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CONTROL FUNCTIONS AND TOTAL BOUNDEDNESS IN THE SPACE L_0

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Abstract. In this paper we introduce a parameter ω_∞ in the space of functions L_0. The parameter measures the lack of equimeasurability using a sequence of functions which controls the oscillations of every function of a given subset of L_0. We estimate the Hausdorff measure of noncompactness in terms of ω_∞ and the parameter σ (see [3]) and characterize the totally bounded subsets of L_0. A criterion of compactness given in [5] for subsets of the space $BC(\Omega, R)$ is extended to the case of the space $BTC(\Omega, M)$.

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1. Introduction

Let Ω denote a nonempty set and (M, d) a pseudometric space. The space L_0 is a space of functions from Ω to M and depends on a submeasure η defined on the power set of Ω. For a suitable choice of η, the space L_0 coincides with the space $\mathcal{B}$ of the closure of the space of all simple functions with respect to the topology of uniform convergence.

The next theorem, proved in [5], characterizes the relatively compact subsets of the Banach space $BC(\Omega, R)$ of real bounded and continuous functions defined on a topological space Ω, endowed with the sup norm.

Theorem 1.1. ([5, Proposition 5]) *A bounded subset A of $BC(\Omega, R)$ is relatively compact if and only if there exists a sequence of bounded functions $\{\psi_j\}$ such that*

$$|f(s) - f(t)| \leq \sum_{j=1}^{\infty} |\psi_j(s) - \psi_j(t)|$$

for $s, t \in \Omega$, $f \in A$ and the series is uniformly convergent.

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The introduction of some numerical parameters in spaces of measurable functions and their comparison with the Hausdorff measure of noncompactness has allowed many authors to generalize some classical compactness results (see for example [2], [3], [6],[8]). In connection to Theorem 1.1 in this paper we introduce a parameter ω_∞ which measures for a given subset of L_0 the lack of equimeasurability using a sequence of functions which controls the oscillations of every function of the set. We compare the parameter ω_∞ with the two parameters σ and ω introduced in [3]. We obtain inequalities (Corollary 3.6) which summarize the results. We derive some estimates of the Hausdorff measure of noncompactness in terms of ω_∞ and σ, which allow us to characterize the totally bounded subset of L_0. We observe that the results generalize to the case in which M is a uniform space.

In the case the submeasure η is σ-subadditive we obtain a better formulation of the inequality $\omega_\infty(A) \leq 2\sigma(A) + \omega(A)$. This result in the space $\mathcal{B}$ generalizes Theorem 1.1 to subsets of the space $BTC(\Omega, M)$ of all continuous functions, from a topological space Ω to M, for which $f(\Omega)$ is totally bounded.

If M is a normed space, $\sigma + \omega_\infty$ is a measure of noncompactness in the space $\mathcal{B}$ with respect to which the Sadovskii fixed point theorem can be formulated.

2. Preliminaries

Let (X, ρ) be a pseudometric space. For $x_0 \in X$ and $r > 0$, denote by $B(x_0, r) = \{x \in X : \rho(x_0, x) \leq r\}$ the closed ball with center x_0 and radius r. Let $Y \subseteq X$. The symbol $\mathrm{diam}Y = \sup\{\rho(x, y) : x, y \in Y\}$ stands for the diameter of Y. The *Hausdorff measure of noncompactness* $\gamma(Y)$ is the infimum of all $\epsilon > 0$ such that Y has a finite ϵ-net in X, i.e. there is a finite subset $\{y_1, \cdots, y_h\}$ of X such that $Y \subseteq \cup_{i=1}^{h} B(y_i, \epsilon)$.

We recall the definitions of the spaces L_0 and $\mathcal{B}$ (see [3]). Let Ω be a nonempty set, $M = (M, d)$ a pseudometric space and $\eta : \mathcal{P}(\Omega) \to [0, +\infty]$ a submeasure defined on the power set of Ω. The space $F = \{f : \Omega \to M\}$ is endowed with the pseudometric defined for all $f, g \in F$ by

$$\rho_\eta(f, g) = \inf\{a > 0 : \eta(\{x \in \Omega : d(f(x), g(x)) \geq a\}) \leq a\},$$

where we assume $\inf \emptyset = +\infty$. Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be an algebra. A function $s \in F$ is called $\mathcal{A}$-*simple* if there are a finite number of elements $m_1, \cdots, m_n \in M$ such that $s(\Omega) = \{m_1, ..., m_n\}$ and $s^{-1}(m_i) \in \mathcal{A}$, for $i = 1, \cdots, n$. Then $L_0 = L_0(\mathcal{A}, \Omega, M, \eta)$ is the closure of the set of all $\mathcal{A}$-simple functions in (F, ρ_η).

Moreover, we define $\rho_\infty : F \times F \to [0, +\infty]$ by setting

$$\rho_\infty(f, g) = \sup\{d(f(x), g(x)) : x \in \Omega\}$$

and denote by $\mathcal{B} = \mathcal{B}(\mathcal{A}, \Omega, M)$ the closure of the set of all $\mathcal{A}$-simple functions in (F, ρ_∞). As $\rho_\eta(f, g) \leq \rho_\infty(f, g)$, for all $f, g \in F$, we always have $\mathcal{B} \subseteq L_0$. We define $\eta_\infty(G) = 0$ if $G = \emptyset$ and $\eta_\infty(G) = +\infty$ if $\emptyset \neq G \in \mathcal{P}(\Omega)$, then $\rho_\infty = \rho_{\eta_\infty}$. Therefore, when $\eta = \eta_\infty$ the space $\mathcal{B}$ coincides with L_0.

3. Inequalities in the space L_0

Definition 3.1 Let $A \subseteq L_0$. For any $j = 1, 2, \cdots$, we define:

$$\omega_j(A) = \inf\{\ \epsilon > 0 :\ \exists\ \psi_1, \psi_2, \cdots, \psi_j \in L_0 \text{ such that, } \forall f \in A, \text{ there exists } D_f \subseteq \Omega \text{ with } \eta(D_f) \leq \epsilon \text{ and, } \forall s, t \in \Omega \setminus D_f,\ d(f(s), f(t)) \leq \epsilon + \sum_{i=1}^{j} d(\psi_i(s), \psi_i(t))\}$$

and

$$\omega_\infty(A) = \inf\{\ \epsilon > 0 :\ \exists \text{ a sequence } \{\psi_j\} \text{ in } L_0 \text{ such that, } \forall f \in A, \text{ there exists } D_f \subseteq \Omega \text{ with } \eta(D_f) \leq \epsilon \text{ and, } \forall s, t \in \Omega \setminus D_f,\ d(f(s), f(t)) \leq \epsilon + \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t))$$
$$\text{and the series is uniformly convergent in } \Omega \times \Omega, \text{ i.e. } \forall\ \delta > 0\ \exists\ j_0 \in N \text{ such that } \textstyle\sum_{j=j_0+1}^{\infty} d(\psi_j(s), \psi_j(t)) < \delta,\ \forall s, t \in \Omega.\}$$

The functions ψ_j are called *control functions.*

We observe that $\omega_\infty(A) \leq \omega_{j+1}(A) \leq \omega_j(A)$ for any $j = 1, 2, \cdots$. Indeed, given $\epsilon > 0$, find $\psi_1, \psi_2, \cdots, \psi_j \in L_0$ using the definition of $\omega_j(A)$. Then, for any arbitrarily chosen $x \in M$, it suffices to set $\psi_i(s) = x$ for $i = j+1, j+2, \cdots$.

Proposition 3.2. *Assume* $\operatorname{diam} M = +\infty$ *and let A be a subset of L_0. Then $\omega_1(A) = \omega_j(A) = \omega_\infty(A)$, for every $j = 2, 3, \cdots$.*

Proof. Let $j \geq 2$ be fixed. To prove the proposition it suffices to show that $\omega_1(A) \leq \omega_j(A)$ and $\omega_1(A) \leq \omega_\infty(A)$.

In order to prove the first inequality, choose $\delta > 0$ and find $\psi_1, \cdots \psi_j \in L_0$ and for $f \in A$ a subset D'_f of Ω with $\eta(D'_f) < \omega_j(A) + \frac{\delta}{2}$ such that

$$d(f(s), f(t)) \leq \omega_j(A) + \frac{\delta}{2} + \sum_{i=1}^{j} d(\psi_i(s), \psi_i(t)) \tag{1}$$

for $s, t \in \Omega \setminus D'_f$. For any $i = 1, \cdots, j$, choose an $\mathcal{A}$-simple function s_i such that $\rho_\eta(s_i, \psi_i) < \frac{\delta}{4j}$. Set $D_{\delta,i} = \{t \in \Omega : d(s_i(t), \psi_i(t)) \geq \frac{\delta}{4j}\}$ and let $D_\delta = \cup_{i=1}^{j} D_{\delta,i}$. Then $\eta(D_\delta) \leq \frac{\delta}{4}$ and for every $t \in \Omega \setminus D_\delta$ and $i = 1, \cdots, j$ we have

$$d(s_i(t), \psi_i(t)) \leq \frac{\delta}{4j}. \tag{2}$$

Let $A_1, \cdots, A_m$ be a partition of Ω in $\mathcal{A}$ such that the function s_i, for $i = 1, \cdots, j$, is constant on each A_h, for $h = 1, \cdots, m$. Fix a point $x_h \in A_h$ for every $h = 1, \cdots, m$, and set

$$c = \max_{1 \le h,k \le m} \max_{1 \le i \le j} d(s_i(x_h), s_i(x_k)).$$

Since $\operatorname{diam} M = +\infty$ we can choose $y_1, \cdots, y_m \in M$ such that

$$d(y_h, y_k) \ge jc,$$

for any $1 \le h, k \le m$ with $h \ne k$. We define an $\mathcal{A}$-simple function φ on Ω by setting

$$\varphi(t) = y_h, \qquad \text{for } t \in A_h \quad (h = 1, \cdots, m).$$

Then for $s, t \in \Omega$, with $s \in A_h$ and $t \in A_k$ for some $h \ne k$, we have

$$\sum_{i=1}^{j} d(s_i(s), s_i(t)) \le jc \le d(\varphi(s), \varphi(t)). \tag{3}$$

By (2) and (3) for $s, t \in \Omega \setminus D_\delta$ we have

$$\sum_{i=1}^{j} d(\psi_i(s), \psi_i(t))$$

$$\le \sum_{i=1}^{j} d(\psi_i(s), s_i(s)) + \sum_{i=1}^{j} d(s_i(s), s_i(t)) + \sum_{i=1}^{j} d(s_i(t), \psi_i(t))$$

$$\le \frac{\delta}{2} + d(\varphi(s), \varphi(t)). \tag{4}$$

Let $f \in A$. Set $D_f = D'_f \cup D_\delta$, then $\eta(D_f) < w_j(A) + \delta$ and by (1) and (4) for $s, t \in \Omega \setminus D_f$ we obtain

$$d(f(s), f(t)) \le w_j(A) + \delta + d(\varphi(s), \varphi(t)).$$

By the arbitrariness of δ, $\omega_1(A) \le \omega_j(A)$.

Now we show that $\omega_1(A) \le \omega_\infty(A)$. Choose $\delta > 0$ and find a sequence of control functions $\{\psi_j\}$ in L_0 and for $f \in A$ a subset D'_f of Ω with $\eta(D'_f) < \omega_\infty(A) + \frac{\delta}{3}$ such that

$$d(f(s), f(t)) \le \omega_\infty(A) + \frac{\delta}{3} + \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t)),$$

for $s, t \in \Omega \setminus D'_f$. Let $j_0 \in N$ such that

$$\sum_{j=j_0+1}^{\infty} d(\psi_j(s), \psi_j(t)) < \frac{\delta}{3}.$$

Using the previous argument find a set D_δ with $\eta(D_\delta) \le \frac{\delta}{6}$ and an $\mathcal{A}$-simple function φ on Ω such that

$$\sum_{j=1}^{j_0+1} d(\psi_j(s), \psi_j(t)) \le \frac{\delta}{3} + d(\varphi(s), \varphi(t))$$

for $s, t \in \Omega \setminus D_\delta$.

Set $D_f = D'_f \cup D_\delta$ then $\eta(D_f) < \omega_\infty(A) + \delta$ and

$$d(f(s), f(t)) \le \omega_\infty(A) + \delta + d(\varphi(s), \varphi(t)),$$

for all $f \in A$ and $s, t \in \Omega \setminus D_f$. By the arbitrariness of δ, $\omega_1(A) \le \omega_\infty(A)$ and the proof is complete. □

The following example shows that if diam$M < +\infty$, then Proposition 3.2 fails to hold.

Example 3.3 Let $M = [0,1]$ and $\Omega = [0, +\infty)$. Let $s_1 = \chi_{[0,1]}$ and $s_2 = \chi_{[0,2]}$ be the characteristic functions of the intervals $[0,1]$ and $[0,2]$, respectively.

Set $A = \{s_1, s_2\} \subseteq \mathcal{B}(\mathcal{P}(\Omega), [0, +\infty), [0,1])$, then $\omega_2(A) = 0$ but it is easy to verify $\omega_1(A) \ne 0$. □

We recall the definition of the parameters $\omega(A)$ and $\sigma(A)$ given in [3]. Let $A \subseteq L_0$, then

$$\begin{aligned}\omega(A) = \inf\{\ \epsilon > 0 :\ & \text{there exists a finite partition } \{A_1, \cdots, A_n\} \text{ of } \Omega \text{ in } \mathcal{A} \\ & \text{such that, } \forall f \in A, \text{ there exists } D_f \subseteq \Omega \text{ with } \eta(D_f) \le \epsilon \\ & \text{and } \operatorname{diam} f(A_i \setminus D_f) \le \epsilon, \text{ for } i = 1, 2, \cdots, n\},\end{aligned}$$

$$\begin{aligned}\sigma(A) = \inf\{\ \epsilon > 0 :\ & \exists\, M_0 \subseteq M \text{ with } \gamma(M_0) \le \epsilon \text{ such that, } \forall f \in A, \\ & \text{there exists } E_f \subseteq \Omega \text{ with } \eta(E_f) \le \epsilon \text{ and } f(\Omega \setminus E_f) \subseteq M_0\}.\end{aligned}$$

Theorem 3.4. *Let $A \subseteq L_0$. Then $\omega(A) \le \omega_\infty(A)$.*

Proof. Let $\delta > 0$ be given. By the definition of $\omega_\infty(A)$, choose a sequence of control functions $\{\psi_j\}$ in L_0 and, for each $f \in A$, a set $D'_f \subseteq \Omega$ with $\eta(D'_f) < \omega_\infty(A) + \frac{\delta}{3}$, such that for all $s, t \in \Omega \setminus D'_f$ we have

$$d(f(s), f(t)) \le \omega_\infty(A) + \frac{\delta}{3} + \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t)), \tag{5}$$

where the series is uniformly convergent in $\Omega \times \Omega$. Take $j_0 \in N$ such that for all $s, t \in \Omega$ we have

$$\sum_{j=j_0+1}^{\infty} d(\psi_j(s), \psi_j(t)) < \frac{\delta}{3}. \tag{6}$$

Observe that $\omega\{\psi_1, \psi_2, \cdots, \psi_{j_0}\} = 0$, hence there exists a partition $\{A_1, \cdots, A_n\}$ of Ω in $\mathcal{A}$ and for each $j = 1, 2, \cdots, j_0$ there exists $D_j \subseteq \Omega$ with $\eta(D_j) < \frac{\delta}{3j_0}$ such that for $s, t \in A_i \setminus D_j$ we have

$$d(\psi_j(s), \psi_j(t)) \le \frac{\delta}{3j_0}$$

for each $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, j_0$. Consequently, for all $s, t \in A_i \setminus \cup_{j=1}^{j_0} D_j$, we have

$$\sum_{j=1}^{j_0} d(\psi_j(s), \psi_j(t)) \le \frac{\delta}{3}. \tag{7}$$

Let $f \in A$ be fixed. Set $D_f = D'_f \cup (\cup_{j=1}^{j_0} D_j)$. Observe that $\eta(D_f) < \omega_\infty(A) + \delta$. By (5)-(7) we obtain,

$$\mathrm{diam} f(A_i \setminus D_f) \le \omega_\infty(A) + \delta,$$

for each $i = 1, 2, \cdots, n$. Consequently, $\omega(A) \le \omega_\infty(A) + \delta$, for any $\delta > 0$, which completes the proof. □

Theorem 3.5. *Let $A \subseteq L_0$. Then $\lim_{j\to\infty} \omega_j(A) \le 2\sigma(A) + \omega(A)$.*

Proof. Let $\delta > 0$ be given. To prove our result, it is enough to show that for any $\delta > 0$, there exists an integer $n_\delta \ge 1$ for which

$$\omega_{n_\delta}(A) \le 2\sigma(A) + \omega(A) + \delta. \tag{8}$$

To this end, choose a partition $\{A_1, A_2, \cdots, A_{n_\delta}\}$ of Ω in $\mathcal{A}$ and, for each $f \in A$, a set $D'_f \subseteq \Omega$ with $\eta(D'_f) < \omega(A) + \frac{\delta}{2}$ such that

$$\mathrm{diam} f(A_i \setminus D'_f) \le \omega(A) + \frac{\delta}{2}$$

for $i = 1, 2, \cdots, n_\delta$.

Then, applying the definition of $\sigma(A)$, we can select a set $M_0 \subseteq M$ such that $\gamma(M_0) < \sigma(A) + \frac{\delta}{2}$ and

$$f(\Omega \setminus E_f) \subseteq M_0.$$

for any $f \in A$, where $E_f \subseteq \Omega$ and $\eta(E_f) < \sigma(A) + \frac{\delta}{2}$.

Denote by $\xi_1, \cdots, \xi_h$ points of M for which $M_0 \subseteq \cup_{i=1}^h B(\xi_i, \sigma(A) + \frac{\delta}{2})$. Choose $x_0, y_0 \in \{\xi_1, \cdots, \xi_h\}$ such that

$$\max_{1 \le i,j \le h} d(\xi_i, \xi_j) = d(x_0, y_0).$$

Set

$$\psi_i(s) = \begin{cases} x_0 & \text{if } s \in A_i \\ y_0 & \text{if } s \notin A_i \end{cases}$$

for $i = 1, 2, \cdots, n_\delta$. Notice that $\sum_{i=1}^{n_\delta} d(\psi_i(s), \psi_i(t))$ is equal to $2d(x_0, y_0)$ if $s \in A_i, t \in A_j$ for some $1 \le i, j \le n_\delta$ with $i \neq j$, and is equal to zero if $s, t \in A_i$ for some $1 \le i \le n_\delta$.

Let $f \in A$ be fixed. Set $D_f = E_f \cup D'_f$. Then $\eta(D_f) < \sigma(A) + \omega(A) + \delta$. Take any $s, t \in \Omega \setminus D_f$. If $s \in A_i$ and $t \in A_j$ for some $1 \le i, j \le n_\delta$ with $i \neq j$, choose $1 \le i(s), i(t) \le h$ such that $f(s) \in B(\xi_{i(s)}, \sigma(A) + \frac{\delta}{2})$ and $f(t) \in B(\xi_{i(t)}, \sigma(A) + \frac{\delta}{2})$. Then

$$d(f(s), f(t)) \le d(f(s), \xi_{i(s)}) + d(f(t), \xi_{i(t)}) + d(\xi_{i(s)}, \xi_{i(t)})$$

$$\le 2\sigma(A) + \delta + d(x_0, y_0) \le 2\sigma(A) + \delta + \sum_{i=1}^{n_\delta} d(\psi_i(s), \psi_i(t)). \tag{9}$$

If $s, t \in A_i$ for some $1 \le i \le n_\delta$, we have

$$d(f(s), f(t)) \le \omega(A) + \frac{\delta}{2}. \tag{10}$$

By (9) and (10) we obtain (8), and the proof is complete. □

Combining Theorem 3.4 and Theorem 3.5 we obtain the following corollary.

Corollary 3.6. *Let $A \subseteq L_0$. Then*

$$\omega(A) \le \omega_\infty(A) \le 2\sigma(A) + \omega(A).$$

Moreover, if $\sigma(A) = 0$, then $\omega(A) = \omega_\infty(A)$.

By [3, Theorem 2.1], for any subset A of L_0, we have

$$\max\{\frac{1}{2}\omega(A),\ \sigma(A)\} \le \gamma(A) \le \sigma(A) + \omega(A).$$

Therefore, by Corollary 3.6, we obtain the following estimates for the Hausdorff measure of noncompactness in terms of the parameters σ and ω_∞.

Corollary 3.7. *Let $A \subseteq L_0$. Then*

$$\max\{\frac{1}{4}\omega_\infty(A),\ \sigma(A)\} \le \gamma(A) \le \sigma(A) + \omega_\infty(A).$$

In particular, A is totally bounded if and only if $\sigma(A) = \omega_\infty(A) = 0$.

In [3, Section 4] measures of noncompactness have been extended to subsets of a uniform space (cf. [7, Definition 1.2.1]).

Precisely, let $G = (G, \mathcal{U})$ be a uniform space and the uniformity $\mathcal{U}$ be generated by a family of pseudometrics $\{d_i,\ i \in I\}$. Let W be a nonempty set and suppose $\mathcal{P}(W)$ endowed with a Frèchet-Nicodym topology generated by a family of submeasures $\{\eta_j,\ j \in J\}$. Set, for any $f, g \in F = \{f : W \to G\}$,

$$\rho_{ij} = \inf\{a > 0 : \eta_j(\{x \in W : d_i(f(x), g(x)) \ge a\}) \le a\}.$$

Let $\mathcal{A} \subseteq \mathcal{P}(W)$ be an algebra. Denote by L_{ij} the closure of the set of all $\mathcal{A}$-simple functions in (F, ρ_{ij}). On the other hand if $\mathcal{U}_0$ is the uniformity generated by the family of pseudometrics $\{\rho_{ij},\ (i,j) \in I \times J\}$ we denote by $L_0(\mathcal{A}, W, G, \mathcal{U}_0)$ the closure of the set of all $\mathcal{A}$-simple functions in $(F, \mathcal{U}_0)$. Then $L_0(\mathcal{A}, W, G, \mathcal{U}_0) \subseteq L_{ij}$ for all $(i,j) \in I \times J$.

Definition 3.8 *For $A \subseteq L_0(\mathcal{A}, W, G, \mathcal{U}_0)$, we define $\gamma(A)$, $\sigma(A)$ and $\omega_\infty(A)$ as functions from $I \times J$ to $[0, \infty]$ by setting*

$$\gamma(A)(i,j) = \gamma^{ij}(A)$$
$$\sigma(A)(i,j) = \sigma^{ij}(A)$$
$$\omega_\infty(A)(i,j) = \omega_\infty^{ij}(A),$$

where $\omega_\infty^{ij}(A)$, $\sigma^{ij}(A)$ and $\gamma^{ij}(A)$ denote the values of the corresponding parameters in the space L_{ij}.

We consider the natural partial ordering in the set of all functions from $I \times J$ to $[0, \infty]$, i.e. $h_1 \le h_2$ if and only if $h_1(i,j) \le h_2(i,j),\ \forall (i,j) \in I \times J$. Since $L_0(\mathcal{A}, W, G, \mathcal{U}_0)$ is dense in (L_{ij}, ρ_{ij}) the Hausdorff measure of noncompactness of a subset $A \subseteq L_0(\mathcal{A}, W, G, \mathcal{U}_0)$ calculated in (L_{ij}, ρ_{ij}) coincides with $\gamma_{ij}(A)$ calculated in (L_0, ρ_{ij}). Therefore by Corollary 3.7 we get the following inequalities.

Corollary 3.9 *Let A be a subset of $L_0(\mathcal{A}, W, G, \mathcal{U}_0)$. Then we have*

$$\max\{\frac{1}{4}\omega_\infty(A),\ \sigma(A)\} \le \gamma(A) \le \sigma(A) + \omega_\infty(A).$$

From now till the end of this section, we will consider $L_0 = L_0(\mathcal{A}, \Omega, M, \eta)$ again, as at the beginning of the section.

The next theorem is one of our main results. It improves the inequality $\omega_\infty(A) \le 2\sigma(A) + \omega(A)$ (Theorem 3.5).

Theorem 3.10. *Let $A \subseteq L_0$.*
(i) Assume that $\inf\{d(x,y) : x,y \in M$ *with* $d(x,y) > 0\} = 0$, *and let η be σ-subadditive. If $\omega(A) = 0$, then for any $\delta > 0$ there exists a sequence $\{\psi_j\}$ in L_0 such that for all $f \in A$ there exists a set $D_f \subseteq \Omega$ with $\eta(D_f) < \sigma(A) + \delta$ for which*

$$d(f(s), f(t)) \leq 2\sigma(A) + \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t))$$

for all $s,t \in \Omega \setminus D_f$, and the series is uniformly convergent in $\Omega \times \Omega$.

(ii) Assume $\inf\{d(x,y) : x,y \in M$ *with* $d(x,y) > 0\} = \alpha > 0$.
If $\omega(A) < \alpha$, then for any $\delta > 0$ there exists a sequence $\{\psi_j\}$ in L_0 such that for all $f \in A$ there exists a set $D_f \subseteq \Omega$ with $\eta(D_f) < \sigma(A) + \omega(A) + \delta$ for which

$$d(f(s), f(t)) \leq 2\sigma(A) + \omega(A) + \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t))$$

for all $s,t \in \Omega \setminus D_f$, and the series is uniformly convergent in $\Omega \times \Omega$.

Proof. Choose $\overline{x}, \overline{y} \in M$ such that $d(\overline{x}, \overline{y}) > 0$. Fix $\delta > 0$ with $\delta < 2d(\overline{x}, \overline{y})$. Applying the definition of $\sigma(A)$ select a set $M_0 \subseteq M$ such that $\gamma(M_0) < \sigma(A) + \frac{\delta}{2}$, and for $f \in A$ let $E_f \subseteq \Omega$ be so chosen that $\eta(E_f) < \sigma(A) + \frac{\delta}{2}$ and

$$f(\Omega \setminus E_f) \subseteq M_0.$$

Let $\xi_1, \cdots, \xi_h$ be the points of M for which $M_0 \subseteq \cup_{i=1}^{h} B(\xi_i, \sigma(A) + \frac{\delta}{2})$. Choose x_0, $y_0 \in \{\xi_1, \cdots, \xi_n\}$ such that

$$\max_{1 \leq i,j \leq h} d(\xi_i, \xi_j) = d(x_0, y_0). \tag{11}$$

Now we prove (i). For any $k = 1, 2, \cdots$, we select the points x_k and y_k in M such that the sequence $d(x_k, y_k)$ is decreasing and

$$\sum_{k=1}^{\infty} d(x_k, y_k) \leq \frac{\delta}{4}.$$

Set $\delta_k = 2d(x_k, y_k)$, for $k = 1, 2, \cdots$. Since $\omega(A) = 0$, we can choose a partition $\{A^k_{n_{k-1}+1}, A^k_{n_{k-1}+2}, \cdots, A^k_{n_k}\}$ (where $n_0 = 0$) of Ω in $\mathcal{A}$ and, for each $f \in A$, a set $D^k_f \subseteq \Omega$ with $\eta(D^k_f) < \delta_k$ such that

$$\operatorname{diam} f(A^k_j \setminus D^k_f) \leq \delta_k$$

for $j = n_{k-1} + 1, n_{k-1} + 2, \cdots, n_k$.

Then we set

$$\psi_j(s) = \begin{cases} \overline{x} & \text{if } s \in A_j^1 \\ \overline{y} & \text{if } s \notin A_j^1 \end{cases}$$

for $j = 1, 2, \cdots, n_1$, and

$$\psi_{n_1+j}(s) = \begin{cases} x_{k-1} & \text{if } s \in A_j^k \\ y_{k-1} & \text{if } s \notin A_j^k \end{cases}$$

for $k = 1, 2, \cdots,$ and $j = n_{k-1} + 1, n_{k-1} + 2, \cdots, n_k$.

We notice that $\sum_{j=1}^{n_1} d(\psi_j(s), \psi_j(t))$ is equal to zero if $s, t \in A_j^1$ for some $1 \leq j \leq n_1$, and is equal to $2d(\overline{x}, \overline{y})$ if $s \in A_i^1, t \in A_j^1$ for some $1 \leq i, j \leq n_1$ with $i \neq j$. Analogously, $\sum_{j=n_{k-1}+1}^{n_k} d(\psi_{n_1+j}(s), \psi_{n_1+j}(t))$ is equal to zero if $s, t \in A_j^k$ for some $n_{k-1} + 1 \leq j \leq n_k$, and is equal to δ_{k-1} (where $\delta_0 = 2d(x_0, y_0)$) if $s \in A_i^k, t \in A_j^k$ for some $n_{k-1} + 1 \leq i, j \leq n_k$ with $i \neq j$.

Let $f \in A$ be fixed. Set $D_f = E_f \cup (\cup_{k=1}^{\infty} D_f^k)$. Then $\eta(D_f) < \sigma(A) + \delta$. Take any $s, t \in \Omega \setminus D_f$.

If $s \in A_i^1$ and $t \in A_j^1$ for some $1 \leq i, j, \leq n_1$ with $i \neq j$, choose $1 \leq i(s), i(t) \leq h$ such that $f(s) \in B(\xi_{i(s)}, \sigma(A) + \frac{\delta}{2})$ and $f(t) \in B(\xi_{i(t)}, \sigma(A) + \frac{\delta}{2})$. Then by (11) and our choice of δ we have

$$d(f(s), f(t)) \leq d(f(s), \xi_{i(s)}) + d(f(t), \xi_{i(t)}) + d(\xi_{i(s)}, \xi_{i(t)})$$

$$\leq 2\sigma(A) + \delta + d(x_0, y_0) \leq 2\sigma(A) + \sum_{j=1}^{2n_1} d(\psi_j(s), \psi_j(t)). \tag{12}$$

If for any $k = 1, 2, \cdots,$ we can find an index $n_{k-1} + 1 \leq j(k) \leq n_k$ such that $s, t \in A_{j(k)}^k$, then for all $f \in A$, and any k, we have

$$d(f(s), f(t)) \leq \delta_k.$$

Consequently,

$$d(f(s), f(t)) = 0. \tag{13}$$

Now suppose, there is an index $1 \leq i \leq n_1$ such that $s, t \in A_i^1$ and there is a $k \geq 2$ such that $s \in A_i^k$ and $t \in A_j^k$, for some $n_{k-1}+1 \leq i, j \leq n_k$ with $i \neq j$. Let $\overline{k}$ the first of those k's. Since $s, t \in A_j^{\overline{k}-1}$, for some index $n_{\overline{k}-2} + 1 \leq j \leq n_{\overline{k}-1}$, we have

$$d(f(s), f(t)) \leq \delta_{\overline{k}-1}.$$

Thus we have

$$(14) \qquad d(f(s), f(t)) \leq \sum_{j=n_{\overline{k}-1}+1}^{n_{\overline{k}}} d(\psi_{n_1+j}(s), \psi_{n_1+j}(t)).$$

By (12)-(14) we obtain, for any $f \in A$ and any $s, t \in \Omega \setminus D_f$, the desired inequality

$$d(f(s), f(t)) \leq 2\sigma(A) + \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t)).$$

The proof of (ii) goes in the same manner as in (i) by using a single partition of Ω in $\mathcal{A}$. We may assume that δ satisfies $\omega(A) + \frac{\delta}{2} < \alpha$.

Find a partition $\{A_1, A_2, \cdots, A_n\}$ of Ω in $\mathcal{A}$ and for each $f \in A$ a set $D'_f \subseteq \Omega$ with $\eta(D'_f) < \omega(A) + \frac{\delta}{2}$ such that $\text{diam} f(A_i \setminus D'_f) \leq \omega(A) + \frac{\delta}{2}$. Hence, by the hypothesis and our choice of δ, we have for $i = 1, 2, \cdots, n$

$$\text{diam} f(A_i \setminus D'_f) = 0.$$

Then we set

$$\psi_j(s) = \begin{cases} \overline{x} & \text{if } s \in A_j \\ \overline{y} & \text{if } s \notin A_j \end{cases}$$

for $j = 1, 2, \cdots, n_1$, and

$$\psi_j(s) = \begin{cases} x_0 & \text{if } s \in A_j \\ y_0 & \text{if } s \notin A_j \end{cases}$$

for $j = n_1 + 1, n_1 + 2, \cdots, 2n_1$.

Let $f \in A$ be fixed. Set $D_f = E_f \cup D'_f$. Then $\eta(D_f) < \sigma(A) + \omega(A) + \delta$. Take any $s, t \in \Omega \setminus D_f$.

If $s \in A_i$, $t \in A_j$ for $1 \leq i, j \leq n$ with $i \neq j$ by (12) we have

$$d(f(s), f(t)) \leq 2\sigma(A) + \sum_{j=1}^{2n_1} d(\psi_j(s), \psi_j(t)).$$

If $s, t \in A_i$ for some $1 \leq i \leq n$, we have already observed that $d(f(s), f(t)) = 0$, which completes the proof. □

The following corollary characterizes the subsets of L_0 for which $\omega_\infty(A) = 0$.

Corollary 3.11. *Let $A \subseteq L_0$ with $\sigma(A) = 0$.*
Assume either $\inf\{d(x, y) : x, y \in M$ *with* $d(x, y) > 0\} = 0$*, and η to be σ-subadditive or* $\inf\{d(x, y) : x, y \in M$ *with* $d(x, y) > 0\} = \alpha > 0$.

Then $\omega_\infty(A) = 0$ if and only if for any $\delta > 0$ there exists a sequence $\{\psi_j\}$ in L_0 such that for all $f \in A$ there exists a set $D_f \subseteq \Omega$ with $\eta(D_f) < \delta$ for which

$$d(f(s), f(t)) \leq \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t))$$

for all $s, t \in \Omega \setminus D_f$, and the series is uniformly convergent in $\Omega \times \Omega$.

Indeed, by the definition of $\omega_\infty(A)$ we readily see that the condition is sufficient. The necessity follows from Theorem 3.10. □

4. The space $\mathcal{B}$

In this section we consider the case when $\eta = \eta_\infty$. Then $L_0 = \mathcal{B}$, and for $A \subseteq \mathcal{B}$ the parameters $\omega_\infty(A)$, $\omega(A)$ and $\sigma(A)$ can be defined in a simpler manner. Namely,

$$\begin{aligned}\omega_\infty(A) = \inf\{\ \epsilon > 0 :\ \exists\ &\text{a sequence } \{\psi_j\} \text{ in } \mathcal{B} \text{ such that, } \forall f \in A \text{ and } \forall s, t \in \Omega \\ &d(f(s), f(t)) \leq \epsilon + \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t)) \\ &\text{and the series is uniformly convergent in } \Omega \times \Omega\}\end{aligned}$$

$$\begin{aligned}\omega(A) = \inf\{\ \epsilon > 0 : &\text{there exists a finite partition } \{A_1, A_2, \cdots, A_n\} \text{ of } \Omega \text{ in } \mathcal{A} \\ &\text{such that, } \forall f \in A,\ \operatorname{diam} f(A_i) \leq \epsilon,\ \text{ for } i = 1, 2, \cdots, n\}\end{aligned}$$

$$\begin{aligned}\sigma(A) = \inf\{\ \epsilon > 0 :\ &\exists\ M_0 \subseteq M \text{ with } \gamma(M_0) \leq \epsilon \text{ such that, } \forall f \in A, \\ &f(\Omega) \subseteq M_0\}.\end{aligned}$$

Theorem 4.1. *Let $A \subseteq \mathcal{B}$. Then there exists a sequence $\{\psi_j\}$ in $\mathcal{B}$ such that*

$$d(f(s), f(t)) \leq \max\{2\sigma(A), \omega(A)\} + \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t))$$

for all $s, t \in \Omega$, $f \in A$, and the series is uniformly convergent in $\Omega \times \Omega$.

Proof. The theorem follows in a straightforward manner from the proof of Theorem 3.10 when rewritten using the above formulation of the parameters. We have to precise that, using the same notation of Theorem 3.10, in the first part of the proof, for any $k = 1, 2, \cdots$, we have to choose a partition $\{A^k_{n_{k-1}+1}, A^k_{n_{k-1}+2}, \cdots, A^k_{n_k}\}$ of Ω in $\mathcal{A}$ such that

$$\operatorname{diam} f(A^k_j) \leq \omega(A) + \delta_k$$

for $j = n_{k-1} + 1, n_{k-1} + 2, \cdots, n_k$. □

Remark 4.2. If $\sigma(A) = 0$, by Theorem 4.1, corresponding to the value $\omega_\infty(A)$, there exists a sequence $\{\psi_j\}$ in $\mathcal{B}$ such that

$$d(f(s), f(t)) \leq \omega_\infty(A) + \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t)) \tag{15}$$

for all $s, t \in \Omega$, $f \in A$, and the series is uniformly convergent in $\Omega \times \Omega$. The sequence $\{\psi_j\}$ cannot be replaced by a finite number of functions. Indeed, [5, Example 4] provides the example of a subset A of $\mathcal{B}(\mathcal{P}(\Omega), \Omega, R)$ with $\omega_\infty(A) = 0$, for which it is not possible to replace the series in (15) by a finite sum. □

Then we obtain the following estimates of γ in $\mathcal{B}$.

Corollary 4.3. *Let $A \subseteq \mathcal{B}$, then*

$$\max\{\frac{1}{2}\omega_\infty(A),\ \sigma(A)\} \leq \gamma(A) \leq \sigma(A) + \omega_\infty(A).$$

Remark 4.4. Let $M = R^n$ and $A \subseteq \mathcal{B}$. In this case it has been observed in [3, p. 579] that $\gamma(A) \leq \sigma(A) + \frac{1}{2}\omega(A)$, therefore by Corollary 4.3 we have

$$\max\{\frac{1}{2}\omega_\infty(A),\ \sigma(A)\} \leq \gamma(A) \leq \sigma(A) + \frac{1}{2}\omega_\infty(A). \qquad \square$$

In the sequel of this section Ω is a topological space. Denote by $BTC(\Omega, M)$ the space of all continuous functions from Ω to M for which $f(\Omega)$ is totally bounded. Then

$$BTC(\Omega, M) \subseteq BT(\Omega, M) = \mathcal{B}(\mathcal{P}(\Omega), \Omega, M).$$

Let A be a subset of $BTC(\Omega, M)$. We want to consider the Hausdorff measures of noncompactness $\gamma_{BTC}(A)$ and $\gamma(A)$. It is easy to check that

$$\gamma(A) \leq \gamma_{BTC}(A) \leq 2\gamma(A). \tag{16}$$

Therefore we obtain

$$\max\{\frac{1}{2}\omega_\infty(A),\ \sigma(A)\} \leq \gamma_{BTC}(A) \leq 2\sigma(A) + 2\omega_\infty(A).$$

We point out that the estimates in (16) are the best possible even when Ω is a compact metric space and $M = R^n$.
Indeed, on the one hand $\gamma(A) = \frac{1}{2}\omega(A)$, on the other hand, by [4, Theorem 7.1.2], we have $\gamma_C(A) = \frac{1}{2}\omega_0(A)$, where $\omega_0(A)$ is the uniform parameter of equicontinuity. Therefore we have $\omega(A) \leq \omega_0(A) \leq 2\omega(A)$ (cf. [3, Prop. 5.1]) and [3, Examples 5.1 (a),(b)] show that the estimates are the best possible.

Corollary 4.5. *Let $A \subseteq BTC(\Omega, M)$ with $\sigma(A) = 0$. Then A is totally bounded if and only if there exists a sequence $\{\psi_j\}$ in $\mathcal{B}$ such that*

$$d(f(s), f(t)) \leq \sum_{j=1}^{\infty} d(\psi_j(s), \psi_j(t))$$

for all $s, t \in \Omega$, $f \in A$ and the series is uniformly convergent in $\Omega \times \Omega$.

Remark 4.6. If $M = R$, then $BC(\Omega, R) = BTC(\Omega, R)$. Therefore Corollary 4.5 yields Theorem 1.1. □

5. Sadovskii fixed point Theorem

Let M be a normed space, then $\mathcal{B}(\mathcal{A}, \Omega, M)$ is a normed space when endowed with the sup norm. It can be checked that in this setting σ and ω_∞ satisfy the following properties.

Proposition 5.1. *Let A, B be subsets of $\mathcal{B}(\mathcal{A}, \Omega, M)$. Then the following conditions hold:*

(i) $A \subseteq B$ implies $\sigma(A) \leq \sigma(B)$ and $\omega_\infty(A) \leq \omega_\infty(B)$;
(ii) $\sigma(\overline{A}) = \sigma(A)$ and $\omega_\infty(\overline{A}) = \omega_\infty(A)$;
(iii) $\sigma(coA) = \sigma(A)$ and $\omega_\infty(coA) = \omega_\infty(A)$;
(iv) $\sigma(A \cup \{f\}) = \sigma(A)$ and $\omega_\infty(A \cup \{f\}) = \omega_\infty(A)$ for every $f \in \mathcal{B}$.

Let $\varphi = \sigma + \omega_\infty$. Then φ is monotone and invariant under the passage to convex closure. Moreover, by (iv) φ is additively-nonsingular. Then the following fixed point theorem of Sadovskii holds (see [1, Theorem 1.5.11]).

Theorem 5.2. *Let K be a nonempty complete and convex subset of $\mathcal{B}$ for which $\varphi(K)$ is finite. Let $T : K \to K$ be a condensing mapping, i.e.*

$$\varphi(T(A)) < \varphi(A)$$

for any subset A of K which is not totally bounded. Then T has at least a fixed point in K.

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