbf{A}.$$

Observe that, with the assumption (3.2), one has that $\sigma_n(\mathbf{A}) \in \mathcal{P}(\ell^2(H))$ and also $P_r(\mathbf{A}) = \sum_{l\in\mathbb{Z}} r^{|l|}\mathbf{D}_l \in \mathcal{A}(\ell^2(H))$ since $\sup_l\|D_l\| < \infty$. It is also important to observe the following.

Remark 3.3.3 *Any matrix $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ satisfies condition (3.2). This is clear since, for $j,k\in\mathbb{N}$, one can write*

$$\|T_{k,j}\| = \sup_{\substack{\|x\|=1\\ \|y\|=1}} \ll \mathbf{A}x\mathbf{e}_j, y\mathbf{e}_k \gg .$$

Lemma 3.3.4 *Let* $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *and* k_n *a summability kernel, and denote* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$. *Then,* $M_n(\mathbf{A}) \xrightarrow[n\to\infty]{WOT} \mathbf{A}$. *In other words,*

$$\ll M_n(\mathbf{A})\mathbf{x}, \mathbf{y} \gg \xrightarrow[n\to\infty]{} \ll \mathbf{A}\mathbf{x}, \mathbf{y} \gg \quad \textit{for all } \mathbf{x}, \mathbf{y} \in \ell^2(H).$$

Proof: As $M_n(\mathbf{A}) = \sum_{l\in\mathbb{Z}} \hat{k}_n(l)\mathbf{D}_l$, we have that $\mathbf{A} - M_n(\mathbf{A}) = \sum_{l\in\mathbb{Z}}(1 - \hat{k}_n(l))\mathbf{D}_l$. Let $x, y \in H$ and $j, k \in \mathbb{N}$. Then

$$\ll (\mathbf{A} - M_n(\mathbf{A}))x\mathbf{e}_j, y\mathbf{e}_k \gg = (1 - \hat{k}_n(j-k))\langle T_{kj}x, y\rangle.$$

Since k_n is a summability kernel, we know that $\lim_n k_n * g = g \ \ \forall g \in L^1(\mathbb{T})$, which gives that $\hat{k}_n(l) \to 1$ as $n \to \infty$ $\forall l \in \mathbb{Z}$, and we conclude that $\ll (\mathbf{A} - M_n(\mathbf{A}))x\mathbf{e}_j, y\mathbf{e}_k \gg \xrightarrow[n\to\infty]{} 0$.

By linearity the same holds for $\mathbf{x}, \mathbf{y} \in c_{00}(H)$ and for general values we use the standard approximation argument. Let $\mathbf{x} \in \ell^2(H)$ and $\mathbf{y} \in \ell^2(H)$ and take sequences $(\mathbf{x}_N)_N \subset c_{00}(H)$ and $(\mathbf{y}_N)_N \subset c_{00}(H)$ such that $\|\mathbf{x} - \mathbf{x}_N\| \to 0$ and $\|\mathbf{y} - \mathbf{y}_N\| \to 0$. Then,

$$\begin{aligned} |\ll (\mathbf{A} - M_n(\mathbf{A}))\mathbf{x}, \mathbf{y} \gg| &= |\ll (\mathbf{A} - M_n(\mathbf{A}))(\mathbf{x} - \mathbf{x}_N), \mathbf{y} \gg| \\ &+ |\ll (\mathbf{A} - M_n(\mathbf{A}))\mathbf{x}_N, \mathbf{y} - \mathbf{y}_N \gg| \\ &+ |\ll (\mathbf{A} - M_n(\mathbf{A}))\mathbf{x}_N, \mathbf{y}_N \gg| \\ &\le \|\mathbf{A} - M_n(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))}(\|\mathbf{x} - \mathbf{x}_N\|\|\mathbf{y}\| + \|\mathbf{x}_N\|\|\mathbf{y} - \mathbf{y}_N\|) \\ &+ \ll (\mathbf{A} - M_n(\mathbf{A}))\mathbf{x}_N, \mathbf{y}_N \gg . \end{aligned}$$

Notice that from a combined use of Proposition 3.3.2 and (2.1.4), we have

$$\|M_n(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} \le \|\mathbf{M}_n\|_{\mathcal{M}_r(\ell^2(H))}\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|k_n\|_{L^1(\mathbb{T})}\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \tag{3.3}$$

and the estimates $\|\mathbf{A} - M_n(\mathbf{A})\| \le (1 + C)\|\mathbf{A}\|$ and $\|\mathbf{x}_N\| \le \|\mathbf{x}\|$ allow us to finish the proof letting $N \to \infty$. ∎

Proposition 3.3.5 *Let* $\mathbf{A}$ *be a matrix with entries in* $\mathcal{B}(H)$ *and* $\{k_n\}$ *a summability kernel, and denote* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$*. Then:*

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H)) \Leftrightarrow \sup_n \|M_n(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} < \infty$.

(ii) $\mathbf{A} \in \mathcal{M}_r(\ell^2(H)) \Leftrightarrow \sup_n \|M_n(\mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} < \infty$.

(iii) $\mathbf{A} \in \mathcal{M}_l(\ell^2(H)) \Leftrightarrow \sup_n \|M_n(\mathbf{A})\|_{\mathcal{M}_l(\ell^2(H))} < \infty$.

Proof: (i) If $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ then the estimate (3.3) gives that $\sup_n \|M_n(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} < \infty$. Assume now that $\sup_n \|M_n(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} < \infty$. As a consequence of Lemma 3.3.4, one obtains that

$$\ll \mathbf{A}\mathbf{x}, \mathbf{y} \gg = \lim_n \ll M_n(\mathbf{A})\mathbf{x}, \mathbf{y} \gg .$$

This gives that $\ll \mathbf{A}\mathbf{x}, \mathbf{y} \gg \leq C\|\mathbf{x}\|\|\mathbf{y}\|$ and hence $\mathbf{A} \in \mathcal{B}(\ell^2(H)))$.

(ii) Assume that $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$. Since $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A} = \mathbf{A} * \mathbf{M}_{k_n}$ we have:

$$\|\mathbf{M}_{k_n} * \mathbf{A}\|_{\mathcal{M}(\ell^2(H))} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \cdot \|\mathbf{M}_{k_n}\|_{\mathcal{M}_r(\ell^2(H))} = \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \|k_n\|_{L^1(\mathbb{T})}.$$

And taking the supremum with respect to n, we conclude that

$$\sup_n \|M_n(\mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} \leq C\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} < \infty$$

Conversely, assume $\sup_n \|M_n(\mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} < \infty$, and take $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. We have

$$\|M_n(\mathbf{B} * \mathbf{A})\|_{\mathcal{B}(\ell^2(H))} \leq \sup_n \|M_n(\mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} \cdot \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}.$$

And using again (as seen in Lemma 3.3.4) that $M_n(\mathbf{B} * \mathbf{A})$ converges in the weak operator topology to $\mathbf{B} * \mathbf{A}$, we have that $\mathbf{B} * \mathbf{A} \in \mathcal{B}(\ell^2(H))$, which means that $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$.

(iii) is proven using the same ideas. ∎

3.4 The space of continuous matrices

As it was defined in the introductory section, $C(\ell^2(H))$ is the subspace of $\mathcal{B}(\ell^2(H))$ formed by those matrices that can be approximated in the operator norm by matrices with a finite number of diagonals, called polynomials in $\mathcal{P}(\ell^2(H))$. Throughout the section, we shall see some results that will make clear the reason why we choose to refer to this type of matrices as "continuous matrices".

Given $\mathbf{A} = (T_{kj})$ we define

$$f_{\mathbf{A}}(t) = \mathbf{M}_t * \mathbf{A} = (e^{i(j-k)t} T_{kj}), \quad t \in [-\pi, \pi)$$

Of course if $\mathbf{A} \in \mathcal{P}(\ell^2(H))$ one has

$$f_{\mathbf{A}}(t) = \sum_{l \in \mathbb{Z}} \mathbf{D_l} e^{ilt} \in P(\mathbb{T}, \mathcal{B}(\ell^2(H))).$$

It turns out that this function takes values in spheres of $\mathcal{B}(\ell^2(H))$ or $\mathcal{M}_r(\ell^2(H))$, whenever $\mathbf{A}$ belongs to $\mathcal{B}(\ell^2(H))$ or $\mathcal{M}_r(\ell^2(H))$.

Proposition 3.4.1 *Let* $\mathbf{A} = (T_{kj})$ *be a matrix and* $t \in [-\pi, \pi)$.

(i) If $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *then* $\|f_{\mathbf{A}}(t)\|_{\mathcal{B}(\ell^2(H))} = \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}$.

(ii) If $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ *then* $\|f_{\mathbf{A}}(t)\|_{\mathcal{M}_r(\ell^2(H))} = \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}$.

(iii) If $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ *then* $\|f_{\mathbf{A}}(t)\|_{\mathcal{M}_l(\ell^2(H))} = \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}$.

Proof: (i) It is a consequence of Proposition 3.3.2 and Bennett's theorem. Indeed,

$$\begin{aligned}
\|f_{\mathbf{A}}(t)\|_{\mathcal{B}(\ell^2(H))} &= \|\mathbf{A} * (e^{i(j-k)t} \cdot Id)_{j,k}\|_{\mathcal{B}(\ell^2(H))} \\
&\leq \|(e^{i(j-k)t} \cdot Id)_{j,k}\|_{\mathcal{M}(\ell^2(H))} \cdot \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \\
&\overset{\text{Prop. 3.3.2}}{=} \|\left(e^{i(j-k)t}\right)_{j,k}\|_{\mathcal{M}(\ell^2)} \cdot \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \\
&\overset{\text{Bennett}}{=} \|\sum_k e^{i(j-k)t}\|_{M(\mathbb{T})} \cdot \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \\
&= \|\delta_{-t}\|_{M(\mathbb{T})} \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}
\end{aligned}$$

To get the other inequality, we take into account that $\mathbf{A} = f_{\mathbf{A}}(t) * (e^{i(k-j)t} \cdot Id)_{j,k\geq 0}$, so we have

$$\begin{aligned}
\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} &\leq \|f_{\mathbf{A}}(t)\|_{\mathcal{B}(\ell^2(H))} \cdot \|(e^{i(k-j)t} \cdot Id)_{j,k}\|_{\mathcal{M}(\ell^2(H))} \\
&\overset{\text{Prop. 3.3.2}}{=} \|\left(e^{i(j-k)t}\right)_{j,k}\|_{\mathcal{M}(\ell^2)} \cdot \|f_{\mathbf{A}}(t)\|_{\mathcal{B}(\ell^2(H))} \\
&\overset{\text{Bennett}}{=} \|\sum_k e^{i(k-j)t}\|_{M(\mathbb{T})} \cdot \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \\
&= \|\delta_t\|_{M(\mathbb{T})}\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}.
\end{aligned}$$

(ii) Use the same argument, but with multiplier norm instead.

(iii) Follows from (ii) taking adjoints or repeating the argument above. ∎

Proposition 3.4.2 *Let $\mathbf{A} = (T_{kj})$ with $T_{kj} = 0$ for each $j \neq 2k$ and $T_{k,2k} = I$ for $k \in \mathbb{N}$. Then $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ and the function $t \to f_{\mathbf{A}}(t)$ is not strongly measurable with values in $\mathcal{B}(\ell^2(H))$.*

Proof: Since $\mathbf{A}\mathbf{x} = (x_{2k})_k$, it is clear that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$, moreover $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = 1$. However, if we take $x = (x_i)_i \in \ell^2(H)$, we notice that

$$(f_{\mathbf{A}}(t) - f_{\mathbf{A}}(s))(\mathbf{x}) = \left((e^{ikt} - e^{iks}) \cdot x_{2k}\right)_k .$$

Therefore, if $t \neq s$,

$$\begin{aligned}
\|f_{\mathbf{A}}(t) - f_{\mathbf{A}}(s)\|_{\mathcal{B}(\ell^2(H))} &= \sup_{\sum_i \|x_i\|^2 \leq 1} \sqrt{\sum_k |e^{i(t-s)} - 1|^2 \|x_{2k-1}\|^2} \\
&= \sup_{k\in\mathbb{N}} |e^{ik(t-s)} - 1| \geq \sqrt{2}.
\end{aligned}$$

This gives that $\{f_{\mathbf{A}}(t) : t \in [-\pi, \pi]\}$ is not separable in $\mathcal{B}(\ell^2(H))$. Therefore, $t \to f_{\mathbf{A}}(t)$ is not strongly measurable by Pettis's measurability theorem. ∎

Proposition 3.4.3 *Let* $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *and* $\mathbf{x} \in \ell^2(H)$. *Then the map*

$$t \to f_{\mathbf{A}}(t)(\mathbf{x}) = \left(e^{-ikt} \sum_{j=1}^{\infty} T_{kj}(x_j) e^{ijt} \right)_k$$

belongs to $C(\mathbb{T}, \ell^2(H))$.

Proof: Let us consider first $\mathbf{x} = x\mathbf{e}_j \in \ell^2(H)$ for some $x \in H$ and $j \in \mathbb{N}$. In this case, $f_{\mathbf{A}}$ looks as follows:

$$f_{\mathbf{A}}(t)(x\mathbf{e}_j) = \left(T_{k,j}(x) e^{i(j-k)t}\right)_k .$$

Fix $\varepsilon > 0$. As a consequence of $\mathbf{A}$ being in $\mathcal{B}(\ell^2(H))$, we have that the series $\sum_k \|T_{k,j}(x)\|^2$ is convergent. Then, we can select $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \|T_{k,j}(x)\|^2 < \varepsilon/8$. Let $\varepsilon_j = \frac{\varepsilon}{2\cdot(N-1)\cdot \sup\limits_{k<N} \|T_{k,j}(x)\|}$, and let δ_j defined in such a way that

$$\sup_{k<N} |e^{i(j-k)u} - 1|^2 < \varepsilon_j, \text{ for all } |u| < \delta_j.$$

Hence, whenever $s, t \in [-\pi, \pi)$ satisfy that $|s-t| < \delta_j$, we have

$$\begin{aligned}
\|f_{\mathbf{A}}(t)(x\mathbf{e}_j) - f_{\mathbf{A}}(s)(x\mathbf{e}_j)\|^2_{\ell^2(H)} &= \sum_k \|T_{k,j}(x)(e^{i(j-k)t} - e^{i(j-k)s})\|^2 = \\
&= \sum_{k=1}^{N-1} \|T_{k,j}(x)(e^{i(j-k)t} - e^{i(j-k)s})\|^2 \\
&+ \sum_{k=N}^{\infty} \|T_{k,j}(x)(e^{i(j-k)t} - e^{i(j-k)s})\|^2 \\
&\leq \sup_{k<N} \|T_{k,j}(x)\| \cdot \sum_{k=1}^{N-1} |e^{i(j-k)t} - e^{i(j-k)s}|^2 + 4 \cdot \varepsilon/8 \\
&\leq \sup_{k<N} \|T_{k,j}(x)\| \cdot \sum_{k=1}^{N-1} |e^{i(j-k)(t-s)} - 1|^2 + \varepsilon/2 \ < \varepsilon.
\end{aligned}$$

So, we have checked that $f_{\mathbf{A}}(t)(x\mathbf{e}_j)$ is continuous. Therefore $f_{\mathbf{A}}(t)(\mathbf{x})$ is also continuous for $\mathbf{x} \in c_{00}(H)$. To obtain the general case, simply consider $\mathbf{x} \in \ell^2(H)$, and select a sequence $(\mathbf{x}_N)_N := ((x_i)_{i=1}^N)_N \subset c_{00}(H)$ converging to $\mathbf{x}$. Then

$$f_{\mathbf{A}}(t)(\mathbf{x}) = f_{\mathbf{A}}(t)(\mathbf{x} - \mathbf{x}_N) + f_{\mathbf{A}}(t)(\mathbf{x}_N)$$

and observe that using (i) in Proposition 3.4.1 we have that

$$\sup_t \|f_{\mathbf{A}}(t)(\mathbf{x} - \mathbf{x}_N)\| \le \sup_t \|f_{\mathbf{A}}(t)\| \|\mathbf{x} - \mathbf{x}_N\| = \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \cdot \|\mathbf{x} - \mathbf{x}_N\| \xrightarrow[N\to\infty]{} 0.$$

Therefore, $f_{\mathbf{A}}(t)(\mathbf{x})$ is continuous since it is a uniform limit of the continuous functions $f_{\mathbf{A}}(t)(\mathbf{x}_N)$. ∎

Theorem 3.4.4 *Let* $\mathbf{A} = (T_{k,j})_{k,j}$ *be a matrix with entries in* $\mathcal{B}(H)$*, satisfying that* $\sup_{k,j}\|T_{k,j}\| < \infty$*. The following statements are equivalent:*

1) $\mathbf{A} \in C(\ell^2(H))$*.*

2) $\lim_{n\to\infty} M_n(\mathbf{A}) = \mathbf{A}$ *in* $\mathcal{B}(\ell^2(H))$ *where* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$ *and* $\{k_n\} \subseteq L^1(\mathbb{T})$ *is a summability kernel.*

3) $\lim_{n\to\infty} \sigma_n(\mathbf{A}) = \mathbf{A}$ *in* $\mathcal{B}(\ell^2(H))$*.*

4) $t \to f_{\mathbf{A}}(t)$ *is a* $\mathcal{B}(\ell^2(H))$*-valued continuous function.*

Proof: 1)⇒ 2). Let $\varepsilon > 0$. Since $\mathbf{A}$ is a continuous matrix, we can select $\mathbf{P} = (S_{k,j})_{k,j} = \sum_{l=-N}^{N} D_l \in \mathcal{P}(\ell^2(H))$ a polynomial matrix satisfying $\|\mathbf{A} - \mathbf{P}\|_{\mathcal{B}(\ell^2(H))} < \varepsilon/3C$, where $C = \sup_n \|k_n\|_{L^1(\mathbb{T})}$. Then,

$$\begin{aligned} \|M_n(\mathbf{P}) - \mathbf{P}\|_{\mathcal{B}(\ell^2(H))} &= \|\sum_{l=-N}^{N} (\hat{k_n}(l) - 1) D_l\|_{\mathcal{B}(\ell^2(H))} \\ &\le \sup_{k,j}\|S_{k,j}\| \cdot (2N+1) \cdot \max_{|l|\le N} |\hat{k_n}(l) - 1|. \end{aligned}$$

Since $\{k_n\}$ is a summability kernel, we have that $\lim_n k_n * g = g \ \ \forall g \in L^1(\mathbb{T})$, and therefore $\lim_n \hat{k}_n(l) = 1 \ \forall l \in \mathbb{Z}$. As a consequence, we can choose $n_0 \in \mathbb{N}$ such that $|\hat{k}_n(l) - 1| < \frac{\varepsilon}{3(2N+1)\sup_{k,j}\|S_{k,j}\|}$ $\forall n \ge n_0$ and $\forall |l| \le N$. This way, we have that $\|M_n(\mathbf{P}) -$

$\mathbf{P}\|_{\mathcal{B}(\ell^2(H))} < \varepsilon/3$. Finally, for $n \geq n_0$,

$$\begin{aligned}\|M_n(\mathbf{A}) - \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} &\leq \|\mathbf{M}_n * (\mathbf{A} - \mathbf{P})\|_{\mathcal{B}(\ell^2(H))} \\ &+ \|\mathbf{M}_n * \mathbf{P} - \mathbf{P}\|_{\mathcal{B}(\ell^2(H))} + \|\mathbf{P} - \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq \\ &\leq \|\mathbf{M}_n\|_{\mathcal{M}(\ell^2(H))} \cdot \|\mathbf{A} - \mathbf{P}\|_{\mathcal{B}(\ell^2(H))} + \varepsilon/3 + \varepsilon/3 = \\ &\leq \|k_n\|_{L^1} \cdot \varepsilon/3C + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

2)$\Rightarrow$ 3). This is obvious, since the Fejér Kernel is an example of summability kernel.

3)$\Rightarrow$ 4). Notice that $\sigma_n(f_{\mathbf{A}}(t)) = f_{\sigma_n(\mathbf{A})}(t) \in \mathcal{P}(\mathbb{T}, \mathcal{B}(\ell^2(H)))$. As a consequence of Proposition 3.4.1, we have

$$\begin{aligned}\sup_t \|f_{\sigma_n(\mathbf{A})}(t) - f_{\mathbf{A}}(t)\|_{\mathcal{B}(\ell^2(H))} &= \sup_t \|f_{\sigma_n(\mathbf{A})-\mathbf{A}}(t)\|_{\mathcal{B}(\ell^2(H))} \\ &= \|\sigma_n(\mathbf{A}) - \mathbf{A}\|_{\mathcal{B}(\ell^2(H))},\end{aligned}$$

so we obtain that $f_{\mathbf{A}}$ is a uniform limit of $\mathcal{B}(\ell^2(H))$-valued polynomials, so it is a continuous function.

4) $\Rightarrow$ 1). Since $f_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(\ell^2(H)))$, we can approximate it in norm by polynomials in $\mathcal{P}(\mathbb{T}, \mathcal{B}(\ell^2(H)))$. Invoking Proposition 3.4.1 again yields

$$\|\mathbf{P} - \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \sup_t \|f_{\mathbf{P}}(t) - f_{\mathbf{A}}(t)\|_{\mathcal{B}(\ell^2(H))},$$

which gives the result. ∎

Corollary 3.4.5 *$\mathcal{A}(\ell^2(H))$ is dense in $C(\ell^2(H))$.*

Proof: Given $\mathbf{A} \in C(\ell^2(H))$ one has that $P_r(\mathbf{A}) = \mathbf{M}_{P_r} * \mathbf{A}$ converges to $\mathbf{A}$ in $\mathcal{B}(\ell^2(H))$ by the previous proposition. Also, observe that the fact that $\mathbf{A}$ belongs to $\mathcal{B}(\ell^2(H))$ ensures that it verifies (3.2), as seen in Remark 3.3.3. Since $P_r(\mathbf{A}) = \sum_l r^{|l|}\mathbf{D_l}$, then $P_r(\mathbf{A}) \in \mathcal{A}(\ell^2(H))$ for each $0 < r < 1$, and the result is complete. ∎

Remark 3.4.6 *$C(\ell^2(H))$ is an ideal of $\mathcal{B}(\ell^2(H))$, that is to say if $\mathbf{A} \in C(\ell^2(H))$ and $\mathbf{B} \in \mathcal{B}(\ell^2(H))$ then $\mathbf{A} * \mathbf{B} \in C(\ell^2(H))$ and $\mathbf{B} * \mathbf{A} \in C(\ell^2(H))$. To see this, it is enough to observe that $\sigma_n(\mathbf{A} * \mathbf{B}) = \sigma_n(\mathbf{A}) * \mathbf{B}$ and $\sigma_n(\mathbf{B} * \mathbf{A}) = \mathbf{B} * \sigma_n(\mathbf{A})$.*

Definition 3.4.7 *We shall denote by $(C(\ell^2(H)), C(\ell^2(H)))_l$ the set of those matrices $\mathbf{A}$ such that*

$$\mathbf{A} * \mathbf{B} \in C(\ell^2(H)) \quad \forall \mathbf{B} \in C(\ell^2(H)).$$

A similar definition can be given for $(C(\ell^2(H)), C(\ell^2(H)))_r$ when the matrix $\mathbf{A}$ is multiplying from the right hand side.

Theorem 3.4.8 *$\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ (respectively $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$) if and only if $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_l$ (respectively $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_r$).*

Proof: We shall prove it only for left Schur multipliers, the other case is similar. Assume that $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ and $\mathbf{B} \in C(\ell^2(H))$. To check that $\mathbf{A} * \mathbf{B} \in C(\ell^2(H))$ observe that $\sigma_n(\mathbf{A} * \mathbf{B}) = \mathbf{A} * \sigma_n(\mathbf{B})$ and take limits as $n \to \infty$.

Let us suppose now that $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_l$, and take $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. Observe that $\sigma_n(\mathbf{B}) \in \mathcal{P}(\ell^2(H)) \subset C(\ell^2(H))$. Therefore, we can apply the hypothesis to obtain that for all $n \in \mathbb{N}$,

$$\begin{aligned} \|\sigma_n(\mathbf{A} * \mathbf{B})\|_{\mathcal{B}(\ell^2(H))} &= \|\mathbf{A} * \sigma_n(\mathbf{B})\|_{\mathcal{B}(\ell^2(H))} \\ &\leq \|\mathbf{A}\|_{(C(\ell^2(H)), C(\ell^2(H)))_l} \cdot \|\sigma_n(\mathbf{B})\|_{\mathcal{B}(\ell^2(H))} \\ &\leq \|\mathbf{A}\|_{(C(\ell^2(H)), C(\ell^2(H)))_l} \cdot \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}. \end{aligned}$$

In particular, this gives that for all $\mathbf{x}, \mathbf{y} \in \ell^2(H)$ with norm 1,

$$| \ll \sigma_n(\mathbf{A} * \mathbf{B})(\mathbf{x}), \mathbf{y} \gg | \leq \|\mathbf{A}\|_{(C(\ell^2(H)), C(\ell^2(H)))_l} \cdot \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \quad \forall n \in \mathbb{N}.$$

Lemma 3.3.4 implies that

$$| \ll \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \gg | \leq \|\mathbf{A}\|_{(C(\ell^2(H)), C(\ell^2(H)))_l} \cdot \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))},$$

and therefore, taking supremums with respect to $\mathbf{x}$ and $\mathbf{y}$, we get

$$\|\mathbf{A} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \leq \|\mathbf{A}\|_{(C(\ell^2(H)),C(\ell^2(H)))_l} \cdot \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))},$$

so $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ and $\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} \leq \|\mathbf{A}\|_{(C(\ell^2(H)),C(\ell^2(H)))_l}$. The proof is complete ∎

3.4.1 The Toeplitz case

Toeplitz matrices shall be the main focus of this subsection. To abbreviate, the notations $\mathcal{B}(\ell^2(H))_{\mathcal{T}} = \mathcal{B}(\ell^2(H)) \cap \mathcal{T}$ and $C(\ell^2(H))_{\mathcal{T}} = C(\ell^2(H)) \cap \mathcal{T}$ will be used.

As it was pointed out in Chapter 2 (Section 2.3), the space of operators $\mathcal{B}(L^1(\mathbb{T}), \mathcal{B}(H))$ can be identified with the space of measures $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ using $\mu(C) = \Phi(\chi_C)$ for any measurable set C. Hence

$$\mathcal{B}(L^1(\mathbb{T}), \mathcal{B}(H)) = (L_1(\mathbb{T})\hat{\otimes}(H\hat{\otimes}H))^* = V^\infty(\mathbb{T}, \mathcal{B}(H))$$

Also in Chapter 2 (more precisely, in Theorem 2.5.1) it was shown that $\mathbf{A} = (T_{kj}) \in \mathcal{B}(\ell^2(H))_{\mathcal{T}} \cap \mathcal{T}$ if and only if there exists $\mu_{\mathbf{A}} \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ such that $\widehat{\mu_{\mathbf{A}}}(j-k) = T_{kj}$ for $j, k \in \mathbb{N}$. We are going to prove the corresponding result for continuous matrices, namely that $\mathbf{A} \in C(\ell^2(H))_{\mathcal{T}} \cap \mathcal{T}$ if and only if there exists $g_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(H))$ such that $\widehat{g_{\mathbf{A}}}(j-k) = T_{kj}$ for $j, k \in \mathbb{N}$. However, this time we shall give a direct proof of the result, not utilizing vector measures. The next lemma will be important.

Lemma 3.4.9 *Let $f \in C(\mathbb{T}, \mathcal{B}(H))$, and consider $\mathbf{A}_f = (T_{k,j})_{k,j}$ with $T_{k,j} := \hat{f}(j-k)$. Then, $\mathbf{A}_f \in C(\ell^2(H))_{\mathcal{T}}$, with*

$$\|\mathbf{A}_f\|_{\mathcal{B}(\ell^2(H))} = \|f\|_{C(\mathbb{T},\mathcal{B}(H))}$$

Proof: Let $\mathbf{x}, \mathbf{y} \in \ell^2(H)$. Observe that $\sum_j x_j e^{-ijt} \otimes \sum_k y_k e^{-ikt} \in L^1(\mathbb{T}, H\hat{\otimes}H)$. Indeed,

$$\int_0^{2\pi} \|\sum_j x_j e^{-ijt} \otimes \sum_k y_k e^{-ikt}\|_{H\hat{\otimes}H} \frac{dt}{2\pi} = \int_0^{2\pi} \|\sum_j x_j e^{-ijt}\| \|\sum_k y_k e^{-ikt}\| \frac{dt}{2\pi}$$

$$\begin{aligned}
&\leq \left(\int_0^{2\pi} \|\sum_j x_j e^{-ijt}\|^2 \frac{dt}{2\pi}\right)^{1/2} \left(\int_0^{2\pi} \|\sum_k y_k e^{-ikt}\|^2 \frac{dt}{2\pi}\right)^{1/2}\\
&= \|\mathbf{x}\|_{\ell^2(H)}\|\mathbf{y}\|_{\ell^2(H)}.
\end{aligned}$$

By means of the identification $\mathcal{B}(H) = (H\hat{\otimes}H)^*$, we are able to write

$$\begin{aligned}
|\ll \mathbf{A}_f\mathbf{x}, \mathbf{y} \gg| &= |\sum_{k,j}\langle T_{k,j}x_j, y_k\rangle| = |\sum_{k,l}\langle T_l x_{l+k}, y_k\rangle|\\
&= \left|\sum_l T_l \left(\sum_k x_{l+k}\otimes y_k\right)\right|\\
&= \left|\int_0^{2\pi}\left(\sum_l \hat{f}(l)e^{ilt}\right)\left(\sum_l\left(\sum_k x_{l+k}\otimes y_k\right)e^{-ilt}\right)\frac{dt}{2\pi}\right|\\
&= \left|\int_0^{2\pi} f(t)\left(\left(\sum_j x_k e^{-ijt}\right)\otimes\left(\sum_k y_k e^{-ikt}\right)\right)\frac{dt}{2\pi}\right|\\
&\leq \int_0^{2\pi} \|f(t)\|_{\mathcal{B}(H)}\left\|\left(\sum_j x_k e^{-ijt}\right)\otimes\left(\sum_k y_k e^{-ikt}\right)\right\|_{H\hat{\otimes}H}\frac{dt}{2\pi}\\
&= \|f\|_{C(\mathbb{T},\mathcal{B}(H))}\int_0^{2\pi}\|\sum_j x_k e^{-ijt}\|\|\sum_k y_k e^{-ikt}\|\frac{dt}{2\pi}\\
&= \|f\|_{C(\mathbb{T},\mathcal{B}(H))}\|\mathbf{x}\|_{\ell^2(H)}\|\mathbf{y}\|_{\ell^2(H)},
\end{aligned}$$

which yields $\|\mathbf{A}_f\|_{\mathcal{B}(\ell^2(H))} \leq \|f\|_{C(\mathbb{T},\mathcal{B}(H))}$. Let us prove the other inequality. In order to do that, select $(x\alpha_j)_j$ and $(y\beta_k)_k$, where $x, y \in H$ are unitary and $(\alpha_j)_j, (\beta_k)_k$ are elements in the unit sphere of ℓ^2. Hence

$$\begin{aligned}
|\langle \mathbf{A}_f(x\alpha_j), y\beta_k\rangle| &= \left|\sum_{k,j}\langle T_{j-k}x, y\rangle\alpha_j\beta_k\right| = \left|\sum_l\langle T_l x, y\rangle\sum_k \alpha_{l+k}y_k\right| =\\
&= \left|\int_0^{2\pi}\left(\sum_l\langle T_l x, y\rangle e^{ilt}\right)\left(\sum_l\sum_k \alpha_{l+k}\beta_k e^{-ilt}\right)\right|\frac{dt}{2\pi} =
\end{aligned}$$

$$= \left| \int_0^{2\pi} \left(\sum_l \langle T_l x, y \rangle e^{ilt} \right) \left(\sum_j \alpha_j e^{ijt} \sum_k \beta_k e^{-ikt} \right) \right| \frac{dt}{2\pi}$$

Now, taking into account that by duality we have

$$\sup_t \left\| \sum_l \langle T_l x, y \rangle e^{ilt} \right\| = \sup_{\|g\|_{L^1(\mathbb{T})}=1} \left| \int \sum_l \langle T_l x, y \rangle e^{ilt} g(t) \frac{dt}{2\pi} \right|,$$

and also recalling the factorization $L^2(\mathbb{T}) \cdot L^2(\mathbb{T}) = L^1(\mathbb{T})$, we can obtain that

$$\sup_t \left\| \sum_l \langle T_l x, y \rangle e^{ilt} \right\| = \sup_{\substack{\|(\alpha_j)_j\|=1 \\ \|(\beta_k)_k\|=1}} |\langle \mathbf{A}_f(x\alpha_j), y\beta_k \rangle| \leq \|\mathbf{A}_f\|_{\mathcal{B}(\ell^2(H))} \|x\| \|y\|.$$

Therefore,

$$\|f\|_{C(\mathbb{T},\mathcal{B}(H))} = \sup_t \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \left\| \sum_l \langle T_l x, y \rangle e^{ilt} \right\| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \sup_t \left\| \sum_l \langle T_l x, y \rangle e^{ilt} \right\| \leq \|\mathbf{A}_f\|_{\mathcal{B}(\ell^2(H))}.$$

This shows that $\mathbf{A}_f \in \mathcal{B}(\ell^2(H))_{\mathcal{T}}$ and $\|f\|_{C(\mathbb{T},\mathcal{B}(H))} = \|\mathbf{A}_f\|_{\mathcal{B}(\ell^2(H))}$. To obtain that $\mathbf{A}_f \in C(\ell^2(H))$ simply observe that if P is a polynomial in $\mathcal{P}(\mathbb{T}, \mathcal{B}(H))$ then $\mathbf{A}_P \in \mathcal{P}(\ell^2(H))$ and the proof is complete using an approximation argument. ∎

When the Toeplitz case is considered, we have the next result, which gives a characterization of the space $C(\ell^2(H)) \cap \mathcal{T}$, stating that these matrices can be identified with the space of continuous functions.

Theorem 3.4.10 *Let $(T_l)_{l\in\mathbb{Z}}$ be a sequence of operators in $\mathcal{B}(H)$ and let $\mathbf{A} = (T_{j-k})_{k,j}$. Then, $\mathbf{A} \in C(\ell^2(H))_{\mathcal{T}}$ if and only if there exists $g_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(H))$ such that $\widehat{g_{\mathbf{A}}}(l) = T_l$. Moreover, $\|g_{\mathbf{A}}\|_{C(\mathbb{T},\mathcal{B}(H))} = \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}$.*

Proof: Start assuming that $\mathbf{A} \in \mathcal{P}(\ell^2(H))$. Then consider $g_{\mathbf{A}} = \sum_{l=-N}^{N} T_l e^{ikt} \in \mathcal{P}(\mathbb{T}, \mathcal{B}(H))$ and, due to Lemma 3.4.9, we obtain that $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|\mathbf{A}_{g_{\mathbf{A}}}\|_{\mathcal{B}(\ell^2(H))} = \|g_{\mathbf{A}}\|_{C(\mathbb{T},\mathcal{B}(H))}$.

Let us prove now the general case. If $\mathbf{A} \in C(\ell^2(H))$, we use that $\sigma_n(\mathbf{A}) \in \mathcal{P}(\ell^2(H))$ converges to $\mathbf{A}$ in $\mathcal{B}(\ell^2(H))$. In particular, $(\sigma_n(\mathbf{A}))_n$ is a Cauchy sequence in $\mathcal{B}(\ell^2(H))$. Using now Lemma 3.4.9 we get that for any n, m, $\|\sigma_n(\mathbf{A}) - \sigma_m(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} = \|g_{\sigma_n(\mathbf{A})} - g_{\sigma_m(\mathbf{A})}\|_{C(\mathbb{T},\mathcal{B}(H))}$. This shows that the sequence $(g_n)_n := (g_{\sigma_n(\mathbf{A})})_n$ is a Cauchy sequence in $C(\mathbb{T}, \mathcal{B}(H))$, and therefore it has a limit $g_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(H))$. Clearly $\widehat{g_{\mathbf{A}}}(l) = T_l$ for each $l \in \mathbb{Z}$ due to the fact $\widehat{g_n}(l) \to \widehat{g_{\mathbf{A}}}(l)$. Also

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \lim_n \|\sigma_n(A)\|_{\mathcal{B}(\ell^2(H))} = \lim_n \|g_n\|_{C(\mathbb{T},\mathcal{B}(H))} = \|g_{\mathbf{A}}\|_{C(\mathbb{T},\mathcal{B}(H))}.$$

Let us prove the converse. Assume that there exists $g_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(H))$ such that $\hat{g_{\mathbf{A}}}(l) = T_l$. Invoking Lemma 3.4.9, we obtain that $\mathbf{A}_{g_{\mathbf{A}}} \in C(\ell^2(H))_{\mathcal{T}}$ and $\|g_{\mathbf{A}}\|_{C(\mathbb{T},\mathcal{B}(H))} = \|\mathbf{A}_{g_{\mathbf{A}}}\|\mathcal{B}(\ell^2(H))$. Since $T_{kj} = T_{j-k} = \widehat{g_{\mathbf{A}}}(j-k)$ we have that $\mathbf{A}_{g_{\mathbf{A}}} = \mathbf{A}$ and the proof is complete. ∎

Now, we shall present another proof of Theorem 2.5.9 from Chapter 2, using Theorem 3.4.8 and Theorem 3.4.10.

Theorem 3.4.11 *Let $(T_l)_{l\in\mathbb{Z}}$ be a sequence of operators in $\mathcal{B}(H)$ and let $\mathbf{A} = (T_{j-k})_{k,j}$. If $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ then $\Psi_A \in \mathcal{B}(C(\mathbb{T}, H), H)$ and*

$$\|\Psi_A\|_{\mathcal{B}(C(\mathbb{T},H),H)} \leq \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}.$$

Proof: By hypothesis, Theorem 3.4.8 and Theorem 3.4.10, we know that for any $\sum_l S_l e^{ilt} \in \mathcal{P}(\mathbb{T}, \mathcal{B}(H))$, we have

$$\sup_{t\in[-\pi,\pi]} \left\| \sum_l T_l S_l e^{ilt} \right\|_{\mathcal{B}(H)} \leq \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} \sup_{t\in[-\pi,\pi]} \left\| \sum_l S_l e^{ilt} \right\|_{\mathcal{B}(H)}. \tag{3.4}$$

We need to prove that for any $\sum_l x_k e^{ilt} \in \mathcal{P}(\mathbb{T}, H)$,

$$\left\| \sum_l T_l x_l \right\| \leq \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} \left\| \sum_l x_l e^{ilt} \right\|_{C(\mathbb{T},H)}. \tag{3.5}$$

To do that, let $x_0 \in H$ with $\|x_0\| = 1$, and define $S_k = x_0 \otimes x_k$. One has that $x_k = S_k(x_0)$, $\sum_l S_k e^{ilt} \in \mathcal{P}(\mathbb{T}, \mathcal{B}(H))$ with

$$\left\|\sum_l x_l e^{ilt}\right\|_{C(\mathbb{T},H)} = \left\|\sum_l S_l e^{ilt}\right\|_{C(\mathbb{T},B(H))}.$$

By using (3.4), for any $x \in H$, it yields

$$\begin{aligned}
\left|\sum_l \langle T_l(x_l), x\rangle\right| &= \left|\left\langle \sum_l T_l S_l(x_0), x \right\rangle\right| \\
&\leq \|x\| \left\|\sum_l T_l S_l\right\|_{\mathcal{B}(H)} \\
&\leq \|x\| \sup_{t\in[-\pi,\pi]} \left\|\sum_l T_l S_l e^{ilt}\right\|_{\mathcal{B}(H)} \\
&\leq \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} \|x\| \sup_{t\in[-\pi,\pi]} \left\|\sum_l S_l e^{ilt}\right\|_{\mathcal{B}(H)} \\
&\leq \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} \|x\| \sup_{t\in[-\pi,\pi]} \left\|\sum_l x_l e^{ilt}\right\|.
\end{aligned}$$

This gives (3.5). Now, if we use the density of polynomials, we have shown that $\Psi_{\mathbf{A}}$ extends to a bounded linear operator from $C(\mathbb{T}, H)$ into H with $\|\Psi_{\mathbf{A}}\|_{\mathcal{B}(C(\mathbb{T},H),H))} \leq \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}$, and this concludes the proof. ∎

I turns out that, if a Toeplitz matrix $\mathbf{A}$ is considered to act as a multiplier just on Toeplitz matrices, the SOT-measures actually describe this class of multipliers. This is the content of the following theorem.

Theorem 3.4.12 *Let $(T_l)_{l\in\mathbb{Z}}$ be a sequence of operators in $\mathcal{B}(H)$ and let $\mathbf{A} = (T_{j-k})_{k,j}$. Then, $\mathbf{A} \in (\mathcal{B}(\ell^2(H))_{\mathcal{T}}, \mathcal{B}(\ell^2(H))_{\mathcal{T}})_l$ if and only if $\Psi_{\mathbf{A}} \in \mathcal{B}(C(\mathbb{T}, H), H)$. Furthermore*

$$\|\Psi_{\mathbf{A}}\|_{\mathcal{B}(C(\mathbb{T},H),H)} = \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}.$$

Proof: In Theorem 3.4.11 we proved the direct implication. Let us see how the converse is proven.

Assume that $\Psi_{\mathbf{A}}$ extends to an element in $\mathcal{B}(C(\mathbb{T},H),H)$. Due to Theorem 3.4.8, we don't need to consider all multipliers, but just focus on showing that $\mathbf{A}*\mathbf{B}\in C(\ell^2(H))$ for any $\mathbf{B}\in C(\ell^2(H))_{\mathcal{T}}$. Or equivalently, that

$$\|\mathbf{A}*\mathbf{P}\|_{\mathcal{B}(\ell^2(H))}\leq\|\Psi_{\mathbf{A}}\|_{\mathcal{B}(C(\mathbb{T},H),H)}\|\mathbf{P}\|_{\mathcal{B}(\ell^2(H))},\quad \mathbf{P}\in\mathcal{P}(\ell^2(H)). \tag{3.6}$$

By Proposition 3.4.10, using the identification between $\mathbf{B}$ and $g_{\mathbf{B}}$, and taking into account that $g_{\mathbf{A}*\mathbf{B}}=g_{\mathbf{A}}*g_{\mathbf{B}}$, what needs to be proved is that if $P(t)=\sum_l S_l\varphi_l\in P(\mathbb{T},\mathcal{B}(H))$, then

$$\sup_{t\in[-\pi,\pi]}\left\|\sum_l T_lS_le^{ilt}\right\|_{\mathcal{B}(H)}\leq\|\Psi_{\mathbf{A}}\|_{\mathcal{B}(C(\mathbb{T},H),H)}\|P\|_{C(\mathbb{T},\mathcal{B}(H))}. \tag{3.7}$$

Now observe that

$$\begin{aligned}
\sup_{t\in[-\pi,\pi]}\left\|\sum_l T_lS_le^{ilt}\right\|_{\mathcal{B}(H)} &= \sup_{t\in[-\pi,\pi],\|x\|=1}\left\|\sum_l T_l(S_l(x))e^{ilt}\right\|\\
&= \sup_{t\in[-\pi,\pi],\|x\|=1}\left\|\Psi_{\mathbf{A}}(\sum_l S_l(x)e^{ilt}\varphi_l)\right\|\\
&\leq \sup_{t\in[-\pi,\pi],\|x\|=1}\|\Psi_{\mathbf{A}}\|\sup_{s\in[-\pi,\pi]}\left\|\sum_l S_l(x)e^{il(t-s)}\right\|\\
&\leq \sup_{t\in[-\pi,\pi],\|x\|=1}\|\Psi_{\mathbf{A}}\|\cdot\|x\|\cdot\sup_{s\in[-\pi,\pi]}\left\|\sum_l S_le^{il(t-s)}\right\|_{\mathcal{B}(H)}\\
&= \|\Psi_{\mathbf{A}}\|\cdot\|P\|_{C(\mathbb{T},\mathcal{B}(H))}.
\end{aligned}$$

The proof is now completed. ∎

Corollary 3.4.13 *Let $\mathbf{A}=(T_{j-k})_{k,j}$ be a Toeplitz matrix. Then*

$\mathbf{A}\in(\mathcal{B}(\ell^2(H))_{\mathcal{T}},\mathcal{B}(\ell^2(H))_{\mathcal{T}})_r$ if and only if there exists $\mu_{\mathbf{A}}\in M_{SOT}(\mathbb{T},\mathcal{B}(H))$ such that $\widehat{\mu_{\mathbf{A}}}(l)=T_{-l}$ for all $l\in\mathbb{Z}$. Furthermore, $\|\mu_{\mathbf{A}}\|_{M_{SOT}(\mathbb{T},\mathcal{B}(H))}=\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}$.

Proof: Consider $\mathbf{A}^* = (T^*_{k-j})_{k,j}$. Taking into account that

$$\mathbf{A} \in \big(B(\ell^2(H))_{\mathcal{T}}, B(\ell^2(H))_{\mathcal{T}}\big)_r \Leftrightarrow \mathbf{A}^* \in \big(B(\ell^2(H))_{\mathcal{T}}, B(\ell^2(H))_{\mathcal{T}}\big)_l$$

and combining the use of theorems 3.4.11 and 3.4.12, we have that $\mathbf{A} \in (\mathcal{B}(\ell^2(H))_{\mathcal{T}}, \mathcal{B}(\ell^2(H))_{\mathcal{T}})_r$ if and only if $\Phi_{\mathbf{A}^*} \in \mathcal{B}(C(\mathbb{T}, H), H)$, with $\|\Phi_{\mathbf{A}^*}\|_{\mathcal{B}(C(\mathbb{T},H),H)} = \|\mathbf{A}^*\|_{\mathcal{M}_l(\ell^2(H))}$. Denote by ν the element of $\mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ associated to $\Phi_{\mathbf{A}^*}$, whose coefficients are $\hat{\nu}(j-k) = T^*_{k-j}$. Invoking Proposition 2.3.7 (see Chapter 2), we get that $\mu_{\mathbf{A}} := \nu^* \in M_{SOT}(\mathbb{T}, \mathcal{B}(H))$, with $\widehat{\mu_{\mathbf{A}}}(j-k) = (T^*_{k-j})^* = T_{k-j}$ and $\|\mu_{\mathbf{A}}\|_{M_{SOT}(\mathbb{T},\mathcal{B}(H))} = \|\Phi_{\mathbf{A}^*}\|_{\mathcal{B}(C(\mathbb{T},H),H)} = \|\mathbf{A}^*\|_{\mathcal{M}_l(\ell^2(H))} = \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}$, which finishes the proof. ∎

3.4.2 A matricial version of the disc algebra

If X is a complex Banach space, we denote $\mathcal{H}(\mathbb{D}, X)$ the space of X-valued holomorphic functions. $H^\infty(\mathbb{D}, X)$ will stand for the Banach space of bounded analytic functions on the unit disc with values in X, and finally $A(\mathbb{D}, X)$ denotes the disc algebra, consisting in functions $f : \mathbb{D} \to X$ that are holomorphic and also extend to a continuous function on the closure of $\mathbb{D}$, with the norm

$$\|f\|_{H^\infty(\mathbb{D},X)} = \sup\{\|f(z)\| \ \mid z \in \mathbb{D}\},$$

$$\|f\|_{A(\mathbb{D},X)} = \sup\{\|f(z)\| \ \mid z \in \overline{\mathbb{D}}\}.$$

In this section we present a version of these spaces in the framework of matrices with entries in $\mathcal{B}(H)$.

If we assume that $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{U}$ satisfies the condition (3.2), it can be guaranteed that

$$F_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} \mathbf{D_l} z^l \in \mathcal{H}(\mathbb{D}, \mathcal{B}(\ell^2(H)))$$

is a well defined holomorphic function. It follows from the definitions that

$$F_{\mathbf{A}}(re^{it}) = \mathbf{M}_{P_r} * f_{\mathbf{A}}(t) = \sum_{l=0}^{\infty} \mathbf{D}_l r^l e^{ilt}.$$

Similar reasoning gives that if $\mathbf{A} = (T_{j-k}) \in \mathcal{U} \cap \mathcal{T}$ then $\tilde{F}_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} T_l z^l \in \mathcal{H}(\mathbb{D}, \mathcal{B}(H))$.

Theorem 3.4.14 *Let $\mathbf{A} = (T_{kj}) \in \mathcal{U}$ satisfying condition (3.2). Then*

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ if and only if $F_{\mathbf{A}} \in H^\infty(\mathbb{D}, \mathcal{B}(\ell^2(H)))$. Furthermore, $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|F_{\mathbf{A}}\|_{H^\infty(\mathbb{D},\mathcal{B}(\ell^2(H)))}$.

(ii) $\mathbf{A} \in C(\ell^2(H))$ if and only if $F_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(\ell^2(H)))$.

Proof: (i) If we use part (i) in Proposition 3.3.5 for $k_n = P_{r_n}$ for a sequence r_n converging to 1, we obtain that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ if and only if $\|F_{\mathbf{A}}\|_{H^\infty(\mathbb{D},\mathcal{B}(\ell^2(H)))} = \sup_n \|P_{r_n}(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} < \infty$, with equality of norms.

(ii) Theorem 3.4.4 tells us that $\mathbf{A} \in C(\ell^2(H))$ if and only if $f_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{B}(\ell^2(H)))$. Using now that $F_{\mathbf{A}}(re^{it}) = P_r(f_{\mathbf{A}}(t))$ and invoking part (i) we have that

$$\|P_r(\mathbf{A}) - \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|P_r * f_{\mathbf{A}} - f_{\mathbf{A}}\|_{C(\mathbb{T},\mathcal{B}(\ell^2(H)))},$$

which gives the result. ∎

Similar ideas can be used to obtain the following corollary.

Corollary 3.4.15 *Let $\mathbf{A} = (T_{j-k}) \in \mathcal{U} \cap \mathcal{T}$ satisfying that*

$$\sup_l \|T_l\| < \infty.$$

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ if and only if $\tilde{F}_{\mathbf{A}} \in H^\infty(\mathbb{D}, \mathcal{B}(H))$. Moreover $\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \|\tilde{F}_{\mathbf{A}}\|_{H^\infty(\mathbb{D},\mathcal{B}(H))}$.

(ii) $\mathbf{A} \in C(\ell^2(H))$ if and only if $\tilde{F}_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(H))$.

We leave it without proof for the moment. In Chapter 4, a version of it with additional items will be presented, and then the proof will be given.

Chapter 4

Integrable matrices

"Art is an attempt to integrate evil."

—Simone de Beauvoir.

4.1 Preliminaries

In Chapter 3, the class of matrices $C(\ell^2(H))$ (also called "continuous matrices") with entries in the space $\mathcal{B}(H)$ was introduced and we observed that it played an important role when it came to the study of Schur multipliers. In this chapter, we shall follow a similar approach to define the notion of "integrable matrices" by means of the notion of "polynomial" matrices.

We recall that given a matrix $\mathbf{A} = (T_{kj})$ with entries $T_{kj} \in \mathcal{B}(H)$, we say that $\mathbf{A}$ is a "polynomial", also denoted $\mathbf{A} \in \mathcal{P}(\ell^2(H))$, whenever there exist $N, M \in \mathbb{N}$ such that $\mathbf{A} = \sum_{l=-N}^{M} \mathbf{D_l}$ and

$$\sup_{k,j} \|T_{kj}\| < \infty. \tag{4.1}$$

We already know that condition (4.1) is needed for any polynomial to define a multiplier. Indeed, at Proposition 3.3.1 we saw that fact for right Schur multipliers (a similar argument also proves it for left Schur multipliers). So, if $\mathbf{A} = (T_{k,j}) \in \mathcal{M}_r(\ell^2(H)) \cup$

$\mathcal{M}_l(\ell^2(H))$ then

$$\sup_{k,j} \|T_{kj}\| \leq \min\{\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}, \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}\}. \tag{4.2}$$

Let us give an alternative and shorter proof of this fact, by using elementary matrices $E_{k_0,j_0}(S)$ with $S \in \mathcal{B}(H)$, whose entries are the zero operator unless $k = k_0$ and $j = j_0$ and $T_{k_0,j_0} = S$. They satisfy that $\|E_{k_0,j_0}(S)\|_{\mathcal{B}(\ell^2(H))} = \|S\|_{\mathcal{B}(H)}$. Observing that $\mathbf{A} * E_{k_0,j_0}(S) = E_{k_0,j_0}(T_{k_0,j_0}S)$ and $E_{k_0,j_0}(S) * \mathbf{A} = E_{k_0,j_0}(ST_{k_0,j_0})$, and taking $S = x \otimes y$ for $x, y \in H$, one has

$$T_{k_0,j_0}(x \otimes y) = x \otimes T_{k_0,j_0}(y), \quad (x \otimes y)T_{k_0,j_0} = T^*_{k_0,j_0}(x) \otimes y,$$

and therefore it is obtained that $\|T_{k_0,j_0}\| \leq \min\{\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}, \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}\}$, which gives (4.2) since (k_0, j_0) was chosen arbitrarily.

It is now time to define the class of integrable matrices, which will be the main focus of this chapter.

Definition 4.1.1 *We define $\mathcal{L}^1_l(\ell^2(H))$ (respectively $\mathcal{L}^1_r(\ell^2(H))$) as the closure of $\mathcal{P}(\ell^2(H))$ in $\mathcal{M}_l(\ell^2(H))$ (respectively $\mathcal{M}_r(\ell^2(H))$). We use the notation $\mathcal{L}^1(\ell^2(H)) = \mathcal{L}^1_l(\ell^2(H)) \cap \mathcal{L}^1_r(\ell^2(H))$.*

This chapter, besides the introductory section, containts two sections. The first section starts analyzing the definition of integrable matrix. Some examples of this class of matrices will be presented, and also we shall show an equivalent formulation by means of the Schur product with Toeplitz matrices given by summability kernels. More precisely, it will be shown that $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$ if and only if $P_r(\mathbf{A})$ converges to $\mathbf{A}$ in $\mathcal{M}_r(\ell^2(H))$ or $\sigma_n(\mathbf{A})$ converges to $\mathbf{A}$ in $\mathcal{M}_r(\ell^2(H))$. Here, $P_r(\mathbf{A})$ and $\sigma_n(\mathbf{A})$ are Schur products with matrices given by the Poisson or the Féjer kernels, as it was seen in the previous chapter.

The last section deals with some vector-valued functions related to $\mathbf{A}$ and studies the properties between them. Two different options are explored: the first one considers a matrix-valued function $f_{\mathbf{A}}(t) = \mathbf{M}_t * \mathbf{A}$, where $\mathbf{M}_t = (e^{i(j-k)t})$ for any matrix $\mathbf{A}$; the second one defines, for each operator-valued function $\mathbf{f}$, a Toeplitz matrix $\mathbf{A_f} = (\widehat{\mathbf{f}}(j-k))$. We prove that $t \to f_{\mathbf{A}}(t)$ is continuous as a $\mathcal{M}_r(\ell^2(H))$-valued function only when $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$, and that $\mathbf{A_f} \in \mathcal{L}^1_r(\ell^2(H))$ whenever $\mathbf{f} \in L^1(\mathbb{T}, \mathcal{B}(H))$. Furthermore, we characterize the space $\mathcal{L}^1_r(\ell^2(H)) \cap \mathcal{T}$. To close the chapter, we consider the situation of upper triangular matrices, presenting its relationship with Hardy spaces.

4.2 The space $\mathcal{L}^1(\ell^2(H))$

The first thing we will do is to look at the norm of $\mathbf{D_l}$, $\mathbf{R_k}$ and $\mathbf{C_j}$ in the space of Schur multipliers.

Example 4.2.1 *Let $\mathbf{A} = (T_{kj})$ and let $l \in \mathbb{Z}$ and $k, j \in \mathbb{N}$. Then*

(i) $\mathbf{D_l} \in \mathcal{M}(\ell^2(H))$ iff $\sup_k \|T_{k,k+l}\| < \infty$ iff $\mathbf{D_l} \in \mathcal{B}(\ell^2(H))$. Moreover

$$\|\mathbf{D_l}\|_{\mathcal{M}(\ell^2(H))} = \|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} = \sup_{k \geq -\min\{l,0\}+1} \|T_{k,k+l}\|.$$

(ii) $\mathbf{C_j} \in \mathcal{M}_l(\ell^2(H))$ iff $\sup_k \|T_{k,j}\| < \infty$. Moreover

$$\|\mathbf{C_j}\|_{\mathcal{M}_l(\ell^2(H))} = \sup_k \|T_{k,j}\|.$$

(iii) $\mathbf{R_k} \in \mathcal{M}_r(\ell^2(H))$ iff $\sup_j \|T_{k,j}\| < \infty$. Moreover

$$\|\mathbf{R_k}\|_{\mathcal{M}_r(\ell^2(H))} = \sup_j \|T_{k,j}\|.$$

Proof: (i) It is straightforward to see that $\|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} = \sup_{k \geq -\min\{l,0\}+1} \|T_{k,k+l}\|$, and it is left to the reader. Observe that for $\mathbf{B} = (S_{kj})$ one has that $\mathbf{D_l} * \mathbf{B} = \mathbf{D_l}'$ where $\mathbf{D_l}' = (T_{k,k+l}S_{k,k+l})_k$. Therefore

$$\|\mathbf{D_l} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} = \sup_{k \geq -\min\{l,0\}+1} \|T_{k,k+l}S_{k,k+l}\| \leq \sup_{k \geq -\min\{l,0\}+1} \|T_{k,k+l}\| \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}.$$

Similarly for $\mathbf{B} * \mathbf{D_l}$. This fact, in combination with (4.2) ends the proof of (i).

(ii) Notice that $\|\mathbf{C_j}\|_{\mathcal{B}(\ell^2(H))} = \sup_{\|x\|=1} (\sum_{k=1}^\infty \|T_{kj}(x)\|^2)^{1/2} < \infty$. Also, if we take

$\mathbf{B} = (S_{kj})$, we have that $\mathbf{C_j} * \mathbf{B} = \mathbf{C_j}'$, where $\mathbf{C_j}' = T_{k,j}S_{k,j}$. Hence

$$\begin{aligned}
\|\mathbf{C_j} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} &= \sup_{\|x\|=1} (\sum_{k=1}^{\infty} \|T_{k,j}S_{k,j}(x)\|^2)^{1/2} \\
&\leq \sup_k \|T_{k,j}\| \sup_{\|x\|=1} (\sum_{k=1}^{\infty} \|S_{k,j}(x)\|^2)^{1/2} \\
&\leq \sup_k \|T_{k,j}\| \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}.
\end{aligned}$$

The result follows now using again (4.2).

(iii) It is a consequence of (ii) by taking adjoints. ∎

Example 4.2.2 *Using that $\|\mathbf{A}\|_{\mathcal{M}(\ell^2(H))} \leq \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))}$ (see Chapter 2, Theorem 2.4.9), we first note that $C(\ell^2(H)) \subset \mathcal{L}^1(\ell^2(H))$.*

More examples can be produced easily if we resort to the scalar case.

Example 4.2.3 *If $A = (a_{kj}) \in \mathcal{L}^1(\ell^2)$ and $T \in \mathcal{B}(H)$, then $\mathbf{A} = (a_{k,j}T) \in \mathcal{L}^1(\ell^2(H))$.*

This follows using Proposition 3.3.2 from Chapter 3, which showed that if $A = (a_{kj}) \in \mathcal{M}(\ell^2)$ and $T \in \mathcal{B}(H)$, then $\mathbf{A} = (a_{k,j}T) \in \mathcal{M}(\ell^2(H))$ and

$$\|\mathbf{A}\|_{\mathcal{M}(\ell^2(H))} = \|A\|_{\mathcal{M}(\ell^2)} \|T\|_{\mathcal{B}(H)}. \tag{4.3}$$

Given $\eta \in M(\mathbb{T})$, we are going to keep the notation used in the previous chapter by denoting $\mathbf{M}_\eta$ the Toeplitz matrix given by

$$\mathbf{M}_\eta = (\hat{\eta}(j-k)Id)_{k,j} \in \mathcal{T}$$

where $Id : H \to H$ is the identity operator. Due to Bennett's theorem (see Theorem 2.1.5) and (4.3), one sees that $\mathbf{M}_\eta \in \mathcal{M}(\ell^2(H))$ and $\|\mathbf{M}_\eta\|_{\mathcal{M}(\ell^2(H))} = \|\eta\|_{M(\mathbb{T})}$. The cases $\eta = \delta_{-t}$ or $d\eta = f dt$ with $f \in L^1(\mathbb{T})$ will be denoted by $\mathbf{M}_t$ and $\mathbf{M}_f$ respectively, that is $\mathbf{M}_t = (e^{i(j-k)t} Id)$ and $\mathbf{M}_f = (\hat{f}(j-k)Id)$. The result above gives more examples of matrices in $\mathcal{L}^1(\ell^2(H))$.

Example 4.2.4 *If $f \in L^1(\mathbb{T})$ and $T \in \mathcal{B}(H)$ then $\mathbf{A} = (\hat{f}(j-k)T) \in \mathcal{L}^1(\ell^2(H))$.*

Recall that a family $\{k_\varepsilon\}_{\varepsilon>0} \subset L^1(\mathbb{T})$ is called a "summability kernel" if it satisfies

1) $\frac{1}{2\pi}\int_{-\pi}^{\pi} k_\varepsilon(t)dt = 1$ for all $\varepsilon > 0$.

2) $\sup_{\varepsilon>0} \frac{1}{2\pi}\int_{-\pi}^{\pi} |k_\varepsilon(t)|dt = C < \infty$.

3) $\forall 0 < \delta < \pi$ one has $\frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} k_\varepsilon(t)dt \underset{\varepsilon\to 0}{\longrightarrow} 0$.

Classical examples that will be used in the sequel are the Féjer kernel (for $\varepsilon = \frac{1}{n}$)

$$K_n(t) = \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt}$$

and the Poisson kernel (for $\varepsilon = 1-r$)

$$P_r(t) = \sum_{k\in\mathbb{Z}} r^{|k|}e^{ikt}.$$

We shall use the notation $\sigma_n(\mathbf{A}) = \mathbf{M}_{K_n} * \mathbf{A}$ and $P_r(\mathbf{A}) = \mathbf{M}_{P_r} * \mathbf{A}$ for $\mathbf{A} = (T_{k,j})$.

Notice that under the assumption (4.1), one has that $\sigma_n(\mathbf{A})$ is clearly in $\mathcal{P}(\ell^2(H))$. Also, under the same assumption, $P_r(\mathbf{A}) \in C(\ell^2(H))$. To see it, observe that we have $\sup_l \|\mathbf{D}_l\|_{\mathcal{B}(\ell^2(H))} < \infty$. This gives that the series $\sum_{l\in\mathbb{Z}} \mathbf{D}_l r^{|l|}$ is absolutely convergent, hence convergent, and therefore $P_r(\mathbf{A})$ is the limit of its partial sums, which are polynomial matrices. Hence, $P_r(\mathbf{A}) \in C(\ell^2(H))$.

In Chapter 3 (Proposition 3.3.5) we saw that given a matrix $\mathbf{A}$ with entries in $\mathcal{B}(H)$ and a summability kernel $\{k_n\}$, if we denote $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$ then

$$\mathbf{A} \in \mathcal{B}(\ell^2(H)) \iff \sup_n \|M_n(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} < \infty, \tag{4.4}$$

$$\mathbf{A} \in \mathcal{M}_r(\ell^2(H)) \iff \sup_n \|M_n(\mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} < \infty \tag{4.5}$$

and similar result for left Schur multipliers.

In the next theorem, we are going to show that the space of those matrices $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ such that $M_n(\mathbf{A})$ converges to $\mathbf{A}$ in $\mathcal{M}_r(\ell^2(H))$ corresponds to $\mathcal{L}^1_r(\ell^2(H))$. The proof follows the same arguments as Theorem 3.4.4, but for the sake of completeness, we include the proof of this one too.

Theorem 4.2.5 *Let $\mathbf{A}$ be a matrix whose entries are in $\mathcal{B}(H)$ and satisfying (4.1). The following are equivalent:*

1) $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$.

2) $\lim_{n\to\infty} M_n(\mathbf{A}) = \mathbf{A}$ *in* $\mathcal{M}_r(\ell^2(H))$ *where* $M_n(\mathbf{A}) = \mathbf{M}_{k_n} * \mathbf{A}$ *and* $\{k_n\} \subseteq L^1(\mathbb{T})$ *is a summability kernel.*

3) $\lim_{n\to\infty} \sigma_n(\mathbf{A}) = \mathbf{A}$ *in* $\mathcal{M}_r(\ell^2(H))$.

4) $\lim_{r\to 1} P_r(\mathbf{A}) = \mathbf{A}$ *in* $\mathcal{M}_r(\ell^2(H))$.

Proof: 1)$\Rightarrow$ 2). Let $\varepsilon > 0$, and select $\mathbf{P} = (S_{k,j})_{k,j} = \sum_{l=-N}^{N} \mathbf{D_l} \in \mathcal{P}(\ell^2(H))$ such that $\|\mathbf{A} - \mathbf{P}\|_{\mathcal{M}(\ell^2(H))} < \varepsilon/3C$ where $C = \sup_n \|k_n\|_{L^1(\mathbb{T})} \geq 1$. Then, taking into account part (i) in Example 4.2.1,

$$\begin{aligned}
\|M_n(\mathbf{P}) - \mathbf{P}\|_{\mathcal{M}_r(\ell^2(H))} &= \left\| \sum_{l=-N}^{N} (\hat{k_n}(l) - 1)\mathbf{D_l} \right\|_{\mathcal{M}_r(\ell^2(H))} \\
&\leq \sup_{k,j} \|S_{k,j}\| \cdot (2N+1) \cdot \max_{|l|\leq N} |\hat{k_n}(l) - 1|
\end{aligned}$$

We know, since $\{k_n\}$ is a summability kernel, that $\hat{k}_n(l) \to 1$ as $n \to \infty$ $\forall l \in \mathbb{Z}$. Therefore we are able to choose $n_0 \in \mathbb{N}$ such that $|\hat{k}_n(l) - 1| < \frac{\varepsilon}{3(2N+1)\sup_{k,j}\|S_{k,j}\|}$ $\forall n \geq n_0$ and $\forall |l| \leq N$. It yields that $\|M_n(\mathbf{P}) - \mathbf{P}\|_{\mathcal{M}_r(\ell^2(H))} < \varepsilon/3$. Finally, for $n \geq n_0$,

$$\begin{aligned}
\|M_n(\mathbf{A}) - \mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} &\leq \|\mathbf{M}_n * (\mathbf{A} - \mathbf{P})\|_{\mathcal{M}_r(\ell^2(H))} \\
&+ \|\mathbf{M}_n * \mathbf{P} - \mathbf{P}\|_{\mathcal{M}_r(\ell^2(H))} + \|\mathbf{P} - \mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \\
&\leq \|\mathbf{M}_n\|_{\mathcal{M}_r(\ell^2(H))} \cdot \|\mathbf{A} - \mathbf{P}\|_{\mathcal{M}_r(\ell^2(H))} + \varepsilon/3 + \varepsilon/3 \\
&\leq \|k_n\|_{L^1(\mathbb{T})} \cdot \varepsilon/3C + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Notice that we have used that $\|\mathbf{M}_n\|_{\mathcal{M}_r(\ell^2(H))} = \|k_n\|_{L^1(\mathbb{T})}$ according to (2.1.5) and (4.3), and the easy to see fact that $\|\mathbf{A} * \mathbf{B}\|_{\mathcal{M}_r(\ell^2(H))} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{B}\|_{\mathcal{M}_r(\ell^2(H))}$.

The implications 2)$\Rightarrow$ 3) & 4) and 3)$\Rightarrow$ 1) are clear since the Féjer and Poisson kernels are particular examples of summability kernels and $\sigma_n(\mathbf{A}) \in \mathcal{P}(\ell^2(H))$.

4) $\Rightarrow$ 1). Note that, using condition (4.1), the series $\sum_{l\in\mathbb{Z}} \mathbf{D_l} r^{|l|}$ is absolutely convergent in $\mathcal{B}(\ell^2(H))$ and in particular $P_r(\mathbf{A}) \in \mathcal{L}^1_r(\ell^2(H))$ for each $0 < r < 1$. Therefore, its limit also belongs to $\mathcal{L}^1_r(\ell^2(H))$.

Corollary 4.2.6 *Let* $\mathbf{A} = (T_{kj})$ *satisfying condition (4.1), and let* $j \in \mathbb{N}$. *Then* $\mathbf{C_j} \in \mathcal{L}^1_l(\ell^2(H))$ *iff* $\lim_{k\to\infty} \|T_{k,j}\| = 0$.

Proof: Notice that one has

$$(\mathbf{C_j} - \sigma_n(\mathbf{C_j}))_{k,j} = \frac{|k-j|}{n+1} T_{k,j} \quad k \leq j+n,$$

$$(\mathbf{C_j} - \sigma_n(\mathbf{C_j}))_{k,j} = T_{k,j} \quad k > j+n.$$

Now, using Theorem 4.2.5 (part (iii)) and Example 4.2.1 (part (ii)), the result follows. ∎

Observe that if $\mathbf{A} = \sum_l \mathbf{D_l}$ satisfies (4.1), one has that $\mathbf{D_l} \in \mathcal{L}^1(\ell^2(H))$ for each $l \in \mathbb{Z}$ since $\mathbf{D_l} - \sigma_n(\mathbf{D_l}) = \frac{|l|}{n+1}\mathbf{D_l}$ for $n \geq |l|$. Also, if $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ then for each $l \in \mathbb{Z}$

$$\|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} = \|\mathbf{D_l}\|_{\mathcal{M}_r(\ell^2(H))} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}. \tag{4.6}$$

We see that a version of the Riemann-Lebesgue Lemma in this framework holds.

Proposition 4.2.7 *(Riemann-Lebesgue Lemma) If* $A = \sum_l \mathbf{D_l} \in \mathcal{L}^1_r(\ell^2(H))$, *then*

$$\|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} \xrightarrow[|l|\to\infty]{} 0.$$

Proof: Let $\varepsilon > 0$ and select $n_0 \in \mathbb{N}$ such that $\|\sigma_{n_0}(\mathbf{A}) - \mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} < \varepsilon$. For $|l| > n_0$, we have that $\sigma_{n_0}(\mathbf{D_l}) = 0$. Now, using (4.6), we obtain

$$\begin{aligned}\|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} &\leq \|\sigma_{n_0}(\mathbf{D_l}) - \mathbf{D_l}\|_{\mathcal{M}_r(\ell^2(H))} \\ &\leq \|\sigma_{n_0}(\mathbf{A}) - \mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} < \varepsilon,\end{aligned}$$

and the proof is complete. ∎

In the previous chapter we introduced $\mathcal{A}(\ell^2(H))$ as the analogue to the Wiener algebra, that is matrices $\mathbf{A} = \sum_{l\in\mathbb{Z}} \mathbf{D_l}$ such that $\sum_{l\in\mathbb{Z}} \|\mathbf{D_l}\|_{\mathcal{B}(\ell^2(H))} < \infty$. Since $P_r(\mathbf{A}) \in \mathcal{A}(\ell^2(H))$ for any $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$, we obtain the following corollary.

Corollary 4.2.8 *$\mathcal{A}(\ell^2(H))$ and $C(\ell^2(H))$ are dense in $\mathcal{L}^1(\ell^2(H))$.*

The following remark is easy to verify.

Remark 4.2.9 *$\mathcal{L}^1_r(\ell^2(H))$ is a right ideal of $\mathcal{M}_r(\ell^2(H))$, that is to say if $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$ and $\mathbf{B} \in \mathcal{M}_r(\ell^2(H))$ then $\mathbf{B} * \mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$.*

Definition 4.2.10 *We write $(\mathcal{B}(\ell^2(H)), C(\ell^2(H)))_l$ for the set of matrices $\mathbf{A}$ such that*

$$\mathbf{A} * \mathbf{B} \in C(\ell^2(H)) \quad \forall \mathbf{B} \in \mathcal{B}(\ell^2(H)).$$

One can give similar definitions for $(C(\ell^2(H)), C(\ell^2(H)))_l$ and for right Schur multipliers.

We recall that in Theorem 3.4.8, we showed that $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ (respectively $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$) if and only if $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_l$ (resp. $\mathbf{A} \in (C(\ell^2(H)), C(\ell^2(H)))_r$).

Corollary 4.2.11 *$\mathcal{L}^1_r(\ell^2(H)) \subset (\mathcal{B}(\ell^2(H)), C(\ell^2(H)))_r$ and similar result for left multipliers.*

Proof: Assume that $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$ and take $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. Since $\sigma_n(\mathbf{B} * \mathbf{A}) = \mathbf{B} * \sigma_n(\mathbf{A})$, we can write

$$\|\sigma_n(\mathbf{B} * \mathbf{A}) - \mathbf{B} * \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq \|(\sigma_n(\mathbf{A}) - \mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}.$$

The right hand side tends to 0 when $n \to \infty$ by hypothesis. Therefore, $\mathbf{B} * \mathbf{A} \in C(\ell^2(H))$ and the result follows. ∎

4.3 Relations between matrices and functions

As it has been pointed out in previous chapters, to each regular vector measure $\mu \in \mathcal{M}(\mathbb{T}, \mathcal{B}(H))$ defined on the Borel sets of $\mathbb{T}$ and taking values in $\mathcal{B}(H)$, we can associate a Toeplitz matrix $\mathbf{A}_\mu = (T_{k,j})$ that is given by

$$T_{k,j} = \hat{\mu}(j-k), \quad k, j \in \mathbb{N}, \tag{4.7}$$

where $\hat{\mu}(l) = \int_0^{2\pi} e^{-ilt} d\mu(t) \in \mathcal{B}(H)$ for $l \in \mathbb{Z}$.

If $d\mu = \mathbf{g}dm$ for some function $\mathbf{g} : \mathbb{T} \to \mathcal{B}(H)$, we shall simply denote it by $\mathbf{A}_\mathbf{g}$. In that case in which $\mathbf{g} \in P(\mathbb{T}, \mathcal{B}(H))$, then clearly $\mathbf{A}_\mathbf{g} \in \mathcal{P}(\ell^2(H))$. We refer to [8, 18, 20, 36] for the results on vector-valued Fourier analysis, vector measures and projective tensor products to be used in the sequel.

The following operator-valued function was introduced in Chapter 3 (Section 3.4) for each matrix $\mathbf{A} = (T_{kj})$:

$$f_\mathbf{A}(t) = (e^{i(j-k)t} T_{kj}), \quad t \in [-\pi, \pi).$$

We recall that if $\mathbf{A} \in \mathcal{P}(\ell^2(H))$, clearly $f_\mathbf{A}(t) = \sum_{l \in \mathbb{Z}} \mathbf{D_l} e^{ilt}$ belongs to $P(\mathbb{T}, \mathcal{B}(\ell^2(H)))$. Also, noticing that $f_\mathbf{A}(t) = \mathbf{M}_t * \mathbf{A}$ and that $\mathbf{M}_t \in \mathcal{M}(\ell^2(H)) \cap \mathcal{T}$ with $\|\mathbf{M}_t\|_{\mathcal{M}(\ell^2(H))} = 1$, then one sees that $f_\mathbf{A}(t)$ takes its values into $\mathcal{B}(\ell^2(H))$, $\mathcal{M}_r(\ell^2(H))$ or $\mathcal{L}^1_r(\ell^2(H))$ whenever the matrix $\mathbf{A}$ belongs to $\mathcal{B}(\ell^2(H))$, $\mathcal{M}_r(\ell^2(H))$ or $\mathcal{L}^1_r(\ell^2(H))$ respectively, and preserves the norm for each $t \in [0, 2\pi)$. In particular, whenever $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$, we have

$$\sup_{t \in [0,2\pi)} \|f_\mathbf{A}(t)\|_{\mathcal{M}_r(\ell^2(H))} = \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}. \tag{4.8}$$

In Chapter 3, we explored the properties of $f_{\mathbf{A}}$ as a $\mathcal{B}(\ell^2(H))$-valued function. For example, we saw that in general, if $\mathbf{A} \in \mathcal{B}(\ell^2(H))$, then $t \to f_{\mathbf{A}}(t)$ was not strongly measurable as a $\mathcal{B}(\ell^2(H))$-valued mapping (see Proposition 3.4.2). We also proved that the function $t \to f_{\mathbf{A}}(t)$ being continuous as a $\mathcal{B}(\ell^2(H))$-valued function equals to the matrix $\mathbf{A}$ being in $C(\ell^2(H))$ (see Theorem 3.4.4). This time, let us take a look at the properties of $f_{\mathbf{A}}$ as a multiplier-valued function.

Proposition 4.3.1 *(i) There exists $\mathbf{A} \in \mathcal{M}(\ell^2(H))$ such that $f_{\mathbf{A}}$ is not strongly measurable as a $\mathcal{M}(\ell^2(H))$-valued function.*

(ii) There exist $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ and $\mathbf{B} \in \mathcal{B}(\ell^2(H))$ such that the map $t \to f_{\mathbf{A}}(t) * \mathbf{B}$ is not strongly measurable as a $\mathcal{B}(\ell^2(H))$-valued function.

Proof: (i) Select $\mathbf{A} = \mathbf{1}$ where we use $\mathbf{1}$ for the unit element in $\mathcal{M}(\ell^2(H))$, that is, the matrix given by given by $\mathbf{1}_{k,j} = Id$ for the identity operator $Id : H \to H$. Note that $f_{\mathbf{1}}(t) = \mathbf{M}_t = (e^{i(j-k)t} Id)$ is not $\mathcal{M}(\ell^2(H))$-valued strongly measurable. Indeed, utilizing (4.3), we get that

$$\|f_{\mathbf{1}}(t) - f_{\mathbf{1}}(s)\|_{\mathcal{M}(\ell^2(H))} = \|\delta_{-t} - \delta_{-s}\|_{M(\mathbb{T})} = 2, \quad t \neq s.$$

This means that the range of $f_{\mathbf{1}}$ is not separable. Now, Pettis's measurability Theorem (see [20]) tells us that $t \to f_{\mathbf{A}}(t)$ can't be strongly measurable.

(ii) Take $\mathbf{A} = \mathbf{1}$ and $\mathbf{B} = (T_{kj})$ where $T_{kj} = 0$ for each $j \neq 2k$ and $T_{k,2k} = Id$ for $k \in \mathbb{N}$, and realize that $f_{\mathbf{1}} * \mathbf{B} = f_{\mathbf{B}}$, which according to Proposition 3.4.2 is not a strongly measurable function with values in $\mathcal{B}(\ell^2(H))$. ∎

Proposition 4.3.2 *Let $\mathbf{A} = (T_{kj}) \in \mathcal{M}_l(\ell^2(H))$ and $\mathbf{B} = (S_{kj}) \in \mathcal{B}(\ell^2(H))$.*

If either $\mathbf{B} \in C(\ell^2(H))$ or $\mathbf{A} \in (\mathcal{B}(\ell^2(H)), C(\ell^2(H)))_l$, then $t \to f_{\mathbf{A}}(t) * \mathbf{B}$ is continuous with values in $\mathcal{B}(\ell^2(H))$.

As a particular case, if $\mathbf{x}, \mathbf{y} \in \ell^2(H)$ then the map

$$f_{\mathbf{A}}(t) * (\mathbf{x} \otimes \mathbf{y}) = \left(e^{i(j-k)t} x_j \otimes T_{kj}(y_k)\right)_{k,j}$$

is continuous from $\mathbb{T}$ *into* $\mathcal{B}(\ell^2(H))$.

Proof: Observing that $f_{\mathbf{A}}(t) * \mathbf{B} = f_{\mathbf{A}*\mathbf{B}}(t)$ and $\mathbf{A} * \mathbf{B} \in C(\ell^2(H))$ in the two scenarios suggested, the result follows by just invoking Theorem 3.4.4 in each case. ∎

Here we have another characterization of matrices in $\mathcal{L}^1_l(\ell^2(H))$.

Theorem 4.3.3 *Let* $\mathbf{A}$ *be a matrix with entries in* $\mathcal{B}(H)$. *Then* $\mathbf{A} \in \mathcal{L}^1_l(\ell^2(H))$ *iff the associated function* $t \to f_{\mathbf{A}}(t)$ *is a* $\mathcal{M}_l(\ell^2(H))$*-valued continuous function.*

Proof: Notice that, as seen in Proposition 3.4.1, $\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} = \|f_{\mathbf{A}}(t)\|_{\mathcal{M}_l(\ell^2(H))}$ for any $t \in \mathbb{T}$. Furthermore, $\sigma_n(f_{\mathbf{A}}(t)) = f_{\sigma_n(\mathbf{A})}(t) \in P(\mathbb{T}, \mathcal{B}(\ell^2(H)))$, so we have

$$\begin{aligned} \sup_t \|f_{\sigma_n(\mathbf{A})}(t) - f_{\mathbf{A}}(t)\|_{\mathcal{M}_l(\ell^2(H))} &= \sup_t \|f_{\sigma_n(\mathbf{A})-\mathbf{A}}(t)\|_{\mathcal{M}_l(\ell^2(H))} \\ &= \|\sigma_n(\mathbf{A}) - \mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}. \end{aligned}$$

Now, using the well known fact that $f_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{M}_l(\ell^2(H)))$ iff $\sigma_n(f_{\mathbf{A}})$ converges to $f_{\mathbf{A}}$ in $C(\mathbb{T}, \mathcal{M}_l(\ell^2(H)))$, we get the equivalence between both conditions. ∎

4.3.1 The Toeplitz case

In Example 4.2.4, we saw that $\mathbf{A}_f = (\hat{f}(j-k)T) \in \mathcal{L}^1(\ell^2(H)) \cap \mathcal{T}$ whenever $f \in L^1(\mathbb{T})$ and $T \in \mathcal{B}(H)$. In the following proposition, we shall verify that this is actually true for operator-valued integrable functions. This result could also be obtained from the inclusion $M(\mathbb{T}, \mathcal{B}(H)) \subset \mathcal{M}_r(\ell^2(H)) \cap \mathcal{T}$ (see Theorem 2.5.6), but the proof provided below is direct.

Proposition 4.3.4 *Let* $\mathbf{f} \in L^1(\mathbb{T}, \mathcal{B}(H))$, *and consider* $\mathbf{A_f} = (\widehat{\mathbf{f}}(j-k)) = (T_{j-k})$. *Then* $\mathbf{A_f} \in \mathcal{L}^1(\ell^2(H))$ *and*

$$\|\mathbf{A_f}\|_{\mathcal{M}(\ell^2(H))} \leq \|\mathbf{f}\|_{L^1(\mathbb{T},\mathcal{B}(H))}.$$

Proof: First, recall the identification $\mathcal{B}(H) = (H\hat{\otimes}H)^*$ provided by the formula $T(x \otimes y) = \langle Tx, y\rangle$. Let us start assuming $\mathbf{f} \in P(\mathbb{T}, \mathcal{B}(H))$, and after we will deal with the general case. Let $\mathbf{x} = (x_j), \mathbf{y} = (y_k) \in \ell^2(H)$ and $\mathbf{B} = (S_{k,j}) \in \mathcal{B}(\ell^2(H))$. We can write

$$
\begin{aligned}
|\ll (\mathbf{A_f}*\mathbf{B})\mathbf{x}, \mathbf{y} \gg| &= \Big|\sum_{k,j}\langle T_{k,j}S_{k,j}x_j, y_k\rangle\Big| = \Big|\sum_{k,l}\langle T_l S_{k,k+l}x_{l+k}, y_k\rangle\Big| \\
&= \Big|\sum_l T_l\Big(\sum_k S_{k,k+l}x_{l+k}\otimes y_k\Big)\Big| \\
&= \left|\int_0^{2\pi}\Big(\sum_l \hat{\mathbf{f}}(l)e^{ilt}\Big)\left(\sum_l\Big(\sum_k S_{k,k+l}x_{l+k}\otimes y_k\Big)e^{-ilt}\right)\frac{dt}{2\pi}\right| \\
&= \left|\int_0^{2\pi}\mathbf{f}(t)\left(\sum_k\Big(\sum_j S_{k,j}x_je^{-ijt}\Big)\otimes y_ke^{ikt}\right)\frac{dt}{2\pi}\right| \\
&\le \int_0^{2\pi}\|\mathbf{f}(t)\|_{\mathcal{B}(H)}\left\|\sum_k\Big(\sum_j S_{k,j}x_je^{-ijt}\Big)\otimes y_ke^{ikt}\right\|_{H\hat{\otimes}H}\frac{dt}{2\pi} \\
&\le \int_0^{2\pi}\|\mathbf{f}(t)\|_{\mathcal{B}(H)}\sum_k\|\sum_j S_{k,j}x_je^{-ijt}\|_H\|y_k\|_H\frac{dt}{2\pi} \\
&\le \|\mathbf{f}\|_{L^1(\mathbb{T},\mathcal{B}(H))}\sup_{t\in[0,2\pi)}\left(\sum_k\|\sum_j S_{k,j}x_je^{-ijt}\|_H^2\right)^{1/2}\|\mathbf{y}\|_{\ell^2(H)} \\
&\le \|\mathbf{f}\|_{L^1(\mathbb{T},\mathcal{B}(H))}\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\|\mathbf{x}\|_{\ell^2(H)}\|\mathbf{y}\|_{\ell^2(H)}.
\end{aligned}
$$

So we have that $\|\mathbf{A_f}\|_{\mathcal{M}_l(\ell^2(H))} \le \|\mathbf{f}\|_{L^1(\mathbb{T},\mathcal{B}(H))}$. To get the corresponding inequality for right Schur multipliers, $\|\mathbf{A_f}\|_{\mathcal{M}_r(\ell^2(H))} \le \|\mathbf{f}\|_{L^1(\mathbb{T},\mathcal{B}(H))}$, just observe that for $\mathbf{f}^*(t) = (\mathbf{f}(t))^*$ one has $\mathbf{f}^* \in L^1(\mathbb{T}, \mathcal{B}(H))$ with $\|\mathbf{f}^*\|_{L^1(\mathbb{T},\mathcal{B}(H))} = \|\mathbf{f}\|_{L^1(\mathbb{T},\mathcal{B}(H))}$ and $\widehat{\mathbf{f}^*}(l) = (\hat{\mathbf{f}}(-l))^*$ for all $l \in \mathbb{Z}$. Since $\mathbf{A}_\mathbf{f}^* = \mathbf{A}_{\mathbf{f}^*}$ we get the other estimate.

To obtain the general case $\mathbf{f} \in L^1(\mathbb{T}, \mathcal{B}(H))$, we use an approximation argument using the fact that the polynomials are dense in $L^1(\mathbb{T}, \mathcal{B}(H))$. ∎

We recall that for $C(\ell^2(H))$, in Lemma 3.4.9 of the previous chapter, we proved that if $\mathbf{f} \in C(\mathbb{T}, \mathcal{B}(H))$ then $\mathbf{A_f} \in C(\ell^2(H))$, and moreover $\|\mathbf{A_f}\|_{\mathcal{B}(\ell^2(H))} = \|\mathbf{f}\|_{C(\mathbb{T},\mathcal{B}(H))}$. This fact will be put to use in the proof of the next proposition.

Definition 4.3.5 *Let $P \in P(\mathbb{T}, \mathcal{B}(H))$, say $P(t) = \sum_l T_l e^{ilt}$ for some $(T_l)_{l\in\mathbb{Z}} \in c_{00}(\mathcal{B}(H))$. We denote*

$$\|P\|_{L^1_{SOT}} = \sup_{\|x\|=1} \int_0^{2\pi} \left\| \sum_l T_l(x) e^{ilt} \right\| \frac{dt}{2\pi}.$$

The next result can be derived from the inclusion $\mathcal{M}_r(\ell^2(H)) \cap \mathcal{T} \subset M_{SOT}(\mathbb{T}, \mathcal{B}(H))$ (which was proven in Chapter 2, Theorem 2.5.9), but here we shall give an independent proof that makes use of Lemma 3.4.9.

Proposition 4.3.6 *Let $(T_l)_{l\in\mathbb{Z}} \in c_{00}(\mathcal{B}(H))$ and a polynomial $P(t) = \sum_l T_l e^{ilt}$. Then, $\mathbf{A}_P \in \mathcal{P}(\ell^2(H))$ with $\|P\|_{L^1_{SOT}} \leq \|\mathbf{A}_P\|_{\mathcal{M}_r(\ell^2(H))}$.*

Proof: For each $\mathbf{B} \in \mathcal{P}(\ell^2(H)) \cap \mathcal{T}$ that is given by $Q(t) = \sum_l S_l e^{ilt} \in P(\mathbb{T}, \mathcal{B}(H))$, if we apply Lemma 3.4.9 to $\mathbf{B} * \mathbf{A}_P$ and $\mathbf{B}$, we are able to write

$$\sup_{t\in[-\pi,\pi]} \left\| \sum_l S_l T_l e^{ilt} \right\|_{\mathcal{B}(H)} \leq \|\mathbf{A}_P\|_{\mathcal{M}_r(\ell^2(H))} \sup_{t\in[-\pi,\pi]} \left\| \sum_l S_l e^{ilt} \right\|_{\mathcal{B}(H)}. \tag{4.9}$$

Recall that $L^1(\mathbb{T}, H) \subset M(\mathbb{T}, H) = (C(\mathbb{T}, H))^*$, that is for $g \in L^1(\mathbb{T}, H)$ then $\Phi_g(\sum_l x_l \varphi_l) = \sum_l \langle x_l, \hat{g}(l) \rangle$, defined for polynomials $\sum_l x_l \varphi_l \in P(\mathbb{T}, H)$ extends to a functional in $(C(\mathbb{T}, H))^*$ with $\|g\|_{L^1(\mathbb{T},H)} = \|\Phi_g\|_{(C(\mathbb{T},H))^*}$. Taking this into account,

$$\|P\|_{L^1_{SOT}} = \sup\{|\sum_l \langle x_l, T_l(x) \rangle| : \|x\| = 1, \|\sum_l x_l \varphi_l\|_{C(\mathbb{T},H)} = 1\}.$$

Now, select $x \in H$ with $\|x\| = 1$ and a sequence $(x_l) \subset H$ such that $\|\sum_l x_l \varphi_l\|_{C(\mathbb{T},H)} = 1$. Define the operator $S_l = x_l \otimes x$ for $l \in \mathbb{Z}$. Note that $\langle T_l(x), x_l \rangle x = S_l(T_l x)$, and also

$\sum_l S_l e^{ilt} \in C(\mathbb{T}, \mathcal{B}(H))$. Indeed,

$$\sup_{t\in[-\pi,\pi]} \left\| \sum_l S_l e^{ilt} \right\|_{\mathcal{B}(H)} = \sup_{t\in[-\pi,\pi],\|z\|=1} \left\| \sum_l \langle z e^{ilt}, x_l \rangle x \right\|_H = \sup_{t\in[-\pi,\pi]} \left\| \sum_l x_l e^{-ilt} \right\|_H .$$

By using (4.9), we can write

$$\begin{aligned} \left| \sum_l \langle T_l(x), x_l \rangle \right| &= \left\| \sum_l S_l T_l(x) \right\|_H \\ &\leq \left\| \sum_l S_l T_l \right\|_{\mathcal{B}(H)} \\ &\leq \sup_{t\in[-\pi,\pi]} \left\| \sum_l S_l T_l e^{ilt} \right\|_{\mathcal{B}(H)} \\ &\leq \|\mathbf{A}_P\|_{\mathcal{M}_r(\ell^2(H))} \sup_{t\in[-\pi,\pi]} \left\| \sum_l S_l e^{ilt} \right\|_{\mathcal{B}(H)} \\ &= \|\mathbf{A}_P\|_{\mathcal{M}_r(\ell^2(H))} \sup_{t\in[-\pi,\pi]} \left\| \sum_l x_l e^{-ilt} \right\|_H . \end{aligned}$$

This completes the proof of the result. ∎

We write $\tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H))$ for the closure of polynomials under the $\| \cdot \|_{L^1_{SOT}}$. A combination of Proposition 4.3.4 and Proposition 4.3.6 gives the following result.

Corollary 4.3.7 $L^1(\mathbb{T}, \mathcal{B}(H)) \subset \mathcal{L}^1_r(\ell^2(H)) \subset \tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H))$.

We conclude the subsection with the following theorem that characterizes the space $\mathcal{L}^1_r(\ell^2(H))$.

Theorem 4.3.8

$$\mathcal{L}^1_r(\ell^2(H)) \cap \mathcal{T} = \tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H)).$$

Proof: The fact that $\mathcal{L}^1_r(\ell^2(H)) \cap \mathcal{T} \subset \tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H))$ is a consequence of Proposition 4.3.6. Let us prove that, given $p = \sum_j T_j e^{ijt} \in \mathcal{P}(\mathbb{T}, \mathcal{B}(H))$, then $\|A_p\|_{\mathcal{M}_r(\ell^2(H))} \leq$

$\|p\|_{L^1_{SOT}}$, where A_p is its associated matrix $A_p = (T_{j-k})_{k,j} \in \mathcal{P}(\ell^2(H)) \cap \mathcal{T}$. This will imply that $\tilde{L}^1_{SOT}(\mathbb{T}, \mathcal{B}(H)) \subset \mathcal{L}^1_r(\ell^2(H)) \cap \mathcal{T}$.

Denote $C_{\mathcal{T}} = C(\ell^2(H)) \cap \mathcal{T}$. Notice that it is enough to prove that $\|A_p\|_{(C_{\mathcal{T}}, B_{\mathcal{T}})_r} \leq \|p\|_{L^1_{SOT}}$, since $(C, B) = (C, C) = (B, B)$ (also $(C_\tau, B_\tau) = (C_\tau, C_\tau) = (B_\tau, B_\tau)$).

In order to do that, take $p = \sum_j T_j e^{ijt} \in \mathcal{P}(\mathbb{T}, \mathcal{B}(H))$, and consider its associated matrix $A_p = (T_{j-k})_{k,j} \in \mathcal{P}(\ell^2(H)) \cap \mathcal{T}$. Take $q = \sum_j S_j e^{ijt} \in \mathcal{P}(\mathbb{T}, \mathcal{B}(H))$. Observe that

$$
\begin{aligned}
\|A_q * A_p\|_{\mathcal{B}(\ell^2(H))} &= \|A_{q*p}\|_{\mathcal{B}(\ell^2(H))} \overset{\text{Lem. 3.4.9}}{=} \|q * p\|_{C(\mathbb{T}, \mathcal{B}(H))} \\
&= \sup_t \sup_{\|x\|=1} \left\| \sum_j S_j T_j(x) e^{ijt} \right\|_H \\
&= \sup_t \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \left| \sum_j \langle T_j x, S_j^* y \rangle e^{ijt} \right| \\
&= \sup_t \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \left| \int \left\langle \sum_k T_k(x) e^{iks}, \sum_j S_j^*(y) e^{ij(t-s)} \right\rangle \frac{ds}{2\pi} \right| \\
&\leq \sup_t \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \int \|p(s)(x)\| \|q^*(t-s)(y)\| \frac{ds}{2\pi} \\
&\leq \sup_{\|x\|=1} \|q^*\|_{C(\mathbb{T}, \mathcal{B}(H))} \int \|p(s)(x)\| \frac{ds}{2\pi} \\
&= \|A_q\|_{\mathcal{B}(\ell^2(H))} \|p\|_{L^1_{SOT}},
\end{aligned}
$$

and the proof is over. ∎

4.3.2 The upper triangular case

In this subsection, we shall expand a few on what it was studied at Subsection 3.4.2 on upper triangular matrices. We recall that if X is a complex Banach space we write $\mathcal{H}(\mathbb{D}, X)$ for the space of X-valued holomorphic functions, $H^\infty(\mathbb{D}, X)$ for the Banach space of bounded analytic functions on the unit disc with values in X, and $A(\mathbb{D}, X)$ stands for the disc algebra that is the closure of analytic polynomials in $H^\infty(\mathbb{D},X)$, with the norm

$$\|F\|_{H^\infty(\mathbb{D},X)} = \sup\{\|F(z)\| \ \mid z \in \mathbb{D}\}.$$

Also we denote by $H^1(\mathbb{D},X)$ the space of functions $F \in \mathcal{H}(\mathbb{D},X)$ such that

$$\|F\|_{H^1(\mathbb{D},X)} = \sup_{0<r<1} \int_0^{2\pi} \|F(re^{it})\| \frac{dt}{2\pi} < \infty.$$

$H^1(\mathbb{T},X)$ will denote the closure of analytic polynomials under this norm, which turns out to coincide with functions in $f \in L^1(\mathbb{T},X)$ such that $\hat{f}(l) = 0$ for $l < 0$. It is well known that given $F \in H^1(\mathbb{D},X)$ and $F_r(z) = F(rz)$ then $F_r \in H^1(\mathbb{T},X)$. Furthermore, one has that $F_r \to F$ in $H^1(\mathbb{D},X)$ if and only if $F \in H^1(\mathbb{T},X)$.

However, in general $H^1(\mathbb{T},X)$ does not coincide with $H^1(\mathbb{D},X)$. The property for that to happen is the so called Analytic Radon-Nikodym property (in short $ARNP$) introduced in [16]. It is easy to check that $c_0 \subset \mathcal{B}(H)$, and therefore $\mathcal{B}(H)$ fails to have the $ARNP$. In particular $H^1(\mathbb{T},\mathcal{B}(H)) \subsetneq H^1(\mathbb{D},\mathcal{B}(H))$.

We refer the reader to the books [24, 28] for possible results to be used, since we only need the basic theory, which extends to the vector-valued setting from the scalar-valued framework.

We recall the definition of $F_{\mathbf{A}}$ for an upper triangular matrix.

Definition 4.3.9 *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{U}$. *Define*

$$F_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} \mathbf{D_l} z^l = (z^{(j-k)} T_{kj}), \quad |z| < 1.$$

Directly from the definitions, we see that

$$F_{\mathbf{A}}(re^{it}) = \mathbf{M}_{P_r} * \mathbf{M}_t * \mathbf{A} = \mathbf{M}_{P_r} * f_{\mathbf{A}}(t) = \sum_{l=0}^{\infty} \mathbf{D}_l r^l e^{ilt}. \tag{4.10}$$

Remark 4.3.10 *If* $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{U}$ *is a matrix satisfying condition (4.1), it follows that* $F_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} \mathbf{D_l} z^l$ *is a well defined holomorphic function in* $\mathcal{H}(\mathbb{D}, \mathcal{B}(\ell^2(H)))$ *(since* $\sup_l \|D_l\| < \infty$ *and* $\sum_{l=0}^{\infty} |z|^l$ *is a convergent series for* $|z| < 1$*).*

Proposition 4.3.11 *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{U}$ *satisfying the condition (4.1).*

(i) $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ *if and only if* $F_{\mathbf{A}} \in H^\infty(\mathbb{D}, \mathcal{M}_r(\ell^2(H)))$*. Moreover*

$$\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} = \|F_{\mathbf{A}}\|_{H^\infty(\mathbb{D}, \mathcal{M}_r(\ell^2(H)))}.$$

(ii) $\mathbf{A} \in (\mathcal{B}(\ell^2(H)), C(\ell^2(H)))_r$ *if and only if* $F_{\mathbf{B}*\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(\ell^2(H)))$ *for all* $\mathbf{B} \in \mathcal{B}(\ell^2(H))$*. Moreover*

$$\|\mathbf{A}\|_{(\mathcal{B}(\ell^2(H)), C(\ell^2(H)))_r} = \sup\{\|F_{\mathbf{B}*\mathbf{A}}\|_{H^\infty(\mathbb{D}, \mathcal{B}(\ell^2(H)))} : \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))} = 1\}.$$

(iii) $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$ *if and only if* $F_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{M}_r(\ell^2(H)))$*. Moreover*

$$\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} = \|F_{\mathbf{A}}\|_{A(\mathbb{D}, \mathcal{M}_r(\ell^2(H)))}.$$

Proof: (i) Looking at (4.10), one gets

$$\|F_{\mathbf{A}}(re^{it})\|_{\mathcal{M}_r(\ell^2(H))} \leq \|\mathbf{M}_{P_r}\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{M}_t\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} = \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))},$$

which gives that $\|F_{\mathbf{A}}\|_{H^\infty(\mathbb{D}, \mathcal{M}_r(\ell^2(H)))} \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}$.

Conversely, using (4.5) for $k_n = P_{r_n}$ for a sequence r_n converging to 1 yields

$$\|F_{\mathbf{A}}\|_{H^\infty(\mathbb{D}, \mathcal{M}_r(\ell^2(H)))} = \sup_n \|P_{r_n}(\mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} = \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}.$$

(ii) It is a direct consequence of applying Theorem 3.4.14 to $\mathbf{B} * \mathbf{A}$ for all $\mathbf{B} \in \mathcal{B}(\ell^2(H))$
.

(iii) Recall that Theorem 4.3.3 tells us that $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$ if and only if $f_{\mathbf{A}} \in C(\mathbb{T}, \mathcal{M}_r(\ell^2(H)))$. Now take into account that $F_{\mathbf{A}}(re^{it}) = P_r(f_{\mathbf{A}}(t))$. A combined use of

Part (i), and the observation

$$\|P_r(\mathbf{A}) - \mathbf{A}\|_{\mathcal{M}(\ell^2(H))} = \|P_r * f_{\mathbf{A}} - f_{\mathbf{A}}\|_{C(\mathbb{T},\mathcal{M}_r(\ell^2(H)))}$$

gives us that $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$ is equivalent to $\|F_{\mathbf{A}}\|_{H^\infty(\mathbb{D},\mathcal{M}_r(\ell^2(H)))}$ together with $F_{\mathbf{A}}$ being continuous at the boundary. This gives the result. ∎

Definition 4.3.12 *Let* $\mathbf{A} = (T_{j-k}) \in \mathcal{U} \cap \mathcal{T}$*, we write*

$$G_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} T_l z^l, \quad |z| < 1.$$

The assumption $\sup_{l\geq 0} \|T_l\| < \infty$ gives that $G_{\mathbf{A}}(z) = \sum_{l=0}^{\infty} T_l z^l \in \mathcal{H}(\mathbb{D}, \mathcal{B}(H))$. In particular, for each $0 < r < 1$

$$(G_{\mathbf{A}})_r(e^{it}) = G_{\mathbf{A}}(re^{it}) = \sum_{l=0}^{\infty} T_l r^l e^{ilt} \in C(\mathbb{T}, \mathcal{B}(H)).$$

The following theorem adds information to the one provided by Corollary 3.4.15, and now we include its proof.

Theorem 4.3.13 *Let* $\mathbf{A} = (T_{j-k}) \in \mathcal{U} \cap \mathcal{T}$ *with* $\sup_{l\geq 0} \|T_l\| < \infty$.

(i) $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *if and only if* $G_{\mathbf{A}} \in H^\infty(\mathbb{D}, \mathcal{B}(H))$.

(ii) $\mathbf{A} \in C(\ell^2(H))$ *if and only if* $G_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(H))$.

(iii) If $G_{\mathbf{A}} \in H^1(\mathbb{D}, \mathcal{B}(H))$ *then* $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$.

(iv) If $G_{\mathbf{A}} \in H^1(\mathbb{T}, \mathcal{B}(H))$ *then* $\mathbf{A} \in \mathcal{L}^1_r(\ell^2(H))$.

Proof: Let us prove (i) and (ii) first. Observe that since $\widehat{(G_{\mathbf{A}})_r}(j-k) = T_{j-k} r^{j-k}$, the Toeplitz matrix associated to $(G_{\mathbf{A}})_r$ is actually $\mathbf{A}_{(G_{\mathbf{A}})_r} = P_r(\mathbf{A})$. Now, the continuity of $(G_{\mathbf{A}})_r$ allows us to apply Lemma 3.4.9, which gives $\|P_r(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} = \|(G_{\mathbf{A}})_r\|_{C(\mathbb{T},\mathcal{B}(H))}$. This, together with (4.4), gives that

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \sup_{0<r<1} \|P_r(\mathbf{A})\|_{\mathcal{B}(\ell^2(H))} = \|G_{\mathbf{A}}\|_{H^\infty(\mathbb{D},\mathcal{B}(H))}.$$

Finally, since $G_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(H))$ iff $(G_{\mathbf{A}})_r \to G_{\mathbf{A}}$ in $H^\infty(\mathbb{D}, \mathcal{B}(H))$, we have that $G_{\mathbf{A}} \in A(\mathbb{D}, \mathcal{B}(H))$ if and only if $\mathbf{A} \in C(\ell^2(H))$.

(iii) Invoking Proposition 4.3.4, we obtain

$$\|P_r(\mathbf{A})\|_{\mathcal{M}_r(\ell^2(H))} \leq \|(G_{\mathbf{A}})_r\|_{L^1(\mathbb{T}, \mathcal{B}(H))}$$

and applying (4.5), we can conclude that $\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \leq \|G_{\mathbf{A}}\|_{H^1(\mathbb{D}, \mathcal{B}(H))}$.

(iv) This item follows from (iii) since

$$\|P_r(\mathbf{A}) - \mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \leq \|(G_{\mathbf{A}})_r - G_{\mathbf{A}}\|_{H^1(\mathbb{D}, \mathcal{B}(H))},$$

and we take limits as $r \to 1$ to conclude the proof. ∎

Chapter 5

Block matrices and new Schur and Kronecker products

"The sculptor produces the beautiful statue by chipping away such parts of the marble block as are not needed - it is a process of elimination."

—Elbert Hubbard.

5.1 Preliminaries

In the previous three chapters, we have been working with a Schur-type product for matrices with operator entries based on the composition of operators. We recall its definition: given two matrices $\mathbf{A} = (T_{kj})$ and $\mathbf{B} = (S_{kj})$ with entries $T_{kj}, S_{kj} \in \mathcal{B}(H)$ we defined their Schur product as the entry-wise product

$$\mathbf{A} * \mathbf{B} = (T_{kj} S_{kj}) \tag{5.1}$$

where $T_{kj} S_{kj}$ denoted the composition of the operators T_{kj} and S_{kj}.

Other options to generalize the Schur product in the framework of operator entries are possible, and in this chapter we present a new one that we believe it is natural and also related to the classical Schur product, and we give an insight on some of its properties. Also, along the same line, a new Kronecker-type product will be explored.

Besides this introductory part, the chapter contains two more sections. The first one starts presenting our definition of the new Schur-type product $\circledast$ for matrices with operator entries (see Definition 5.2.1) and shows some of its basic properties. Subsection 5.2.1 is devoted to prove that our new product endows $\mathcal{B}(\ell^2(H))$ with a structure of Banach algebra, which is achieved in Theorem 5.2.4 after dealing with some particular cases. In the process, we obtain some corollaries that show how to compute the operator and multiplier norms of block matrices in terms of scalar matrices (see Corollary 5.2.5 and Corollary 5.2.6). Subsection 5.2.2 provides two applications of the previous results: the first one is a theorem of correspondence between multipliers for the product $\circledast$ and multipliers for the classical product $*$ (see Theorem 5.2.7), whereas the second one (Theorem 5.2.8) shows a way of constructing a countable amount of measures belonging to certain vector measure spaces, starting from a single function of $L^\infty(\mathbb{T})$.

Section 5.3.2 starts presenting a Kronecker-type product for block matrices also based on the Schur product, and continues discussing about traces of block matrices, in conjunction with the two products introduced. It is in Section 5.3.3 where we define a couple of interesting spaces of matrices that play an important role regarding the submultiplicativity of the trace operator, and also we show an application consisting on a version of an exponential matrix for our Schur product. Finally, Section 5.3.4 gives some remarks regarding traces of products of matrices where both the new Schur and Kronecker products are combined.

5.2 About the new Schur-type product

Let us define now the new Schur product for matrices with operator entries that we will be using in the sequel. We shall denote $Mat(X)$ the space of matrices (possibly infinite matrices) with entries in X.

Definition 5.2.1 (Product $\circledast$) *Let $\mathbf{A}, \mathbf{B} \in Mat(\mathcal{B}(H))$. Denote $\mathbf{A} = (T_{k,j})_{k,j}$ and $\mathbf{B} = (S_{k,j})_{k,j}$. Then, we introduce the notion of the product $\circledast$ as*

$$\mathbf{A} \circledast \mathbf{B} = (T_{k,j} * S_{k,j})_{k,j}$$

where "$*$" stands for the classical Schur product. Here, we are identifying $T_{k,j}$ and $S_{k,j}$ with their associated matrices.

Notice that when $T_{k,j}, S_{k,j}$ are just elements of $\mathbb{C}$ $\forall k, j$, this product coincides with the classical Schur product. Observe also that, if $\mathbf{A}, \mathbf{B}$ only have one entry (that is, they actually are infinite matrices with complex entries), then $\circledast$ is just the classical Schur product again.

Here we list some basic properties satisfied by the product $\circledast$.

- **Property 1)** $\circledast$: $Mat(\mathcal{B}(H)) \times Mat(\mathcal{B}(H)) \longrightarrow Mat(\mathcal{B}(H))$ is a bilinear map.

- **Property 2)** (Associativity). $\mathbf{A} \circledast (\mathbf{B} \circledast \mathbf{C}) = (\mathbf{A} \circledast \mathbf{B}) \circledast \mathbf{C}$.

- **Property 3)** (Distributivity with respect to the sum).

  - $(\mathbf{A} + \mathbf{B}) \circledast \mathbf{C} = (\mathbf{A} \circledast \mathbf{C}) + (\mathbf{B} \circledast \mathbf{C})$
  - $\mathbf{A} \circledast (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \circledast \mathbf{B}) + (\mathbf{A} \circledast \mathbf{C})$.

- **Property 4)** (Mixed associativity). $\alpha(\mathbf{A} \circledast \mathbf{B}) = (\alpha\mathbf{A}) \circledast \mathbf{B} = \mathbf{A} \circledast (\alpha\mathbf{B})$, for every complex number α.

- **Property 5)** (Commutativity). $\mathbf{A} \circledast \mathbf{B} = \mathbf{B} \circledast \mathbf{A}$.

Taking into account that the space $Mat(\mathcal{B}(H))$ with the operations "+" (sum of matrices) and "·" (product by scalar) is a vector space, then it becomes a commutative algebra when equipped with $\circledast$ due to properties **2)-5)**. It shall be proved soon that $(\mathcal{B}(\ell^2(H)), \circledast)$ is actually a Banach algebra.

5.2.1 Sub-multiplicativity

Our main goal in this section will be to prove that the operator norm is sub-multiplicative for this product, to get that our product $\circledast$ actually provides a Banach algebra structure to $(\mathcal{B}(\ell^2(H)), \circledast)$. The proof of this is given in Theorem 5.2.4, but first we deal with some particular cases. Considering the isometric isomorphism between separable Hilbert spaces and ℓ^2 spaces, the following results will be written in terms of the latter.

Theorem 5.2.2 $\left(\mathcal{B}\left(\ell^2\left(\ell^2_n(\mathbb{C})\right)\right), \circledast\right)$ *is a commutative Banach algebra* $\forall n \in \mathbb{N}$.

Proof: The properties of associativity, mixed associativity, commutativity and distributivity are checked very easily. The unit element is the matrix where every entry is the unit matrix for the Schur product (all its entries equal to 1).

Now, take an infinite block matrix $\mathbf{A} = (T_{i,j})_{i,j} \in Mat(\mathcal{B}(\ell^2_n(\mathbb{C})))$ from $\mathcal{B}\left(\ell^2\left(\ell^2_n(\mathbb{C})\right)\right)$. Consider $\mathbf{x} = (x_j)_j$ an infinite vector where each component is a vector of n coordinates, $x_j = (x_j^l)_{l=1}^n$. We have

$$\mathbf{Ax} = \begin{pmatrix} \sum_j T_{1,j}x_j \\ \\ \sum_j T_{2,j}x_j \\ \\ \sum_j T_{3,j}x_j \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_j (\sum_{l=1}^n T_{1,j}^{k,l} x_j^l)_{k=1,n} \\ \\ \sum_j (\sum_{l=1}^n T_{2,j}^{k,l} x_j^l)_{k=1,n} \\ \\ \sum_j (\sum_{l=1}^n T_{3,j}^{k,l} x_j^l)_{k=1,n} \\ \vdots \end{pmatrix} = \begin{pmatrix} (\sum_j \sum_{l=1}^n T_{1,j}^{k,l} x_j^l)_{k=1,n} \\ \\ (\sum_j \sum_{l=1}^n T_{2,j}^{k,l} x_j^l)_{k=1,n} \\ \\ (\sum_j \sum_{l=1}^n T_{3,j}^{k,l} x_j^l)_{k=1,n} \\ \vdots \end{pmatrix}.$$

Denote by $\mathbf{A}'$ the scalar matrix defined by letting free all elements from the entries of the matrix $\mathbf{A}$, and let it act on the vector $\mathbf{x}'$ constructed in the same way from the vector

$\mathbf{x}$. Then

$$\mathbf{A}'\mathbf{x}' = \begin{pmatrix} T_{1,1}^{1,1} & \cdots & T_{1,1}^{1,n} & T_{1,2}^{1,1} & \cdots & T_{1,2}^{1,n} & \cdots \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ T_{1,1}^{n,1} & \cdots & T_{1,1}^{n,n} & T_{1,2}^{n,1} & \cdots & T_{1,2}^{n,n} & \cdots \\ T_{2,1}^{1,1} & \cdots & T_{2,1}^{1,n} & T_{2,2}^{1,1} & \cdots & T_{2,2}^{1,n} & \cdots \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix} \begin{pmatrix} x_1^1 \\ \vdots \\ x_1^n \\ x_2^1 \\ \vdots \\ x_2^n \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_j \sum_{l=1}^n T_{1,j}^{1,l} x_j^l \\ \vdots \\ \sum_j \sum_{l=1}^n T_{1,j}^{n,l} x_j^l \\ \\ \sum_j \sum_{l=1}^n T_{2,j}^{1,l} x_j^l \\ \vdots \\ \sum_j \sum_{l=1}^n T_{2,j}^{n,l} x_j^l \\ \vdots \end{pmatrix}.$$

Observe that the norm of both $\mathbf{Ax}$ and $\mathbf{A}'\mathbf{x}'$ is the same, and we conclude that

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(\ell^2_n(\mathbb{C})))} = \|\mathbf{A}'\|_{\mathcal{B}(\ell^2(\mathbb{C}))}.$$

Therefore, if we take now a matrix $\mathbf{B}$ in the space $\mathcal{B}\left(\ell^2\left(\ell^2_n(\mathbb{C})\right)\right)$, we obtain (using this argument and the fact that $(\mathcal{B}(\ell^2(\mathbb{C})), *)$ is a Banach algebra) that

$$\begin{aligned} \|\mathbf{A} \circledast \mathbf{B}\|_{\mathcal{B}(\ell^2(\ell^2_n(\mathbb{C})))} &= \|\mathbf{A}' * \mathbf{B}'\|_{\mathcal{B}(\ell^2(\mathbb{C}))} \leq \|\mathbf{A}'\|_{\mathcal{B}(\ell^2(\mathbb{C}))} \|\mathbf{B}'\|_{\mathcal{B}(\ell^2(\mathbb{C}))} = \\ &= \|\mathbf{A}\|_{\mathcal{B}(\ell^2(\ell^2_n(\mathbb{C})))} \|\mathbf{B}\|_{\mathcal{B}(\ell^2(\ell^2_n(\mathbb{C})))}. \quad \blacksquare \end{aligned}$$

From now on, we shall keep the “prime” notation used in Theorem 5.2.2 to denote the operation of transforming a matrix $\mathbf{A}$ with operator entries, into a matrix $\mathbf{A}'$ with scalar entries, where those entries come from freeing the entries of the original matrix.

Notice that nothing prevents us from applying the argument of Theorem 5.2.2 to a matrix $\mathbf{A}$ whose entries are rectangular matrices. We have the restriction (that appears naturally due to the way the vector $\mathbf{x}$ is splitted) that matrices in a same column of $\mathbf{A}$ need to have the same number of columns, and matrices in the same row of $\mathbf{A}$ need to have the same number of rows. In other words, we have:

Theorem 5.2.3 *Take $(n_i)_i$ and $(m_j)_j$ two sequences of natural numbers. Consider matrices $(T_{i,j})_{i,j}$, where the entry (i,j) is a bounded operator that has a $n_i \times m_j$ matrix associated. Such matrices map elements of $(\ell^2_{m_1} \oplus \ell^2_{m_2} \oplus \cdots)$ to the space $(\ell^2_{n_1} \oplus \ell^2_{n_2} \oplus \cdots)$. The set of these matrices that define bounded linear operators, together with the operation $\circledast$, is a commutative Banach algebra.*

Proof: Reasoning as in Theorem 5.2.2, we get that

$$\|\mathbf{A}\|_{\mathcal{B}\left((\ell^2_{m_1}\oplus\ell^2_{m_2}\oplus\cdots)\,,\,(\ell^2_{n_1}\oplus\ell^2_{n_2}\oplus\cdots)\right)} = \|\mathbf{A}'\|_{\mathcal{B}(\ell^2(\mathbb{C}))},$$

and taking $\mathbf{B} \in \mathcal{B}\left((\ell^2_{m_1} \oplus \ell^2_{m_2} \oplus \cdots),(\ell^2_{n_1} \oplus \ell^2_{n_2} \oplus \cdots)\right)$,

$$\begin{aligned}
& \|\mathbf{A} \circledast \mathbf{B}\|_{\mathcal{B}\left((\ell^2_{m_1}\oplus\ell^2_{m_2}\oplus\cdots)\,,\,(\ell^2_{n_1}\oplus\ell^2_{n_2}\oplus\cdots)\right)} = \|\mathbf{A}' * \mathbf{B}'\|_{\mathcal{B}(\ell^2(\mathbb{C}))} \\
\leq\ & \|\mathbf{A}'\|_{\mathcal{B}(\ell^2(\mathbb{C}))}\|\mathbf{B}'\|_{\mathcal{B}(\ell^2(\mathbb{C}))} \\
=\ & \|\mathbf{A}\|_{\mathcal{B}\left((\ell^2_{m_1}\oplus\ell^2_{m_2}\oplus\cdots)\,,\,(\ell^2_{n_1}\oplus\ell^2_{n_2}\oplus\cdots)\right)} \cdot \|\mathbf{B}\|_{\mathcal{B}\left((\ell^2_{m_1}\oplus\ell^2_{m_2}\oplus\cdots)\,,\,(\ell^2_{n_1}\oplus\ell^2_{n_2}\oplus\cdots)\right)}. \quad \blacksquare
\end{aligned}$$

Now, let us see how to extend it to the general case in which the entries of the matrices are also infinite matrices.

Theorem 5.2.4 *Consider matrices of the type $(T_{i,j})_{i,j}$, where the entry (i,j) is a bounded linear operator from ℓ^2 to ℓ^2. The set of those matrices of this type that define bounded linear operators, with the operation $\circledast$, is a commutative Banach algebra.*

Proof: We recall that the norm of a matrix can be calculated as the supremum of the norms of rectangular projections of the original matrix. Indeed, if $\mathbf{A} = (T_{k,j})_{\substack{k\in\mathbb{N}\\ j\in\mathbb{N}}}$, and we denote $F_M(\mathbf{A}) = (T_{k,j})_{\substack{k\leq M\\ j\in\mathbb{N}}}$ and $C_N(\mathbf{A}) = (T_{k,j})_{\substack{k\in\mathbb{N}\\ j\leq N}}$, we have

$$\begin{aligned}
\|\mathbf{A}\| &= \sup_{\substack{\mathbf{x}\in c_{00}\\ \|\mathbf{x}\|=1}} \|\mathbf{A}\mathbf{x}\| = \sup_{\substack{\mathbf{x}\in c_{00}\\ \|\mathbf{x}\|=1}} \sup_M \|F_M(\mathbf{A})\mathbf{x}\| \\
&= \sup_{\|\mathbf{x}\|=1} \sup_N \sup_M \|C_N(F_M(\mathbf{A}))\mathbf{x}\| \\
&= \sup_N \sup_M \|C_N(F_M(\mathbf{A}))\|.
\end{aligned}$$

where c_{00} denotes the space of sequences having only finitely many nonzero elements. Therefore it shall be enough to observe matrices of the type $(T_{k,j})_{\substack{k\leq M\\ j\leq N}}$ (although each $T_{k,j}$ still has an infinite matrix associated). We have to analyse this situation:

$$\begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,N} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ T_{M,1} & T_{M,2} & \cdots & T_{M,N} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}.$$

First of all, the norm of each y_i can be calculated as the supremum of the norms of its projections. This generates row cuts in the operators of the i-th row of the matrix. Therefore, we have

$$\|y_i\| = \sup_{m_i} \|P_{m_i}(y_i)\| = \sup_{m_i} \|F_{m_i}(T_{i,1})(x_1) + \cdots + F_{m_i}(T_{i,N})(x_N)\|.$$

On the other hand, notice that we can approximate each one of these norms with vectors x's of finite size, and this will translate into column cuts for the T's. So, using that $c_{0,0}(H)$ is dense in H and the triangle inequality, we obtain

$$\|y_i\| \leq \sup_{m_i, n_1, \ldots, n_N} \|F_{m_i}(T_{i,1})(P_{n_1}(x_1)) + \cdots + F_{m_i}(T_{i,N})(P_{n_N}(x_N))\|.$$

Also, it is obvious that the following inequality holds too

$$\sup_{\|x\|=1} \|y_i\| = \sup_{\|x\|=1} \sup_{m_i} \|F_{m_i}(T_{i,1})(x_1) + \cdots + F_{m_i}(T_{i,N})(x_N)\| \geq$$

$$\geq \sup_{\substack{\|x\|=1 \\ m_i, n_1, \ldots, n_N}} \|F_{m_i}(T_{i,1})(P_{n_1}(x_1)) + \cdots + F_{m_i}(T_{i,N})(P_{n_N}(x_N))\|.$$

This yields, combining both inequalities, that

$$\sup_{\|x\|=1} \|y_i\| = \sup_{\substack{\|x\|=1 \\ m_i, n_1, \ldots, n_N}} \|F_{m_i}(T_{i,1})(P_{n_1}(x_1)) + \cdots + F_{m_i}(T_{i,N})(P_{n_N}(x_N))\|.$$

In other words,

$$\sup_{\|x\|=1} \|y_i\| = \sup_{\substack{\|x\|=1 \\ m_i, n_1, \ldots, n_N}} \|C_{n_1}(F_{m_i}(T_{i,1}))(x_1) + \cdots + C_{n_N}(F_{m_i}(T_{i,N}))(x_N)\|.$$

Hence, we have seen that to obtain the norm of the matrix $\mathbf{A}$ it is enough to compute

$$\sup_{\substack{\|\mathbf{x}\|=1 \\ n_1, \cdots, n_N \\ m_1, \cdots, m_M}} \left\| \begin{pmatrix} C_{n_1}F_{m_1}T_{1,1} & C_{n_2}F_{m_1}T_{1,2} & \cdots & C_{n_N}F_{m_1}T_{1,N} \\ C_{n_1}F_{m_2}T_{2,1} & C_{n_2}F_{m_2}T_{2,2} & \cdots & C_{n_N}F_{m_2}T_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n_1}F_{m_M}T_{M,1} & C_{n_2}F_{m_M}T_{M,2} & \cdots & C_{n_N}F_{m_M}T_{M,N} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \right\| \tag{5.2}$$

where only finite matrices whose entries are also finite matrices are involved.

So, all in all, the norm of an infinite matrix whose entries are operators from ℓ^2 to ℓ^2 can be obtained as a supremum of norms of finite rectangular matrices whose entries are operators with finite matrices associated, also of rectangular size.

In order to simplify the notation, let us introduce the family

$$\mathcal{F} = \{(P, Q, a_1, \cdots, a_N, b_1, \cdots, b_M),\ P, Q \in \mathbb{N}, a_i, b_j \in \mathbb{N}\ \forall 1 \leq i \leq N, 1 \leq j \leq M\}.$$

Now, $I = (N, M, N_1, \cdots, n_N, m_1, \cdots, m_M) \in \mathcal{F}$, and we can refer to the matrix in equation 5.2 as A_I. We are ready to conclude the proof. Let $\mathbf{A} = (T_{k,j})_{\substack{k\in\mathbb{N}\\ j\in\mathbb{N}}}$ and $\mathbf{B} = (S_{k,j})_{\substack{k\in\mathbb{N}\\ j\in\mathbb{N}}}$ be matrices as in the statement of the theorem. Taking into account all the process explained, we have

$$\|\mathbf{A} \circledast \mathbf{B}\| = \sup_{I\in\mathcal{F}} \|(\mathbf{A} \circledast \mathbf{B})_I\| = \sup_{I\in\mathcal{F}} \|\mathbf{A}_I \circledast \mathbf{B}_I\| \overset{\text{Thm. 5.2.3}}{\leq} \sup_{I\in\mathcal{F}} \|\mathbf{A}_I\| \|\mathbf{B}_I\| = \|\mathbf{A}\| \|\mathbf{B}\|. \quad \blacksquare$$

The arguments used in Theorem 5.2.4 allow us a to prove the following results of independent interest, that relate the norm of matrices with operator entries with norms of matrices with scalar entries.

Corollary 5.2.5 *Let $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. Then, its norm can be computed as*

$$\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))} = \sup_{I\in\mathcal{F}} \|(\mathbf{B}_I)'\|_{\mathcal{B}_I}$$

where I has the form $(N, M, n_1, \cdots, n_N, m_1, \cdots, m_M)$, the matrix $\mathbf{B}_I$ is constructed as described in the proof of Theorem 5.2.4, and we use the notation $\mathcal{B}_I$ for the space $\mathcal{B}\left((\ell^2_{n_1} \oplus \cdots \oplus \ell^2_{n_N}), (\ell^2_{m_1} \oplus \cdots \oplus \ell^2_{m_M})\right)$.

Proof: Applying the computations from Theorem 5.2.4, we get that the norm of $\mathbf{B}$ can be obtained as

$$\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))} = \sup_{I\in\mathcal{F}} \|\mathbf{B}_I\|_{\mathcal{B}\left(\ell^2_N(H), \ell^2_M(H)\right)}.$$

Now it is enough to apply the equality of norms between block matrices (with entries of finite size) and scalar matrices seen at Theorem 5.2.3 to get that for each $I \in \mathcal{F}$, we have that $\|\mathbf{B}_I\|_{\mathcal{B}\left(\ell^2_N(H), \ell^2_M(H)\right)} = \|(\mathbf{B}_I)'\|_{\mathcal{B}\left((\ell^2_{n_1}\oplus\cdots\oplus\ell^2_{n_N}), (\ell^2_{m_1}\oplus\cdots\oplus\ell^2_{m_M})\right)}$, which concludes the result. $\blacksquare$

We also have the multiplier version of the previous result. We will denote by $\mathcal{M}^{\circledast}(X, Y)$ the space of multipliers from X to Y with respect to the product $\circledast$, and in the same way as when deal with usual multipliers, we can denote $\mathcal{M}^{\circledast}(X) := \mathcal{M}^{\circledast}(X, X)$.

Corollary 5.2.6 *Let $\mathbf{A} \in \mathcal{M}^{\circledast}(\mathcal{B}(\ell^2(H))$. Then, its multiplier norm can be computed as*

$$\|\mathbf{A}\|_{\mathcal{M}^{\circledast}(\mathcal{B}(\ell^2(H)))} = \sup_{I \in \mathcal{F}} \|(\mathbf{A}_I)'\|_{\mathcal{M}_I}$$

where I has the form $(N, M, n_1, \cdots, n_N, m_1, \cdots, m_M)$, the matrix $\mathbf{A}_I$ is constructed as described in the proof of Theorem 5.2.4 and we use the notation $\mathcal{M}_I$ for the space $\mathcal{M}\left(\mathcal{B}\left((\ell^2_{n_1} \oplus \cdots \oplus \ell^2_{n_N}),\ (\ell^2_{m_1} \oplus \cdots \oplus \ell^2_{m_M})\right)\right)$.

Proof: First, let us see that the inequality $\|\mathbf{A}\|_{\mathcal{M}^{\circledast}(\mathcal{B}(\ell^2(H)))} \geq \sup_{I \in \mathcal{F}} \|(\widetilde{\mathbf{A}_I})'\|_{\mathcal{M}_I}$ is satisfied.

$$\begin{aligned}
\|\mathbf{A}\|_{\mathcal{M}^{\circledast}(\mathcal{B}(\ell^2(H)))} &= \sup_{\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}=1} \|\mathbf{A} \circledast \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \\
&\overset{\text{Cor. 5.2.5}}{=} \sup_{\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}=1} \sup_{I \in \mathcal{F}} \|(\mathbf{A}_I)' * (\mathbf{B}_I)'\|_{\mathcal{B}_I} \\
&= \sup_{I \in \mathcal{F}} \sup_{\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}=1} \|(\mathbf{A}_I)' * (\mathbf{B}_I)'\|_{\mathcal{B}_I} \\
&\geq \sup_{I \in \mathcal{F}} \sup_{\|(\mathbf{B}_I)'\|_{\mathcal{B}_I}=1} \|(\mathbf{A}_I)' * (\mathbf{B}_I)'\|_{\mathcal{B}_I} \\
&= \sup_{I \in \mathcal{F}} \|(\mathbf{A}_I)'\|_{\mathcal{M}_I},
\end{aligned}$$

where the last inequality follows from the fact that, for each $I \in \mathcal{F}$, we can appropriately select a set, namely S, of matrices from $\mathcal{B}(\ell^2(H))$ in a way that $\mathcal{B}_I \subset \{(\mathbf{R}_I)'\,; \mathbf{R} \in S\}$. On the other hand, we have

$$\begin{aligned}
\|\mathbf{A}\|_{\mathcal{M}^{\circledast}(\mathcal{B}(\ell^2(H)))} &\overset{\text{Cor. 5.2.5}}{=} \sup_{\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}=1} \sup_{I \in \mathcal{F}} \|(\mathbf{A}_I)' * (\mathbf{B}_I)'\|_{\mathcal{B}_I} \\
&\leq \sup_{I \in \mathcal{F}} \left(\|(\mathbf{A}_I)'\|_{\mathcal{M}_I} \cdot \sup_{\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}=1} \|(\mathbf{B}_I)'\|_{\mathcal{B}_I} \right) \\
&\leq \sup_{I \in \mathcal{F}} (\|(\mathbf{A}_I)'\|_{\mathcal{M}_I}) \cdot \sup_{I \in \mathcal{F}} \left(\sup_{\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}=1} \|(\mathbf{B}_I)'\|_{\mathcal{B}_I} \right) \\
&= \sup_{I \in \mathcal{F}} (\|(\mathbf{A}_I)'\|_{\mathcal{M}_I}) \cdot \sup_{\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}=1} \left(\sup_{I \in \mathcal{F}} \|(\mathbf{B}_I)'\|_{\mathcal{B}_I} \right) \\
&\overset{\text{Cor. 5.2.5}}{=} \sup_{I \in \mathcal{F}} \|(\mathbf{A}_I)'\|_{\mathcal{M}_I} \cdot \sup_{\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}=1} \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}
\end{aligned}$$

$$= \sup_{I\in\mathcal{F}} \|(\mathbf{A}_I)'\|_{\mathcal{M}_I},$$

which gives the other inequality and finishes the proof. ∎

5.2.2 Some applications

In the previous subsection we showed how to relate a matrix with operator entries to certain matrix with scalar entries, and vice versa, and it turned out that the norm of these matrices, in their corresponding spaces, was the same. With this fact in mind, we can present a result (Theorem 5.2.7) that establishes a connection between the multipliers for the product $\circledast$ and the multipliers for the classical Schur product.

Additionally, in Theorem 5.2.8, we shall see how the idea of splitting in blocks that we have used previously allows us to give a method to construct a countable amount of measures in V^∞ with values in spaces of operators, starting from a single element of $L^\infty(\mathbb{T})$.

Theorem 5.2.7 (A correspondence between multipliers for $*$ and $\circledast$)

(i) Let $A = (a_{i,j})_{i,j}$ be a matrix from $\mathcal{M}(\ell^2)$. Then, given $n \geq 1$, the matrix $\mathbf{A}^{n)}$ formed by taking $n \times n$ size blocks in A is a matrix with operator entries that defines an element of $\mathcal{M}^{\circledast}\left(\mathcal{B}\left(\ell^2\left(\ell^2_n(\mathbb{C})\right)\right)\right)$. The matrix $\mathbf{A}^{n)}$ looks as follows:

$$\begin{pmatrix}
\begin{array}{|cccc|}\hline a_{1,1} & a_{1,2} & \cdots & a_{1,n}\\ a_{2,1} & \cdots & \cdots & a_{2,n} \\ \vdots&\vdots&\vdots&\vdots\\ a_{n,1}&\cdots&\cdots&a_{n,n}\\\hline\end{array} & \begin{array}{|cccc|}\hline a_{1,n+1} & a_{1,n+2} & \cdots & a_{1,2n}\\ a_{2,n+1} & \vdots & \vdots & a_{2,2n} \\ \vdots&\vdots&\vdots&\vdots\\ a_{n,n+1}&\cdots&\cdots&a_{n,2n}\\\hline\end{array} & \cdots \\
\begin{array}{|cccc|}\hline a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n}\\ a_{n+2,1}&\ddots&\ddots&a_{n+2,n}\\ \vdots&\ddots&\ddots&\vdots\\ a_{2n,1}&\cdots&\cdots&a_{2n,n}\\\hline\end{array} & \ddots & \\
\vdots & &
\end{pmatrix}.$$

Observe that in the event that A is Toeplitz, $\mathbf{A}^{n)}$ is too.

(ii) Let $\mathbf{A}^{n)}$ be an infinite matrix whose entries are $n \times n$ matrices, with $\mathbf{A}^{n)}$ in the space $\mathcal{M}^{\circledast}\left(\mathcal{B}\left(\ell^2\left(\ell^2_n(\mathbb{C})\right)\right)\right)$. Then, the matrix A with scalar entries, obtained by freeing the entries of $\mathbf{A}^{n)}$, defines an element of $\mathcal{M}(\ell^2)$. Also, if $\mathbf{A}^{n)}$ was Toeplitz, A needs not be.

Proof: Statements (i) and (ii) follow by this chain of equalities:

$$\begin{aligned}
&\|\mathbf{A}^{n)}\|_{\mathcal{M}^{\circledast}(\mathcal{B}(\ell^2(\ell^2_n(\mathbb{C}))))} \\
&\quad \sup_{\|\mathbf{B}^{n)}\|_{\mathcal{B}(\ell^2(\ell^2_n(\mathbb{C})))}=1} \|\mathbf{A}^{n)} \circledast \mathbf{B}^{n)}\|_{\mathcal{B}(\ell^2(\ell^2_n(\mathbb{C})))} \\
&\overset{\text{Thm. 5.2.2}}{=} \sup_{\|\mathbf{B}^{n)}\|_{\mathcal{B}(\ell^2(\ell^2_n(\mathbb{C})))}=1} \|\left(\mathbf{A}^{n)} \circledast \mathbf{B}^{n)}\right)'\|_{\mathcal{B}(\ell^2)} \\
&= \sup_{\|\mathbf{B}^{n)'}\|_{\mathcal{B}(\ell^2)}=1} \|\mathbf{A}^{n)'} * \mathbf{B}^{n)'}\|_{\mathcal{B}(\ell^2)} \\
&= \|\mathbf{A}^{n)'}\|_{\mathcal{M}(\ell^2)} = \|A\|_{\mathcal{M}(\ell^2)}. \quad \blacksquare
\end{aligned}$$

Let us present the second application of the arguments of the previous subsection.

Theorem 5.2.8 (Construction of measures of V^∞ from functions of L^∞)

Consider $f(t) := \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{ikt} \in L^\infty(\mathbb{T})$. Then, given $N \in \mathbb{N}$, we have that the distribution

$$f_N(t) \sim \sum_{k=-\infty}^{\infty} T_k^{N)} e^{ikt}$$

belongs to $V^\infty\left(\mathbb{T}, \mathcal{B}(\ell^2_N(\mathbb{C}))\right)$, where $T_k^{N)}$ is a Toeplitz matrix given by the sequence $(\widehat{f}(Nk+j))_{j=-N+1}^{j=N-1}$.

Proof: Since f defines an element of $L^\infty(\mathbb{T})$, Toeplitz's Theorem (see 2.1.4) gives that its associated Toeplitz matrix, namely A, is in $\mathcal{B}(\ell^2)$,

$$A = \begin{pmatrix} \widehat{f}(0) & \widehat{f}(1) & \widehat{f}(2) & \widehat{f}(3) & \\ \widehat{f}(-1) & \ddots & \ddots & \ddots & \ddots \\ \widehat{f}(-2) & \ddots & \ddots & \ddots & \ddots \\ \widehat{f}(-3) & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \in \mathcal{B}(\ell^2).$$

Now fix $N \in \mathbb{N}$, and construct from A a matrix $\mathbf{A}^{N)}$ composed by boxes of size $N \times N$, like this:

$$\left(\begin{array}{ccc} \boxed{\begin{array}{cccc} \widehat{f}(0) & \widehat{f}(1) & \cdots & \widehat{f}(N-1) \\ \widehat{f}(-1) & \ddots & \ddots & \widehat{f}(N-2) \\ \vdots & \ddots & \ddots & \ddots \\ \widehat{f}(1-N) & \cdots & \cdots & \widehat{f}(0) \end{array}} & \boxed{\begin{array}{cccc} \widehat{f}(N) & \widehat{f}(N+1) & \cdots & \widehat{f}(2N-1) \\ \widehat{f}(N-1) & \ddots & \ddots & \widehat{f}(2N-2) \\ \vdots & \ddots & \ddots & \ddots \\ \widehat{f}(1) & \cdots & \cdots & \widehat{f}(N) \end{array}} & \cdots \\ \boxed{\begin{array}{cccc} \widehat{f}(-N) & \widehat{f}(-N+1) & \cdots & \widehat{f}(-1) \\ \widehat{f}(-N-1) & \ddots & \ddots & \widehat{f}(-2) \\ \vdots & \ddots & \ddots & \ddots \\ \widehat{f}(1-2N) & \cdots & \cdots & \widehat{f}(-N) \end{array}} & \ddots & \\ \vdots & & \end{array} \right)$$

The norm of A coincides with the norm of $\mathbf{A}^{N)}$ (see the proof of Theorem 5.2.2), therefore $\mathbf{A}^{N)} \in \mathcal{B}\left(\ell^2\left(\ell^2_N(\mathbb{C})\right)\right)$. Now we invoke Theorem 2.5.1 for matrices with operator entries, and we have that the distribution associated to $\mathbf{A}^{N)}$, namely f_N, defines an element of $V^\infty\left(\mathbb{T}, \mathcal{B}(\ell^2_N(\mathbb{C}))\right)$. Clearly $\widehat{f_N}(k) = T_k^{N)}$, and the proof is completed. ∎

5.3 Traces of block matrices

From now on, we will denote by $\mathcal{M}_{n,m}(X)$ the space of matrices of size $n \times m$ with entries in X. If X is also a space of matrices, we will use the expression "block matrices" to refer to the elements of $\mathcal{M}_{n,m}(X)$. We recall that the trace of a matrix $A = (a_{k,j})_{k,j} \in \mathcal{M}_{n\times n}(\mathbb{C})$

is the sum of its diagonal elements, that is

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{i,i}.$$

We refer the reader to the papers of Das and Vashisht (see [17]) and Taskara and Gumus (see [56]), where the authors investigated traces of Schur and Kronecker products. One of our goals in the chapter is to generalize some of those results to the context of block matrices, for new versions of Schur and Kronecker products.

Consider now $A = (a_{k,j})_{k,j} \in \mathcal{M}_{n,m}(\mathbb{C})$ and $B = (b_{k,j})_{k,j} \in \mathcal{M}_{p,q}(\mathbb{C})$. Their Kronecker product, denoted by $A \otimes B$, is defined as follows:

$$A \otimes B := \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,m}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,m}B \end{pmatrix} \in \mathcal{M}_{np,mq}(\mathbb{C}).$$

We can't help to recall that both the Schur product and the Kronecker product are very useful for applications in fields such as matrix theory, matricial analysis or statistics. For instance, we refer the reader to [40], where Magnus and Neudecker gave some results and statistical applications regarding the Schur and Kronecker products; and also to [44], where Persson and Popa use the Schur product as a tool in the area of matricial harmonic analysis to develop theories and results on matrix spaces parallel to their scalar counterparts, some of which we have extended or generalized to the framework of matrices with operator entries throughout this thesis.

We already defined the new Schur product for block matrices (see for example Definition 5.2.1), and we recall here its definition particularized to the case of finite matrices, which is the one that we shall pay attention to from now on.

Definition 5.3.1 *Let* $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C}))$ *and consider another matrix of the same dimensions,* $\mathbf{B} = (S_{k,j})_{k,j} \in \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C}))$. *We define the Schur product* *of* $\mathbf{A}$ *and* $\mathbf{B}$ *as*

$$\mathbf{A} \circledast \mathbf{B} := (T_{k,j} * S_{k,j})_{k,j} \in \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C})),$$

where $T_{k,j} * S_{k,j}$ denotes the classical Schur product of the matrices $T_{k,j}$ and $S_{k,j}$. If $m, n = 1$ or $N, M = 1$, this product coincides with the classical Schur product.

5.3.1 A Kronecker-type product

Consider now a matrix $T \in \mathcal{M}_{n,m}(\mathbb{C})$ and a block matrix $\mathbf{B} = (B_{k,j})_{k,j} \in \mathcal{M}_{N,M}(\mathcal{M}_{n,m}(\mathbb{C}))$. We define a block Kronecker product of T and $\mathbf{B}$ as $T \boxtimes \mathbf{B} = (T * B_{k,j})_{k,j}$. Taking this into account, we can define our Kronecker product of two block matrices as follows.

Definition 5.3.2 *Let* $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C}))$ *and also* $\mathbf{B} = (S_{k,j})_{k,j} \in \mathcal{M}_{P\times Q}(\mathcal{M}_{n\times m}(\mathbb{C}))$. *We define their Kronecker product,* $\mathbf{A} \boxtimes \mathbf{B}$, *as*

$$\mathbf{A} \boxtimes \mathbf{B} := (T_{k,j} \boxtimes \mathbf{B})_{k,j} \in \mathcal{M}_{NP\times MQ}(\mathcal{M}_{n\times m}(\mathbb{C})).$$

It is easy to observe that this product is not commutative, but the Schur product for block matrices is. Notice that if $m, n = 1$, this Kronecker product becomes the Kronecker product of matrices with complex entries; and if $P, Q, N, M = 1$, the classical Schur product is recovered. We point out again that, of course, other natural definitions of a block Kronecker product exist (see for example [29]).

Here, we compare some of the basic properties that we mentioned the new Schur product has, with the ones that the new Kronecker product satisfies, in the case of finite matrices. Let $N, M, P, Q, n, m \in \mathbb{N}$.

- **Property 1)** The products $\circledast$ and $\boxtimes$ establish bilinear maps between spaces of block matrices:

$$\circledast : \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C})) \times \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C})) \longrightarrow \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C}))$$

$$\boxtimes : \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C})) \times \mathcal{M}_{P\times Q}(\mathcal{M}_{n\times m}(\mathbb{C})) \longrightarrow \mathcal{M}_{NP\times MQ}(\mathcal{M}_{n\times m}(\mathbb{C})).$$

- **Property 2)** (Associativity).
  - $\mathbf{A} \circledast (\mathbf{B} \circledast \mathbf{C}) = (\mathbf{A} \circledast \mathbf{B}) \circledast \mathbf{C}$.
  - $\mathbf{A}\boxtimes(\mathbf{B}\boxtimes\mathbf{C}) = (\mathbf{A}\boxtimes\mathbf{B})\boxtimes\mathbf{C}$.
- **Property 3)** (Distributivity with respect to the sum).
  - $(\mathbf{A} + \mathbf{B}) \circledast \mathbf{C} = (\mathbf{A} \circledast \mathbf{C}) + (\mathbf{B} \circledast \mathbf{C})$.
  - $\mathbf{A} \circledast (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \circledast \mathbf{B}) + (\mathbf{A} \circledast \mathbf{C})$.
  - $(\mathbf{A} + \mathbf{B})\boxtimes\mathbf{C} = (\mathbf{A}\boxtimes\mathbf{C}) + (\mathbf{B}\boxtimes\mathbf{C})$.
  - $\mathbf{A}\boxtimes(\mathbf{B} + \mathbf{C}) = (\mathbf{A}\boxtimes\mathbf{B}) + (\mathbf{A}\boxtimes\mathbf{C})$.
- **Property 4)** (Mixed associativity). For every $\alpha \in \mathbb{C}$,
  - $\alpha(\mathbf{A} \circledast \mathbf{B}) = (\alpha\mathbf{A}) \circledast \mathbf{B} = \mathbf{A} \circledast (\alpha\mathbf{B})$.
  - $\alpha(\mathbf{A}\boxtimes\mathbf{B}) = (\alpha\mathbf{A})\boxtimes\mathbf{B} = \mathbf{A}\boxtimes(\alpha\mathbf{B})$.
- **Property 5)** (Commutativity of $\circledast$).
  - $\mathbf{A} \circledast \mathbf{B} = \mathbf{B} \circledast \mathbf{A}$.

The fact that the space of block matrices with the operations "+" (sum of matrices) and "·" (product by scalar) is a vector space, makes it become an algebra when equipped with $\boxtimes$ due to properties 2)-4), and a commutative algebra when it is equipped with $\circledast$ instead, due to properties 2)-5).

5.3.2 Remarks on traces of block matrices

In this subsection, we start studying some equalities and inequalities involving traces of block matrices and the products defined above. Since we are interested in traces, from now on we restrict ourselves to the context of square block matrices whose entries are also square matrices, with entries in $\mathbb{R}$, and the notation for the spaces of matrices shall be abbreviated in the following way: $\mathcal{M}_N(\mathcal{M}_n) := \mathcal{M}_{N\times N}(\mathcal{M}_{n\times n}(\mathbb{R}))$. First of all, take into account that the trace of a block matrix is computed by summing the traces of its diagonal elements. That is, if $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n)$, then

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} \operatorname{tr}(T_{i,i}) = \sum_{i=1}^{N}\sum_{l=1}^{n} T_{i,i}(l,l),$$

where the trace after the first equality is the usual trace for matrices with scalar entries.

Proposition 5.3.3 *Let* $\mathbf{A} \in \mathcal{M}_N(\mathcal{M}_n)$ *and* $\mathbf{B} \in \mathcal{M}_M(\mathcal{M}_n)$ *with* $\mathbf{A} = (T_{k,j})_{k,j}$ *and* $\mathbf{B} = (S_{k,j})_{k,j}$. *Then*

(i) If $M = N$, $\operatorname{tr}(\mathbf{A} \circledast \mathbf{B}) = \sum_{i=1}^{N}\sum_{l=1}^{n} T_{i,i}(l,l)S_{i,i}(l,l)$.

(ii) $\operatorname{tr}(\mathbf{A}\boxtimes\mathbf{B}) = \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{l=1}^{n} T_{i,i}(l,l)S_{j,j}(l,l)$.

(iii) If $M = N$, $\operatorname{tr}(\mathbf{A}\boxtimes\mathbf{B}) = \operatorname{tr}(\mathbf{A} \circledast \mathbf{B}) + \sum_{\substack{i=1,j=1\\ i\neq j}}^{N}\sum_{l=1}^{n} T_{i,i}(l,l)S_{j,j}(l,l)$.

Proof:

$$\text{(i) } \operatorname{tr}(\mathbf{A} \circledast \mathbf{B}) = \sum_{i=1}^{N} \operatorname{tr}(\mathbf{A} \circledast \mathbf{B})_{i,i} = \sum_{i=1}^{N}\sum_{l=1}^{n} (T_{i,i} * S_{i,i})(l,l)$$
$$= \sum_{i=1}^{N}\sum_{l=1}^{n} T_{i,i}(l,l)S_{i,i}(l,l).$$

$$\text{(ii) } \operatorname{tr}(\mathbf{A}\boxtimes\mathbf{B}) = \sum_{i=1}^{N} \operatorname{tr}(T_{i,i}\boxtimes\mathbf{B}) = \sum_{i=1}^{N}\sum_{j=1}^{M} \operatorname{tr}(T_{i,i} * S_{j,j})$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{l=1}^{n} T_{i,i}(l,l)S_{j,j}(l,l).$$

(iii) Follows from (i) and (ii). ∎

Remark 5.3.4 *Although we know that given* $\mathbf{A}, \mathbf{B}$ *square block matrices in general one has* $\mathbf{A} \boxtimes \mathbf{B} \neq \mathbf{B} \boxtimes \mathbf{A}$, *it is an immediate consequence of part b) in Proposition 5.3.3 that* $\mathrm{tr}(\mathbf{A} \boxtimes \mathbf{B})$ *is always equal to* $\mathrm{tr}(\mathbf{B} \boxtimes \mathbf{A})$. *Of course,* $\mathrm{tr}(\mathbf{A} \circledast \mathbf{B}) = \mathrm{tr}(\mathbf{B} \circledast \mathbf{A})$ *since the matrices in question actually coincide.*

In a recent paper, Das and Vashisht (see [17]) presented some equalities and inequalities regarding traces of Schur products of matrices. Our products also behave well with traces, and verify the natural generalizations of those equalities/inequalites, as we shall see in the rest of the chapter. The following proposition lists some examples of basic properties regarding traces of these products.

Proposition 5.3.5 *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{M}_N(\mathcal{M}_n)$. *Denote* $\mathbf{A} = (T_{k,j})_{\substack{k \leq N \\ j \leq N}}$ *and* $\mathbf{B} = (S_{k,j})_{\substack{k \leq N \\ j \leq N}}$. *We have the following properties.*

(i) $\mathrm{tr}\left((\mathbf{A}+\mathbf{B}) \circledast (\mathbf{A}-\mathbf{B})\right)) = \mathrm{tr}(\mathbf{A} \circledast \mathbf{A}) - \mathrm{tr}(\mathbf{B} \circledast \mathbf{B})$.

(ii) $\mathrm{tr}\left((\mathbf{A} \pm \mathbf{B}) \circledast (\mathbf{A} \pm \mathbf{B})\right)) = \mathrm{tr}(\mathbf{A} \circledast \mathbf{A}) \pm 2\,\mathrm{tr}(\mathbf{A} \circledast \mathbf{B}) + \mathrm{tr}(\mathbf{B} \circledast \mathbf{B})$.

(iii) $\mathrm{tr}\left((\mathbf{A}+\mathbf{B}) \boxtimes (\mathbf{A}-\mathbf{B})\right)) = \mathrm{tr}(\mathbf{A} \boxtimes \mathbf{A}) - \mathrm{tr}(\mathbf{B} \boxtimes \mathbf{B})$.

(iv) $\mathrm{tr}\left((\mathbf{A} \pm \mathbf{B}) \boxtimes (\mathbf{A} \pm \mathbf{B})\right)) = \mathrm{tr}(\mathbf{A} \boxtimes \mathbf{A}) \pm 2\,\mathrm{tr}(\mathbf{A} \boxtimes \mathbf{B}) + \mathrm{tr}(\mathbf{B} \boxtimes \mathbf{B})$.

(v) If $T_{k,k}$ *and* $S_{k,k}$ *have positive real diagonals* $\forall 1 \leq k \leq N$, *then we have*

$$\mathrm{tr}\left(\overbrace{(\mathbf{A}+\mathbf{B}) \circledast \cdots \circledast (\mathbf{A}+\mathbf{B}))}^{m}\right) \leq \sum_{s=1}^{m} \binom{m}{s} (\mathrm{tr}(\mathbf{A}))^{m-s} \cdot (\mathrm{tr}(\mathbf{B}))^{s}.$$

Proof: Most of the items are consequence of the properties of mixed associativity and distributivity with respect to the sum that the Kronecker and the Schur product have, and also the linearity of the trace and Remark 5.3.4. As examples, we will prove (i) and (v).

(i) $\mathrm{tr}\left((\mathbf{A}+\mathbf{B}) \circledast (\mathbf{A}-\mathbf{B})\right)) =$

$$= \sum_{k=1}^{N} \sum_{j=1}^{n} (T_{k,k}(j,j) + S_{k,k}(j,j)) \cdot (T_{k,k}(j,j) - S_{k,k}(j,j))$$

$$= \sum_{k=1}^{N} \sum_{j=1}^{n} T_{k,k}(j,j)^2 - S_{k,k}(j,j)^2$$

$$= \sum_{k=1}^{N} \operatorname{tr}(T_{k,k} * T_{k,k}) - \operatorname{tr}(S_{k,k} * S_{k,k})$$

$$= \operatorname{tr}(\mathbf{A} \circledast \mathbf{A}) - \operatorname{tr}(\mathbf{B} \circledast \mathbf{B}).$$

$$\text{(v)}\ \ \operatorname{tr}\left(\overbrace{(\mathbf{A}+\mathbf{B}) \circledast \cdots \circledast (\mathbf{A}+\mathbf{B})}^{m}\right) = \sum_{k=1}^{N} \sum_{j=1}^{n} (T_{k,k}(j,j) + (S_{k,k}(j,j))^m$$

$$\leq \sum_{k=1}^{N} (\operatorname{tr}(T_{k,k}) + \operatorname{tr}(S_{k,k}))^m \leq \left(\sum_{k=1}^{N} \operatorname{tr}(T_{k,k}) + \operatorname{tr}(S_{k,k}) \right)^m$$

$$= (\operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B}))^m = \sum_{s=1}^{m} \binom{m}{s} (\operatorname{tr}(\mathbf{A}))^{m-s} \cdot (\operatorname{tr}(\mathbf{B}))^s. \qquad \blacksquare$$

Remark 5.3.6 *Recall that the arithmetic mean of a sequence $(\alpha_l)_{l=1}^m$ is greater or equal than its geometric mean, that is*

$$\frac{\sum_{l=1}^m \alpha_l}{m} \geq \left(\prod_{l=1}^{m} \alpha_l \right)^{\frac{1}{m}}.$$

Proposition 5.3.7 *Let $p \in \mathbb{N}$, and consider a finite sequence of block matrices $(\mathbf{A}^{s)})_{s=1}^p \subset \mathcal{M}_N(\mathcal{M}_n)$. Then, we have the following inequality:*

$$\operatorname{tr}(\circledast_{s=1}^p \mathbf{A}^{s)}) \leq \operatorname{tr}\left(\circledast^p \sum_{s=1}^{p} \frac{\mathbf{A}^{s)}}{p} \right).$$

Proof:

$$\operatorname{tr}\left(\circledast^p \sum_{s=1}^{p} \frac{\mathbf{A}^{s)}}{p} \right) = \operatorname{tr}\left(\left(\frac{\mathbf{A}^{1)} + \cdots + \mathbf{A}^{p)}}{p} \right) \circledast \overset{p}{\cdots} \circledast \left(\frac{\mathbf{A}^{1)} + \cdots + \mathbf{A}^{p)}}{p} \right) \right) =$$

$$= \sum_{i=1}^{N} \operatorname{tr}\left(\left(\frac{\mathbf{A}^{1)}+\cdots+\mathbf{A}^{p)}}{p}\right) \circledast \overset{p}{\cdots} \circledast \left(\frac{\mathbf{A}^{1)}+\cdots+\mathbf{A}^{p)}}{p}\right)\right)_{i,i} =$$

$$= \sum_{i=1}^{N}\sum_{l=1}^{n} \left(\frac{\mathbf{A}^{1)}_{i,i}(l,l)+\cdots+\mathbf{A}^{p)}_{i,i}(l,l)}{p}\right)^{p}$$

$$\overset{\text{Rmk. 5.3.6}}{\geq} \sum_{i=1}^{N}\sum_{l=1}^{n}\prod_{s=1}^{p} \mathbf{A}^{s)}_{i,i}(l,l) = \sum_{i=1}^{N} \operatorname{tr}\left(\circledast_{s=1}^{p} \mathbf{A}^{s)}_{i,i}\right)$$

$$= \sum_{i=1}^{N} \operatorname{tr}\left(\left(\circledast_{s=1}^{p} A^{s)}\right)_{i,i}\right) = \operatorname{tr}\left(\circledast_{s=1}^{p} \mathbf{A}^{s)}\right).$$

∎

5.3.3 Trace sub-multiplicativity and the spaces $\mathcal{M}_N^S(\mathcal{M}_n)$ and $\mathcal{M}_N^+(\mathcal{M}_n)$

In this subsection we are interested in exploring the relation between the value of the trace of Schur or Kronecker products of matrices and the product of the traces of the original matrices. First of all, recall that in the case of matrices with scalar entries, it is well known that for the Kronecker product one has $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \cdot \operatorname{tr}(B)$. However, for block matrices this is not the case, neither for the product $\boxtimes$ nor for the product $\circledast$, as the following example shows.

Example 5.3.8 *Consider the following matrices from $\mathcal{M}_2(\mathcal{M}_2)$:*

$$\mathbf{A} = \begin{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}.$$

Their block Schur product is the matrix

$$\mathbf{A} \circledast \mathbf{B} = \begin{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} & \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix} \in \mathcal{M}_2(\mathcal{M}_2),$$

and their block Kronecker product is the matrix

$$\mathbf{A} \boxtimes \mathbf{B} = \begin{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix} & \begin{pmatrix} -2 & 0 \\ 0 & -5 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix} & \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 2 \\ 3 & -1 \end{pmatrix} & \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} & \begin{pmatrix} -6 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} & \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix} \in \mathcal{M}_4(\mathcal{M}_2).$$

Then, $\mathrm{tr}(\mathbf{A}) = 3$, $\mathrm{tr}(\mathbf{B}) = -4$, *but* $\mathrm{tr}(\mathbf{A} \circledast \mathbf{B}) = -7 \neq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B})$ *and* $\mathrm{tr}(\mathbf{A} \boxtimes \mathbf{B}) = -6 \neq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B})$.

Notice that Example 5.3.8 also served the purpose of revealing that the inequalities $\mathrm{tr}(\mathbf{A} \boxtimes \mathbf{B}) \leq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B})$ and $\mathrm{tr}(\mathbf{A} \circledast \mathbf{B}) \leq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B})$ are not true in general. However, it is natural to ask ourselves if the trace operator might be sub-multiplicative for these products under certain restrictions. The following two spaces of block matrices will be relevant for that matter.

Definition 5.3.9 *Given* $N, n \in \mathbb{N}$, *we define the following subsets of* $\mathcal{M}_N(\mathcal{M}_n)$*:*

$$\mathcal{M}_N^S(\mathcal{M}_n) := \{(T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n) \ / \ \sum_{k=1}^{N} T_{k,k}(l,l) \geq 0, \ \forall\, 1 \leq l \leq n\},$$

$$\mathcal{M}_N^+(\mathcal{M}_n) := \{(T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n) \ / \ T_{k,k}(l,l) \geq 0, \ \forall\, 1 \leq k \leq N, \ \forall\, 1 \leq l \leq n\}.$$

Of course, $\mathcal{M}_N^+(\mathcal{M}_n) \subsetneq \mathcal{M}_N^S(\mathcal{M}_n)$.

The trace operator is actually sub-multiplicative when acting on matrices that fulfill conditions related to the spaces of matrices from Definition 5.3.9, as we show in the next theorem.

Theorem 5.3.10 *Let* $\mathbf{A} = (T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n)$ *and* $\mathbf{B} = (S_{k,j})_{k,j} \in \mathcal{M}_M(\mathcal{M}_n)$.

(i) If $M = N$, $\mathbf{A} \in \mathcal{M}_N^S(\mathcal{M}_n)$ *and* $\mathbf{B} \in \mathcal{M}_N^+(\mathcal{M}_n)$, *then*

$$\mathrm{tr}(\mathbf{A} \circledast \mathbf{B}) \leq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B}).$$

(ii) If $\mathbf{A} \in \mathcal{M}_N^S(\mathcal{M}_n)$ *and* $\mathbf{B} \in \mathcal{M}_M^S(\mathcal{M}_n)$, *then*

$$\mathrm{tr}(\mathbf{A} \boxtimes \mathbf{B}) \leq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B}).$$

Proof: (i) Firstly, since $\mathbf{A} \in \mathcal{M}_N^S(\mathcal{M}_n)$, we have that

$$\sum_{k=1}^{N} T_{k,k}(l,l) \leq \sum_{l=1}^{n} \sum_{k=1}^{N} T_{k,k}(l,l) = \mathrm{tr}(\mathbf{A}).$$

Additionally, $\mathbf{B}$ belonging to $\mathcal{M}_N^+(\mathcal{M}_n)$ yields the following inequality:

$$\sup_i S_{i,i}(l,l) \leq \sum_{i=1}^{N} S_{i,i}(l,l) \leq \sum_{l=1}^{n} \sum_{i=1}^{N} S_{i,i}(l,l) \leq \mathrm{tr}(\mathbf{B}).$$

Therefore, we have

$$\begin{aligned} \mathrm{tr}(\mathbf{A} \circledast \mathbf{B}) &\overset{\text{Prop. 5.3.3 } (i)}{=} \sum_{i=1}^{N} \sum_{l=1}^{n} T_{i,i}(l,l) S_{i,i}(l,l) \\ &\leq \sum_{l=1}^{n} \sup_i S_{i,i}(l,l) \sum_{i=1}^{N} T_{i,i}(l,l) \\ &\leq \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B}). \end{aligned}$$

(ii) In a similar fashion, we obtain the upper estimate for the trace of the block Kronecker product:

$$\begin{aligned}
\operatorname{tr}(\mathbf{A} \boxtimes \mathbf{B}) &\overset{\text{Prop. 5.3.3 } (ii)}{=} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{l=1}^{n} T_{i,i}(l,l) S_{j,j}(l,l) \\
&= \sum_{l=1}^{n} \left(\sum_{i=1}^{N} T_{i,i}(l,l) \right) \cdot \left(\sum_{j=1}^{M} S_{j,j}(l,l) \right) \\
&\leq \left(\sup_{l} \sum_{j=1}^{M} S_{j,j}(l,l) \right) \cdot \sum_{l=1}^{n} \left(\sum_{i=1}^{N} T_{i,i}(l,l) \right) \\
&\leq \left(\sum_{l=1}^{n} \sum_{j=1}^{M} S_{j,j}(l,l) \right) \cdot \operatorname{tr}(\mathbf{A}) \leq \operatorname{tr}(\mathbf{A}) \cdot \operatorname{tr}(\mathbf{B}). \quad \blacksquare
\end{aligned}$$

Corollary 5.3.11 *(i) Let* $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_m \in \mathcal{M}_N^+(\mathcal{M}_n)$. *Then*

$$\operatorname{tr}(\mathbf{A}_1 \circledast \mathbf{A}_2 \circledast \cdots \circledast \mathbf{A}_m) \leq \prod_{i=1}^{m} \operatorname{tr}(\mathbf{A}_i).$$

(ii) Let $\mathbf{B}_i \in \mathcal{M}_{N_i}^+(\mathcal{M}_n)$, $1 \leq i \leq m$. *Then*

$$\operatorname{tr}(\mathbf{B}_1 \boxtimes \mathbf{B}_2 \boxtimes \cdots \boxtimes \mathbf{B}_m) \leq \prod_{i=1}^{m} \operatorname{tr}(\mathbf{B}_i).$$

Proof: (i) Notice that for any $1 \leq i \leq m$, one has

$$(\mathbf{A}_1 \circledast \mathbf{A}_2 \circledast \cdots \circledast \mathbf{A}_i)_{k,k}(l,l) = (\mathbf{A}_1)_{k,k}(l,l) \cdot (\mathbf{A}_2)_{k,k}(l,l) \cdot \ldots \cdot (\mathbf{A}_i)_{k,k}(l,l) \geq 0$$

for all k, l such that $1 \leq k \leq N$ and $1 \leq l \leq n$, since by hypothesis we have that all matrices $\mathbf{A}_i$ are in $\mathcal{M}^+$. Therefore, $\mathbf{A}_1 \circledast \mathbf{A}_2 \circledast \cdots \circledast \mathbf{A}_i$ is also in $\mathcal{M}^+$ for each i. Now, the inequality is a consequence of this observation, part (i) of Theorem 5.3.10 and an induction argument.

(ii) Follows the same line as (i), because the matrix $\mathbf{B}_1 \boxtimes \cdots \boxtimes \mathbf{B}_i$ is also in $\mathcal{M}^+$ for each $1 \leq i \leq m$ noticing that its diagonals are just Schur products of diagonals of matrices that are all of them in $\mathcal{M}^+$ by hypothesis. Therefore we are able to apply part (ii) of Theorem 5.3.10, and an induction argument concludes the proof. ∎

We present now a small application of the previous inequality. We will take a look at the trace of a version of the exponential of a block matrix based on our block Schur product. Let $\mathbf{A} \in \mathcal{M}_N(\mathcal{M}_n)$ be a block matrix. We define $e_S^{\mathbf{A}}$ as follows:

$$e_S^{\mathbf{A}} := \sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{j} \mathbf{A}}{j!},$$

where $\circledast_{i=1}^{0} \mathbf{A} := I \in \mathcal{M}_N(\mathcal{M}_n)$ is just the identity matrix for the block Schur product, whose trace is $\mathrm{tr}(I) = Nn$. Observe that $e_S^{\mathbf{A}}$ is well defined, since taking multiplier norms we have

$$\left\| \sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{j} \mathbf{A}}{j!} \right\| \leq \sum_{j=0}^{\infty} \left\| \frac{\circledast_{i=1}^{j} \mathbf{A}}{j!} \right\| \leq \sum_{j=0}^{\infty} \frac{\|\mathbf{A}\|^j}{j!} = e^{\|\mathbf{A}\|}.$$

Notice that in the second inequality we used the trivial fact that the product $\circledast$ endows the space of multipliers from $\mathcal{M}_N(\mathcal{M}_n)$ to $\mathcal{M}_N(\mathcal{M}_n)$ with a structure of Banach algebra. Furthermore, we already saw that the product $\circledast$ also endows the space of bounded linear operators represented by elements of $\mathcal{M}_N(\mathcal{M}_n)$ with a structure of Banach algebra with the operator norm (see Subsection 5.2.1). So we could also have taken operator norms to justify that $e_S^{\mathbf{A}}$ is well defined. The only extra thing to take into account in that case is that the operator norm of $I \in \mathcal{M}_N(\mathcal{M}_n)$, however finite, is not equal to 1.

Now, letting $d = Nn - 1$, the trace of $e_S^{\mathbf{A}}$ can be bounded from above when $\mathbf{A} \in \mathcal{M}_N^+(\mathcal{M}_n)$ as follows:

$$\mathrm{tr}(e_S^{\mathbf{A}}) = \mathrm{tr}\left(\sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{j} \mathbf{A}}{j!} \right) = \sum_{j=0}^{\infty} \frac{\mathrm{tr}(\circledast_{i=1}^{j} \mathbf{A})}{j!} \overset{\text{Cor. 5.3.11 (i)}}{\leq} Nn + \sum_{j=1}^{\infty} \frac{\mathrm{tr}(\mathbf{A})^j}{j!} = d + e^{\mathrm{tr}(\mathbf{A})}. \tag{5.3}$$

Proposition 5.3.12 *Let $\mathbf{A}, \mathbf{B} \in \mathcal{M}_N(\mathcal{M}_n)$ such that $\mathbf{A} + \mathcal{B} \in \mathcal{M}_N^+(\mathcal{M}_n)$, and let $d = Nn - 1$. Then*

$$\mathrm{tr}(e_S^{\mathbf{A}} \circledast e_S^{\mathbf{B}}) \leq d + e^{tr(\mathbf{A})} \cdot e^{tr(\mathcal{B})}.$$

Proof: Since the product $\circledast$ is commutative, we can write

$$\begin{aligned}
e_S^{\mathbf{A}} \circledast e_S^{\mathbf{B}} &= \left(\sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{j} \mathbf{A}}{j!} \right) \left(\sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{j} \mathbf{B}}{j!} \right) \\
&= \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{m} \mathbf{A} \circledast_{i=1}^{j} \mathbf{B}}{m! j!} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{\circledast_{i=1}^{m} \mathbf{A} \circledast_{i=1}^{l-m} \mathbf{B}}{m!(l-m)!} \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=0}^{l} \frac{l!}{m!(l-m)!} \circledast_{i=1}^{m} \mathbf{A} \circledast_{i=1}^{l-m} \mathbf{B} \\
&= \sum_{l=0}^{\infty} \frac{\circledast_{i=1}^{l} (\mathbf{A} + \mathbf{B})}{l!} = e_S^{\mathbf{A}+\mathbf{B}}.
\end{aligned}$$

With that, and applying inequality (5.3) to $\mathbf{A} + \mathbf{B}$, we obtain:

$$\mathrm{tr}(e_S^{\mathbf{A}} \circledast e_S^{\mathbf{B}}) = \mathrm{tr}(e_S^{\mathbf{A}+\mathbf{B}}) \overset{(5.3)}{\leq} d + e^{\mathrm{tr}(\mathbf{A}+\mathbf{B})} = d + e^{(\mathrm{tr}(\mathbf{A})+\mathrm{tr}(\mathbf{B}))} = d + e^{\mathrm{tr}(\mathbf{A})} \cdot e^{\mathrm{tr}(\mathbf{B})}.$$

∎

The following corollary is a direct consequence of applying induction to Proposition 5.3.12, and it gives an upper estimate for the trace of a finite Schur product of exponentials of matrices.

Corollary 5.3.13 *Let* $\{\mathbf{A}_i\}_{i=1}^{m} \subset \mathcal{M}_N(\mathcal{M}_n)$ *such that* $\sum_{i=1}^{m} \mathbf{A}_i \in \mathcal{M}_N^{+}(\mathcal{M}_n)$, *and let* $d = Nn - 1$. *Then, we have*

$$\mathrm{tr}(\circledast_{i=1}^{m} e_S^{\mathbf{A}_i}) \leq d + \prod_{i=1}^{m} e^{tr(\mathbf{A}_i)}.$$

5.3.4 Trace inequalities combining both products

In this last subsection, we present a result that provides upper estimates for the traces of block matrices generated by combined Kronecker and Hadamard products, in terms of the trace of matrices where only one of the products is involved. The utility of these lies in the fact that it is easier to compute the latter ones.

Theorem 5.3.14 *Let* $\mathbf{A}_i, \mathbf{B}_i \in \mathcal{M}_N^+(\mathcal{M}_n)$*, for* $1 \leq i \leq m$*. Then, we have*

(i) $\operatorname{tr}\Big((\mathbf{A}_1 \boxtimes \mathbf{A}_2 \boxtimes \cdots \boxtimes \mathbf{A}_m) \circledast (\mathbf{B}_1 \boxtimes \mathbf{B}_2 \boxtimes \cdots \boxtimes \mathbf{B}_m) \Big) \leq \prod_{i=1}^m \operatorname{tr}(\mathbf{A}_i \circledast \mathbf{B}_i)$.

(ii) $\operatorname{tr}\Big((\mathbf{A}_1 \circledast \mathbf{A}_2 \circledast \cdots \circledast \mathbf{A}_m) \boxtimes (\mathbf{B}_1 \circledast \mathbf{B}_2 \circledast \cdots \circledast \mathbf{B}_m) \Big) \leq \prod_{i=1}^m \operatorname{tr}(\mathbf{A}_i \boxtimes \mathbf{B}_i)$.

Proof: First of all, observe that direct computations show that the product $\circledast$ and the product $\boxtimes$ are related by the following equation, that we shall use below:

$$(\mathbf{A} \boxtimes \mathbf{B}) \circledast (\mathbf{C} \boxtimes \mathbf{D}) = (\mathbf{A} \circledast \mathbf{C}) \boxtimes (\mathbf{B} \circledast \mathbf{D}). \tag{5.4}$$

(i) To prove it, we use an induction argument. For $m = 1$, the result is clear. Let us assume that the result is true for $m = s$, and let us prove that it is also true for $m = s+1$. We can write

$$\begin{aligned} &\operatorname{tr}\Big((\mathbf{A}_1 \boxtimes \mathbf{A}_2 \boxtimes \cdots \boxtimes \mathbf{A}_{s+1}) \circledast (\mathbf{B}_1 \boxtimes \mathbf{B}_2 \boxtimes \cdots \boxtimes \mathbf{B}_{s+1}) \Big) \\ &\overset{(5.4)}{=} \operatorname{tr}\Big(\big((\mathbf{A}_1 \boxtimes \mathbf{A}_2 \boxtimes \cdots \boxtimes \mathbf{A}_s) \circledast (\mathbf{B}_1 \boxtimes \mathbf{B}_2 \boxtimes \cdots \boxtimes \mathbf{B}_s) \big) \boxtimes (\mathbf{A}_{s+1} \circledast \mathbf{B}_{s+1}) \Big). \end{aligned}$$

Now, hypothesis gives that $\mathbf{A}_{s+1} \circledast \mathbf{B}_{s+1}$ is in a space $\mathcal{M}^+$. The matrix $(\mathbf{A}_1 \boxtimes \mathbf{A}_2 \boxtimes \cdots \boxtimes \mathbf{A}_s) \circledast (\mathbf{B}_1 \boxtimes \mathbf{B}_2 \boxtimes \cdots \boxtimes \mathbf{B}_s)$ is also in that space, because all of its diagonal entries are just Schur products of the diagonal entries of the matrices $\mathbf{A}_i$, $\mathbf{B}_j$, which were all of them in $\mathcal{M}^+$ by hypothesis. Hence, a use of part (ii) of Theorem 5.3.10 and induction hypothesis allows us to conclude that

$$\begin{aligned} &\operatorname{tr}\Big(\big((\mathbf{A}_1 \boxtimes \mathbf{A}_2 \boxtimes \cdots \boxtimes \mathbf{A}_s) \circledast (\mathbf{B}_1 \boxtimes \mathbf{B}_2 \boxtimes \cdots \boxtimes \mathbf{B}_s) \big) \boxtimes (\mathbf{A}_{s+1} \circledast \mathbf{B}_{s+1}) \Big) \\ &\overset{\text{Thm. 5.3.10 (ii)}}{\leq} \operatorname{tr}\Big((\mathbf{A}_1 \boxtimes \mathbf{A}_2 \boxtimes \cdots \boxtimes \mathbf{A}_s) \circledast (\mathbf{B}_1 \boxtimes \mathbf{B}_2 \boxtimes \cdots \boxtimes \mathbf{B}_s) \Big) \cdot \operatorname{tr}(\mathbf{A}_{s+1} \circledast \mathbf{B}_{s+1}) \\ &\leq \prod_{i=1}^{s} \operatorname{tr}(\mathbf{A}_i \circledast \mathbf{B}_i) \cdot \operatorname{tr}(\mathbf{A}_{s+1} \circledast \mathbf{B}_{s+1}) = \prod_{i=1}^{s+1} \operatorname{tr}(\mathbf{A}_i \circledast \mathbf{B}_i). \end{aligned}$$

(ii) The proof is analogous, but uses part (i) of Theorem 5.3.10 instead. ∎

Bibliography

[1] Ahmed, N., A note on Radon-Nikodym theorem for operator valued measures and its applications., *Commun. Korean Math. Soc.* **28** (2013), no. 2, 285–295.

[2] Alexandrov, A. B. ; Peller, V.V., Hankel and Toeplitz-Schur multipliers, *Math. Ann.* **324** (2002), 277–327.

[3] Badea, C.; Paulsen, V., Schur multipliers and operator-valued Foguel-Hankel operators, *Indiana Univ. Math. J.* **50** (2001), 1509–1522.

[4] Badrikian, A.; Johnson, G.W.; Yoo, I., The composition of operator-valued measurable functions is measurable, *Proc. Amer. Math. Soc.* **123** (1995), no. 6, 1815–1820.

[5] Bahrami, F.; Bayati Eshkaftaki, A.; Manjegani, S.M, Operator-valued Bochner integrable functions and Jensen's inequality, *Georgian Math. J.* **20** (2013), no. 4, 625–640.

[6] Barza, S.; Persson, L.E.; Popa, N., A matriceal analogue of Fejer's theory, *Math. Nachr.* **260** (2003), 14–20.

[7] Bennett, G., Schur multipliers, *Duke Math. J.* **44** (1977), 603—639.

[8] Blasco, O., Fourier Analysis on vector measures on locally compact abelian groups, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* (2016), **110**, 2, 519–539.

[9] Blasco, O.; García-Bayona, I., A class of Schur multipliers of matrices with operator entries, *Mediterranean J. Math.*, to appear.

[10] Blasco, O.; García-Bayona, I., New spaces of matrices with operator entries, *Quaest. Math.*, 2019. DOI: 10.2989/16073606.2019.1605416.

[11] Blasco, O.; García-Bayona, I., Remarks on measurability of operator-valued functions, *Mediterranean J. Math.*, **13** (2016), 5147–5162.

[12] Blasco, O.; García-Bayona, I., Schur Product with Operator-valued Entries, *Taiwanese J. Math.*, advance publication, 30 November 2018. DOI:10.11650/tjm/181110. https://projecteuclid.org/euclid.twjm/1543546839.

[13] Blasco, O.; Van Neerven, J., Spaces of operator-valued functions measurable with respect to the strong operator topology. Vector Measures, Integration and related topics. *Oper. Theory Adv. Appl.* (2010), **201**, 65–78.

[14] Böttcher, A; Grudsky, S. *Toeplitz Matrices, Asymptotic Linear Algebra, and Functional Analysis.* Hindustan Book Agency, New Delhi, 2000 and Birkhäuser Verlag, Basel,Boston, Berlin, 2000.

[15] Bruzual, R.; Dominguez, M., Representation of operator valued measurable indefinite functions., *Acta Cient. Venezolana* **52** (2001), no. 3, 223–225.

[16] Bukhvalov, A.V.; Danilevich, A.A., Boundary properties of analytic and harmonic functions with values in Banach spaces, *Mat. Zametki* **31** (1982), 203–214, English translation: Math. Notes **31** (1982), 104–110.

[17] Das, P.K.; Vashisht, L.K., Traces of Hadamard and Kronecker products of matrices, *Math. Appl.* **6** (2017), 143-150.

[18] Diestel, J.; Fourie, J.; Swart, J., *The metric theory of tensor products. Grothendieck's Résumé Revisited* , American Math. Soc, 2008.

[19] Diestel,J.; Jarchow, H.; Tonge, A., *Absolutely summing operators*, Cambridge University Press, 1995.

[20] Diestel, J.; Uhl, J.J, *Vector measures*, American Mathematical Society, Providence, (1977).

[21] Dinculeanu, N., *Vector Measures*, VEB Deutscher Verlag der Wissenschaften. Berlin, 1966.

[22] Dunford, N., On one parameter groups of linear transformations, *Annals of Math.* (2) **39** (1938), 569–573.

[23] Dunford, N.; Schwartz, J., *Linear Operators, part I: General Theory.* Interscience, New York, (1958).

[24] Duren, P.L., *Theory of H^p spaces*, Academic Press, New York and London, 1970.

[25] Edwards, R.E., *Functional Analysis. Theory and Applications.* Dover Pub. Inc, New York (1995).

[26] Fourie, J.H., On operator-valued measurable functions, Vector measures, integration and related topics, *Oper. Theory Adv. Appl.* **201** (2010), 205–214.

[27] García-Bayona, I., Traces of Schur and Kronecker products for block matrices, *Khayyam J. Math.*, 2019. DOI: 10.22034/kjm.2019.84207.

[28] Garnett, J.B., *Bounded analytic functions*, Academic Press, New York and London, 1981.

[29] Günter, M.; Klotz, L., Schur's theorem for a block Hadamard product, *Linear Algebra Appl.* **437** (2012), 948-956.

[30] Halmos, P.R., *Finite Dimensional Vector Spaces*, second edition, Van Nostrand, Princeton, New Jersey, 1958.

[31] Halmos, P.R., *Measure Theory.* New York: Springer-Verlag, 1974.

[32] Hengsen, W., A simpler proof of Singer's representation theorem, *Proc. Amer. Math. Soc.* **124** (1996), 3211-3212.

[33] Hille, E.; Phillips, R.S., *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc, Providence, R. I., 1957.

[34] Horn, R.A., The Hadamard product, *Proc. Symp. Appl. Math.* **40** (1990), 87-169.

[35] Horn, R.A.; Mathias, R.; Nakamura, Y., Inequalities for unitarily invariant norms and bilinear matrix products, *Linear and multilinear algebra*, **30** No.4 (1991), 303-314.

[36] Hytonen, T.; van Neerven, J.; Veraar, M.; Weis, L., *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory.* Series of Modern Surveys in Mathematics, 63. Springer, Cham, 2016. xvi+614 pp.

[37] Johnson, G.W., The product of strong operator measurable functions is strong operator measurable, *Proc. Amer. Math. Soc.* **117** (1993), 1097-1104.

[38] Kwapien, S.; Pelczyinski, A., The main triangle projection in matrix spaces and its applications, *Studia Math.* **34** (1970), 43–68.

[39] Livshits, L., Block-matrix generalizations of infinite-dimensional Schur products and Schur multipliers, *Linear and multilinear algebra*, **38** No.1-2 (1994), 59-78.

[40] Magnus, J.R.; Neudecker. H, *Matrix Differential Calculus with Applications in Statistics and Econometrics*, 2nd ed., Wiley, Chichester, UK, 1999. (1995), 17-56.

[41] Marcoci, A.N.; Marcoci, L.G., A new class of linear operators on and Schur multipliers for them, *J. Funct. Spaces Appl.* **5** (2007), no.2, 151-165.

[42] Michael, E., On a theorem of Rudin and Klee, *Proc. Amer. Math. Soc.* **12** (1961), 921.

[43] Paulsen, V., *Completely bounded maps and operator algebras*, Cambridge studies in advanced mathematics, **78**, Cambridge University Press, 2002.

[44] Persson, L-E.; Popa, N., *Matrix Spaces and Schur Multipliers: Matriceal Harmonic Analysis*, NJ : World Scientific, 2014. 208 s.

[45] Pettis, B.J., On integration in vector spaces, *Amer. Math. Soc.*, **44** (1938) 277–304.

[46] Pommerenke, C., Univalent Functions, Hubert, 1975.

[47] Reed, M.; Simon, B., *Methods of modern mathematical physics,* Vol. I, Functional analysis, revised enlarged ed., Academic Press, New York, 1980.

[48] Rudin, M.E.; Klee Jr, V.L., A note on certain function spaces, *Arch. Math. (Basel)* **7** (1957), 469–470.

[49] Ryan, R.A., *Introduction to tensor products of Banach Spaces*, Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2002.

[50] Rybakov, V.I., Theorem of Bartle, Dunford, and Schwartz concerning vector measures, *Mat. Zametki*, 7:2 (1970), 247–254; *Math. Notes*, 7:2 (1970), 147–151.

[51] Schur, J., Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Verandlichen., *J. Reine Angew. Math.* **140** (1911), 1–28.

[52] Shapiro, H.S.; Shields, A.L., On some interpolations problems for analytic functions, *Amer. J. Math.* **83** (1961), 513–532.

[53] Singer, I., Linear functionals on the space of continuous mappings of a compact Hausdorf space into a Banach space (in Russian), *Rev. Roum. Math. Pures Appl.* **2** (1957), 301–315.

[54] Singer, I., Sur les applications linéaires intégrales des espaces de fonctions continues, *Rev. Roum. Math. Pures Appl.* **4** (1959), 391–401.

[55] Styan, G.P.H., Hadamard products and multivariate statistical analysis, *Linear Algebra* **6** (1973), 217–240.

[56] Taskara, N.; Gumus, I.H., On the traces of Hadamard and Kronecker products of matrices, *Selçuk J. Appl. Math.* **14** (2013), 31-36.

[57] Toeplitz, O., Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veranderlichen, *Math. Annalen* **70** (1911), 351-376.