Certainty equivalence principle in stochastic differential games: an inverse problem approach

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Abstract

This paper aims to characterize a class of stochastic differential games which satisfy the certainty equivalence principle, beyond the cases with quadratic, linear or logarithmic value functions. We focus on scalar games with linear dynamics in the players' strategies and with separable payoff functionals. Our approach uses an inverse problem to find a strictly concave utility function for which the game satisfies the certainty equivalence principle. Besides establishing necessary and sufficient conditions, the results obtained in this paper are also a tool for discovering new closed-form solutions, as we show in two specific applications: a generalization of a dynamic advertising model, and in a game of noncooperative exploitation of a productive asset.

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1 Introduction

Many problems in economics, management and operations research are naturally modeled as differential games. See Long[1] and Jorgensen and Zaccour[2,3] for recent updates of the literature on differential games. The aim of this paper is to identify games for which the certainty equivalence principle holds. According to Theil,[4] certainty equivalence means that a decision agent who maximizes expected utility and takes actions based on the information available at the time of taking the decision, may neglect the disturbances and to suppose that the uncertain elements are settled at their mean values. This is an important property, as it means that the equilibrium of the deterministic game is robust, in the sense that it continues to be optimal, even if the system becomes exposed to zero mean random shocks in the state variable. We say that a Markov Perfect Nash Equilibrium (MPNE henceforth) is robust if it is also an MPNE of an associated stochastic differential game.

It is well known that linear quadratic stochastic dynamic games in which the random source is independent from both the state variable and the player's strategies, satisfy the certainty equivalence principle[*]. The question we address in this paper is whether the certainty equivalence principle holds in other classes of differential games, and how we can identify them. There are some results in this direction already. Games with linear value functions satisfy this principle. This is because the second-order derivatives of the value function are null, so the Hamilton-Jacobi-Bellman equation system is the same for both the deterministic and the stochastic game. Relevant examples are Sorger,[6] where a non-linear marketing game of advertising is studied, and Yeung,[7] where a class of games with linear value functions is identified. In Kuwana,[8] it is shown that logarithmic utility is the only utility specification that satisfies this property in Merton's model with partially observable drift[†]. Our investigations show that, in a variety of games, beyond those with logarithmic, quadratic or linear value functions, the certainty equivalence principle holds. Our starting point, Theorem 2.1 establishes a necessary condition. It points

[*] This is true only for the equilibrium based on linear strategies. Tsutsui and Mino[5] is one of the first papers dealing with nonlinear equilibria in linear quadratic deterministic differential games.

[†] In the class of games we analyze here, all variables are observable for every player.

out that, for an MPNE to be robust, it must be the case that changes in shadow prices have constant variance. Theorem 2.1 is complemented with Theorem 3.1, which solves an inverse problem to determine utility functions that make the certainty equivalence principle hold.

The study of inverse optimum problems in economics has a long history, starting with Hahn[9] and Kurz;[10] Chang[11] extended the approach to the stochastic optimal growth model, and He and Huang[12] discussed a quite general inverse Merton's model. When a policy function can be rationalized by a well behaved utility function (i.e., strictly concave), it means that the prescribed behavior is consistent with an optimizing behavior. Inverse problems are easily handled with the Euler-Lagrange system of equations that directly characterizes the MPNE. Working with these equations in stochastic differential games constitutes a novel approach introduced in Josa-Fombellida and Rincón-Zapatero.[13] It presents some advantages when studying questions like those we address here. The Euler-Lagrange system is obtained from the Hamilton-Jacobi-Bellman system upon differentiation with respect to the state variables[‡].

Theorem 3.1 applies to a class of scalar games with linear dynamics in the strategies of the players. This class encompasses many interesting differential games. For instance, our method allows us to extend the aforementioned dynamic advertising model studied in Sorger[6§] and Prasad and Sethi.[17] In addition, we study the noncooperative management of a stochastic productive asset. We find that games with a linear recruitment function and constant elasticity of variance (a CEV model) satisfy certainty equivalence for constant relative risk aversion (CRRA) utility functions with a suitable coefficient of risk aversion. Our approach allows us to discover new solutions in closed-form that, to our knowledge, are not available in the literature[¶].

The paper is organized as follows. Section 2 is devoted to the definition of the game and to presenting some general results, including the Euler-Lagrange equations and the necessary

[‡]Rincón-Zapatero[14] deals with deterministic games, allowing for non-smooth MPNE. See Josa-Fombellida and Rincón-Zapatero[15] for a derivation of the Euler-Lagrange equations from the Maximum Principle in optimal control problems.

[§]The model is the duopoly extension of a model first proposed in Sethi[16] in a single-player framework.

[¶]We have limited ourselves to the cases where we are able to find closed-form solutions, but Theorem 3.1 is a general result.

condition established in Theorem 2.1. Section 3 studies a class of scalar games with linear dynamics in the players' strategies that satisfies the certainty equivalence principle. Theorem 3.1 gives sufficient conditions for the existence of strictly concave utility functions for which the MPNE is robust. An explicit expression for the player's value function is also provided. Section 4 studies two applications: a general advertising game and a productive asset game.

2 Description of the game and general results

In this section, we briefly formulate a general stochastic differential game and provide results of Euler-Lagrange equations that will be used along the paper. Standard references for differential games are Mehlmann,[18] Başar and Olsder[19] or Dockner et al.[20] We consider an N–person differential game over a bounded or unbounded time interval. In the former case, T denotes the final date. The state process, $X(s) \in \mathcal{X} \subseteq \mathbb{R}^n$, satisfies the system of stochastic differential equations (SDEs henceforth)

$$dX(s) = f(s, X(s), u(s))\, ds + \sigma(s, X(s))\, dw(s), \quad t \le s \le T. \tag{1}$$

Players' strategies are denoted by $u^i \in U^i \subseteq \mathbb{R}^n$, $i = 1, \ldots, N$, and $u = (u^1, \ldots, u^N)$ is a profile of strategies. As it is common in many games, we will assume that the equilibrium strategies are interior to U^i for each $i = 1, \ldots, N$. The random source is given by a d–dimensional Brownian motion $w(s)$ defined on a suitable probabilistic space. The instantaneous utility function of player i is L^i and the bequest function, S^i. Given initial conditions $(t, x) \in [0, T] \times \mathcal{X}$ and an admissible profile u, the payoff function of each player to be maximized is

$$J^i(t, x; u) = \mathrm{E}_{tx} \left\{ \int_t^T e^{-\rho^i (s-t)}\, L^i(s, X(s), u(s))\, ds + e^{-\rho^i (T-t)} S^i(T, X(T)) \right\}, \tag{2}$$

where E_{tx} denotes conditional expectation, given the initial condition $X(t) = x$. All functions that describe the game are of class C^2, and $\rho^i \ge 0$ is the rate of discount, which is supposed to be strictly positive for all i in the infinite horizon case. In this case, the bequest functions S^i are null.

Definition 2.1 (Admissible strategies) *A profile u is admissible if $u^i(t) \in U^i$, all $t \in [0,T]$, for $i = 1, \dots, N$ and*

(i) *for every (t,x),* (1) *admits a pathwise unique strong solution;*

(ii) *for each $i = 1, \dots, N$, there exists a function ϕ^i with ϕ^i_t, ϕ^i_{tx} and ϕ^i_{xx} continuous, such that $u^i(s) = \phi^i(s, X(s))$ for every $s \in [0,T]$.*

Let $\mathcal{U}^i$ be the set of admissible strategies of player i and let $\mathcal{U} = \mathcal{U}^1 \times \cdots \times \mathcal{U}^N$.

Definition 2.2 (MPNE) *An N–tuple of strategies $\phi \in \mathcal{U}$ is called a Markov Perfect Nash Equilibrium if for every $(t,x) \in [0,T] \times \mathcal{X}$, for every $u^i \in \mathcal{U}^i$*

$$J^i(t, x; (u^i|\phi_{-i})) \leq J^i(t, x; \phi),$$

for all $i = 1, \dots, N$.

In the above definition, $(u^i|\phi_{-i})$ denotes $(\phi^1, \dots, \phi^{i-1}, u^i, \phi^{i-1}, \dots, \phi^N)$.

Along the paper, we will use subscripts to denote partial derivatives, and primes to denote scalar derivatives. Also, ∂_z denotes total derivation with respect to the variable z; and for a matrix A, $\operatorname{tr}(A)$ is the trace of A, $A^\top$ denotes the transpose of A and $A^{-\top}$ denotes the transpose of the inverse of A, A^{-1}.

Given an MPNE ϕ, the value function of player i is $V^i(t,x) := J^i(t,x;(\phi^i))$, the deterministic Hamiltonian is $H^i(s,x,u,p^i) := L^i(s,x,u) + (p^i)^\top f(s,x,u)$, and the costate function is

$$\Gamma^i(t,x,u) := -f_{u^i}^{-\top} L^i_{u^i}(t,x,u), \tag{3}$$

$i = 1, \dots, N$.

As shown in Josa-Fombellida and Rincón-Zapatero,[13] an interior and smooth MPNE ϕ satisfies the Euler-Lagrange system of differential equations

$$\begin{aligned} -\rho^i \Gamma^i_j(t,x,\phi(t,x)) \quad &+ \quad \partial_t \Gamma^i_j(t,x,\phi(t,x)) + \partial_{x_j} H^i(t,x,\phi(t,x),\Gamma^i(t,x,\phi(t,x))) \\ &+ \quad \tfrac{1}{2} \partial_{x_j} \operatorname{tr}\left(\sigma(t,x)\sigma(t,x)^\top \partial_x \Gamma^i(t,x,\phi(t,x))\right) = 0, \end{aligned} \tag{4}$$

with final conditions $\phi^i(T,x) = \varphi^i(x)$, for $j = 1, \ldots, N$, given implicitly by

$$\Gamma^i(T, x, \phi(T,x)) + S^i_x(T,x) = 0,$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, n$.

In the case where the game is autonomous, of infinite horizon, and the MPNE is stationary, the term $\partial_t \Gamma^i(x, \phi(x)) = 0$ vanishes.

Now, consider the associated deterministic game by taking $\sigma = 0$, so that the state equation becomes

$$dX(s) = f(s, X(s), u(s))ds, \quad s \in [t,T], \tag{5}$$

with initial condition $X(t) = x$. The objective of the ith player is to maximize in $u^i \in \mathcal{U}^i$

$$J^i(t, x; u^i|u_{-i}) = \int_t^T e^{-\rho^i(s-t)} L^i(s, X(s), u(s))ds + e^{-\rho^i(T-t)} S^i(T, X(T)), \tag{6}$$

$i = 1, \ldots, N$, once the remaining players have fixed their strategies $u_{-i} \in \mathcal{U}_{-i}$.

Definition 2.3 (Robust MPNE) *We say that an MPNE of the associated deterministic game* (5), (6) *is robust if it is also an MPNE of the stochastic game* (1), (2).

Theorem 2.1 *Suppose that ϕ is a robust MPNE of the deterministic game. Then there exist functions $A_i(t)$, such that for all $i = 1, \ldots, N$, for all $x \in \mathcal{X}$, for all $t \in [0,T)$*

$$\operatorname{tr}\left((\sigma\sigma^\top)(t,x)\partial_x \Gamma^i(t, x, \phi(t,x))\right) = A_i(t). \tag{7}$$

Proof. System (4) also characterizes the deterministic game, which is obtained when σ is the null matrix. This is because the maximization condition of the Hamiltonian, for both the deterministic and the stochastic game, is the same, since σ is independent of the strategies of the players. This implies that $\Gamma^i(t, x, u)$ is also the costate variable of player i in the deterministic game. Hence, if ϕ is an MPNE of both the deterministic and the stochastic game, then the vector

$$\partial_x \operatorname{tr}\left((\sigma\sigma^\top)(t,x)\partial_x \Gamma^i(t, x, \phi(t,x))\right)$$

must be null for all $i = 1, \ldots, N$; hence, $\operatorname{tr}\left((\sigma\sigma^\top)(t,x)\partial_x \Gamma^i(t, x, \phi(t,x))\right)$ may depend only on t. □

3 A class of differential games satisfying the certainty equivalence principle

From now on, we will focus on the infinite horizon game and on a particular class of stochastic differential games that satisfy the certainty equivalence principle. To carry out this identification, we solve an inverse problem, which consists of, given the rest of the elements that define the game, finding well behaved utility functions (*i.e* smooth and strictly concave in each players' strategies) for which the certainty equivalence principle is satisfied. To do so, we use the Euler-Lagrange equation (4) for both the deterministic and the stochastic games, and the necessary condition established in Theorem 2.1.

The particular inverse problem we study can be described as follows. The functions involved have the degree of smoothness required in the previous section. Let the evolution of the scalar state variable be

$$dX(t) = \left(-\sum_{i=1}^{N} a_i(X(t))u^i(t) + b(X(t)) \right) dt + \sigma(X(t))dw(t), \tag{8}$$

where the functions a_i and b are given and with a continuous derivative. We assume that the functions a_i are not null. Moreover, they are monotone (not necessarily strictly) for all $i = 1, \ldots, N$. The objective is to find a strictly concave utility function $\ell_i(u^i)$ such that, given the functions $h_i(x)$, $i = 1, \ldots, N$, the game with payoffs

$$\mathrm{E}_x \int_0^{\infty} e^{-\rho^i t} \Big(\ell_i(u^i(t)) + h_i(X(t)) \Big) dt \tag{9}$$

admits a robust MPNE. Hence, the utility function is $L^i(x, u^i) = \ell_i(u^i) + h_i(x)$. Note that the game is autonomous, so we have eliminated the time dependence in the interval of integration and in the expectation. In what follows we will denote

$$\Theta(x) = \int^x \frac{1}{\sigma^2(v)} dv, \tag{10}$$

the antiderivative of $1/\sigma^2(x)$, with null constant.

Summing up, given the tuple $((\ell_i), (h_i), (a_i), b, \sigma)$, the inverse problem consists of finding utility functions $\ell_1, \ldots, \ell_N$ with suitable properties, such that the game admits the same MPNE as the associated deterministic game, $((\ell_i),(h_i),(a_i),b,0)$. The next proposition establishes that a robust and smooth MPNE must satisfy a system of linear differential equations, where the expression of the unknowns functions $\ell_1,\dots,\ell_N$ do not appear explicitly.

Proposition 3.1 *If the game* $(\ell_i),(h_i),(a_i),b,\sigma)$ *admits a robust MPNE* ϕ*, then there exist constants* A^i, B^i *such that* ϕ *satisfies the system of linear linear differential equations*

$$\begin{aligned} & (A^i\Theta(x)+B^i)\sum_{j\neq i}^N(\phi^j)'(x)a_j(x) \\ = & -\sum_{j=1}^N\Big(\tfrac{A^i}{\sigma^2(x)}a_j(x)+(A^i\Theta(x)+B^i)a_j'(x)\Big)\phi^j(x) \\ & +(b'(x)-\rho^i)(A^i\Theta(x)+B^i)+h_i'(x)+\tfrac{A^i}{\sigma^2(x)}b(x), \end{aligned} \tag{11}$$

for all $x\in\mathcal{X}$*, for* $i=1,\dots,N$*.*

Proof. If ϕ is a robust MPNE, then it satisfies the Euler-Lagrange equation (4) with $\sigma=0$, which for this game becomes (we omit the dependence of the functions on x)

$$-\rho^i\Gamma^i(x,\phi^i)+\partial_x\left(\ell_i(\phi^i)+h_i(x)+\Gamma^i(x,\phi^i)(-\sum_{i=1}^N a_i(x)\phi^i+b(x))\right)=0, \tag{12}$$

where

$$\Gamma^i(x,u^i)=\frac{\ell_i'(u^i)}{a_i(x)}. \tag{13}$$

Moreover, a robust MPNE satisfies, by Theorem 2.1,

$$\partial_x\Gamma^i(x,\phi^i)=\frac{A^i}{\sigma^2(x)} \tag{14}$$

for some constant A^i, and for all $i=1,\dots,N$. Then, integrating in the expression above

$$\Gamma^i(x,\phi^i(x))=A^i\Theta(x)+B^i$$

for another arbitrary constant B^i. From (13), $\ell_i'(\phi^i)=a_i(x)(A^i\Theta(x)+B^i)$. Plugging (13) and

(14) into (12), we obtain that ϕ satisfies the linear system of differential equations

$$\begin{aligned} 0 &= -\rho^i(A^i\Theta + B^i) + \ell_i'(\phi^i)(\phi^i)' + h_i' + \partial_x\Gamma^i \left(-\sum_{j=1}^N a_j\phi^j + b\right) \\ &\quad +\Gamma^i\left(-\sum_{j=1}^N a_j'\phi^j - \sum_{j=1}^N a_j(\phi^j)' + b'\right) \\ &= -\rho^i(A^i\Theta + B^i) + (A^i\Theta + B^i)a_i(\phi^i)' + h_i' + \frac{A^i}{\sigma^2}\left(-\sum_{j=1}^N a_j\phi^j + b\right) \\ &\quad +(A^i\Theta + B^i)\left(-\sum_{j=1}^N a_j'\phi^j - \sum_{j=1}^N a_j(\phi^j)' + b'\right). \end{aligned}$$

Rearranging terms, we obtain (11). □

Note that (11) is linear because the dynamics (8) is linear in the strategies of the players. We could have set a general dynamics, at the cost of dealing with a nonlinear system for the robust MPNE. As we wish, not only to give a theoretical result, but to find explicitly utilities and equilibria, we make the simplifying assumption (8). Another advantage of our game problem specification is that, as the system is linear, suitable assumptions on the coefficients would guarantee the existence of a global solution, feature that is needed for the sufficient condition given in the following result. We use the notation ζ^i for the inverse of ϕ^i, that is, $x = \zeta^i(\phi^i(x))$, for $x \in \mathcal{X}$, for $i = 1, \ldots, N$. The inverse of ϕ^i exists under the assumptions of the theorem below, and the monotonicity of functions a_i and $a_i\Theta$. for all $i = 1, \ldots, N$, are imposed in order to obtain functions ℓ_i that are strictly concave.

Theorem 3.1 *Suppose that $a_i\Theta$ has a continuous derivative for all $i = 1, \ldots, N$, where Θ is defined in* (10) *and that the functions a_i and $a_i\Theta$ are monotone, with at least one of them strictly monotone, for all $i = 1, \ldots, N$. Let ϕ be a solution of the system* (11) *for which each ϕ^i is twice continuously differentiable and strictly monotone in $\mathcal{X}$, such that* (8)*, with initial condition $X(t) = x$, admits a unique strong solution X^ϕ for each $(t, x) \in [0, \infty) \times \mathbb{R}$. Let*

$$\ell_i(u^i) = \int^{u^i} \Big(A^i\Theta(\zeta^i(v)) + B^i\Big)a_i(\zeta^i(v))dv, \tag{15}$$

with A^i, B^i constants, and let V^i be defined by

$$\begin{aligned}\rho^i V^i(x) &= \ell_i(\phi^i(x)) + h_i(x) - \Big(A^i\Theta(x) + B^i\Big)\sum_{j=1}^{N} a_j(x)\phi^j(x) \\ &\quad +(A^i\Theta(x) + B^i)b(x) + \frac{A^i}{2},\end{aligned} \tag{16}$$

and suppose that the following transversality condition holds: for all $u^i \in \mathcal{U}^i$,

$$\liminf_{T\to\infty} e^{-\rho^i T}\mathrm{E}_{tx} V^i(X^{u^i|\phi_{-i}}(T)) \geq 0, \tag{17}$$

and

$$\limsup_{T\to\infty} e^{-\rho^i T}\mathrm{E}_{tx} V^i(X^{\phi}(T)) \leq 0, \tag{18}$$

for $i = 1, \ldots, N$. Then

(a) The function ℓ_i is twice continuously differentiable and strictly concave for suitable constants A^i, B^i, for all $i = 1, \ldots, N$.

(b) The infinite horizon game $((\ell_i), (h_i), (a_i), b, \sigma)$ has ϕ as a robust MPNE.

(c) The function V^i is the (strictly concave) value function of player i, for all $i = 1, \ldots, N$.

Proof. It is clear that ℓ_i is twice continuously differentiable, for $i = 1, \ldots, N$. Taking the derivative in (15) we obtain

$$\ell_i'(u^i) = a_i(\zeta^i(u^i))(A^i\Theta(\zeta^i((u^i)) + B^i).$$

Deriving again and collecting terms, we have

$$\ell_i''(u^i) = (\zeta^i)'(u^i)\Big(A^i(a_i\Theta)'(\zeta^i((u^i)) + B^i a_i'(\zeta^i(u^i))\Big). \tag{19}$$

We want to show that it is possible to choose suitable constants A^i and B^i such that $\ell_i''(u^i) < 0$, for $i = 1, \ldots, N$. This will prove (a). Note that $(\zeta^i)'$ has the same sign of $(\phi^i)'$. By assumption, this is positive or negative for all $x \in \mathcal{X}$, since ϕ^i is strictly monotone. Assume, without loss of generality, that $(\zeta^i)' > 0$ for some player $i \in \{1, \ldots, N\}$. Since that both $a_i\Theta$ and a_i are monotone, and that at least one of these two functions is strictly monotone, it is possible to select constants $A^i \neq 0$ and $B^i \neq 0$ such that $A^i(a_i\Theta)' \leq 0$ and $B^i a_i' \leq 0$ for all $x \in \mathcal{X}$, and at least one of these two expressions is negative for all $x \in \mathcal{X}$. Thus, $\ell_i''(u^i) < 0$ as claimed. The case $(\zeta^i)' < 0$ is handled in the same way. This shows (a). We will prove (b) and (c) at once. We start by showing that (16) defines a solution of the Hamilton-Jacobi-Bellman equations (HJB henceforth) of the stochastic differential game. To this end, let us compute the first and second derivatives of V. Note that, in fact, recalling that $\Gamma^i = A^i\Theta + B^i$, we have

$$\rho^i V^i(x) = \ell_i(\phi^i) + h_i(x) + \Gamma^i(x, \phi^i)\left(-\sum_{j=1}^{N} a_j(x)\phi^j + b(x)\right) + \frac{A^i}{2}. \tag{20}$$

Hence, given that ϕ satisfies (12)

$$-\rho^i \Gamma^i(x, \phi^i) + \partial_x \rho^i V^i(x) = 0,$$

we have $(V^i)'(x) = \Gamma^i(x, \phi(x)) = A^i\Theta(x) + B^i$ and $(V^i)''(x) = \frac{A^i}{\sigma^2(x)}$. Then

$$\begin{aligned}
&-\rho V^i + H^i(x, \phi, (V^i)') + \frac{1}{2}\sigma^2 (V^i)'' \\
&\quad = -\rho^i V^i + \ell_i(\phi^i) + h_i(x) + \left(-\sum_{i=1}^{N} a_i(x)\phi^i + b(x)\right)(V^i)' + \frac{1}{2}\sigma^2 (V^i)'' \\
&\quad = -\rho^i V^i + \ell_i(\phi^i) + h_i(x) + \left(-\sum_{i=1}^{N} a_i(x)\phi^i + b(x)\right)\Gamma^i + \frac{A^i}{2},
\end{aligned}$$

which is null by (20). Hence, we have proved

$$-\rho^i V^i(x) + H^i(x, \phi, (V^i)') + \frac{1}{2}\sigma^2 (V^i)'' = 0.$$

On the one hand, we have by definition that $\Gamma^i(x, \phi(x)) = \frac{\ell_i'(\phi(x))}{a_i(x)}$; on the other hand, $(V^i)'(x) = \Gamma^i(x, \phi(x))$. Hence, $\ell_i'(\phi^i) - a_i(x)(V^i)' = 0$ or, in terms of the Hamiltonian, $H^i_{u^i}(x, (u^i|\phi_{-i}), (V^i)') = 0$ for all u^i, for all $i = 1, \ldots, N$. Since the Hamiltonian

$$H^i(x, (u^i|\phi_{-i}), (V^i)') = \ell_i'(u^i) + h_i(x) + \left(-u^i - \sum_{j \neq i1}^{N} a_j(x)\phi^j + b(x)\right)(V^i)'$$

is strictly concave in u^i because ℓ_i is strictly concave, a critical point of the function $u^i \longmapsto H^i(x, u^i|\phi_{-i}, (V^i)'(x))$ is a (unique) global maximum. Hence

$$H^i(x, (u^i|\phi_{-i}), (V^i)'(x)) = \max_{u^i} H^i(x, (u^i|\phi_{-i}), (V^i)'(x))$$

for all u^i, $i = 1, \dots, N$. Thereby, the HJB equation for a MPNE,

$$0 = -\rho^i V^i + \max_{u^i} H^i(x, (u^i, \phi_{-i}), (V^i)') + \frac{1}{2}\sigma^2 (V^i)'',$$

holds. Finally, the transversality conditions (17) and (18) allow us to apply a Verification Theorem, see Fleming and Soner[21] Ch. III Th. 9.1—turning minimizing to maximizing— or Dockner et al.[20] Th. 8.5. □

Remark 3.1 (Control Problem) *Note that in control problems, $N = 1$,* (11) *is an algebraic equation, not a differential one. Introducing the notation $a_i = a$, $h_i = h$, $A^i = A$ and $B^i = B$ for all $i = 1, \dots, N$, we have, after solving for ϕ*

$$\phi(x) = \frac{(b' - \rho)(A\Theta(x) + B) + h'(x) + \frac{A}{\sigma^2(x)} b(x)}{\frac{A}{\sigma^2(x)} a(x) + (A\Theta(x) + B)a'(x)},$$

as a candidate for robust control.

Remark 3.2 (Symmetric Game) *Consider a symmetric game with $N > 1$ players and let ϕ be a robust symmetric MPNE. As in the remark above, we denote all functions and constants defining the game without indexes, as well as the introduced constants A and B. Observe that, with the assumptions of Theorem 3.1, the sign of $A\Theta(x) + B$ is well defined, positive or negative for all x. The linear ODE* (11) *takes the form*

$$\phi'(x) = P(x)\phi(x) + Q(x), \tag{21}$$

where the coefficients are

$$\begin{aligned} P(x) &= \frac{N}{1-N}\left(\frac{A}{\sigma^2(x)(A\Theta(x) + B)} + \frac{a'(x)}{a(x)}\right); \\ Q(x) &= \frac{1}{(N-1)a(x)}\left(b'(x) - \rho + \frac{h'(x)}{A\Theta(x) + B} + \frac{Ab(x)}{\sigma^2(x)(A\Theta(x) + B)}\right). \end{aligned}$$

An integrating factor is $|A\Theta(x) + B|^{\frac{N}{1-N}} a(x)^{\frac{N}{1-N}}$. In consequence, the general solution of (21) *is*

$$\phi(x) = |A\Theta(x) + B|^{\frac{N}{N-1}} a(x)^{\frac{N}{N-1}} \left(\int^x Q(z)|A\Theta(z) + B|^{\frac{N}{1-N}} a(z)^{\frac{N}{1-N}} dz + C\right),$$

where C is an arbitrary constant. This is a candidate for robust MPNE, for the game with the utility function ℓ_i as given in (15)*. We will use this formula in Section 4.2 below.*

Remark 3.3 (Linear Value Functions) *It has been proven in Theorem 3.1 that* $(V^i)'(x) = \Gamma^i(x) = A^i\Theta(x) + B^i$. *Hence, the value function is linear in* x *for player* i *if it is possible to choose* $A^i = 0$. *Note that it is not possible to take* $\Theta(x)$, *defined in* (10), *constant because* $1/\sigma^2(x) \neq 0$. *It is important to note that, from* (19) *in the proof of the theorem, the selection* $A^i = 0$ *is possible only if* a_i' *does not vanish; otherwise, the function* ℓ_i *constructed in the theorem is not strictly concave, and there is no guarantee that the solution of the system* (11), *be a Nash equilibrium of the Hamiltonians of the players. See Section 4.1 below for a game with linear value function.*

4 Examples

We analyze two stochastic differential games with a different structure. The value function of the first game is linear, which explains why the MPNE satisfies certainty equivalence. The second game's value function is of the CRRA family.

4.1 A dynamic advertising game

Consider the stochastic differential game of competitive dynamic advertising of two firms studied in Sorger[6] in its infinite horizon formulation. Two firms compete for market shares through advertising effort. We denote the market share of firm 1 at time t by $X(t)$ and assume that the size of the total market is constant over time. Normalizing the total market to 1, we obtain that $1 - X(t)$ is the market share of firm 2 at time t. Let us denote then $X_1(t) = X(t)$ and $X_2(t) = 1 - X(t)$, so $X_i(t)$ represents the market shares of firm i at time t. The objective functional for firm i is

$$J^i(t,x;u^1,u^2) = \mathrm{E}_{tx}\int_0^\infty e^{-\rho^i t}\left(q_i X_i(t) - C_i(u^i(t))\right)dt,$$

$i = 1, 2$, and dynamics

$$\begin{aligned} dX(t) &= \Big(\delta_1(1-X(t))^{\frac{1}{1+m_1}}u^1(t) - \delta_2 X(t)^{\frac{1}{1+m_2}}u^2(t) - \delta(2X(t)-1)\Big)dt \\ &\quad +\sigma(X(t))dw(t), \end{aligned}$$

$X(0) = x \in \mathcal{X} = [0,1]$, where $m_i > 0$, for $i = 1,2$. The cost function C_i has to be determined so that the MPNE of the game is robust. The model specification in Sorger[6] is obtained with $C_i(u^i) = c_i \frac{(u^i)^2}{2}$ and $m_1 = m_2 = 1$, $\delta_1 = \delta_2 = 1$, $\delta = 0$. Prasad and Sethi[17] allows for $\delta_1, \delta_2, \delta > 0$. In the above, $u^i(t)$ is the advertising rate at time t, $\rho^i > 0$ is the constant discount rate and $q_i > 0$ is the constant revenue per unit of market share of firm i, for $i = 1,2$. The diffusion parameter $\sigma(x) \geq 0$ satisfies $\sigma(0) = \sigma(1) = 0$. The state dynamics reflects two facts, already present in the classical Vidale-Wolfe advertising model, Vidale and Wolfe:[22] (i) a concave saturation effect in the capture of new costumers, and (ii) a positive (*resp.* negative) effect of own (*resp.* competitor) advertising spending. In comparison with the original game, we allow here for asymmetric market responses, even if the advertising effectiveness parameters δ_1 and δ_2 are equal. The churn parameter $\delta > 0$, accounts for declining effects in market shares due to other causes than advertising from the competitor firm, such as product obsolescence or lack of product differentiation.

In the notation of Section 3, $a_1(x) = -\delta_1(1-x)^{\frac{1}{1+m_1}}$, $a_2(x) = \delta_2 x^{\frac{1}{1+m_2}}$, $b(x) = -\delta(2x-1)$ and $h_1(x) = q_1 x$, $h_2(x) = q_2(1-x)$. By (3)

$$\Gamma^1(x,u^1) = C_1'(u^1)\delta_1(1-x)^{-\frac{1}{1+m_1}}, \quad \Gamma^2(x,u^2) = -C_2'(u^2)\delta_2 x^{-\frac{1}{1+m_2}}.$$

Linear value functions require (see Remark 3.3)

$$C_1'(u^1(x)) = B^1(1-x)^{\frac{1}{1+m_1}}, \quad C_2'(u^2(x)) = -B^2 x^{\frac{1}{1+m_2}}$$

for suitable constants $B^1 > 0$, $B^2 < 0$, so (7) is satisfied independently of $\sigma(x)$. We still have to check that the these two strategies solve the system (11), which in this particular game become (with $A^i = 0$)

$$\begin{aligned}
B^1\delta_2 x^{\frac{1}{1+m_2}}(u^2)' &= -B^1\frac{\delta_1}{1+m_1}(1-x)^{\frac{-m_1}{1+m_1}}u^1 - B^1\frac{\delta_2}{1+m_2}x^{\frac{-m_2}{1+m_2}}u^2 \\
&+ B^1(-2\delta-\rho^1) + q_1, \\
-B^2\delta_1(1-x)^{\frac{1}{1+m_1}}(u^1)' &= -B^2\frac{\delta_1}{1+m_1}(1-x)^{\frac{-m_1}{1+m_1}}u^1 - B^2\frac{\delta_2}{1+m_2}x^{\frac{-m_2}{1+m_2}}u^2 \\
&+ B^2(-2\delta-\rho^2) - q_2.
\end{aligned}$$

The structure of these equations suggests a solution (u^1, u^2) of the form $u^1(x) = \eta_1(1-x)^{\frac{m_1}{1+m_1}}$, $u^2(x) = \eta_2 x^{\frac{m_2}{1+m_2}}$, with $\eta_i > 0$, $i = 1, 2$. After substitution and collection of terms, the above differential system reduces to the following pair of algebraic relations

$$\begin{aligned} B^1\left(\eta_2\delta_2 + \eta_1\tfrac{\delta_1}{1+m_1} + 2\delta + \rho^1\right) - q_1 &= 0, \\ B^2\left(\eta_1\delta_1 + \eta_2\tfrac{\delta_2}{1+m_2} + 2\delta + \rho^2\right) + q_2 &= 0. \end{aligned} \tag{22}$$

Let $\zeta^1(v) = 1 - (\frac{v}{\eta_1})^{\frac{1+m_1}{m_1}}$ and $\zeta^2(v) = (\frac{v}{\eta_2})^{\frac{1+m_2}{m_2}}$, the inverse functions of u^1 and u^2, respectively. From (15), we have the cost functions (remember that $A^i = 0$)

$$\begin{aligned} -C_1(u^1) &= B^1\int^{u^1} a_1(\zeta^1(v))dv = -B^1\delta_1\frac{1}{\eta_1^{\frac{1}{m_1}}}\frac{(u_1)^{1+\frac{1}{m_1}}}{1+\frac{1}{m_1}}; \\ -C_2(u^2) &= B^2\int^{u^2} a_2(\zeta^2(v))dv = B^2\delta_2\frac{1}{\eta_2^{\frac{1}{m_2}}}\frac{(u_2)^{1+\frac{1}{m_2}}}{1+\frac{1}{m_2}}. \end{aligned}$$

If we denote $c_1 = B^1\frac{\delta_1}{\eta_1^{\frac{1}{m_1}}}$ and $c_2 = -B^2\frac{\delta_2}{\eta_2^{\frac{1}{m_2}}}$, then both $c_1, c_2 > 0$. Solving for B^1 and B^2 and plugging these values into the system (22), we obtain

$$\begin{aligned} c_1\eta_1^{\frac{1}{m_1}}\left(\eta_2\delta_2 + \eta_1\frac{\delta_1}{1+m_1} + 2\delta + \rho^1\right) - \delta_1 q_1 &= 0, \\ c_2\eta_2^{\frac{1}{m_2}}\left(\eta_1\delta_1 + \eta_2\frac{\delta_2}{1+m_2} + 2\delta + \rho^2\right) - \delta_2 q_2 &= 0. \end{aligned}$$

Given c_i, m_i, ρ^i, q_i, δ_i, $i = 1, 2$ and δ, the existence of positive solutions η_1, η_2 of this algebraic system guarantees the existence of a robust MPNE $\phi(x) = (\eta_1(1-x)^{\frac{m_1}{1+m_1}}, \eta_2 x^{\frac{m_2}{1+m_2}})$. This is because the SDE for the optimal path X^ϕ is linear, hence a unique strong solution exists. It is straightforward to check the rest of the conditions of Theorem 3.1, including the transversality conditions, since the value functions are linear, as can be easily realized from (16).

4.2 A stochastic productive asset game

In this section, we consider an N player symmetric noncooperative differential game where each player i consumes at rate $c^i \geq 0$ from a stochastic productive asset X. The payoff functional is

$$J^i(t, x; (c^1, \ldots, c^N) = E_{tx}\left\{\int_t^\infty e^{-\rho^i(s-t)}\ell(c^i(s))ds\right\},$$

subject to

$$dX(s) = \left(F(X(s)) - \sum_{i=1}^{N} c^i(s)\right) ds + \sigma(X(s))dw(s), \qquad X(t) = x > 0. \tag{23}$$

Function F is the recruitment/production function. The class of admissible strategies $\mathcal{U}^i$ for each player is as in Definition 2.1, with the additional condition that $X \geq 0$ almost sure is required, that is to say $\mathcal{X} = [0, \infty)$. Further conditions on $\mathcal{U}^i$ will be given in each of the specific cases we consider below. This game has been analyzed in Josa-Fombellida and Rincón-Zapatero,[13] where we provide necessary and sufficient conditions for existence of a unique and smooth MPNE of the finite horizon game.

According to Remark 3.2, and taking $B = 0$, a robust equilibrium must satisfy (21) with coefficients

$$P(x) = \frac{\frac{N}{1-N}}{\sigma^2(x)\Theta(x)}, \quad Q(x) = \frac{1}{1-N}\left(\rho - F'(x) - \frac{F(x)}{\sigma^2(x)\Theta(x)}\right). \tag{24}$$

Recall that Θ is the primitive of $1/\sigma^2(x)$ with null constant. We consider a linear production function[‖], $F(x) = \mu x$, $\mu \geq 0$, and assume a CEV model, $\sigma(x) = \sigma x^a$, with $1 - \frac{1}{2N} < a < 1$ and $\rho > 2\mu(1-a)$. We have

$$\Theta(x) = \frac{1}{\sigma^2(1-2a)} x^{1-2a}, \quad P(x) = \frac{N}{1-N}\frac{1-2a}{x}, \quad Q(x) = \frac{\rho + 2\mu(a-1)}{1-N}.$$

Function Θ is negative since $a > \frac{1}{2}$, hence we take $A < 0$ and $B = 0$. In this case, by Remark 3.2, $\phi(x) = \beta x + \eta x^{(2a-1)N/(N-1)}$ is a solution of (24), where $\beta = \frac{\rho - 2\mu(1-a)}{1-2N(1-a)}$ is positive given our assumptions, and $\eta \geq 0$. Note that ϕ is smooth, positive and increasing in $(0, \infty)$ and $\phi(0) = 0$. We consider only the case with $\eta = 0$, so the inverse of ϕ is $\zeta(c) = c/\beta$. Substituting into (15) and taking $A = \sigma^2(1-2a)\beta^{1-2a}$, we find that the utility function is of the CRRA class, $L(c) = \frac{c^{2-2a}}{2(1-a)}$. Denoting $\theta \equiv 2a - 1$, we obtain an isoelastic utility function $L(c) = \frac{c^{1-\theta}}{1-\theta}$. The constraints on a imply $1 - \frac{1}{N} < \theta < 1$. In terms of θ, the diffusion coefficient is then

[‖] We have also solved the inverse problem for linear F and σ, as well as for a square root recruitment function, $F(x) = \mu\sqrt{x}$, and linear σ. Readers interested in the details will receive a copy of our computations and proof of optimality upon request.

$\sigma(x) = \sigma x^{(1+\theta)/2}$. At equilibrium, the asset evolves according to the SDE

$$dX^{\phi}(s) \equiv dX(s) = (\mu - N\beta)X(s)ds + \sigma X(s)^{(1+\theta)/2}dw(s), \quad X(0) = x > 0. \tag{25}$$

Regarding the existence of solutions to (25), the functions $1/\sigma^2(x)$ and $x/\sigma^2(x)$ are locally integrable Borel functions in $(0, \infty)$. Indeed, both functions are continuous on any compact subset of $(0, \infty)$, and are thus integrable. Hence, the SDE (25) admits a unique-in-law weak solution, see Karatzas and Shreve[23] Ch. V, Th. 5.15. In fact, since the coefficients are locally Lipschitz, (25) admits a pathwise unique strong solution up to exit time of the interval $(0, \infty)$, see Karatzas and Shereve[23] Ch. IX, Ex. (2.10). To continue with the proof, we restrict the class of admissible strategies $\mathcal{U}^i$ to those elements c^i which satisfy $0 \le c^i \le kx$ for suitable k. For $c^i \in \mathcal{U}^i$, let the process X^{c^i} be given by

$$dX^{c^i}(s) = (\mu X^{c^i}(s) - c^i(s) - (N-1)\beta X^{c^i}(s))ds + \sigma(X^{c^i}(s))^{(1+\theta)/2}dw(s),$$

$X^{c^i}(0) = x >$. Let $\hat{X}$ with $\hat{X}(0) = x$ and

$$d\hat{X}(s) = (\mu - k - (N-1)\beta)\hat{X}(s)ds + \sigma\hat{X}(s)^{(1+\theta)/2}dw(s). \tag{26}$$

By a comparison theorem in Ikeda and Watanabe,[24] $X^{c^i} \ge \hat{X}$. Let $\mu_0 = \mu - k - (N-1)\beta$. By Example 3.2 in Mijatović and Ususov,[25] the exponential

$$M(t) = \exp\left(-\frac{\mu_0}{\sigma}\int_0^t \hat{X}(s)^{(1-\theta)/2}dw(s) - \frac{1}{2}\frac{\mu_0^2}{\sigma^2}\int_0^t \hat{X}(s)^{1-\theta}ds\right)$$

is a uniformly integrable martingale, since $\frac{1+\theta}{2} < 1$. This implies that $\hat{X} \in (0, \infty)$ with probability one, see Theorem 2.1 in Mijatović and Ususov.[25] In consequence, $V(X^{c^i}(T))$ is well defined and $V(X^{c^i}(T)) \ge V(\hat{X}(T)) \ge 0$ a.e., so (17) in Theorem 3.1 trivially holds. On the other hand, $X^{\phi} \le \tilde{X}$, where $d\tilde{X}(s) = \mu\tilde{X}(s)ds + \sigma\tilde{X}(s)^{(1+\theta)/2}dw(s)$, and $\tilde{X}(0) = x$. Note that $e^{-\mu t}\tilde{X}(t) = x + \sigma\int_0^t e^{-\mu s}\tilde{X}^{(1+\theta)/2}(s)dw(s)$ is a nonnegative local martingale, thus a supermartingale, hence $E_x(e^{-\mu T}\tilde{X}(T)) \le x$. This implies that X^{ϕ} does not exit at ∞. Moreover, since the value function is increasing and concave, then

$$E_x(V(X^{\phi}(T)) = \frac{A}{1-\theta}E_x(X^{\phi}(T))^{1-\theta} \le \frac{A}{1-\theta}(E_x\tilde{X}(T))^{1-\theta} \le \frac{A}{1-\theta}x^{1-\theta}e^{\mu(1-\theta)T},$$

by Jensen's inequality. Thus, condition (18) holds, as we have supposed $\rho > \mu(1-\theta)$.

5 Conclusions

This paper studies whether the certainty equivalence property holds in games beyond the well-known linear quadratic case and games with linear or logarithmic value functions. To approach the problem through the value function and the HJB equations is not easy, at least when the diffusion coefficient depends on the state variable, since then—with the exception of games where the value function is linear—the value function will be different in the stochastic and in the deterministic case. Hence, we have chosen to work with the Euler-Lagrange equations, which deal directly with the MPNE. As the MPNE is the same for both games, the deterministic and the stochastic, these equations provide a convenient tool to solve the problem. We have shown how our approach can be used to find closed-form solutions to games for which a solution was not known. We will try to explore this feature in new examples. Also, further research will focus on analyzing games where the uncertainty is other than Brownian uncertainty.

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