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www.elsevier.com/locate/physletbThe path to 0.01% theoretical luminosity precision for the FCC-ee S. Jadach ^{a,*}, W. Płaczek ^b, M. Skrzypek ^a, B.F.L. Ward ^{c,d}, S.A. Yost ^e^a Institute of Nuclear Physics, Polish Academy of Sciences, ul. Radzikowskiego 152, 31-342 Kraków, Poland^b Marian Smoluchowski Institute of Physics, Jagiellonian University, ul. Łojasiewicza 11, 30-348 Kraków, Poland^c Baylor University, Waco, TX, USA^d Max Planck Institute für Physik, München, Germany^e The Citadel, Charleston, SC, USA

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ABSTRACT

The current status of the theoretical precision for the Bhabha luminometry is critically reviewed and pathways are outlined to the requirement targeted by the FCC-ee precision studies. Various components of the pertinent error budget are discussed in detail – starting from the context of the LEP experiments, through their current updates, up to prospects of their improvements for the sake of the FCC-ee. It is argued that, with an appropriate upgrade of the Monte Carlo event generator *BHLUMI* and/or other similar MC programs calculating QED effects in the low angle Bhabha process, the total theoretical error of 0.01% for the FCC-ee luminometry can be reached. A new study of the *Z* and *s*-channel γ exchanges within the angular range of the FCC-ee luminometer using the *BHWIDE* Monte Carlo was instrumental in obtaining the above result. Possible ways of *BHLUMI* upgrade are also discussed.

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1. Introduction

The current status of the theoretical precision for the Bhabha luminometry is critically reviewed and pathways are outlined to the requirement targeted by the FCC-ee precision studies. Various components of the pertinent error budget are discussed in detail – starting from the context of the LEP experiments, through their current updates, up to prospects of their improvements for the sake of the FCC-ee. It is argued that, with an appropriate upgrade of the Monte Carlo event generator *BHLUMI* and/or other similar MC programs calculating QED effects in the low angle Bhabha (*LABH*) process $e^-e^+ \rightarrow e^-e^+$, the total theoretical error of 0.01% for the luminometry at the high luminosity FCC-ee machine [1] can be reached. Possible ways of this upgrade are also discussed.

In Section 2 we recap the main aspects of the theoretical precision in the LEP luminosity measurement and present important components of the corresponding error budget. In Section 3 we present current improvements on some of the above components. In Section 4 we discuss in detail prospects on reaching the 0.01% theory precision for the FCC-ee luminometry and outline ways of upgrading the main Monte Carlo program for this purpose, *BH-*

LUMI, in this respect. In Section 5 the important issue of technical precision is addressed. Finally, in Section 6 we briefly summarize our work.

2. Theoretical uncertainty in LEP luminometry, A.D. 1999

Let us recapitulate the essential aspects of the theory (mainly QED) uncertainty in the LEP luminometry, as seen A.D. 1999. Luminosity measurements of all four LEP collaborations at CERN and also of SLD at SLAC relied on theoretical predictions for the low-angle Bhabha process obtained using the *BHLUMI* Monte Carlo multiphoton event generator featuring a sophisticated QED matrix element with soft photon resummation. Its version 2.01 was published in 1992 (see ref. [2]) and the upgraded version 4.04 was published in ref. [3].

The theoretical uncertainty of the *BHLUMI* Bhabha prediction, initially rated at 0.25% [14], was re-evaluated in 1996 after extensive tests and debugging to be 0.16% [15]. From that time, the code of *BHLUMI* version 4.04 used by all LEP collaborations in their data analysis remains frozen. The following re-evaluation of its precision came from investigations using external calculations outside the *BHLUMI* code. For instance, the 0.11% estimate of ref. [12] was based on better estimations of the QED corrections missing in *BHLUMI* and on improved knowledge of the vacuum polarization contribution. The detailed composition of the final estimate of the theoretical uncertainty $\delta\sigma/\sigma \simeq 0.061\%$ of the *BHLUMI* 4.04

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Table 1

Summary of the total (physical+technical) theoretical uncertainty for a typical calorimetric detector. For LEP1, the above estimate is valid for a generic angular range within 1° – 3° (18–52 mrad), and for LEP2 energies up to 176 GeV and an angular range within 3° – 6° . Total uncertainty is taken in quadrature. Technical precision included in (a).

Type of correction/error	LEP1		LEP2	
	1996	1999	1996	1999
(a) Missing photonic $\mathcal{O}(\alpha^2)$ [4,5]	0.10%	0.027%	0.20%	0.04%
(b) Missing photonic $\mathcal{O}(\alpha^3 L_e^3)$ [6]	0.015%	0.015%	0.03%	0.03%
(c) Vacuum polarization [7,8]	0.04%	0.04%	0.10%	0.10%
(d) Light pairs [9,10]	0.03%	0.03%	0.05%	0.05%
(e) Z and s-channel γ [11,12]	0.015%	0.015%	0.0%	0.0%
Total	0.11% [12]	0.061% [13]	0.25% [12]	0.12% [13]

Table 2

Summary of the total (physical+technical) theoretical uncertainty for a typical calorimetric LEP luminosity detector within the generic angular range of 18–52 mrad. Total error is summed in quadrature.

Type of correction/error	1999	Update 2018
(a) Photonic $\mathcal{O}(L_e \alpha^2)$	0.027% [5]	0.027%
(b) Photonic $\mathcal{O}(L_e^3 \alpha^3)$	0.015% [6]	0.015%
(c) Vacuum polariz.	0.040% [7,8]	0.013% [26]
(d) Light pairs	0.030% [10]	0.010% [18,19]
(e) Z and s-channel γ exchange	0.015% [11,12]	0.015%
(f) Up-down interference	0.0014% [28]	0.0014%
(f) Technical Precision	–	(0.027)%
Total	0.061% [13]	0.038%

prediction, based on published works, is shown in Table 1, following ref. [13]. This value was used in the final LEP1 data analysis in ref. [16]. On the other hand, at LEP2 the experimental error was substantially larger than the QED uncertainty of the Bhabha process listed in Table 1, where we define $L_e = \ln(|t|/m_e^2)$.

All four LEP collaborations were quoting the experimental luminosity errors for LEP1 data below 0.05%, that is below the theoretical error. The best experimental luminosity error 0.034% was quoted by the OPAL collaboration¹ – they also quoted a slightly smaller theory error, 0.054%, thanks to use of improved light-fermion-pair calculations of refs. [18,19]; see also the review article [20] and workshop presentations [21,22].

3. Present status (2018)

From the end of LEP until the present time there has been limited progress on practical calculations for low-angle Bhabha scattering at energies around and above the Z resonance.² A new Monte Carlo generator BabaYaga based on the parton shower algorithm was developed [20,23–25]. It was intended mainly for low energy electron–positron colliders with $\sqrt{s} \leq 10$ GeV, claiming precision at 0.1%, but was not validated for energies near the Z peak.

There was, however, a steady improvement in the precision of the vacuum polarization in the t -channel photon propagator; see the recent review in the FCC-ee workshop [26]. Using the uncertainty $\delta\Delta_{had}^{(5)} = 0.63 \cdot 10^{-4}$ at $\sqrt{-t} = 2$ GeV quoted in Ref. [27] one obtains $\delta\sigma/\sigma = 1.3 \cdot 10^{-4}$. It is shown in the second column in Table 2, marked “Update 2018”. The improvement of the light-pair corrections of refs. [18,19] is also taken into account there.

The important point is that the technical precision, which is marked in parentheses as 0.027%, is not included in the sum, be-

¹ The OPAL collaboration has found all their experimental distributions for low-angle Bhabha data to be in a striking agreement with the BHLUMI Monte Carlo simulation [17].

² This is in spite of a considerable effort on the $\mathcal{O}(\alpha^2)$ so-called “fixed-order” (without resummation) QED calculations for the Bhabha process; see below for more discussion.

Table 3

Anticipated total (physical+technical) theoretical uncertainty for a FCC-ee luminosity calorimetric detector with the angular range being 64–86 mrad (narrow), near the Z peak. Description of photonic corrections in square brackets is related to the 2nd column. The total error is summed in quadrature.

Type of correction/error	Update 2018	FCC-ee forecast
(a) Photonic [$\mathcal{O}(L_e \alpha^2)$] $\mathcal{O}(L_e^2 \alpha^3)$	0.027%	0.1×10^{-4}
(b) Photonic [$\mathcal{O}(L_e^3 \alpha^3)$] $\mathcal{O}(L_e^4 \alpha^4)$	0.015%	0.6×10^{-5}
(c) Vacuum polariz.	0.014% [26]	0.6×10^{-4}
(d) Light pairs	0.010% [18,19]	0.5×10^{-4}
(e) Z and s-channel γ exchange	0.090% [11]	0.1×10^{-4}
(f) Up-down interference	0.009% [28]	0.1×10^{-4}
(f) Technical Precision	(0.027)%	0.1×10^{-4}
Total	0.097%	1.0×10^{-4}

cause according to ref. [12] it is included in the uncertainty of the photonic corrections. Future reduction of the photonic correction error will require a clear separation of the technical precision from other uncertainties and it may turn out to be a dominant one.

4. Path to 0.01% precision for FCC-ee

In the following we shall describe what steps are needed on the path to the $\leq 0.01\%$ precision required for the low-angle Bhabha (LABH) luminometry at the FCC-ee experiments. The last column in Table 3 summarizes this goal component-by-component in the precision forecast for the FCC-ee luminometry. We will also specify all improvements in the next version BHLUMI which could bring us to the FCC-ee precision level.

Before coming to the details of the envisaged improvements in QED calculations for the LABH process, let us recapitulate briefly basic features of the LABH luminometry which have to be kept in mind in QED perturbative calculations for FCC-ee. First of all, the largest photonic QED effects due to multiple real and virtual photon emission are strongly cut-off dependent. Event acceptance of the LABH luminometer is quite complicated, and cannot be dealt with analytically, hence a Monte Carlo implementation of QED perturbative results is mandatory. The LABH detector at FCC-ee will be similar to that of LEP, with calorimetric detection of electrons and photon (not distinguishing them) within the angular range $(\theta_{min}, \theta_{max})$ on opposite sides of the collision point [29]. The detection rings are divided into small cells and the angular range on both sides is slightly different in order to minimize QED effects. The angular range at FCC-ee is planned to be 64–86 mrad (narrow) [29] while at LEP it was typically 28–50 mrad (narrow range, ALEPH/OPAL silicon detector); see Fig. 2 in ref. [15] (also Fig. 16 in ref. [30]) for an idealized detection algorithm of the generic LEP silicon detector. The average t -channel transfer near the Z resonance will be $|\bar{t}|^{1/2} = \langle |t| \rangle^{1/2} \simeq 3.25$ GeV at FCC-ee instead of 1.75 GeV at LEP.³ The important scale factor controlling

³ At 350 GeV, the FCC-ee luminometer will have $|\bar{t}| = 12.5$ GeV.

photonic QED effects, $\gamma = \frac{\alpha}{\pi} \ln \frac{|\vec{t}|}{m_e^2} = 0.042$ for FCC-ee, that is only slightly greater than 0.039 for LEP. On the other hand, the factor $x = |t|/s$ suppressing s -channel contributions will be 1.27×10^{-3} , significantly larger than 0.37×10^{-3} for LEP.

Finally, let us remark that the process $e^+e^- \rightarrow 2\gamma$ is also considered for FCC-ee luminometry, see refs. [20,31] for more discussion on the QED radiative corrections to this process.

4.1. Photonic higher-order and subleading corrections

Photonic corrections (items (a) and (b) in Table 2) are large but they are mainly due to collinear and soft singularities which are known in QED at any perturbative order, hence they can be resummed. The cross section of the LABH luminometer is highly sensitive to emission of real soft and collinear photons. Even relatively soft collinear photon emission in the initial state (ISR) may pull final electrons outside the acceptance angular range, while final-state photons can easily change the shape of the final state “calorimetric cluster”. This is why resummation of the multiple photon effects has to be implemented in an exclusive way, using the method of exclusive exponentiation (EEX), as in BHLUMI [3], or using the parton shower (PS) method as in BabaYaga [23]. It was shown [32] that, for instance, the so-called “fixed-order” $\mathcal{O}(\alpha^2)$ calculations without resummation⁴ are completely inadequate for the LABH luminometry, leaving out uncontrolled QED effects of the order $\sim 0.5\%$ in the angular distribution, and even a few percent in some other important distributions.

Assuming that the technical precision is dealt with separately (see the discussion in the following Section 5), item (a) in Table 2, missing in BHLUMI v. 4.04, scales like $L_e = \ln(|t|/m_e^2)$, where t is the relevant squared momentum transfer. However, this item will disappear from the error budget completely once the EEX matrix element of BHLUMI is upgraded to include $\mathcal{O}(L_e\alpha^2)$ contributions, which are already known and published. In fact, these $\mathcal{O}(L_e\alpha^2)$ corrections consist of 2-real photon contributions, 1-loop corrections to 1-real emission and 2-loop corrections. Efficient numerical and analytic methods of calculating the exact $\mathcal{O}(\alpha^2)$ matrix element (spin amplitudes) for 2 real photons, keeping fermion masses, have been known for decades; see refs. [35,36]. In ref. [37] exact 2-photon amplitudes were compared with the matrix element of BHLUMI.

Truly pioneering work on $\mathcal{O}(L_e\alpha^2, L_e^0\alpha^2)$ virtual corrections to 1-photon distributions was done in ref. [4]. These were calculated neglecting interference terms between e^+ and e^- lines, which near the Z peak are of the order of $(\frac{\alpha}{\pi})^2 \frac{|t|}{s} L_e \sim 10^{-7}$ times some logarithm of the cut-off. Let us note in passing that we know from the s -channel analog in [38] that the pure $\mathcal{O}(L_e\alpha^2)$ correction of this class (neglecting the $\mathcal{O}(L_e^0\alpha^2)$ term) is amazingly compact – it consists of merely a 3-line formula at the amplitude level. Let us add for completeness that the above correction was also calculated numerically in ref. [39].

Finally, in ref. [13], the two-loop $\mathcal{O}(L_e\alpha^2)$ t -channel photon form-factor relevant for the LABH process (keeping in mind $|t|/s$ suppression) continued analytically from the known s -channel result of ref. [40] was added, thus accounting for the complete $\mathcal{O}(L_e\alpha^2)$ photonic correction, known but not included in the MC BHLUMI v4.04. Once the above well-known photonic $\mathcal{O}(L_e\alpha^2)$ part is added in the future upgrade of the EEX matrix element in BHLUMI, the corresponding item will disappear from the list of the projected FCC-ee luminometry uncertainties in Table 3.

In view of the above discussion, it is clear that the major effort of calculating the complete $\mathcal{O}(\alpha^2)$ QED correction to low

and wide-angle Bhabha processes in refs. [33,34,41,42], see also [43–46], is of rather limited practical importance for the LABH luminometry at FCC-ee.⁵ All these works essentially add previously unknown $\mathcal{O}(L_e^0\alpha^2)$ corrections, which are of order $\sim 10^{-5}$. Their size should be checked⁶ using auxiliary programs outside the BHLUMI Monte Carlo, in order to be listed among QED uncertainties in the uncertainty budget as in our Table 3. In any case, we expect corrections of this class to stay well below 10^{-4} , and most likely there will be no need to add the complete $\mathcal{O}(L_e^0\alpha^2)$ corrections to the matrix element of any MC for the LABH process.

Another important photonic correction listed as item (b) in Table 1 as an uncertainty of BHLUMI is the $\mathcal{O}(\alpha^3 L_e^3)$ correction (third order LO). It is already known from Ref. [6,47] and is currently omitted from v. 4.04 of BHLUMI, although already included in the LUMLOG part of BHLUMI. Given its already known size, we would need to implement this third order leading-order result into the EEX matrix element of BHLUMI, and it will disappear from the uncertainty list. Once it is done, the uncertainty due to $\mathcal{O}(\alpha^4 L_e^4)$ and $\mathcal{O}(\alpha^3 L_e^2)$ should be estimated and included in the list of photonic uncertainties of BHLUMI with the upgraded EEX matrix element. We can use the scaling rules indicated in the previous discussion to estimate an error due to missing $\mathcal{O}(\alpha^4 L_e^4)$ as $0.015\% \times \gamma = 0.6 \times 10^{-5}$ near the Z peak. The scale of the missing $\mathcal{O}(\alpha^3 L_e^2)$ is also of a similar order, $\gamma^2 \alpha / \pi \simeq 10^{-5}$, and its actual estimate is currently highly uncertain.

The so called up-down interference between photon emission from e^+ and e^- lines was calculated in ref. [28] at $\mathcal{O}(\alpha^1)$ to be roughly $\delta\sigma/\sigma \simeq 0.07 |t|/s$. At LEP1 its contribution is negligible, see Table 2, but at the FCC-ee luminometer it will be the factor of 10 larger and has to be included in the matrix element of the upgraded BHLUMI. Once it is done, its uncertainty should be again negligible, as indicated in Table 3, where we used $2\gamma \times 0.07 |t|/s$ as a crude estimator of its future uncertainty.

4.2. EEX versus CEEX matrix element

BHLUMI multi-photon distributions obey a clear separation into exact Lorentz invariant phase space and squared matrix element. The matrix element is an independent part of the program and is currently built according to exclusive exponentiation (EEX) based on the Yennie–Frautshi–Suura [48] (YFS) soft photon factorization and resummation performed on the spin-summed squared amplitude. It includes complete $\mathcal{O}(\alpha^1)$ and $\mathcal{O}(L_e^2\alpha^2)$ corrections, neglecting interference terms between electron and positron lines, suppressed by a $|t|/s$ factor.

Let us underline that the above EEX-style matrix element in BHLUMI has not been changed in the upgrades since version 2.01 [2]. As already said, we may continue this practice and introduce the results from Refs. [4,6,13,47] into the EEX matrix element, that is $\mathcal{O}(\alpha^2 L_e)$ and $\mathcal{O}(\alpha^3 L_e^3)$, neglecting again some $\sim |t|/s$ terms.

On the other hand, using the same underlying multi-photon phase space MC generator of BHLUMI and exploiting the results from Refs. [4,6,13,47], one could implement a more sophisticated matrix element of the CEEX [49] type, where CEEX stands for coherent exclusive exponentiation. In the CEEX resummation methodology, soft photon factors are factorized at the amplitude level and the matching with fixed order results is also done at the

⁵ They are more relevant for the wide-angle Bhabha, provided they are included in the MC with soft-photon resummation. However, this is rather problematic, because in all these works soft-real-photon contributions are added to loop corrections *ala* Bloch–Nordsieck, instead of subtracting the well-known virtual form-factor from virtual loop results already at the amplitude level, before squaring them.

⁶ This kind of correction is often enhanced by π^2 factors.

⁴ For instance, see the calculations of refs. [33,34].

amplitude level (before squaring and spin-summing). The big advantage of CEEX over EEX is that the separation of the infrared (IR) parts and matching with the fixed-order result are much simpler and more transparent when done at the amplitude level – all IR cancellations for complicated interferences are managed automatically and numerically. The inclusion of the s -channel Z and photon exchange and t -channel Z exchange including $\mathcal{O}(\alpha)$ corrections, soft photon interference between electron and positron lines, and all that would be much easier to take into account for CEEX than in the case of EEX. However, the inclusion of $\mathcal{O}(\alpha^3 L_e^3)$ in CEEX will have to be worked out and implemented.

Summarizing, the CEEX version would allow a more systematic further development of the program as we move forward with the FCC-ee project. From this perspective, the CEEX version is preferable, although the improvement of the EEX matrix element should be also pursued. See some additional discussion in Sect. 5.

4.3. Error on hadronic vacuum-polarization contribution

The uncertainty of the low-angle Bhabha cross section due to imprecise knowledge of the QED running coupling constant of the t -channel photon exchange is simply $\frac{\delta_{VP}\sigma}{\sigma} = 2\frac{\delta\alpha_{eff}(\bar{t})}{\alpha_{eff}(\bar{t})}$, where \bar{t} is the average transfer of the t -channel photon. For the FCC-ee luminometer, it will be $|\bar{t}|^{1/2} \simeq 3.5 \text{ GeV}$ near the Z peak and $|\bar{t}|^{1/2} \simeq 13 \text{ GeV}$ at 350 GeV.

The uncertainty of $\alpha_{eff}(t)$ is mainly due to the use of the experimental cross section σ_{had} for $e^-e^+ \rightarrow hadrons$ below 10 GeV as an input to the (subtracted) dispersion relations. A comprehensive review of the corresponding methodology and the latest update of the results can be found in refs. [50,51], see also the FCC-ee workshop presentation [26].

In the above works, the hadronic contribution to α_{eff} from the dispersion relation is encapsulated in $\Delta\alpha^{(5)}(-s_0)$, where $2 \text{ GeV} \leq s_0^{1/2} \leq 10 \text{ GeV}$ in order to minimize the dependence on $\sigma_{had}(s)$, such that the main contribution comes from $s^{1/2} \leq 2 \text{ GeV}$. Moreover, prospects of improving experimental data on $\sigma_{had}(s)$ in this energy range are very good also, because the main contribution to the error in the measurement of the muon $g-2$ comes from the same cross section range [50].

The above works are focusing on the parameter range $2 \text{ GeV} \leq s_0^{1/2} \leq 10 \text{ GeV}$, which is accidentally of paramount interest for the FCC-ee luminometry, are part of a wider strategy in refs. [50, 51] of obtaining $\alpha_{eff}(M_Z^2)$ in two steps, where $\Delta\alpha^{(5)}(-s_0)$ is obtained from dispersion relations and the difference $\Delta\alpha^{(5)}(M_Z^2) - \Delta\alpha^{(5)}(-s_0)$ is calculated using the perturbative QCD technique of the Adler function [52]. The error of the above difference due to limited knowledge of α_s , the c and b quark masses and higher-order perturbative QCD effects is small enough, such that the overall uncertainty of $\alpha_{eff}(M_Z^2)$ is smaller than that from the direct use of the dispersion relation.

Taking $s_0^{1/2} = 2.0 \text{ GeV}$ and the value $\Delta\alpha^{(5)}(-s_0) = (64.09 \pm 0.63) \times 10^{-4}$, of ref. [27] as a benchmark, in Table 2 we quote $(\delta_{VP}\sigma)/\sigma = 1.3 \times 10^{-4}$. Thanks to anticipated improvements of data for $\sigma_{had}(s)$, $s^{1/2} \leq 2.5 \text{ GeV}$, one may expect the factor of 2 improvement by the time of the FCC-ee experiments, that is $\delta_{VP}\sigma/\sigma = 0.65 \times 10^{-4}$ near the Z peak, see Table 3.

At the high-energy end of FCC-ee, 350 GeV, due to the increase of the average transfer $|\bar{t}| = 12.5 \text{ GeV}$, one obtains presently from the dispersion relation $\delta\alpha_{eff}/\alpha_{eff} = 1.190 \times 10^{-4}$ and $\delta_{VP}\sigma/\sigma \simeq$

2.4×10^{-4} , and again with the possible improvement of the factor of 2, so that the FCC-ee expectation⁷ is $(\delta_{VP}\sigma)/\sigma \simeq 1.2 \times 10^{-4}$.

There are also alternative proposals for the measurement of $\alpha_{eff}(t)$ not relying (or relying less) on dispersion relations; see refs. [53,54]. Ref. [53] proposed a method for the direct measurement of $\alpha_{eff}(M_Z^2)$ using charge asymmetry in $e^-e^+ \rightarrow \mu^-\mu^+$ near the Z resonance. One may ask whether its precise value can also be used to predict very precisely $\alpha_{eff}(t)$ in the FCC-ee luminometer range $2 \text{ GeV} \leq |t|^{1/2} \leq 10 \text{ GeV}$? It turns out that the uncertainty due to the use of pQCD [26] in the transition from the M_Z scale down to below 10 GeV is about the same as in the traditional methods. However, a direct measurement of $\alpha_{eff}(M_Z^2)$ may serve as an important crosscheck. The other proposal, in ref. [54], of the direct measurement of $\alpha_{eff}(t)$, $t \sim -1 \text{ GeV}^2$, from the elastic scattering of energetic muons on atomic electrons sounds interesting, but requires more studies.

4.4. The uncertainty due to light fermion pairs

Three groups of calculations are available for the light-fermion-pair effect in the low angle Bhabha process: [9,10], [18,19] and [55–59].

The biggest correction, due to additional electron pair production, was calculated in Ref. [18], where process $e^+e^- \rightarrow e^+e^-e^+e^-$ calculated with the help of the ALPHA algorithm [60] was combined with virtual/soft corrections of Refs. [61–63], resulting in the theoretical error on pair correction to be 0.01%.⁸ This value is quoted in Table 2 as the present state of the art for the uncertainty of corrections due to light fermion pair production.

In Refs. [56,57] e^+e^- pair corrections were calculated in a semi-analytic way at NLO accuracy, omitting non-logarithmic corrections and taking virtual corrections from [61,62]. The third order LO correction due to simultaneous emission of the additional e^+e^- pair (Non-Singlet and Singlet) and additional photon were also evaluated. The overall precision of the Bhabha scattering formula of Refs. [56,57] was estimated there to be 0.006%, mainly due to omission of the heavier lepton pairs ($\mu^+\mu^-$, $\tau^+\tau^-$) and quark pairs (0.005%). One can assume conservatively the same 0.006% as the total error on additional pair correction.

In the Ref. [9] the complete LO semi-analytic calculations based on the electron structure functions were presented up to the third order for the Non-Singlet⁹ and Singlet structure functions. Contrary to Ref. [56], results are provided also for the asymmetric acceptances.

The approach of [10] was based on the extension of the YFS [48] scheme of the soft photon resummation to the case of soft e^+e^- pair emission, with relevant real and virtual soft ingredients calculated in [64] (omitting up-down interference, multi-peripheral graphs etc.). The calculation is implemented in the unpublished BHLUMI v. 2.30 MC code. The accuracy of results was estimated to be 0.02% for the asymmetric angular acceptance, i.e. 3.3° – 6.3° and 2.7° – 7.0° , with the energy cut $1 - s'/s < z_{cut} = 0.5$. Ref. [18] has concluded that this precision is even better, 6×10^{-5} for $z_{cut} \leq 0.5$, while for hard emission, $z_{cut} > 0.5$, with significant multi-peripheral component, the precision deteriorates to 0.01%.

What should be done in order to consolidate the above, mostly LEP era, calculations of the fermion pair contribution and to reach even better precision level needed for FCC-ee?

⁷ We thank F. Jegerlehner for elucidating private communications on the above predictions.

⁸ The emission of a μ -pair is also discussed in Ref. [18].

⁹ This is contrary to the incorrect statement in Ref. [56]. Third order NS $e^+e^-\gamma$ corrections are realized in Ref. [9] by second order structure function with the running coupling.

As in ref. [18], for the additional real e^+e^- pair radiation the complete matrix element should be used, because non-bremsstrahlung-type graphs can contribute as much as 0.01% for the cut-off $z_{cut} \sim 0.7$. There are a number of MC generators for the $e^+e^- \rightarrow 4f$ process, developed for the LEP2 physics to be exploited for that purpose.¹⁰

In order to improve on 0.005% uncertainty of Ref. [56], due to the emission of the $\mu^+\mu^-$, $\tau^+\tau^-$, and quark pairs, one may use LO calculation of ref. [9], incorporating lepton pair contributions by means of the modification of the running coupling. A naive rescaling of the electron logarithm (due to the mass of the muon) gives $\ln \frac{|t|}{m_e^2} = 17.5$ and $\ln \frac{|t|}{m_\mu^2} = 6.9$, i.e. for muon pairs we find a suppression factor of $\ln^2 \frac{|t|}{m_\mu^2} / \ln^2 \frac{|t|}{m_e^2} = 0.4^2 = 0.16$ relative to the electron pair. Rescaling the additional e^+e^- pair contribution of 0.05% one obtains an estimate of the muon pair contribution of 0.008%.¹¹ For the tau lepton logarithm $\ln \frac{|t|}{m_\tau^2} = 1.2$ we obtain $\ln^2 \frac{|t|}{m_\tau^2} / \ln^2 \frac{|t|}{m_e^2} = 0.07^2 = 0.005$ suppression factor relative to the electron pair, hence this contribution can be neglected. Adding $\mu^+\mu^-$ pairs to the BHLUMI v. 2.30 code of Ref. [64] would be straightforward. Also in the approach of ref. [18,19] this should be possible.¹² The contribution of light quark pairs (π pairs etc.) can be roughly estimated using quantity $R_{had} = \sigma_{had} / \sigma_\mu \simeq 3$ for the effective hadronic production threshold of the order of 1 GeV. One obtains $R_{had} \ln^2 \frac{|t|}{0.5^2 \text{ GeV}^2} / \ln^2 \frac{|t|}{m_\mu^2} = 0.9$, i.e. this contribution is of the size of the muon pair contribution, that is of the order of 0.008%.¹³

The third group of corrections are the higher order terms. The emission of two (or more) electron pairs is suppressed by another factor $(\frac{\alpha}{\pi} \ln \frac{|t|}{m_e^2})^2 \sim 10^{-3}$ and is negligible. The additional $e^+e^- + n\gamma$ correction is non-negligible. Its evaluation was based either on LO structure functions ([56] (Table 1), [18] (Fig. 8), [9]) or on the YFS [48] soft approximation [10] (Fig. 4), resulting in quite different results and their comparison is rather inconclusive. They are at most of the order of 0.5 to 0.75 of the additional e^+e^- correction (without γ). The remaining non-leading, non-soft additional $e^+e^- + n\gamma$ corrections are suppressed by another $1 / \ln \frac{|t|}{m_e^2} \sim 0.06$ and should be negligible ($\sim 0.003\%$). It would be also possible to calculate the additional $e^+e^- + \gamma$ real emission in a way similar to the existing code for LEP2 physics, [67].

The above improvements can be implemented either directly in the upgraded BHLUMI or using a separate calculation, such as BHLUMI 2.30 [10] code, or external MC programs like these of Refs. [18,19]. To summarize, the proposed future error budget is the following: (1) The contribution of light quark pairs must be calculated with the accuracy of 25%, i.e. 0.0027%. (2) The contribution of the muon pairs will be known to 10%, i.e. to 0.0008%. (3) The non-leading, non-soft additional $e^+e^- + n\gamma$ corrections will be treated as an error of 0.003%. Adding (1)–(3) in quadrature we obtain 0.004%. Applying safety factor of 1.25 we end up with a 0.005% possible pair production uncertainty forecast for the FCC-ee, quoted in Table 3.

¹⁰ One needs to be sure that the collinear configurations of outgoing four electrons are covered, for example like it is done in KoralW [65] which in its latest version 1.53 [66], accounts for photonic radiation to t-channel exchanges as well.

¹¹ This is less optimistic than the estimate in Ref. [56].

¹² The other option is to use the above described general purpose LEP2 4f codes, including also the discussed earlier corresponding virtual corrections.

¹³ This is less optimistic than the estimate in Ref. [56]. Adding in quadrature errors due to muon and light quark pairs one obtains 0.011% rather than 0.006% of Ref. [56]. 0.011% is consistent with the estimate of Ref. [18].

4.5. Z exchange and s-channel photon exchange

In the Bhabha scattering process, in addition to γ exchange in the t channel γ_t , there are also contributions from γ exchange in the s channel γ_s and Z exchange in both t and s channels, Z_t and Z_s . In fact, they all should be added at the amplitude level (Feynman diagram) and then squared to obtain the differential cross section for the Bhabha process, giving rise to several interference contributions. Numerically the most important for the low angle Bhabha (LABH) luminometry, apart from the pure t -channel γ exchange, $\gamma_t \otimes \gamma_t$, are interferences of other contributions with the γ_t amplitude, due to the enhancement factor $\sim s/|t|$. Among these, near the Z peak, the most sizable is the interference $\gamma_t \otimes Z_s$, because of the resonant enhancement. In the context of LEP luminometry it was studied in detail in ref. [11] for two types of detectors: SICAL with an angular coverage of $\sim 1.5^\circ$ – 3° and LCAL with an angular coverage of $\sim 3^\circ$ – 6° . Based on this work, the $\gamma_t \otimes Z_s$ contribution was implemented in BHLUMI 4.02 and its theoretical precision for the LEP luminosity measurement was assessed. We are going to exploit these results and estimate theoretical errors of all other contributions beyond the dominant $\gamma_t \otimes \gamma_t$. Since the angular coverage of the planned FCC-ee luminometer [29] is close to the LCAL one, we shall use the results of ref. [11] obtained for this type of the detector.

The Born-level $\gamma_t \otimes Z_s$ contribution is up to $\sim 1\%$ and changes from being positive below the Z peak to negative above, reaching the maximal absolute value at about ± 1 GeV from the peak. Radiative corrections, dominated by QED, are sizable, up to $\sim 0.5\%$ (up to $\sim 50\%$ of the Born-level contribution) and change in the opposite way, i.e. from negative to positive values when going from below to above the Z peak. BHLUMI includes the QED corrections and running-coupling effects for this contribution within the $\mathcal{O}(\alpha)$ YFS exclusive exponentiation. The theoretical uncertainty for this calculation was estimated at 0.090% for LCAL and is used as an initial estimate of the theoretical error for the FCC-ee luminometry concerning the $\gamma_t \otimes Z_s$ contribution in Table 3.

The other contributions will be estimated by means of relating them to the $\gamma_t \otimes Z_s$ or $\gamma_t \otimes \gamma_t$, using rescaling factors, $|t|/s \approx 1.3 \times 10^{-3}$ and $\tilde{\gamma}_Z = \Gamma_Z / M_Z \approx 2.7 \times 10^{-2}$.

The next most sizable contribution comes from the interference $\gamma_t \otimes \gamma_s$. At the Born level, near the Z peak, it is smaller than the $\gamma_t \otimes Z_s$ contribution by the factor¹⁴ $\sim 4 \tilde{\gamma}_Z \approx 0.1$. Taking $\sim 1\%$ for the Born-level $\gamma_t \otimes Z_s$, we get $\sim 0.1\%$ for $\gamma_t \otimes \gamma_s$. It is included in BHLUMI, so we need to estimate the missing radiative corrections. Since this is smooth near the Z peak, the photonic QED corrections should stay within 10%, for not too tight cuts on radiative photons. The resulting estimate of the theoretical precision of $\gamma_t \otimes \gamma_s$ contribution in BHLUMI for the FCC-ee luminometry is $\sim 0.01\%$.

The resonant pure s -channel Z contribution, $Z_s \otimes Z_s$, at the Born level, is multiplied with respect to the $\gamma_t \otimes Z_s$ term by the factor $\sim |t|/s \times 1/(4 \tilde{\gamma}_Z) \approx 1.3 \times 10^{-2}$, thus its size is $\sim 0.01\%$. It is omitted in the current version of BHLUMI, hence it enters into theoretical error as a whole. However, it can be included rather easily, such that only the missing radiative corrections will matter. Due to the Z -resonance effect, they can reach even $\sim 50\%$ of the Born-level contribution, hence the corresponding theoretical error would be $\sim 0.005\%$.

The t -channel interference $\gamma_t \otimes Z_t$ we estimate multiplying the $\gamma_t \otimes Z_s$ contribution by the $\sim |t|/s \times \tilde{\gamma}_Z \approx 3.5 \times 10^{-5}$ factor. It can be easily implemented in BHLUMI, with the theoretical error due to the missing photonic corrections being below 10^{-5} .

¹⁴ The factor of 4 comes from the ratio of the corresponding coupling constants.

Table 4

Results from BHWIDE for the Z and γ_s exchanges contribution to the FCC-ee luminosity with respect to the $\gamma_t \otimes \gamma_t$ process for the calorimetric LCAL-type detector [14] with the symmetric angular range 64–86 mrad; no acoplanarity cut was applied. MC errors are marked in brackets.

E_{CM} [GeV]	Δ_{tot} [%]	$\delta_{\mathcal{O}(\alpha)}^{QED}$ [%]	$\delta_{h.o.}^{QED}$ [%]	δ_{tot}^{weak} [%]
90.1876	+0.642 (12)	−0.152 (59)	+0.034 (38)	−0.005 (12)
91.1876	+0.041 (11)	+0.148 (59)	−0.035 (38)	+0.009 (12)
92.1876	−0.719 (13)	+0.348 (59)	−0.081 (38)	+0.039 (13)

The pure s -channel $\gamma_s \otimes \gamma_s$ contribution is much smaller in the Z -peak region than the resonant Z exchange. It is suppressed by the factor $\sim (4\tilde{\gamma}_Z)^2 \approx 0.01$ with respect to $Z_s \otimes Z_s$ (which is worth $\sim 0.01\%$), so is of the order of 10^{-6} .

Finally, the $Z_t \otimes Z_t$ contribution is smaller than the dominant $\gamma_t \otimes \gamma_t$ one by the factor $\sim (|t|/s/4)^2 < 10^{-6}$, thus it is completely negligible.

Adding the above theoretical errors in the quadrature, we obtain the total uncertainty (contributions omitted in BHLUMI) due to the Z exchanges and γ_s exchange for the FCC-ee luminometer near the Z peak at the level of 0.090%, quoted as present state of the art in Table 3.

The above uncertainty is completely dominated by the uncertainty of the $\gamma_t \otimes Z_s$ contribution which comes from a rather conservative estimate in ref. [11] based on comparisons of BHLUMI with the MC generator BABAMC [68] and the semi-analytic program ALIBABA [69,70], the latter including higher-order leading-log QED effects. Later on, the new MC event generator BHWIDE [71] was developed for the wide-angle Bhabha scattering including all Born-level contributions for Bhabha process and $\mathcal{O}(\alpha)$ YFS exponentiated EW radiative corrections. The comparison of BHLUMI with BHWIDE for FCC-ee luminometer would help to reduce all the above theoretical errors. In principle, the Born-level but also $\mathcal{O}(\alpha)$ QED matrix elements of BHWIDE could be implemented in BHLUMI. This would reduce the theoretical error for the above group of contributions below 0.01%, as indicated in Table 3. What we can do right now is to examine in a more detail the $\gamma_t \otimes Z_s$ contribution using BHWIDE, in order to get better idea about its future uncertainty. The main advantage of BHWIDE is that its matrix element includes complete $\mathcal{O}(\alpha)$ corrections to γ_s , Z_s and Z_t exchanges, while BHLUMI includes only part of $\mathcal{O}(\alpha)$ corrections due to soft photon resummation.

4.6. Study of Z and s -channel γ exchanges using BHWIDE

We are going to present numerical results obtained with BHWIDE for the calorimetric LCAL-type detector, as described in ref. [14], for the symmetric angular range 64–86 mrad without any cut on acoplanarity (i.e. the number of azimuthal sectors in LCAL was set to 1). For the Z -boson mass and width we used the current PDG values: $M_Z = 91.1876$ GeV and $\Gamma_Z = 2.4952$ GeV. The weak corrections, i.e. the non-QED electroweak (EW) ones, were calculated with the help of the ALIBABA EW library [69,70]. The results, shown in Table 4, were obtained for three values of the centre-of-mass (CM) energy: $E_{CM} = M_Z$, $M_Z \pm 1$ GeV. The last two values were chosen because for these energies the Z -contribution is (close to) the largest – with the opposite sign, see e.g. ref. [11].

The numbers shown in the second column of Table 4 represent the total relative contribution of the Z and γ_s exchanges, $\Delta_{tot} = |\gamma_t + \gamma_s + Z_s + Z_t|^2 / |\gamma_t|^2 - 1$, as predicted by BHWIDE, that is for Born + $\mathcal{O}(\alpha)$ YFS exponentiated matrix elements, including $\mathcal{O}(\alpha)$ EW corrections. As one can see, this contribution is positive below the Z -peak with the size up to $\sim 0.64\%$, gets close to zero near the Z -peak and changes the sign above the Z -peak with the size up to $\sim -0.72\%$. This agrees with our rough estimate given in the

previous subsection that the Born-level contribution is up to about 1%.

These effects are in general consistent with the results of ref. [11], although they are slightly smaller. There are two main reasons for this: (1) the polar angles of the LCAL detector are a bit smaller here than in ref. [11] and (2) here we used the calorimetric acceptance, while the results in Tables 1 and 2 of ref. [11] were obtained for the non-calorimetric acceptance.

In the next three columns we present various interesting components of the radiative corrections in Δ_{tot} . The fixed-order (without exponentiation) $\mathcal{O}(\alpha)$ QED corrections, shown in the third column, are sizable – from about -0.15% below M_Z to about $+0.35\%$ above it, and they have the opposite sign to the Born-level contribution. This also agrees with our estimate given above that the QED correction can reach up to a half of the size of the Born-level effect.

In the fourth column we show the higher-order QED corrections, i.e. the ones beyond the $\mathcal{O}(\alpha)$ QED fixed-order, which result from the YFS exponentiation. They also change their sign near the Z -peak, but in the opposite way to the $\mathcal{O}(\alpha)$ corrections, and their size is about a quarter of the latter. Based on the size of these corrections one can estimate the higher-order QED effects missing in BHWIDE. Since near the Z -peak the dominant are soft-photon corrections which are treated by the YFS exponentiation very accurately, we may expect that those missing effects are much smaller than the ones in the fourth column of Table 4. To estimate their size we can use the factor $\gamma = \frac{\alpha}{\pi} \ln \frac{|t|}{m_e^2} = 0.042$ of Section 4 and the ‘safety’ factor 2 of ref. [11], and apply them to the largest h.o. correction in Table 4, i.e. $0.081\% \times \gamma \times 2 \simeq 0.007\%$.

Note that precision of the MC results in Table 4 is limited by the statistical error of order 0.01–0.06%, because MC sample from BHWIDE in the current version of the program is limited due to the size limit of the Fortran integer numbers. Nevertheless, these results provide useful estimates on the size of the higher order effects.

In the last column of Table 4 the pure EW corrections, i.e. the EW corrections minus the QED ones are shown – implemented within the $\mathcal{O}(\alpha)$ YFS exponentiation scheme. They are at the level of about 0.01% below and at M_Z , while above M_Z they increase up to $\sim 0.04\%$.

To estimate the size of the missing higher-order weak corrections in BHWIDE, we can apply the same factor as for the QED corrections to get the value of $\sim 0.003\%$.

Altogether, we can estimate the physical precision of the Z and γ_s -exchanges contribution to FCC-ee luminometry in BHWIDE by adding linearly (to be conservative) the above two numbers on the missing effects to get $\sim 0.01\%$. Therefore, if the predictions of BHLUMI for the luminosity measurement at FCC-ee are combined with the ones from BHWIDE for this contribution, then the error in the line (e) of Table 3 could be reduced to 0.01%. Of course, this result requires more dedicated numerical tests and cross-checks with independent calculations.

In ref. [72] it was shown that for $\sqrt{s} \gg M_Z$ all the above contributions are below 0.01% – they then can be neglected in the FCC-ee luminometry at energies above the Z peak.

The best method to reduce the uncertainty of the above contributions practically to zero would be to include these Z and γ_s exchanges within the CEEX matrix element at $\mathcal{O}(\alpha^1)$ in BHLUMI. Most likely, it would be enough to add the EW corrections to the LABH process in the form of effective couplings in the Born amplitudes. On the other hand, the new BHLUMI with such a CEEX matrix element would serve as a starting point for a much better wide-angle Bhabha MC generator, similarly as BHLUMI v. 4.04 has served as a starting point for BHWIDE [71].

5. Technical precision

The question of the technical precision is quite nontrivial and difficult. The evaluation of the technical precision of BHLUMI v.4.04 with YFS soft-photon resummation and complete $\mathcal{O}(\alpha^1)$ relies on two pillars: the comparison with semi-analytic calculations done in ref. [47] and comparisons with two hybrid MC programs LUMLOG+OLDBIS and SABSPV, reported in ref. [30]. This precision was established to be 0.027% (together with missing photonic corrections). Note that this was not an ideal solution, because the above two hybrid MCs did not feature complete soft photon resummation and disagreed with BHLUMI by more than 0.17% for sharp cut-offs on the total photon energy.

In fact, after the LEP era, another MC program BabaYaga [23–25], with soft-photon resummation has been developed using a parton shower (PS) technique, and in principle could be used for better validation of the technical precision of both BHLUMI and BabaYaga. In fact, such a comparison with BHWIDE MC [71] was done for $s^{1/2} \leq 10\text{ GeV}$ and the 0.1% agreement was found. It is quite likely that such an agreement persists near $s^{1/2} = M_Z$.

Let us note in passing that the inclusion of the complete $\mathcal{O}(\alpha^1)$ into BabaYaga was done before three technologies of matching fixed-order NLO calculations with a parton shower (PS) algorithm were unambiguously established: MC@NLO [73], POWHEG [74] and KrkNLO [75]. The algorithm of NLO matching in BabaYaga is quite similar to that of KrkNLO.¹⁵

Ideally, in the future validation of the upgraded BHLUMI, in order to get its technical precision at the level 10^{-5} for the total cross section and 10^{-4} for single differential distributions, one would need to compare it with another MC program developed independently, which properly implements the soft-photon resummation, LO corrections up to $\mathcal{O}(\alpha^3 L_e^3)$, and the second-order corrections with the complete $\mathcal{O}(\alpha^2 L_e)$.

In principle, an extension of a program like BabaYaga to the level of NNLO for the hard process, keeping the correct soft-photon resummation, would be the best partner for the upgraded BHLUMI to establish the technical precision of both programs at the 10^{-5} precision level.¹⁶ In the meantime, the comparison between the upgraded BHLUMI with EEX and CEEX matrix elements would also offer a very good test of its technical precision, since the basic multi-photon phase space integration module of BHLUMI was already well tested in ref. [47] and such a test can be repeated at an even higher-precision level.

6. Summary

Summarizing, we conclude that an upgraded new version of BHLUMI with the error budget of 0.01% shown in Table 3 is perfectly feasible. With appropriate resources, such a version of BHLUMI with the $\mathcal{O}(\alpha^2)$ CEEX matrix element and with the precision tag of 0.01%, needed for the FCC-ee physics, could be realized. A new study of the Z and s-channel γ exchanges using BHWIDE MC was instrumental in the above analysis. Keeping in mind that the best experimental error of luminosity measurement achieved at LEP was 0.034% [17], it would be interesting to study whether the systematic error of the designed FCC-ee luminosity detector [29] can match the above anticipated theory precision.

¹⁵ A single MC weight is introducing NLO correction in both methods, but in KrkNLO it sums over real photons, while in BabaYaga it takes product over them. However, it is the same when truncated to $\mathcal{O}(\alpha^1)$. We are grateful to authors of BabaYaga for clarification on this point.

¹⁶ The upgrade of the BHLUMI distributions will be relatively straightforward because its multi-photon phase space is exact [76] for any number of photons.

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