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How to Teach the Pythagorean Theorem: An Analysis of Lesson Plans

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Abstract

This research was conducted among mathematics graduates who participated in a pedagogical formation certificate program. Participants were asked to prepare a lesson plan intended for use in teaching the Pythagorean theorem as part of a ninth grade mathematics course. Eighteen out of 43 participants included a proof of the Pythagorean theorem as a component of their lesson plan. These proofs were classified in three categories: visual proofs (two participants), algebraic proofs (nine participants), and proofs by using triangular similarities (seven participants). In addition, the solved examples, homework, and evaluation questions included in the lesson plans were classified according to TIMSS cognitive levels. Of the 233 questions prepared by 43 participants, 37% of the questions were at the knowledge level, 60% were at the application level, and the remaining 3% were at the reasoning level.

Keywords: Cognitive demand levels, lesson plan, Pythagorean theorem, ninth grade mathematics

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The Pythagorean theorem, possibly the most well-known theorem in mathematics is named after Pythagoras of Samos. Even 2,500 years after his time, -it is likely to be found reference to Pythagoras in any book about mathematics, science, or science history (Bell, 1953; Hodgkin, 2005; Katz, 1998; Pickover, 2009). In addition, there is a wealth of publications regarding Pythagoras's life, his followers (Pythagoreans), and about the times in which they lived (Ferguson, 2008; Joost-Gaugier, 2009; Kahn, 2001; Kaplan and Kaplan, 2011; Martinez, 2012). It is virtually impossible to know exactly how many books and articles have been published regarding the Pythagorean theorem (Chambers, 1999; Crawford, 2001; Givental, 2006; Loomis, 1968; Maor, 2007; Sparks, 2008).

Loomis (1968) collected currently known proofs of the Pythagorean theorem and published the proofs in a single book by dividing them into four sections, including algebraic, geometric, quaternionic, and dynamic proofs. Forty years after Loomis' book, Sparks (2008) carried out a similar project by way of collecting some known proofs of the Pythagorean theorem. Until the present, there has continued to be publication of research articles and books documenting newly discovered proofs and/or generalizations of the Pythagorean theorem (de Lemos, 1995; Strathern, 1997; Veljan, 2000).

Among these publications, the articles that outline how mathematics teachers may approach teaching proofs of the Pythagorean theorem (Chambers, 1999; Crawford, 2001) and articles presenting research regarding how the Pythagorean theorem are taught in different countries (Huang and Leung, 2002; Hugener, Pauli, Reusser, Lipowsky, Rakoczy and Klieme, 2009; Lipowsky, Rakoczy, Pauli, Drollinger-Vetter, Klieme, and Reusser, 2009; Yang, 2009). Considering that teaching and learning processes are social interactions, it is natural that education is influenced not only by a country's cultural and social environment but also by the expectations of its individuals (An, Kulm, and Wu, 2002). Huang and Leung (2002) investigated how teachers from the Czech Republic, Hong Kong, and Shanghai taught the Pythagorean theorem to eighth grade students. Researchers chose sample videos from the TIMSS-R study, one video documenting instruction from a Hong Kong teacher and one from a Czech teacher. In addition, one sample video recorded by the researchers was chosen from a total of 11 videos documenting instruction by a Shanghai teacher. Researchers considered a variety of factors including lesson planning, instruction of the Pythagorean theorem, and the dynamics of in-class communication. The researchers observed that the teachers from Hong Kong and Shanghai spent a greater amount of instructional time on proving the Pythagorean theorem than the Czech teacher (Huang and Leung, 2002).

Huang and Leong (2002) also examined the examples given by the teachers during their lessons in cognitive terms by categorizing them as requiring memorization, interpretation, or exploration. Their classification showed that the Czech teacher utilized interpretation questions in instruction 54 % of the time and memorization 46 % of the time. In addition, it was determined that all instructional examples given by the Hong Kong teacher were at the interpretation level while the Shanghai teacher utilized interpretation 75 % of the time and exploration for the remaining 25 %. A review of the examples utilized by the mathematics teachers revealed that while the Hong Kong teacher's instruction made good use of real-world scenarios, the problems given by both the Hong Kong and Shanghai teachers were found to be difficult by the students (Huang and Leung, 2002).

In a similar study, carried out in Germany and the German-speaking part of Switzerland, 39 lesson videos documenting instruction on the Pythagorean theorem were investigated to identify any correlations that existed between the teaching styles of mathematics teachers and their students' comprehension of the subject (Hugener et al., 2009). In addition, whether a correlation existed between the teachers' approaches and the students' short-term gains was investigated. The teachers' pedagogical approaches were classified as lecturing, developing, or discovering. Among the 39 teachers researched, seven teachers (four from Germany, three from Switzerland) were observed while lecturing, 13 (seven from Germany, six from Switzerland) were observed while developing, and the remaining 19 (nine from Germany, ten from Switzerland) were observed while utilizing the discovering pedagogies. The research results revealed that the teaching styles used in teaching the Pythagorean theorem were not country specific, and the pre-test and post-test administered to students showed that teaching approach used did not influence students' mathematical achievement (Hugener et al., 2009).

Lipowsky et al. (2009) investigated the instruction on the Pythagorean theorem from the perspective of required cognitive demand levels from students, the existence of a supportive classroom environment, and the quality of classroom management. In this research, 38 videos of mathematics lessons from Germany and the German-speaking part of Switzerland were examined. Teachers participating in this study were asked to provide instruction on the Pythagorean theorem during three class hours and in doing so provide at least one proof of the theorem. The knowledge of students participating in this study was assessed through pre-test and post-test. In all videos, the quality of teaching was also assessed in terms of cognitive activation, a supportive learning environment, and classroom management. The analysis showed a correlation between the students' interest in mathematics and high-level of cognitive stimulation and also revealed that students with higher interest in mathematics in particular benefited from the teaching of this type of lesson.

Past research had already revealed that when teachers required low cognitive demand levels from their students during instruction, it was impossible for their students to carry on the subject at a high cognitive demand level (Stein and Smith, 1998). For instance, when a teacher only expects students to memorize the Pythagorean theorem, students are less likely to personally inquire the validity of the theorem and instead only attempt to learn where and how they may apply this knowledge. Furthermore, Stein and Smith (1998) also argued that if teachers made their expectations for high cognitive performance levels from their students clear from the start, students would be prone to respond in a positive manner.

Unlike in Western countries, in the People's Republic of China, lectures in schools may be observed by colleagues (Yang, 2009). Since 1952, when the Teaching Research Group (TRG) was established at all schools throughout China, this has been the custom. These TRG groups have been comprised of teachers in the same discipline, with the intention of providing colleagues a better understanding of how instruction in their discipline could be carried out as well as resolving the problems they might encounter during instruction (Yang, 2009). Yang (2009) also observed how TRG affected lessons given by a young mathematics teacher. In this study, a lesson on the Pythagorean theorem was presented by the youngest and least experienced member of the Mathematics Teaching Research Group (MTRG) at a school that had three sections of eighth grade. The MTRG meetings regarding these instructional observations were recorded and examined. The teacher who would be observed first prepared a lesson plan and presented the lecture. Following the lecture, the MTRG reviewed the lecture given through its video and gave recommendations to the teacher. The teacher then made changes to the lesson plan, based on the MTRG's recommendations. This process was repeated after reviewing the subsequent videos of the lesson in each of the other sections.

Observations revealed that during the first lesson, the teacher spent nearly half of that lesson covering applications of Pythagorean theorem, while she spent most of the second lesson proving the theorem. Following the recommendations of the MTRG, the teacher adjusted the lesson during the lecture to the third class by focusing nearly 15% of the lesson on reviewing area calculation, 40% of the time producing the Pythagorean theorem, 20% of class time on proving the theorem mathematically, 22% of the time on proving the theorem with the help of figures (i.e., the use of a jigsaw puzzle), and the remaining 3% of the time on summarizing the learning. While the teacher placed emphasis on the application of the Pythagorean theorem in the first lesson, following the MTRG meeting, the primary emphasis of the second lesson was on proving the theorem. In addition, in the MTRG meeting following the second lesson, it was recommended that the teacher provide more detailed instruction to students regarding the origins of the Pythagorean theorem. During the debriefing with the MTRG, following the completion of all three lessons, the teacher commented that in preparation of the first lesson, she feared that the Pythagorean theorem would be too difficult for students; and as a result, her motivation then was to place emphasis on the applications of the theorem. The teacher also stated that, following the recommendation of the MTRG, she opted to focus attention on proving the theorem visually and used graph paper to do so. This switch in lesson focus allowed for less time to be spent on the proof of the theorem, and it also appeared that the algebraic details of the proof did not distract students' attention (Yang, 2009).

In another study, 19 videos (eight from Hong Kong, 11 from Shanghai) containing lectures on the Pythagorean theorem were examined (Huang and Leung, 2004). In addition to the investigation of the lectures, the teachers were given questionnaires and the teaching materials utilized in lectures were also reviewed. The research results of this study revealed that teachers created an encouraging classroom environment that allowed their students to better discover the Pythagorean theorem. In addition, researchers also investigated whether or not teachers provided any proofs of the Pythagorean theorem, showing that five teachers from Hong Kong verified the theorem visually while the remaining three teachers from Hong Kong and all of the teachers from Shanghai provided an algebraic proof of the Pythagorean theorem. The study also reported that eight teachers from Shanghai and one teacher from Hong Kong provided students with more than one proof of the Pythagorean theorem during their lectures (Huang and Leung, 2004).

The Pythagorean theorem is a critical subject in mathematics instruction during the second half of a student's primary education. Despite the need to understand various mathematical concepts, such as area from geometry and equation/equality from algebra, in order to solve specific proofs of the Pythagorean theorem, it is important to remember that there are also several visual proofs which can be easily comprehended (Nelsen, 1993). Gaining a more comprehensive understanding of which proof is best to present to students can be a research topic in and of itself.

In Turkey, graduates of mathematics departments are eligible to become high school mathematics teachers following the successful completion of a pedagogical formation certificate program. It could be suggested that since mathematics graduates have undergone a more theoretical approach to various fields in mathematics throughout the entirety of their education, they are more likely to place importance on their students' understanding of mathematical concepts. Furthermore, during their lectures, they may also have higher expectations and place greater importance on their students' comprehension of the proofs of mathematical theorems.

As of 2013, the Turkish Ministry of National Education (MEB, 2013) has required that a proof of the Pythagorean theorem be included in the ninth grade mathematics curriculum. As a result, the intention of this research is not only to investigate which teaching method the mathematics department graduates participating in this study planned to utilize in their ninth grade lessons to introduce the Pythagorean theorem, but also to gauge the importance they attached to mathematical concepts, and proofs. Another question of this research is to determine the cognitive demand levels required in the examples, evaluations, and assessments they had prepared**.**

Method

In this research, document analysis, a qualitative research method was utilized. The process of document analysis research involves the analysis of written materials, which consist of knowledge regarding the topic that is being investigated (Yıldırım and Şimşek, 2000). Written documents, such as course books, curriculum objectives, programs, student records, examinations, and lesson/unit plans, may all be useful materials to analyze in educational research. Importantly, through document analysis, researchers may meet their data collection needs without ever needing to conduct observations and/or interviews (Karasar, 2012).

Participants

The participants for this study were 43 mathematics graduates (22 female and 21 male) taking part in a pedagogical formation certificate program in a school of education at a state university in the Aegean region of the Republic of Turkey

Data Collection Tools

In this study, lesson plans prepared by the study participants were utilized for data collection. During the second term of the pedagogical formation certificate program, the study participants were asked to prepare a lesson plan for their ninth grade mathematics course. The objective of their lesson plan was to “prove and apply the Pythagorean theorem using the right triangle” (MEB, 2013). No bound was put on the number of class hours the participants could use to complete the instruction of this topic. The participants were given one week to prepare their lesson plans. Upon the submission of their lesson plans, participants were asked to respond to a questionnaire, consisting of two questions regarding the lesson plans they had prepared.

Throughout the term, participants attended a special teaching methods course that covered lesson plan preparation and teaching pedagogy. Therefore, the participants were not given additional explanation when tasked with preparing a lesson plan for instruction of the Pythagorean theorem. The goal of this hands-off approach was to allow the participants the freedom to prepare lesson plans as they deemed fit without being affected by the researchers’ objectives.

Following participants’ preparation of their lesson plans for instruction of the Pythagorean theorem, the participants were asked to complete a questionnaire regarding the lesson plan they had created. The questions included in the questionnaire were prepared by the researcher, and then two academicians specialized in the fields of Turkish and mathematics revised the questionnaire. Following the experts’ critique, the questionnaire was finalized, and participants were asked to identify the teaching methodology they intended to utilize in their lesson plan, and whether or not a proof of the Pythagorean theorem was included in the lesson and what their reasons were for not including a proof.

Collection of the Data

The data for this research was collected in the 12th week of the spring semester during the 2014-15 academic year. Participants in this research were mathematics graduates attending a special teaching-methods course as part of a pedagogical formation certificate program. During the 11th week of the spring semester, participants were asked to prepare lesson plans for ninth grade mathematics curriculum objective 9.4.4.1: “prove and apply the Pythagorean theorem using the right triangle” (MEB, 2013). In addition, during the 12th week of the course students were also asked, following the submission of their lesson plans, to complete a questionnaire, consisting of two questions regarding the lesson plans they had prepared. All lesson plans and survey questionnaires were collected and analyzed by the researcher.

Analysis of the Data

In total, 44 lesson plans and questionnaires were completed and submitted by participants. Each lesson plan and questionnaire was analyzed independently by the researcher and ultimately the data from one participant was discarded because the answers this participant provided in the questionnaire were inconsistent with the information in the participant's lesson plan. As a result, the data analyzed for this research was drawn from the remaining 43 participants (22 female and 21 male).

Assessment: Lesson Plans and Questionnaires. The lesson plans obtained from participants were assessed according to the number of hours required to complete this topic and whether or not the lesson plan included a proof of the Pythagorean theorem. Also, according to the TIMSS cognitive demand levels, the worked-out examples, homework and evaluation questions included in the participants' lesson plans were categorized as requiring knowledge, application, and reasoning.

The questionnaires completed by the participants following the submission of their lesson plans were examined to determine their rationale for not including a proof of the Pythagorean theorem in their lesson. If a proof of the theorem was given in the course plan, an assessment was made of the preferred proof type: visual, algebraic, or triangular similarity.

On the basis of the TIMSS classification guidelines, all of the worked-out examples, homework and evaluation questions included in the participants' lesson plans were classified as knowledge-, application-, or reasoning-level questions. According to the TIMSS classification, the questions at the knowledge level should be related to the requisite formulas, operations, and concepts known by students (Mullis, Martin, Ruddock, O'Sullivan and Preuschoff, 2009). In this study, questions requiring mathematical operations that used the Pythagorean relation were as knowledge-level questions.

Questions at the application level are routine questions that students are expected to previously have been exposed to. However, in order to solve such questions, it is important that students not only remember the Pythagorean relation but also perform mental activities such as selection, modeling, and demonstration (Mullis et al., 2009). Reasoning level questions are unfamiliar to students, are complex in nature, and have multiple steps (Mullis et al., 2009).

In order to classify the questions, 25 questions (approximately 10% of the total of 233 questions) from the participants' lesson plans were randomly selected. The 25 randomly selected questions were classified independently by both the researcher and a mathematics education faculty member who specialized in TIMSS questions.

Following their independent classifications, the researcher and mathematics education faculty member made comparisons of their decisions and reached an agreement of 92% reliability. The remaining questions were classified by the researcher and all questions were evaluated on the basis of the TIMSS classification.

Presentation of the Data

Table 1 below shows the sex of the participant versus the number of lesson plans that include/ do not include a proof of the Pythagorean theorem. Various types of proofs of the Pythagorean theorem that were given in the participants' lesson plans are also presented. In addition, examples of knowledge, application, and reasoning types of questions classified according to TIMSS guidelines are provided. Examples of direct responses from the participants' questionnaires are also presented in the results section as direct quotations, and the participant's number and gender are included (e.g., 15F for female participant number 15 and 22M for male participant number 22).

Results

The submitted lesson plans show that the preferred instructional approach to teaching the Pythagorean theorem was lecturing. One participant chose to incorporate both the lecturing and discovery approaches. Examination of the submitted lesson plans revealed that 29 (67%) of the 43 submitted lesson plans were prepared as one-hour lectures, while the remaining 14 (33%) of the submitted lesson plans were prepared as two-hour lectures.

Proof of the Pythagorean Theorem

The researcher's review of the prepared lesson plans indicated that 18 participants (42%) included a proof of the Pythagorean theorem in their lesson plan. Of these, 11 were female and 7 were male. A review of the proofs given in the lesson plans revealed that three types of proofs were utilized by participants: (i) visual proofs (two participants), (ii) algebraic proofs (nine participants), and (iii) proofs in which the similarity of triangles was used (seven participants). The remaining 25 participants (58%) did not include a proof of the Pythagorean theorem in their lesson plans.

Table 1

Sex of the participant versus existence of a proof of the Pythagorean theorem in the participant's lesson plan

	A proof given	**No proof given**	**Total**
Female	11 (50%)	11 (50%)	22
Male	7 (33%)	14 (67%)	21
Total	18 (42%)	25 (58%)	43

Among the 11 female participants who included a proof of the theorem in their lesson plan, one made use of a visual proof, five made use of an algebraic proof, and the remaining five made use of the similarity of triangles proof. Among the seven male participants who included a proof of the theorem in their lesson plan, one made use of a visual proof, four made use of an algebraic proof, and the remaining two made use of the similarity of triangles proof.

Two participants (26M and 32F) opted to utilize a proof of the Pythagorean theorem without any mathematical operations. The proof included in the lesson plan by participant 26M, shown in Figure 1, is considered to be a classic visual proof known as "the bride's chair" (Sparks, 2008, p. 50-51). The bride's chair proof begins with two different squares with sides of *a* and *b* units. Out of these squares two identical right triangles with legs of *a* and *b* units are cut and rearranged. The result is a square with the side length of *c* (i.e., the hypotenuse of the right triangles) is obtained. Thereby, the Pythagorean theorem is proved to the students visually without any calculations by creating a new square whose area equals the sum of the areas of the two original squares.

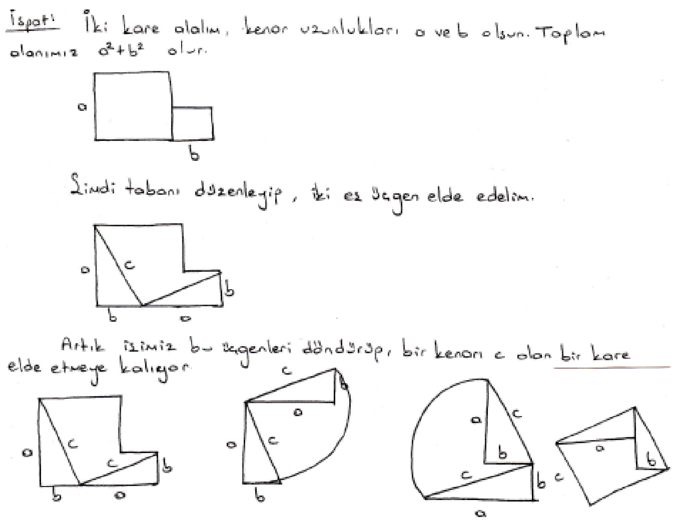

Figure 1. A visual proof of the Pythagorean theorem (26M)

A total of nine participants included proofs in their lesson plans that made use of the areas of squares and right triangles in an algebraic expression. Examples of this type of proof were included in the lesson plans of two participants (8F and 24F). These proofs are given in Figures 2 and 3.

In Figure 2, participant 8F provides an example of an algebraic proof of the Pythagorean theorem, starting with a square that has one of its sides equal to ($a + b$), and then obtaining a new square, that is formed by the hypotenuses of the right triangles in the middle, with a side of c units, by placing the four right triangles, with sides *a* and *b* units, to the corners of a square. The algebraic expression is written on the basis of the principle that the area of the square on the left side is equal to the area of the figure on the right side; thus, the Pythagorean relation is attained by making the required reductions.

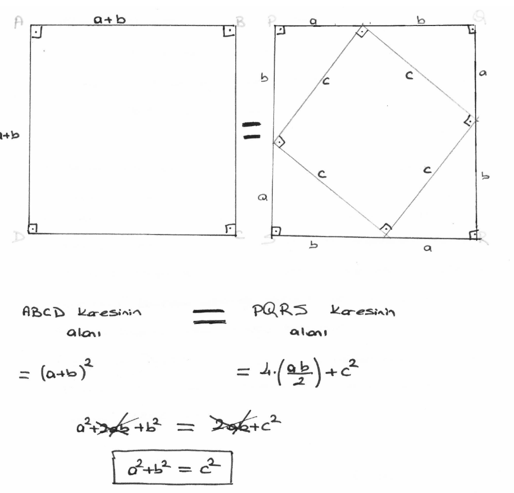

Figure 2. An algebraic proof of the Pythagorean theorem (8F)

In Figure 3, the proof provided also involves algebraic expressions and requires knowing the areas of squares and triangles. Though, in this example, each side of a square, with side c, is matched with a right triangle, the hypotenuse of which is *c,* thus obtaining a smaller square, with side of ($b – a$) in the middle of the figure.

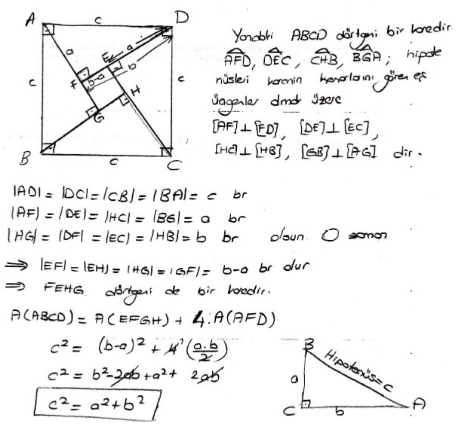

Figure 3. A different algebraic proof of the Pythagorean theorem (24F)

Seven participants opted to include a proof of the Pythagorean theorem in their lesson plan that made use of triangular similarities, which is given in the ninth grade mathematics curriculum before the Pythagorean theorem. The proof given by participant 18F is known as Legendre's proof (Sparks, 2008, p. 58) and is proved by means of the equalities attained by making use of the similarity of the two right triangles that are obtained by the perpendicular drawn from the left corner *C* of the bigger triangle *ABC* to its hypotenuse *AB* (Figure 4).

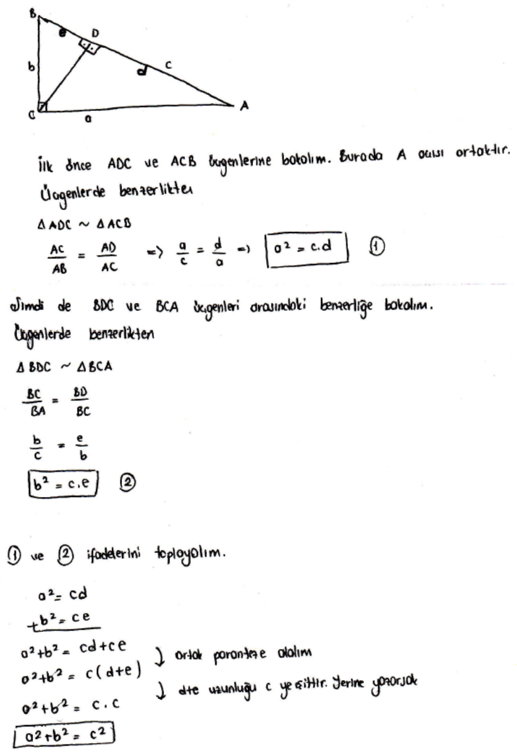

Figure 4. Proving Pythagorean theorem by making use of triangular similarity (18F)

Participants' Rationale for not Including a Proof of the Pythagorean Theorem

The data from this research shows that out of the 43 participants who created lesson plans, only 18 included a proof of the Pythagorean theorem in their lesson plans. A review of participants' responses to the questionnaire they completed following the submission of their lesson plan suggested a division of participants into two groups in terms of their rationale for creating their lesson plan: those who attached more importance to including examples in their lesson plan, and those who believed students would have difficulty comprehending a proof of the Pythagorean theorem. Examples of the participants' responses are provided below:

- I didn't give a proof; however, I thought giving respective examples would be sufficient. (1M)
- I did not give a proof of the theorem clearly, but I tried to explain it through examples. In other words, my lecture was not theoretical, but was applied. (10F)
- I rather focused on examples and thereby try to draw students' attention. (15F)
- I did not put forth a proof of the Pythagorean theorem in academic terms. I tried to explain it with areas of squares. I thought ninth grade students would comprehend it better, and it would be beneficial to give real-life examples, rather than giving symbolic expressions. I enriched it through the use of examples. (11M)
- A proof of the Pythagorean theorem is too long and requires so many details. Proving this theorem to a ninth grade student is too hard and requires quite a high level of knowledge. That is why I did not give a proof of the theorem. (13F)
- Since the students may find it difficult to comprehend the theoretical part and may therefore be alienated from the subject, I did not give a proof of the theorem. Giving the lesson in a more application-oriented way would be more effective for students' comprehension. (3M)
- I did not give a proof of Pythagorean theorem because I attached more importance to the subject and its applications. (19F)
- I chose not to give a proof of the theorem so as not to create any prejudice among the students against the subject. I had come to the conclusion that proofs are intimidating. (22F)
- I did not give a proof. Theoretical issues are not all that fulfilling for the students in the end. I preferred to make use of materials or visual videos. (23F)
- I did not give a proof of the theorem because I planned to have the 9th grade students understand the Pythagorean relation first. (25M)
- I did not give any proof. However, I prepared a lesson plan that is comprehensible and attractive enough for high school students by making use of the presentation method. I further developed the lesson plan with assessment and evaluation questions. (36F)
- I intentionally did not give a proof of the theorem. I thought it would be more beneficial for students to comprehend the Pythagorean relation itself. (39M)

Cognitive Demand Levels of Questions from Lesson Plans

All the worked-out examples, homework and evaluation questions included by the participants in their lesson plans were categorized in terms of TIMSS cognitive demand levels. Out of the 233 questions given in the lesson plans of 43 participants, a total of 86 (37%) were categorized as knowledge-level questions, 141 (60%) were application-level questions, and the remaining six (3%) were reasoning-level questions. Four participants included a proof of the Pythagorean theorem in their lesson plans but opted not to include a worked-out example of the theorem. Three participants, however, submitted lesson plans with 12 worked-out examples, which was the highest number submitted by the participants.

The following section provides examples of worked-out problems, homework and evaluation questions from various cognitive demand levels, taken from participants' lesson plans. Since participants' lesson plans were related to how the Pythagorean theorem can be taught to students, knowledge-level questions were those that could be solved using the Pythagorean relation in a mathematical operation. Figures 5 and 6 show worked-out examples requiring the cognitive demand level of knowing that were included in the lesson plans of Participants 43M and 22F.

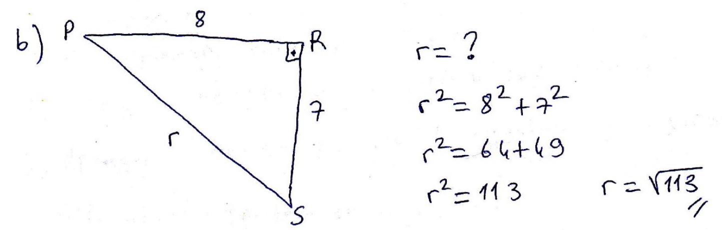

Figure 5. Worked-out example at the knowledge level (43M)

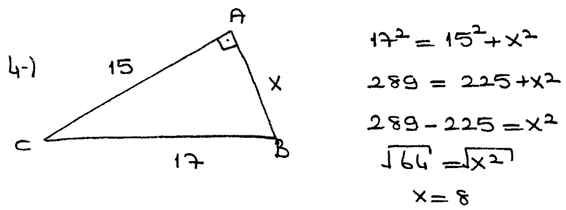

Figure 6. Worked-out example at the knowledge level (22F)

As can be seen from Figures 5 and 6, students were expected to determine the unknown value by placing the given lengths in appropriate positions in the Pythagorean formula. In the submitted lesson plans, 28 (out of 43) participants prepared 86 (37%) worked-out examples, homework and evaluation questions requiring the cognitive demand level of knowing.

Questions categorized as requiring the cognitive demand level of applying are routine questions that cannot be solved by means of a single mathematical operation but are deemed familiar enough that students should already know this question type. Of the worked-out examples, homework and evaluation questions prepared by 38 (out of 43) participants 141 (60%) required the cognitive demand level of applying. Examples of questions categorized as applying that were included in the participants' lesson plans are provided in Figures 7, 8, and 9.

In the example provided in Figure 7, the Pythagorean relation should be applied for three different right triangles separately in order to calculate the lengths *x*, *y* and *z*. This example was included in the lesson plan of participant 20M.

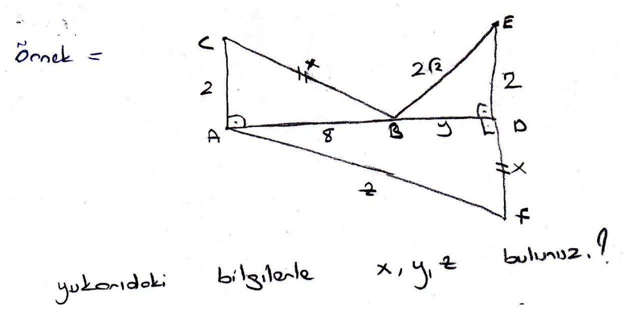

Figure 7. Example at the application level (20M)

Figure 8 shows a homework question at the application level prepared by Participant 5M. In this question, the students should obtain a right triangle by drawing a perpendicular line segment *DH* from point *D* to base *BC* in order to use the Pythagorean theorem. This question can be solved by recognizing that the length of the height *DH* of the right triangle is equal to the length of side *AB* of the trapezoid.

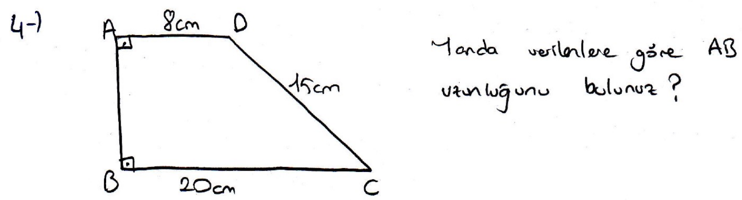

Figure 8. Homework question at the application level (5M)

In the question shown in Figure 9, the students should first draw a perpendicular line segment *AH* from point *A* to base *BC*, and as a result, obtain a right triangle, the hypotenuse of which is the side *AC*. Also of importance is recognizing that the angle *C* is 45° in order to conclude that the right triangle *AHC* thus obtained is an isosceles triangle. Then students may use the Pythagorean relation twice to determine the length of side *AB*.

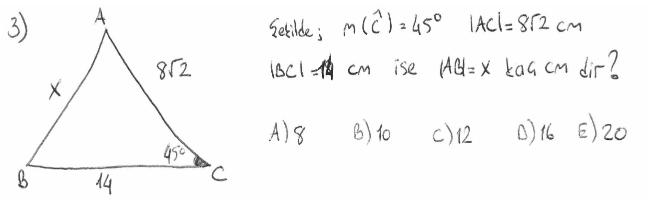

Figure 9. Evaluation question requiring the cognitive demand level of applying (29F)

Of all the worked-out, homework, and assessment questions written only six required the cognitive demand level of reasoning. Two examples of reasoning questions are given in Figures 10 and 11. In the example shown in Figure 10, a homework question included in the lesson plan of Participant 10F, the students are expected to know beyond the Pythagorean theorem in order to solve the question. For example, in the homework question, the students should not only be familiar with calculating the area of a circle but understand that the inscribed angle facing the diameter is a right angle. In the question, the information that angle *B* is a right angle is not provided, and the students' are expected to recognize this through their own reasoning.

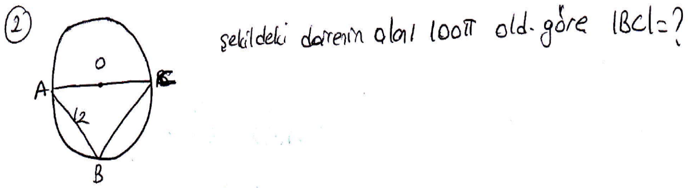

Figure 10. Homework question at the reasoning level (10F)

In Figure 11, a question included in the lesson plan of Participant 11M, the students are expected to draw a perpendicular line segment *AH* from point *A* down to the base *DB* in order to obtain two more right triangles. The length of side *AD* may then be calculated by solving the equalities obtained by means of the Pythagorean relations written for triangles *AHB*, *AHC*, and *AHD*. In order to write these equalities, the students should appoint variables to the sides, the lengths of which are not provided in the question.

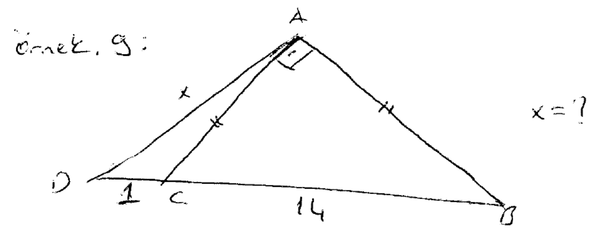

Figure 11. Example at the reasoning level (11M)

Discussion and Conclusions

The research results revealed that 29 out of 43 participants prepared lesson plans that covered this material in one class hour, while 14 of the participants prepared lesson plans that took two class hours. The content knowledge of "right triangles and trigonometry" is provided as part of the ninth grade mathematics curriculum and contains four objectives to be covered within 12 class hours. One of these four objectives is related to the Pythagorean theorem, and participants were asked to prepare lesson plans based on this objective. The other objectives included in this content knowledge are described as (i) defining the trigonometric ratios of the acute angles in right triangles and performing applications, (ii) defining the unit circle and correlating the trigonometric ratios with the coordinates of a point on the unit circle, and (iii) proving the cosine theorem of a triangle and performing applications (MEB, 2013). According to the description and objectives of this curriculum, it is recommended that no less than a total of three classroom hours should be allocated towards the instruction and outcome of the Pythagorean theorem.

An examination of the lesson plans prepared by the participants revealed that the entire group of mathematics teacher candidates had chosen to utilize the lecturing approach in their instruction. These findings differ from the results seen in Hugener et al. (2009), which instead revealed that only seven of 39 math teachers from Germany and the German-speaking part of Switzerland utilized the lecturing approach in their mathematics lessons. Each of these teacher candidates earned a four-year mathematics degree, so seemingly had sufficient understanding of the mathematic concepts and theorems. The participants' lack of creativity in adding to a variety of instructional strategies to their lesson plans may potentially be a result of them receiving insufficient instruction during their pedagogic formation coursework.

Of the 43 participants who submitted lesson plans for the instruction of the Pythagorean theorem, 18 participants included a proof of the theorem in their lesson plans while the remaining 25 participants did not. This occurred even though the participants were provided information regarding the ninth grade mathematics curriculum from the Ministry of National Education (MEB, 2013) that included a proof of the Pythagorean theorem. The teacher candidate who preferred to provide instruction without giving proofs taught students about the Pythagorean theorem by providing specific examples. These participants stated that in their opinion, providing a proof of the Pythagorean theorem would be too difficult for ninth grade students to fully comprehend, and, as a result, believed the more fruitful approach was to solve examples involving the Pythagorean relation. A portion of the 25 participants who did not include a proof of the Pythagorean theorem in their lesson plan claimed that they believed a proof of the Pythagorean theorem would be quite complex for ninth grade students and require a high level of knowledge that might ultimately alienate students from mathematics if they experience too much difficulty comprehending a theoretical proof. An issue for future research consideration is the fact that even though all of the teacher candidates had their undergraduate education in mathematics, they appeared not to have sought out a proof to be included in their instruction that closely matched their students' level of comprehension.

It is inexplicable to the researcher that a person who has studied university level mathematics for four years did not even attempt to find an appropriate proof from the tens of proofs of the Pythagorean theorem that ninth grade students can easily comprehend. In today’s Internet-based world, a quick online search could have easily yielded a wealth of various proofs of the Pythagorean theorem, but this research has shown that only two of the participants provided a visual proof of the Pythagorean theorem, which was acquired from an online search.

Out of the all the worked-out examples, homework and evaluation questions included in the lesson plans, 37% of them were found to be at the knowledge level, while 60% were found to be at the application level. These research results show similarities to the cognitive demand levels required in the examples (i.e., 46% knowledge and 51% application) given by the Czech teacher in Huang and Leung’s (2002) research. It is also important to consider that all the examples given by the Hong Kong teacher in that research were at the cognitive demand level of application; whereas, those of the Shanghai teacher’s were 75% in the application and 25% in the reasoning level. Compared with these, 37% of the questions being at the knowledge level in this current research is considered to be quite high.

The participants who submitted lesson plans for instruction of the Pythagorean theorem without including a proof of the theorem justified their decision by stating that solving examples would better allow students to comprehend the Pythagorean theorem. However, research results point out that learning opportunities provided to students are of great importance for their success (Lipowsky et al., 2009; Stein and Smith, 1998). If a majority of the examples provided to students are at knowledge or application level, and students lack the opportunity to do reasoning-level questions or a proof of the Pythagorean theorem, then these students are being confined to a low-level learning environment.

As a continuation of this study, one could investigate whether or not current in-service middle and high school mathematics teachers can provide an easily comprehensible proof of the Pythagorean theorem to their eighth or ninth grade students. Moreover, mathematics teachers who state that they can provide a comprehensible proof of the Pythagorean theorem may be further queried about which proof they give to their students, as well as why they chose the specific proof that they did.

In terms of scope, the research presented here was a qualitative study carried out among a limited number of mathematics graduates who were registered to a pedagogical formation program at one state university in Turkey. As a result, the findings attained from this research should not be generalized for all mathematics graduates who plan to teach mathematics in high school.

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Pisagor Teoremini Nasıl Öğretirsiniz: Ders Planlarının Analizi

MAKALE TÜRÜ	Başvuru Tarihi	Kabul Tarihi	Erken Görünüm Tarihi
Araştırma Makalesi	19.11.2017	13.03.2018	14.03.2018

Necdet Güner
Pamukkale Üniversitesi

Öz

Bu araştırma pedagojik formasyon sertifika programına katılan matematik bölümü mezunları üzerinde yapılmıştır. Katılımcılardan lise dokuzuncu sınıf matematik dersi programındaki Pisagor teoremi kazanımı için bir ders planı hazırlamaları istenmiştir. Kırküç katılımcının on sekizinin ders planlarında Pisagor teoreminin ispatına yer verdikleri tespit edilmiştir. Ders planlarında verilen ispatlar; görsel ispat (iki katılımcı), cebirsel ispat (dokuz katılımcı) ve üçgen benzerliğinin kullanıldığı ispat (yedi katılımcı) olarak üç kategoride değerlendirilmiştir. Bunun dışında, ders planlarında verilen çözümlü örnek, ev ödevi ve ölçme değerlendirme soruları TIMSS bilişsel düzeylerine göre bilgi, uygulama ve akıl yürütme düzeylerinde sorular olarak sınıflandırılmıştır. Kırküç katılımcının hazırladığı 233 sorunun yaklaşık % 37'sinin bilgi, % 60'ının uygulama ve % 3'ünün akıl yürütme düzeyinde sorular olduğu görülmüştür

Anahtar sözcükler: Bilişsel istem düzeyleri, ders planı, Pisagor teoremi, dokuzuncu sınıf matematik

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Özet

Amaç ve Önem

Ülkemizde matematik bölümü mezunları, eğitim fakültelerinden alacakları pedagojik formasyon sertifikası ile liselerde matematik öğretmeni olma hakkını kazanmaktadırlar. Dört yıl boyunca matematik bölümünde analiz, cebir, topoloji gibi farklı alanlarında teorik eğitim almış olan matematik mezunlarının lise öğrencilerine matematik dersi anlatırken, matematiksel kavramların öğrenciler tarafından algılanmasına daha çok önem verecekleri öngörülebilir. Bunun yanı sıra öğrencilerine anlattıkları matematik konularındaki matematiksel gerçeklerin (teoremlerin) ispatlarının da öğrencileri tarafından anlaşılmasına önem atfedecekleri beklenebilir. Bu çalışmada, matematik bölümü mezunlarının, Pisagor teoremi gibi her lise mezununun hatırlayabileceği bir konu için hazırladıkları ders planlarının incelenmesi amaçlanmıştır.

Yöntem

Bu araştırmada nitel araştırma yöntemlerinden biri olan doküman analizi kullanılmıştır. Araştırmada kullanılan iki veri toplama aracı vardır. Birincisi, katılımcıların hazırladıkları ders planlarıdır. İkincisi ise katılımcıların ders planlarını teslim ettikleri ders saatinde cevaplandırdıkları bir anket formudur. Araştırmada kullanılan veriler 2014 – 2015 akademik yılının bahar döneminin on ikinci haftasında pedagojik formasyon sertifika programına kayıtlı matematik bölümü mezunlarının kayıtlı olduğu özel öğretim yöntemleri dersinde toplanmıştır. Katılımcılardan, on birinci hafta dokuzuncu sınıf matematik dersi öğretim programının bir kazanımı olan “Dik üçgenlerde Pisagor teoremini ispatlar ve uygulamalar yapar.” için bir ders planı hazırlamaları istenmiştir. Ders planlarının toplandığı on ikinci hafta katılımcılardan, hazırladıkları ders planı hakkında sorular içeren bir anket formunun cevaplandırmaları istenmiştir.

Bulgular

Katılımcılardan 18’inin (% 42) hazırladıkları ders planınlarında Pisagor teoreminin bir ispatını verdikleri görülmüştür. Katılımcıların verdikleri ispatlar incelendiğinde üç farklı ispat türünün kullanıldığı tespit edilmiştir. Bunlar (i) görsel ispat, (ii) alan bilgisinin kullanıldığı cebirsel ispat ve (iii) üçgenlerin benzerliğinin kullanıldığı ispat türleri olarak adlandırılmıştır. İki katılımcı görsel ispat, dokuz katılımcı alan bilgisinin kullanıldığı cebirsel ispat ve yedi katılımcı da üçgenlerde benzerliği kullanarak Pisagor teoremini ispatlamayı seçmiştir. Kırk üç katılımcıdan 25’i (% 58) ise hazırladıkları ders planlarında Pisagor teoreminin bir ispatına yer vermemişlerdir. Bu katılımcıların, ders planlarının toplanmasından sonra cevaplandırdıkları anket formuna yazdıkları incelenerek; ispatı vermek yerine daha fazla örnek çözmeye önem verenler ve ispatın öğrenciler için zor olduğunu düşündüğü için ispata yer vermeyenler olarak iki gruba ayrılabileceği görülmektedir. Katılımcıların ders planlarında verdikleri çözümlü örnek, ev ödevi ve ölçme değerlendirme soruları TIMSS bilişsel düzeylerine göre sınıflandırılarak bilgi,

uygulama ve akıl yürütme düzeyinde sorular olarak üçe ayrılmıştır. Ders planlarında, 43 katılımcının verdiği toplam 233 örneğin 86'sı bilgi (% 37), 141'i uygulama (% 60) ve altısı da akıl yürütme (% 3) düzeyinde sorular olarak sınıflandırılmışlardır. Dört katılımcı ders planlarında sadece Pisagor teoreminin bir ispatını verip herhangi bir örneğe yer vermezken, üç katılımcı 12'şer örnek ile en çok örnek içeren ders planlarını teslim etmişlerdir.

Tartışma ve Sonuçlar

Pisagor teoremi için bir ders planı hazırlayan katılımcıların 18'i Pisagor teoreminin bir ispatını vermiştir. Yirmi beş katılımcı ise, Milli Eğitim Bakanlığı'nın verdiği öğretim programında olmasına rağmen teoremin bir ispatına ders programlarında yer vermemişlerdir. Ders planlarında teoremin ispatını vermeden konuyu örneklerle öğretmeyi seçen öğretmen adayları teoremin ispatının dokuzuncu sınıf öğrencileri için çok ağır olduğunu, öğrencilere anlamayacakları bir ispatı vermek yerine konu ile ilgili daha fazla örnek çözmenin kendilerine göre daha verimli bir ders işlemek olduğunu belirtmişlerdir. Bir katılımcı ise öğrencilerin ispatın teorik kısımda zorlanarak matematikten soğuyabileceğini düşündüğünü belirtmiştir. Katılımcıların hepsinin fen edebiyat fakültesi matematik bölümü mezunu olduklarını düşündüğümüzde, katılımcıların neden öğrencilerin seviyesine uygun bir ispat bulmaya çalışmadıklarının sorgulanması gereken bir durum olarak karşımıza çıkmaktadır. Katılımcıların hazırladıkları ders planlarında verdikleri çözümlü örnek, ev ödevi ve ölçme değerlendirme sorularının yaklaşık %37'sinin bilgi, %60'ının ise uygulama düzeyinde sorular olduğu belirlenmiştir. Ders planlarında Pisagor teoreminin bir ispatını vermeyen katılımcılar öğrencilere daha çok örnek çözerek bu konuyu daha iyi anlamalarını sağlamayı amaçladıklarını savunmaktadırlar. Ancak, matematik eğitimi üzerine yapılan araştırmalar öğrencilere sunulan öğrenme imkânlarının öğrencilerin başarısı için önemli olduğunu belirtmektedirler. Öğrencilere verilen örneklerin büyük çoğunluğunun bilgi veya uygulama düzeyinde sorular olması ve öğrencilerin akıl yürütmeye gerek duyacakları soru tipleri ile karşılaşmamaları, teoremin ispatının da verilmemesi ile birlikte öğrenciler çok alt düzey bir öğrenme ortamı ile karşı karşıya kalmaktadırlar. Dört yıl üniversitede matematik eğitimi almış bir kişinin Pisagor teoremi gibi onlarca ispatı olan bir teoremin dokuzuncu sınıf öğrencilerinin kolayca anlayabileceği bir ispatını bulmaya çalışmaması (bugünün teknolojilerini kullanmaması) anlaşılır gibi değildir. Bugün web arama motorları ile yapılacak kısa bir araştırma sonrası Pisagor teoreminin farklı birçok ispatına ulaşılabilmektedir. Katılımcılardan sadece ikisinin web'de yaptıkları aramalar sonucunda Pisagor teoreminin görsel bir ispatını yer verdikleri görülmüştür. Ders planlarında verilen diğer 16 ispatın tamamının farklı dokuzuncu sınıf matematik ders kitaplarında verilen ispatlar olduğu tespit edilmiştir. Her ne kadar Pisagor teoreminin ispatının verilip verilmemesinin öğrencilerin bu konu ile ilgili sınavdaki başarısına bir etkisi olduğu yapılan çalışmalarla gösterilmemiş olsa da öğrencilerin ilgisini çekebilecek uygun bir ispatın verilmesinin, öğrencilerin matematiğe bakış açısını olumlu yönde değiştirebilecek bir adım olacağı düşünülmektedir.