Using interval weights in MADM problems

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Abstract

The choice of weights vectors in multiple attribute decision making (MADM) problems has generated an important literature, and a large number of methods have been proposed for this task. In some situations the decision maker (DM) may not be willing or able to provide exact values of the weights, but this difficulty can be avoided by allowing the DM to give some variability in the weights. In this paper we propose a model where the weights are not fixed, but can take any value from certain intervals, so the score of each alternative is the maximum value that the weighted mean can reach when the weights belong to those intervals. We provide a closed-form expression for the scores achieved by the alternatives so that they can be ranked them without solving the proposed model, and apply this new method to an MADM problem taken from the literature.

Keywords: MADM problems, variable weights, interval weights, SAW method.

1. Introduction

There is a wide variety of problems that can be solved through the use of Multiple Attribute Decision Making (MADM) methods (see, for instance, Greco et al., 2016). Many of these methods require information about the relative importance of each attribute (for a classification of MADM methods according to the available information see, for instance, Hwang & Yoon, 1981, p. 9 and Zavadskas & Turskis, 2011, p. 404), and in many of them it is necessary to provide a weight for each attribute. For this reason, there exist in the literature a large number of procedures to determine the weights of the attributes (see, for instance, Wang & Luo, 2010, Roszkowska, 2013, Chin et al., 2015, and Fu et al., 2018). In accordance with these authors, it is usual to classify the methods into three categories: *subjective methods* (also called by other authors as

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direct explication or *a priori weights*), where the weights of the attributes are calculated by means of the information provided by the decision maker (DM); *objective methods* (also called by other authors as *indirect explication* or *a posteriori weights*), where the weights are determined through the information collected in the decision matrix; and *integrated methods*, where the weights are obtained by using both information sources.

It is worth noting that, in some cases, getting the weights through the decision matrix may have undesirable effects. For instance, Kao (2010) proposes a MADM method where the weights are determined from the decision matrix by using a compromise programming technique, and uses an example given by Jacquet-Lagrèze & Siskos (1982) to illustrate his method. In that example, ten cars have to be ranked taking into account six criteria: maximum speed (km/h), horse power (CV), space (m^2), consumption in town (lt/100 km), consumption at 120 km/h (lt/100 km), and price (francs). The weights obtained applying Kao's method are, respectively, 0.6346, 0.01, 0.01, 0.01, 0.01, and 0.3254. Notice that there exist four weights that are practically zero; so the corresponding criteria have little influence on the ranking of the cars. However, some of those criteria are the space and the consumption, which should have some importance in the ranking of the cars.

Notice also that when the weights are obtained from the decision matrix, the inclusion or exclusion of an alternative may significantly change the importance of the weights. For instance, consider the following example, taken from Deng et al. (2000), where seven textile companies, $A_1,\ldots, A_7$, are evaluated by using four financial ratios, which are identified as the criteria: profitability (C_1), Productivity (C_2), market position (C_3), and debt ratio (C_4)[1]. The performance ratings of each company with respect to the criteria are shown in Table 1.

When the entropy method[2] is used to calculate the weights (see details in Deng et al., 2000), we get $w_1 = 0.541$, $w_2 = 0.125$, $w_3 = 0.277$, and $w_4 = 0.057$ (see Table 4 in Deng et al., 2000). It is worth noting that the first criterion is more important than the rest of the criteria combined; in fact, the first criterion is about twice as important as the second most important criterion, C_3. Suppose now that the DM had not considered company A_3 in its analysis[3]. In this case, the weights obtained with the entropy method have been $w_1 = 0.335$, $w_2 = 0.202$, $w_3 = 0.379$, and $w_4 = 0.084$. Now, the most important criterion is C_3.

[1]The ratings of the debt ratio were adjusted so that it could be treated as a benefit criterion.

[2]The entropy method is a well-known procedure based on the idea that, if all alternatives have similar values with respect to an attribute, then a small weight should be assigned to that attribute (Zeleny, 1982).

[3]This company has the worst result for the first criterion, the second worst result for the third and fourth criteria, and the third worst result for the second criterion (see Table 1 in Deng et al., 2000). Moreover, when the modified TOPSIS proposed by the

Table 1: Performance ratings of companies.

	Profitability (C_1)	Productivity (C_2)	Market position (C_3)	Debt ratio (C_4)
A_1	0.12	49469	0.15	1.21
A_2	0.08	34251	0.14	1.23
A_3	0.04	32739	0.09	1.12
A_4	0.16	44631	0.11	1.56
A_5	0.09	33151	0.13	1.09
A_6	0.15	31408	0.07	1.39
A_7	0.13	30654	0.17	1.16

With regard to the subjective methods, there are several methods that allow obtaining the weight vector from the information provided by the DM (see, for instance, Wang & Luo, 2010; Chin et al., 2015; de Almeida et al., 2016). However, these procedures are not always available because the opinions of the DMs may be vague due to lack of information or knowledge. Sometimes the DM only provides an order of importance among the criteria (note that, according to some authors, there are several reasons to prefer this procedure; see Barron, 1992; Roszkowska, 2013). In this case, the attribute weights are calculated by using the ordinal ranking of the attributes provided by the DM (see, for instance, Roszkowska, 2013; Danielson & Ekenberg, 2014 for a revision on surrogate weights). However, it should be noted that although the DM only provides an ordinal ranking of the attributes, it is necessary weighting the criteria from their ranks, which may cause that the DM does not completely agree with the weights used.

One of the reasons given in the literature for the use of rank ordering weighting methods is that the DM may not be willing or able to provide exact values of the weights. This difficulty can also be avoided by allowing the DM to give some variability in the weights. This idea has been used, for example, in several methods proposed for dealing with incomplete information in weighting models (see, for instance, Weber, 1987; Arbel, 1989; Salo & Hämäläinen, 1992; Edwards & Barron, 1994; Salo & Hämäläinen, 1995; Park & Kim, 1997; Malakooti, 2000; Salo & Punkka, 2005; Mustajoki et al., 2005; Liu et al., 2018; Yu et al., 2019); in ranked voting systems, where each candidate is evaluated with the most favorable scoring vector for her (see, for instance, Cook & Kress, 1990; Llamazares & Peña, 2009, 2013; Llamazares, 2016, 2017, and the

authors is used with four sets of weights, this company is ranked last in the four cases (see Table 5 in Deng et al., 2000).

references therein), and also in the construction of composite indicators (see Nardo et al., 2008, pp. 92–94). Note also that Liu et al. (2019) have recently proposed a model where the ranking of each alternative is determined by the average of three rankings: the minimum and maximum ranking positions generated by several optimization models, and the average ranking position obtained through the Monte Carlo method.

One of the simplest ways to allow the variability of the weights is through intervals, so that each weight w_j can vary in an interval $[a_j, b_j]$. Notice that interval weights have been previously used in this context. For instance, Morais et al. (2014) conduct a study on the areas of a water distribution network on the municipality of Carnaíba, Pernambuco (Brazil). In this study, the authors use the Revised Simos' procedure (see Figueira & Roy, 2002) to obtain the criteria weights for each DM and, after that, for each criterion they consider an interval whose extremes are the minimum and maximum values obtained for the DMs. Likewise, Rezaei (2016) proposes a non-linear minmax model to determine the criteria weights and, given that sometimes your model may have multiple solutions, he suggests using the midpoint of certain interval weights.

In this paper we propose a model where the weights are not fixed, but can take any value from certain intervals, so the score of each alternative is the maximum value that the weighted mean can reach when the weights belong to those intervals. In this way, each alternative is assessed with the most favorable weight vector for it. We also provide a closed-form expression for the scores achieved by the alternatives so that it is possible to rank them without the need to solve the proposed model.

The rest of the paper is organized as follows. In Section 2 we propose our model and give a closed-form expression for the scores obtained by the alternatives. Moreover, we suggest several ways to build the interval weights required in our model. In Section 3 we apply our model to an MADM problem taken from Mulliner et al. (2016). Finally, some concluding remarks are provided in Section 4.

2. The model

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a finite set of alternatives and let $C = \{C_1, \ldots, C_n\}$ be a finite set of criteria in a multiple attribute decision making problem. We suppose that all alternatives score with respect to all criteria are known; and we denote by x_{ij} the performance value of alternative A_i with respect to criterion C_j. Since criteria are usually expressed in different units, a normalization process is generally necessary to ensure that all the values are dimensionless. In this process it is essential to make a distinction between benefit criteria (whose values are always better when larger) and cost criteria (whose values are always better when smaller).[4] Once the normalization process has been carried out, we will denote by $z_{ij} \in [0,1]$ the normalized value of alternative A_i with respect to criterion C_j. The matrix $\mathbf{Z} = (z_{ij})_{m\times n}$ will be call the *decision matrix* (see Table 2).

Table 2: A typical decision matrix.

	C_1	C_2	$\cdots$	C_n
A_1	z_{11}	z_{12}	$\cdots$	z_{1n}
A_2	z_{21}	z_{22}	$\cdots$	z_{2n}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
A_m	z_{m1}	z_{m2}	$\cdots$	z_{mn}

Many MADM methods require a weight vector that reflects the importance of each criterion; that is, a vector $\boldsymbol{w} = (w_1, \dots, w_n) \in [0,1]^n$ such that $\sum_{i=1}^n w_i = 1$. It is usual to suppose that $w_i > 0$ for all $i \in N$, where N denotes the set $\{1, \dots, n\}$. Among the great variety of methods proposed in the MADM field, the simple additive weighting (SAW) method is one of the most often used because of its transparency and simplicity. In the SAW method, the score of each alternative is obtained through the expression

$$Z_i = \sum_{j=1}^n z_{ij} w_j.$$

In our model we consider that each weight w_j can vary in an interval $[a_j, b_j]$, $j = 1, \dots, n$; so that the score of each alternative is the maximum value that the weighted mean can reach considering that the weights vary in the intervals $[a_j, b_j]$; that is,

$$\begin{aligned} Z_i^* = \max \ & \sum_{j=1}^n z_{ij} w_j, \\ \text{s.t. } & a_j \le w_j \le b_j, \quad j = 1, \dots, n, \\ & \sum_{j=1}^n w_j = 1. \end{aligned} \tag{1}$$

Notice that if $\sum_{j=1}^n a_j > 1$ or $\sum_{j=1}^n b_j < 1$, then the feasible set is empty. On the other hand, if $\sum_{j=1}^n a_j = 1$ or $\sum_{j=1}^n b_j = 1$, then the feasible set has only one element. Hence, the constraints $\sum_{j=1}^n a_j < 1 < \sum_{j=1}^n b_j$ are a requirement that we ask to the intervals $[a_j, b_j]$.

[4]A survey of the main methods used in the normalization of the values can be found in Jahan & Edwards, 2015.

In the following theorem we give closed-form expressions for the scores of alternatives when Model (1) is used. In this way, we can know the score obtained for each alternative without the need to solve the model.

Theorem 1. *Consider Model (1). Then*

$$Z_i^* = \sum_{j=1}^{p-1} \left(z_{i[j]} - z_{i[p]}\right) b_{[j]} + z_{i[p]} - \sum_{j=p+1}^{n} \left(z_{i[p]} - z_{i[j]}\right) a_{[j]},$$

where $[\cdot]$ *is a permutation of* N *such that* $z_{i[1]} \geq z_{i[2]} \geq \cdots \geq z_{i[n]}$ *and* $p \in N$ *satisfies*

$$\sum_{j=1}^{p-1} \left(b_{[j]} - a_{[j]}\right) < 1 - \sum_{j=1}^{n} a_j \leq \sum_{j=1}^{p} \left(b_{[j]} - a_{[j]}\right).$$

To illustrate the result given in the above theorem, consider the decision matrix given in Table 3, where we have added two rows: the first contains the interval weights of the criteria while the second shows the amplitude of these intervals.

Table 3: A decision matrix for illustrating Theorem 1.

	C_1	C_2	C_3	C_4	C_5
A_1	0.8	0.9	0.8	1	0.9
A_2	0.9	1	0.9	0.8	0.8
Interval weights	$[0.19, 0.29]$	$[0.16, 0.22]$	$[0.13, 0.23]$	$[0.14, 0.30]$	$[0.10, 0.24]$
Amplitude	0.10	0.06	0.10	0.16	0.14

To calculate the score obtained by alternative A_1 we may proceed as follows:

1. Order the z_{1j} scores from highest to lowest. Notice that in this case there is more than one permutation that provides the same order,[5] so we choose one of them (the subscript indicates the criterion in which the score has been achieved).

$$1_{C_4} \geq 0.9_{C_2} \geq 0.9_{C_5} \geq 0.8_{C_1} \geq 0.8_{C_3}.$$

[5] It is important to note that the score of the alternative does not depend on the permutation chosen (see the footnote 20 in the proof of Theorem 1).

2. Determine the value of p. For the above permutation, the value of p is 3 since

$$0.16 + 0.06 < 1 - 0.72 \leq 0.16 + 0.06 + 0.14.$$

3. Calculate the score of A_1 by using the expression given in Theorem 1:

$$Z_1^* = 0.1 \cdot 0.30 + 0 \cdot 0.22 + 0.9 - 0.1 \cdot 0.19 - 0.1 \cdot 0.13 = 0.898.$$

Using the same procedure for alternative A_2 we have

1. $1_{C_2} \geq 0.9_{C_1} \geq 0.9_{C_3} \geq 0.8_{C_4} \geq 0.8_{C_5}$.
2. $p = 4$ since $0.06 + 0.10 + 0.10 < 0.28 \leq 0.06 + 0.10 + 0.10 + 0.16$.
3. $Z_2^* = 0.2 \cdot 0.22 + 0.1 \cdot 0.29 + 0.1 \cdot 0.23 + 0.8 - 0 \cdot 0.10 = 0.896$.

The extreme case of our model is when the weights can vary in the interval $[0, 1]$. Then, the score of each alternative is the maximum value it attains over all criteria.

Corollary 1. *Consider Model (1) with $a_j = 0$ and $b_j = 1$ for all $j \in N$. Then*

$$Z_i^* = \max_{j=1,\dots,n} z_{ij}.$$

Nevertheless, this extreme case does not seem the most appropriate choice on most occasions. On the one hand, the probability that several alternatives reach the maximum score is greater than when smaller intervals are used. On the other hand, the winning alternative may not be the most appropriate. For instance, consider the decision matrix given in Table 4. According to Corollary 1, when the weights can vary in the interval $[0, 1]$, the scores of the alternatives are $Z_1^* = Z_2^* = 1$ and $Z_3^* = 0.99$. Hence, the alternative A_3 is not the winner, which does not seem very reasonable.

Table 4: Decision matrix with 3 alternatives and 2 criteria.

	C_1	C_2
A_1	1	0
A_2	0	1
A_3	0.99	0.99

When applying our model, it is necessary that the DM have the interval weights. If the DM instead of having the interval weights has weight vectors, we can apply the following strategies:

1. If the DM only has a weight vector, $\boldsymbol{w} = (w_1, \ldots, w_n)$, then, for each weight w_j, the interval can be constructed by taken the weight w_j plus or minus a percentage of w_j. For instance, if we consider $w_j = 0.3$ and a percentage of 10%, the interval is $[0.27, 0.33] = [0.3(1-r), 0.3(1+r)]$, where $r = 0.1$. Note that to avoid that the endpoints of the interval take values less than zero or greater than one, we have to use the expression

$$[\max(0, w_j(1-r)), \min(1, w_j(1+r))],$$

where $r > 0$. But when $r \in (0, 1]$ the intervals are of the form

$$[w_j(1-r), \min(1, w_j(1+r))],$$

and when $r = 1$ they are $[0, \min(1, 2w_j)]$, which means that there may be criteria that do not influence the score of alternatives. Notice also that when $r \in (0, 1]$, if we add the conditions $r \leq (1 - w_j)/w_j$ for all $j \in N$, then $w_j(1+r) \leq 1$ for all $j \in N$ and, consequently, the intervals are of the form $[w_j(1-r), w_j(1+r)]$. In this case the following corollary shows the score obtained by the alternatives.

Corollary 2. *Let* $\mathbf{w}$ *be a weight vector and let* $r \in (0, 1]$ *such that* $r \leq (1 - w_j)/w_j$ *for all* $j \in N$*. If we consider Model (1) with* $a_j = w_j(1-r)$ *and* $b_j = w_j(1+r)$ *for all* $j \in N$*, then*

$$Z_i^* = \sum_{j=1}^{n} z_{ij} w_j + r \sum_{j=1}^{n} \left| z_{ij} - z_{i[p]} \right| w_j,$$

where $[\cdot]$ *is a permutation of* N *such that* $z_{i[1]} \geq z_{i[2]} \geq \cdots \geq z_{i[n]}$ *and* $p \in N$ *satisfies*

$$\sum_{j=1}^{p-1} w_{[j]} < 0.5 \leq \sum_{j=1}^{p} w_{[j]}.$$

It is worth noting that the value of p does not depend on the value of r. Moreover, the score Z_i^* obtained by alternative A_i is that given by the SAW method plus r times the value $\sum_{j=1}^{n} \left| z_{ij} - z_{i[p]} \right| w_j$; that is, $Z_i^* = \text{SAW}_i + rM_i$, where $\text{SAW}_i = \sum_{j=1}^{n} z_{ij} w_j$ and $M_i = \sum_{j=1}^{n} \left| z_{ij} - z_{i[p]} \right| w_j$. Notice also that the graph of Z_i^* as a function of r is a straight line whose slope is M_i.[6]

The fact of knowing the score obtained by each alternative allows us to analyze the relative order between two alternatives: Given two alternatives A_i and A_j with scores $Z_i^* = \text{SAW}_i + rM_i$ and $Z_j^* = \text{SAW}_j + rM_j$, then $Z_i^* \geq Z_j^*$ if

[6]Note also that the expression given for the score Z_i^* is also valid when $r = 0$.

(a) $\text{SAW}_i \geq \text{SAW}_j$ and $\text{SAW}_i + M_i \geq \text{SAW}_j + M_j$.[7]

(b) $\text{SAW}_i > \text{SAW}_j$, $0 < \frac{\text{SAW}_i - \text{SAW}_j}{M_j - M_i} < 1$, and $r \leq \frac{\text{SAW}_i - \text{SAW}_j}{M_j - M_i}$.

(c) $\text{SAW}_i < \text{SAW}_j$, $0 < \frac{\text{SAW}_j - \text{SAW}_i}{M_i - M_j} < 1$, and $r \geq \frac{\text{SAW}_j - \text{SAW}_i}{M_i - M_j}$.

As an immediate consequence of Corollary 2 we get the following result for the case of the weight vector $\boldsymbol{w} = (1/n, \ldots, 1/n)$.

Corollary 3. *Let $\boldsymbol{w} = (1/n, \ldots, 1/n)$ and $r \in (0, 1]$. If we consider Model (1) with $a_j = (1 - r)/n$ and $b_j = (1 + r)/n$ for all $j \in N$, then*

$$Z_i^* = \frac{1}{n} \sum_{j=1}^{n} z_{ij} + \frac{r}{n} \sum_{j=1}^{n} \left| z_{ij} - z_{i[p]} \right|,$$

where $[\cdot]$ is a permutation of N such that $z_{i[1]} \geq z_{i[2]} \geq \cdots \geq z_{i[n]}$ and $p = \lfloor (n+1)/2 \rfloor$; that is, it is $n/2$ if n is even and $(n + 1)/2$ if n is odd.

2. If the DM has several weight vectors where at least two of them are different each other, he/she could follow different strategies:

   (a) Consider interval weights whose extremes are the minimum and maximum weights available for each criterion (see, for instance, Morais et al., 2014).

   (b) Same procedure as the previous one but where outliers have been previously eliminated. For that, the DM has to choose a method to detect outliers.[8] Usual procedures to detect outliers in the case of one-dimensional data are the boxplot rule (Tukey, 1977) and the MAD–median rule (see, for instance, Iglewicz & Hoaglin, 1993, Wilcox, 2012, and Leys et al., 2013).

   (c) Consider interval weights whose extremes are the first and the third quartile of the weights available for each criterion.

   (d) Consider interval weights of the form $[\mu_j - k\sigma_j, \mu_j + k\sigma_j]$, where μ_j is the mean of the weights for criterion j, σ_j is their standard deviation, and $k > 0$. Notice that $\sum_{j=1}^{n} \mu_j = 1$ and, by Chebyshev's inequality (also called the Bienaymé-Chebyshev inequality), we know that at least $1 - 1/k^2$ of the weights are within k standard deviations of the mean; that is,

$$P(|X - \mu_j| \leq k\sigma_j) \geq 1 - \frac{1}{k^2}.$$

[7]In this case $Z_i^* \geq Z_j^*$ for any value of $r \in [0, 1]$.

[8]There is an abundant literature on this topic; see, for instance, Iglewicz & Hoaglin, 1993, Barnett & Lewis, 1994, Wilcox & Keselman, 2003, Seo, 2006, and Aggarwal, 2017.

For instance, at least 50% of the weights fall in the interval $[\mu_j - \sqrt{2}\sigma_j, \mu_j + \sqrt{2}\sigma_j]$, 66% in the interval $[\mu_j - \sqrt{3}\sigma_j, \mu_j + \sqrt{3}\sigma_j]$, 75% in the interval $[\mu_j - 2\sigma_j, \mu_j + 2\sigma_j]$, and 88% in the interval $[\mu_j - 3\sigma_j, \mu_j + 3\sigma_j]$.[9] Notice that $k = \sqrt{3}$ is very interesting since it maximizes the ratio between the minimum number of weights inside the interval and the length of it: It is easy to see that the function

$$f(k) = \frac{1 - \frac{1}{k^2}}{2k\sigma_j} = \frac{1}{2\sigma_j}\frac{k^2 - 1}{k^3}$$

has a maximum in $k = \sqrt{3}$. As discussed above, the choice of excessively large intervals does not seem the most suitable in most cases. Hence, values of k located between 1 and 2 seem the most appropriate.

Notice also that to avoid that the extreme of the intervals take values less than zero or greater than one, we have to use the expression

$$[\max(0, \mu_j - k\sigma_j), \min(1, \mu_j + k\sigma_j)].$$

It is easy to check that the constraints $\mu_j - k\sigma_j \geq 0$ and $\mu_j + k\sigma_j \leq 1$ are satisfied if and only if $k \leq \min(\mu_j, 1 - \mu_j)/\sigma_j$. Therefore, when $k \leq \min_{j \in N} \min(\mu_j, 1 - \mu_j)/\sigma_j$, the intervals are of the form $[\mu_j - k\sigma_j, \mu_j + k\sigma_j]$. In this case the following corollary shows the score obtained by the alternatives.[10]

Corollary 4. *Suppose the DM has several weight vectors where at least two of them are different each other, and let μ_j and σ_j be the mean and the standard deviation of the weights for criterion j, $j \in N$, and let $k > 0$ such that $k \leq \min(\mu_j, 1 - \mu_j)/\sigma_j$ for all $j \in N$. If we consider Model (1) with $a_j = \mu_j - k\sigma_j$ and $b_j = \mu_j + k\sigma_j$ for all $j \in N$, then*

$$Z_i^* = \sum_{j=1}^{n} z_{ij}\mu_j + k \sum_{j=1}^{n} \left| z_{ij} - z_{i[p]} \right| \sigma_j,$$

where $[\cdot]$ is a permutation of N such that $z_{i[1]} \geq z_{i[2]} \geq \cdots \geq z_{i[n]}$ and $p \in N$ satisfies

$$\sum_{j=1}^{p-1} \sigma_{[j]} < 0.5 \sum_{j=1}^{n} \sigma_j \leq \sum_{j=1}^{p} \sigma_{[j]}.$$

[9]Note that in the case of weights with a normal distribution the percentages increase considerably. For instance, $P(|X - \mu_j| \leq \sigma_j) \approx 0.6827$, $P(|X - \mu_j| \leq 2\sigma_j) \approx 0.9545$, and $P(|X - \mu_j| \leq 3\sigma_j) \approx 0.9973$.

[10]We omit the proof because it is similar to that of Corollary 2. In the same way, the comments made after Corollary 2 about the score Z_i^* are also valid for the expression obtained in Corollary 4 changing the role of r by k.

It is important to emphasize that, in addition to the four methods previously mentioned, the DM could consider others depending on the characteristics of the problem. Notice also that the constraints $\sum_{j=1}^{n} a_j < 1 < \sum_{j=1}^{n} b_j$ are satisfied in the first and fourth cases[11] but in the second and third ones they are not guaranteed. For instance, suppose that in a MADM problem with five attributes, a DM has five potential weights vector, as listed in Table 5.

Table 5: Weights vector for five criteria.

C_1	C_2	C_3	C_4	C_5
0.96	0.01	0.01	0.01	0.01
0.01	0.96	0.01	0.01	0.01
0.01	0.01	0.96	0.01	0.01
0.01	0.01	0.01	0.96	0.01
0.01	0.01	0.01	0.01	0.96

It is easy to check that, in all criteria, the value 0.96 is an outlier and the third quartile is the value 0.01. Therefore, by using the second and third methods we have $\sum_{j=1}^{5} b_j = 0.05 < 1$ and, consequently, the feasible set of Model (1) is empty.

To illustrate the above procedures, we consider an example given by Morais et al. (2014), where the authors use the Revised Simos' procedure (see Figueira & Roy, 2002) to obtain the weights of 6 criteria for 5 DMs (see Table 5).

Table 6: Weights given by the DMs (Table 5 in Morais et al., 2014).

	C_1	C_2	C_3	C_4	C_5	C_6
DM_1	0.34	0.20	0.09	0.09	0.14	0.14
DM_2	0.10	0.10	0.15	0.15	0.25	0.25
DM_3	0.17	0.17	0.22	0.26	0.05	0.13
DM_4	0.24	0.17	0.09	0.24	0.09	0.17
DM_5	0.15	0.15	0.15	0.15	0.15	0.25

[11] Remember that $\sum_{j=1}^{n} \mu_j = 1$.

Figure 1: Boxplot of the weights of Table 6.

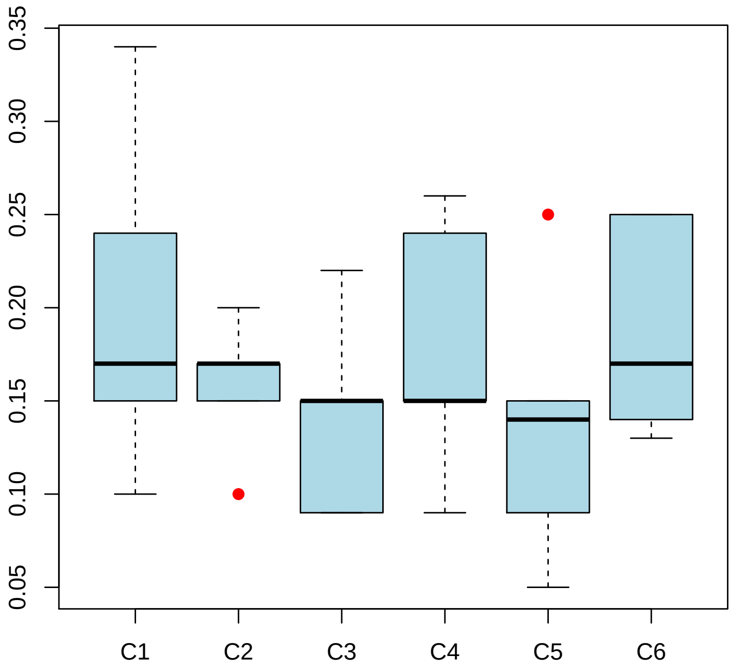


Table 7 shows the interval weights obtained by using the following procedures: all weights (P_1), all weights minus outliers[12] (P_2), the first and the third quartile (P_3), and intervals of the form $[\mu_j - k\sigma_j, \mu_j + k\sigma_j]$, with $k = 1, \sqrt{2}, \sqrt{3}$, and 2 ($P_4$, P_5, P_6, and P_7, respectively).[13]

Note that the standard deviation of the weights is relatively large in some criteria (for instance, $\sigma_1 = 0.083$ and $\sigma_5 = 0.067$).[14] This causes the length of the intervals of the form $[\mu_j - k\sigma_j, \mu_j + k\sigma_j]$ to be relatively large. Notice also that the intervals obtained with $k = \sqrt{3}$ contain all the weights.

3. Application to an MADM problem

The MADM problem that we consider is taken from Mulliner et al. (2016), where several MADM methods were applied to rank 10 Liverpool housing wards by using 20 criteria. Table 1 in Mulliner et al. (2016) collects the weights used and the values obtained when evaluating each alternative with respect to the different criteria. Mulliner et al. (2016) consider the SAW method (called weighted sum model (WSM) in

[12]The outliers are detected using the boxplot rule (see Figure 1).

[13]The values corresponding to P_4, P_5, P_6, and P_7 have been rounded to two decimal places.

[14]The coefficients of variation of these criteria are $CV_1 = 0.416$ and $CV_5 = 0.496$.

Table 7: Interval weights using different procedures.

	C_1	C_2	C_3	C_4	C_5	C_6
P_1	$[0.10, 0.34]$	$[0.10, 0.20]$	$[0.09, 0.22]$	$[0.09, 0.26]$	$[0.05, 0.25]$	$[0.13, 0.25]$
P_2	$[0.10, 0.34]$	$[0.15, 0.20]$	$[0.09, 0.22]$	$[0.09, 0.26]$	$[0.05, 0.15]$	$[0.13, 0.25]$
P_3	$[0.15, 0.24]$	$[0.15, 0.17]$	$[0.09, 0.15]$	$[0.15, 0.24]$	$[0.09, 0.15]$	$[0.14, 0.25]$
P_4	$[0.12, 0.28]$	$[0.12, 0.19]$	$[0.09, 0.19]$	$[0.11, 0.24]$	$[0.07, 0.20]$	$[0.14, 0.24]$
P_5	$[0.08, 0.32]$	$[0.11, 0.20]$	$[0.07, 0.21]$	$[0.09, 0.27]$	$[0.04, 0.23]$	$[0.11, 0.26]$
P_6	$[0.06, 0.34]$	$[0.10, 0.22]$	$[0.06, 0.22]$	$[0.07, 0.29]$	$[0.02, 0.25]$	$[0.10, 0.28]$
P_7	$[0.03, 0.37]$	$[0.09, 0.22]$	$[0.04, 0.24]$	$[0.05, 0.30]$	$[0, 0.27]$	$[0.08, 0.29]$

their paper) with the following normalization. Firstly, they transform cost criteria into benefit ones through $x_j^{\max} + x_j^{\min} - x_{ij}$, where $x_j^{\max}$ and $x_j^{\min}$ are, respectively, the maximum and the minimum criterion value; that is, $x_j^{\max} = \max_i x_{ij}$ and $x_j^{\min} = \min_i x_{ij}$. After that, all data correspond to benefit criteria and values are normalized through

$$z_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}}.$$

However, it is important to emphasize that this normalization may cause a rank reversal problem (see Belton & Gear, 1983; Triantaphyllou, 2000, pp. 11-12 in the context of AHPand Mufazzal & Muzakki, 2018 for a discussion of this problem). Another normalization commonly used in MADM problems is

$$z_{ij} = \frac{x_{ij} - x_j^{\min}}{x_j^{\max} - x_j^{\min}},$$

for benefit criteria, and

$$z_{ij} = \frac{x_j^{\max} - x_{ij}}{x_j^{\max} - x_j^{\min}},$$

for cost criteria. Nevertheless, in the example taken from Mulliner et al. (2016) there are criteria in which all the values are the same (see criteria 3, 12, 15, and 19 in Table 1 of Mulliner et al., 2016). Thus, this normalization cannot be used in these criteria. The normalization that we consider is that given by

$$z_{ij} = \frac{x_{ij}}{x_j^{\max}},$$

for benefit criteria, and

$$z_{ij} = \frac{x_j^{\max} + x_j^{\min} - x_{ij}}{x_j^{\max}} = 1 - \frac{x_{ij} - x_j^{\min}}{x_j^{\max}},$$

for cost criteria (see Norm (9) and Norm (12) in Jahan & Edwards, 2015).[15] The values obtained with this normalization are shown in Table 8.

[15]Notice that this normalization is the one used by Mulliner et al., 2016 in the method they call *Revised AHP 1*. So, the scores obtained when the authors apply this method are the ones that we have obtained with the SAW method (see Table 3 in Mulliner et al., 2016 and Table 9 in this paper).

Table 8: Normalized values of data of Table 1 in Mulliner et al. (2016).

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}
A_1	1	1	1	0.929	0.667	0.367	0.289	1	0.667	0.833	1	1	1	1	1	0.517	0.805	0.882	1	0
A_2	0.725	0.633	1	0.286	0.333	0.933	1	1	0.5	1	0.333	1	1	0.5	1	1	0.782	0.809	1	0.949
A_3	0.765	0.833	1	0.229	0.333	0.767	0.859	1	0.667	0.833	0.667	1	1	0.833	1	0.403	0.769	0.838	1	0.947
A_4	0.725	0.7	1	0.586	0.333	0.9	0.985	1	0.833	0.833	0.667	1	0.833	0.833	1	0.946	0.883	0.779	1	0.968
A_5	0.686	0.7	1	0.214	0.667	0.9	0.867	1	0.667	0.667	1	1	1	0.667	1	0.579	0.959	0.838	1	1
A_6	0.902	0.833	1	0.429	0.667	0.833	0.874	0.667	0.667	0.667	0.333	1	1	0.833	1	0.616	1	0.941	1	0.602
A_7	0.745	0.667	1	0.071	1	0.433	0.807	1	0.667	0.5	0.667	1	1	0.667	1	0.691	0.862	0.926	1	0.144
A_8	0.98	0.633	1	0.786	1	0.367	0.289	1	0.833	0.833	1	1	1	0.833	1	0.753	0.81	0.971	1	0.04
A_9	0.941	0.867	1	0.5	0.333	0.767	0.63	1	0.833	1	0.333	1	1	0.667	1	0.032	0.991	0.897	1	0.364
A_{10}	0.765	0.8	1	1	0.667	1	0.733	1	1	1	1	1	1	0.667	1	0.378	0.922	1	1	0.774

The weights used by Mulliner et al. (2016) were determined from the opinion of 337 housing and planning experts (Mulliner & Maliene, 2012). The experts ranked the criteria from 1 to 10, where 1 meant "not important at all" and 10 meant "most important". The mean scores and the variances obtained for each criterion were the following (Mulliner & Maliene, 2012):

$$\boldsymbol{\mu}' = (8.7, 8.7, 8, 8, 7.1, 6.5, 6.1, 7.4, 6.8, 6.9, 6.3, 6.6, 6.4, 5.5, 6, 6.1, 7.6, 7.2, 5.8, 6.1),$$
$$\boldsymbol{\sigma}'^2 = (2.4, 2.1, 2.6, 2.5, 3.6, 3.7, 4.5, 3.2, 3.6, 3.6, 3.6, 3.7, 3.5, 4.1, 4.1, 4.1, 3.4, 4, 5.2, 4.5).$$

The final weights μ_j were obtained by dividing each mean score μ'_j by 137.8, which is the sum of the mean scores (see Table 1 in Mulliner et al., 2016). Analogously, the standard deviations used in some intervals, σ_j, are obtained by dividing each standard deviation σ'_j by 137.8.[16]

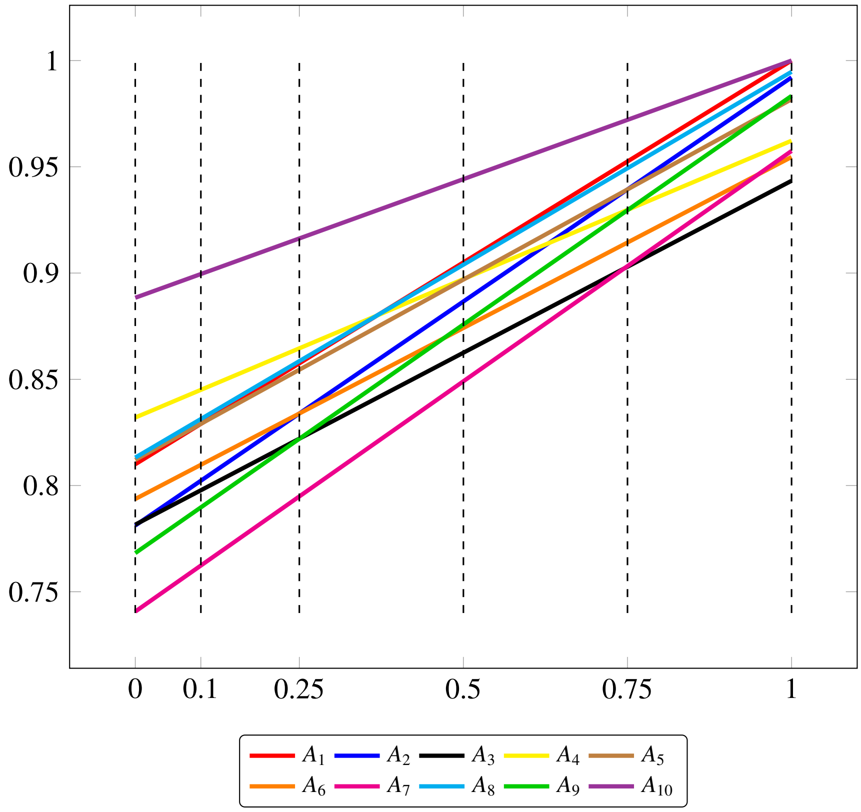
Next we assess the alternatives using the procedure described in the previous section. We consider the cases where the intervals are of the form $[\mu_j(1-r), \mu_j(1+r)]$ and $[\mu_j - k\sigma_j,\ \mu_j + k\sigma_j]$. Table 9 lists the scores of alternatives as functions of r and k (see Corollaries 2 and 4), and Figures 2 and 3 show the graph of these functions when $r \in [0, 1]$ and $k \in [1, 2]$.

Table 9: Scores of the alternatives as functions of r and k.

	r	k
A_1	$0.81001 + 0.18999\,r$	$0.81001 + 0.05842\,k$
A_2	$0.78117 + 0.21091\,r$	$0.78117 + 0.05416\,k$
A_3	$0.78159 + 0.16184\,r$	$0.78159 + 0.04342\,k$
A_4	$0.83200 + 0.13027\,r$	$0.83200 + 0.03416\,k$
A_5	$0.81206 + 0.16971\,r$	$0.81206 + 0.04361\,k$
A_6	$0.79366 + 0.16095\,r$	$0.79366 + 0.04427\,k$
A_7	$0.74067 + 0.21679\,r$	$0.74067 + 0.05912\,k$
A_8	$0.81315 + 0.18156\,r$	$0.81315 + 0.05340\,k$
A_9	$0.76820 + 0.21517\,r$	$0.76820 + 0.06325\,k$
A_{10}	$0.88837 + 0.11163\,r$	$0.88837 + 0.03201\,k$

[16] Remember that if X is a random variable (or observed data) with mean μ_X and standard deviation σ_X, and $Y = bX$, then $\mu_Y = b\mu_X$ and $\sigma_Y = |b|\sigma_X$.

Figure 2: Graphs of the scores of the alternatives when $r \in [0, 1]$.

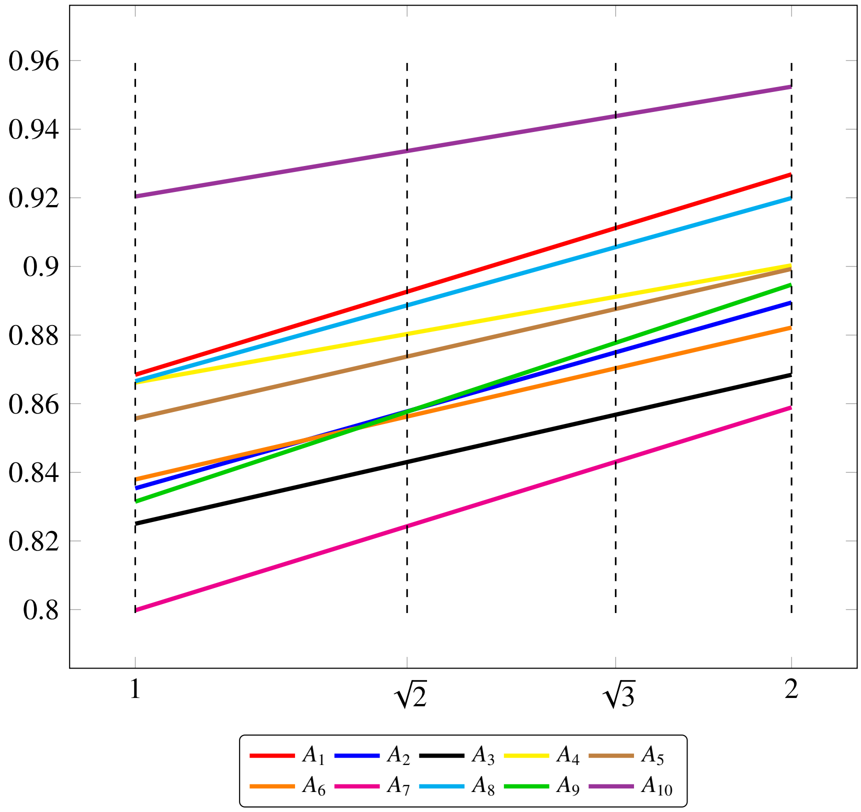


Remember that the independent terms of both families of polynomials are the scores that the SAW method gives to the alternatives. Notice also that there is an important difference between both methods in terms of the size of the slopes.[17] This is because we are using different scales for the variables r and k. For instance, the score obtained by A_1 is the same in both methods when $k/r = 0.18999/0.05842 = 3.25214$, whereas in the case of A_2 is $k/r = 0.21091/0.05416 = 3.8942$, $k/r = 3.72731$ in the case of A_3, etc. Regarding the size of the intervals, $[\mu_j(1-r), \mu_j(1+r)]$ and $[\mu_j - k\sigma_j,\, \mu_j + k\sigma_j]$ are the same when $k/r = \mu_j/\sigma_j$ (5.616 in the case of C_1, 6.0037 in the case of $C_2, \ldots,$ 2.8756 in the case of C_{20}).

It is interesting to note that Figures 2 and 3 allow us to easily appreciate the behavior of the scores of the alternatives when r and k vary. For instance, we can see that the use of intervals of the form $[\mu_j - k\sigma_j,\, \mu_j + k\sigma_j]$, with $k \in [1, 2]$, provides fairly stable rankings: As can be observed in Figure 3, A_1 is always in the

[17] According to the available information in this example, it seems more convenient to use the intervals of the form $[\mu_j - k\sigma_j,\, \mu_j + k\sigma_j]$. Nevertheless, we also consider those of the form $[\mu_j(1-r), \mu_j(1+r)]$ in order to analyze the behavior of our model with both families.

Figure 3: Graphs of the scores of the alternatives when $k \in [1, 2]$.



second position, A_3 in the ninth position, A_4 in the fourth position, A_5 in the fifth position, A_7 in the tenth position, A_8 in the third position, and A_{10} in the first position; that is,

$$A_{10} \succ A_1 \succ A_8 \succ A_4 \succ A_5 \succ \{A_6, A_2, A_9\} \succ A_3 \succ A_7.$$

Table 10 gathers the rankings of the alternatives obtained with our model for different values of r and k ($r \in \{0, 0.1, 0.25, 0.5, 0.75, 1\}$, $k \in \{1, \sqrt{2}, \sqrt{3}, 2\}$), and those obtained with the methods used by Mulliner et al. (2016). It is important to emphasize that the *Revised AHP 1* in Mulliner et al. (2016) is the SAW method with the normalization used in this paper (that is, our model with $r = 0$ or $k = 0$). Moreover, in this example, the *Revised AHP 2* method used by the authors gives the same ranking as the *Revised AHP 1*. Hence, these methods are represented in Table 10 under the column $r = 0$.

Notice that the methods obtained with the proposed model by using $k = \sqrt{3}$ and $k = 2$ (and also $r = 0.5$ and $k = \sqrt{2}$) provide the same rankings. Moreover, all methods rank A_{10} in the first position,[18] and A_7

[18]When $r = 1$, A_1 and A_{10} reach the maximum score and tie for the first position. Hence, they are assigned 1.5; that is, the

Table 10: Ranking of the alternatives for different methods.

	WSM	WPM	TOPSIS	COPRAS	r						k			
					0	0.10	0.25	0.50	0.75	1	1	$\sqrt{2}$	$\sqrt{3}$	2
A_1	4	10	8	6	5	5	4	2	2	1.5	2	2	2	2
A_2	7	6	3	4	8	7	6	6	4	4	7	6	7	7
A_3	8	5	7	8	7	8	8	9	10	10	9	9	9	9
A_4	2	2	2	2	2	2	2	4	6	7	4	4	4	4
A_5	5	3	4	3	4	4	5	5	5	6	5	5	5	5
A_6	6	4	5	7	6	6	7	8	8	9	6	8	8	8
A_7	10	9	9	10	10	10	10	10	9	8	10	10	10	10
A_8	3	7	6	5	3	3	3	3	3	3	3	3	3	3
A_9	9	8	10	9	9	9	9	7	7	5	8	7	6	6
A_{10}	1	1	1	1	1	1	1	1	1	1.5	1	1	1	1

is always ordered in the last positions (see more comments on WSM, WPM, TOPSIS, and COPRAS in Mulliner et al., 2016). Note also the behavior of alternative A_1: with some values of r and k, it achieves the second position (and even ties in the first position when $r = 1$) whereas it is in last position when the WPM method is applied. This is because the score of A_1 in criterion C_{20} is 0 (and the WPM method is based on the geometric weighted mean) and there are ten criteria in which A_1 achieves the maximum score (which benefits A_1 in our model of variable weights).

The similarity between the rankings can be best appreciated when we use the Spearman's and Kendall's correlation coefficients (see Table 11).[19] For instance, we can see that Spearman's correlation coefficients between the methods that use the intervals of the form $[\mu_j - k\sigma_j, \mu_j + k\sigma_j]$ (with $k \in \{1, \sqrt{2}, \sqrt{3}, 2\}$) are very high, always greater than 0.95 (0.86 in the case of Kendall's correlation coefficients).

average of the ranks 1 and 2.

[19]We use both coefficients because there is no clear consensus in the literature on which of the two is more convenient. Note that Kendall's coefficients are, in absolute value, smaller than or equal to Spearman's coefficients. Notice also that Mulliner et al. (2016) calculate the Pearson's correlation coefficients, so their results do not match ours (see Table 5 in Mulliner et al., 2016).

Table 11: Spearman's (in blue) and Kendall's (in red) correlation coefficients calculated from data on Table 10.

	WSM	WPM	TOPSIS	COPRAS	$r = 0$	$r = 0.10$	$r = 0.25$	$r = 0.50$	$r = 0.75$	$r = 1$	$k = 1$	$k = \sqrt{2}$	$k = \sqrt{3}$	$k = 2$
WSM	1	0.564	0.721	0.867	0.976	0.988	0.988	0.891	0.745	0.547	0.939	0.891	0.867	0.867
WPM	0.511	1	0.855	0.733	0.661	0.648	0.539	0.261	0.115	−0.134	0.333	0.261	0.236	0.236
TOPSIS	0.556	0.689	1	0.915	0.721	0.77	0.745	0.503	0.455	0.195	0.539	0.503	0.418	0.418
COPRAS	0.778	0.644	0.778	1	0.855	0.903	0.903	0.77	0.697	0.474	0.758	0.77	0.709	0.709
$r = 0$	0.911	0.6	0.556	0.778	1	0.988	0.952	0.818	0.636	0.419	0.879	0.818	0.806	0.806
$r = 0.10$	0.956	0.556	0.6	0.822	0.956	1	0.976	0.855	0.709	0.492	0.903	0.855	0.83	0.83
$r = 0.25$	0.956	0.467	0.6	0.822	0.867	0.911	1	0.915	0.794	0.608	0.927	0.915	0.879	0.879
$r = 0.50$	0.733	0.244	0.378	0.6	0.644	0.689	0.778	1	0.939	0.851	0.964	1	0.988	0.988
$r = 0.75$	0.556	0.067	0.289	0.422	0.467	0.511	0.6	0.822	1	0.948	0.879	0.939	0.903	0.903
$r = 1$	0.405	-0.09	0.135	0.27	0.315	0.36	0.449	0.674	0.854	1	0.742	0.851	0.839	0.839
$k = 1$	0.822	0.333	0.378	0.6	0.733	0.778	0.778	0.911	0.733	0.584	1	0.964	0.952	0.952
$k = \sqrt{2}$	0.733	0.244	0.378	0.6	0.644	0.689	0.778	1	0.822	0.674	0.911	1	0.988	0.988
$k = \sqrt{3}$	0.689	0.2	0.333	0.556	0.6	0.644	0.733	0.956	0.778	0.629	0.867	0.956	1	1
$k = 2$	0.689	0.2	0.333	0.556	0.6	0.644	0.733	0.956	0.778	0.629	0.867	0.956	1	1

Notice also that the Spearman's and Kendall's correlation coefficients between WPM and our method with $r = 1$ (that is, with intervals of the form $[0, 2\mu_j]$) are both negative, -0.134 and -0.09, respectively. This is due to the different philosophy on which both methods are based: WPM penalizes alternatives with low scores in some criteria whereas our model allows the low scores of some criteria to be taken less into account.

4. Concluding remarks

There are a great variety of methods in the literature to determine the weights of the attributes in MADM problems. Some of them use the information collected in the decision matrix but, as we have seen in this paper, this methodology may have undesirable effects in some cases. Other methods use the information provided by the DM but sometimes, due to lack of information or knowledge, he/she may not be willing or able to provide exact values of the weights. One of the simplest ways to allow the variability of the weights is through intervals. For this reason, in this paper we have proposed a model where the score of each alternative is the maximum values that the weighted mean can reach when the weights can take any value from certain intervals (the maxmax criterion). Moreover, we have given closed-form expressions for the scores obtained by the alternatives and we have suggested several ways to build the interval weights required in our model. It is worth noting that the proposed model is easy to understand and apply, and it takes into account the importance that the DM gives to the criteria but also the good performance that some alternatives may have in certain criteria. We have applied this new methodology to an MADM problem taken from Mulliner et al. (2016), and we have seen that, in this example, the use of intervals of the form $[\mu_j - k\sigma_j, \mu_j + k\sigma_j]$ (with $k = 1, \sqrt{2}, \sqrt{3}$, and 2) provides fairly stable rankings. Lastly, it is worth pointing out that different methodologies than the one used in this paper (such as the maxmin or Hurwicz criteria) could be studied in future research.

Acknowledgements

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Appendix A. Proofs

Proof of Theorem 1. Consider in Model (1) the following change of variables:

$$s_j = w_j - a_j, \quad j = 1, \ldots, n.$$

Then we can write

$$
\begin{aligned}
Z_i^* = \max \ & \sum_{j=1}^{n} z_{ij} s_j + \sum_{j=1}^{n} z_{ij} a_j, \\
\text{s.t. } \ & 0 \leq s_j \leq b_j - a_j, \quad j = 1, \ldots, n, \\
& \sum_{j=1}^{n} s_j = 1 - \sum_{j=1}^{n} a_j.
\end{aligned}
\tag{A.1}
$$

Notice now that Model (A.1) is equivalent to

$$
\begin{aligned}
\widehat{Z_i^*} = \max \ & \sum_{j=1}^{n} z_{ij} s_j, \\
\text{s.t. } \ & 0 \leq s_j \leq d_j, \quad j = 1, \ldots, n, \\
& \sum_{j=1}^{n} s_j = D,
\end{aligned}
\tag{A.2}
$$

where $d_j = b_j - a_j$, $j = 1, \ldots, n$, and $D = 1 - \sum_{j=1}^{n} a_j$. Let $[\cdot]$ be a permutation of N such that $z_{i[1]} \geq z_{i[2]} \geq \cdots \geq z_{i[n]}$; and let $p \in N$ such that

$$
\sum_{l=1}^{p-1} d_{[l]} < D \leq \sum_{l=1}^{p} d_{[l]}.
$$

Note that p always exists because $\sum_{j=1}^{n} a_j < 1 < \sum_{j=1}^{n} b_j$ and, consequently, $0 < D < \sum_{j=1}^{n} d_j$. Then, it is easy to check that a solution of Model (A.2) is[20]

$$
s_{[j]} = \begin{cases} d_{[j]}, & \text{if } j < p, \\ D - \sum_{l=1}^{p-1} d_{[l]}, & \text{if } j = p, \\ 0, & \text{if } j > p. \end{cases}
$$

Therefore,

$$
\widehat{Z_i^*} = \sum_{j=1}^{p-1} z_{i[j]} d_{[j]} + z_{i[p]} \left(D - \sum_{j=1}^{p-1} d_{[j]} \right),
$$

[20]It is worth noting that if all elements z_{ij}, $j = 1, \ldots, n$, are different then the permutation $[\cdot]$ is unique and, consequently, the solution is also unique. However, if the permutation $[\cdot]$ is not unique, there may be several optimal solutions.

and, consequently,

$$
\begin{aligned}
Z_i^* &= \sum_{j=1}^{p-1} z_{i[j]} \left(b_{[j]} - a_{[j]}\right) + z_{i[p]} \left(1 - \sum_{j=1}^{n} a_{[j]} - \sum_{j=1}^{p-1} \left(b_{[j]} - a_{[j]}\right)\right) + \sum_{j=1}^{n} z_{i[j]} a_{[j]} \\
&= \sum_{j=1}^{p-1} \left(z_{i[j]} - z_{i[p]}\right) b_{[j]} + z_{i[p]} - \sum_{j=1}^{p-1} \left(z_{i[j]} - z_{i[p]}\right) a_{[j]} + \sum_{j=1}^{n} \left(z_{i[j]} - z_{i[p]}\right) a_{[j]} \\
&= \sum_{j=1}^{p-1} \left(z_{i[j]} - z_{i[p]}\right) b_{[j]} + z_{i[p]} - \sum_{j=p+1}^{n} \left(z_{i[p]} - z_{i[j]}\right) a_{[j]}.
\end{aligned}
$$

Proof of Corollary 1. Consider Model (1) with $a_j = 0$ and $b_j = 1$ for all $j \in N$. Notice that, in the proof of Theorem 1, we have $D = 1$ and $d_j = 1$ for all $j \in N$. Therefore $p = 1$ and, since $a_j = 0$ for all $j \in N$, we get

$$
Z_i^* = \max_{j=1,\ldots,n} z_{ij}.
$$

Proof of Corollary 2. From Theorem 1 we know that

$$
\begin{aligned}
Z_i^* &= \sum_{j=1}^{p-1} \left(z_{i[j]} - z_{i[p]}\right) w_{[j]} + r \sum_{j=1}^{p-1} \left(z_{i[j]} - z_{i[p]}\right) w_{[j]} + z_{i[p]} \\
&\quad - \sum_{j=p+1}^{n} \left(z_{i[p]} - z_{i[j]}\right) w_{[j]} + r \sum_{j=p+1}^{n} \left(z_{i[p]} - z_{i[j]}\right) w_{[j]} \\
&= \sum_{j=1}^{p-1} z_{i[j]} w_{[j]} + \sum_{j=p+1}^{n} z_{i[j]} w_{[j]} + z_{i[p]} \left(1 - \sum_{j=1}^{p-1} w_{[j]} - \sum_{j=p+1}^{n} w_{[j]}\right) + r \sum_{j=1}^{n} \left|z_{ij} - z_{i[p]}\right| w_j \\
&= \sum_{j=1}^{n} z_{ij} w_j + r \sum_{j=1}^{n} \left|z_{ij} - z_{i[p]}\right| w_j,
\end{aligned}
$$

where $[\cdot]$ is a permutation of N such that $z_{i[1]} \geq z_{i[2]} \geq \cdots \geq z_{i[n]}$ and $p \in N$ satisfies

$$
\sum_{j=1}^{p-1} 2r w_{[j]} < r \leq \sum_{j=1}^{p} 2r w_{[j]};
$$

that is,

$$
\sum_{j=1}^{p-1} w_{[j]} < 0.5 \leq \sum_{j=1}^{p} w_{[j]}.
$$

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