Journal of Algebraic Combinatorics (2019) 50:39–47
https://doi.org/10.1007/s10801-018-0838-y

Containment problem and combinatorics

Ł. Farnik[1] · J. Kabat[2] · M. Lampa-Baczyńska[2] · H. Tutaj-Gasińska[3]

Received: 19 November 2017 / Accepted: 15 August 2018 / Published online: 15 September 2018

Abstract
In this note, we consider two configurations of twelve lines with nineteen triple points (i.e. points where three lines meet). Both of them have the same arrangemental combinatorial features, which means that in both configurations nine of twelve lines have five triple points and one double point, and the remaining three lines have four triple points and three double points. Taking the ideal of the triple points of these configurations we discover that, quite surprisingly, for one of the configurations the containment $I^{(3)} \subset I^2$ holds, while for the other it does not. Hence, for ideals of points defined by arrangements of lines, the (non)containment of a symbolic power in an ordinary power is not determined alone by arrangemental combinatorial features of the configuration. Moreover, for the configuration with the non-containment $I^{(3)} \not\subseteq I^2$, we examine its parameter space, which turns out to be a rational curve, and thus establish the existence of a rational non-containment configuration of points. Such rational examples are very rare.

Keywords Arrangements of lines · Containment problem · Configurations · Triple points · Combinatorial features · Symbolic power

Mathematics Subject Classification 52C30 · 14N20 · 05B30

✉ Ł. Farnik
Lucja.Farnik@gmail.com

J. Kabat
xxkabat@gmail.com

M. Lampa-Baczyńska
lampa.baczynska@wp.pl

H. Tutaj-Gasińska
halszka.tutaj-gasinska@im.uj.edu.pl

1 Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-656 Warszawa, Poland

2 Instytut Matematyki UP, Podchorążych 2, 30-084 Kraków, Poland

3 Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland

1 Introduction

The notion of the symbolic power of an ideal appears recently in many problems. Let $I \subset \mathbb{C}[\mathbb{P}^N] = \mathbb{C}[x_0, \ldots, x_N]$ be a homogeneous ideal. By m-th symbolic power of I, we mean $I^{(m)} = \mathbb{C}[\mathbb{P}^N] \cap \left(\bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} (I^m)_{\mathfrak{p}}\right)$. For a radical ideal I, the Nagata-Zariski theorem says that $I^{(m)}$ is the ideal of all $f \in I$ which vanish to order at least m along the zero-set of I. The main question concerning the symbolic powers may be stated as follows. For which r and m does the containment

$$I^{(m)} \subset I^r$$

hold? Or, more generally, when $M = (x_0, \ldots, x_N)$, for which r, m and j do we have

$$I^{(m)} \subset M^j I^r?$$

Ein, Lazarsfeld and Smith [11], and Hochster with Huneke [15] showed that, for any ideal $I \subset \mathbb{C}[\mathbb{P}^N]$, the containment $I^{(rN)} \subset I^r$ holds.

A few years ago Huneke asked if always $I^{(3)} \subset I^2$, afterwards Harbourne in [1] asked the following question. Let I be an ideal of points in $\mathbb{P}^N$. Does then the containment

$$I^{(rN-(N-1))} \subset I^r$$

hold for all r? Lots of examples suggested that the answer is positive. For an ideal of points in $\mathbb{P}^2$ in particular, the question was if $I^{(2r-1)} \subset I^r$ holds. In the paper [10], the first counterexample for the case $r = 2$, $N = 2$ was presented. Since then, quite a few counterexamples appeared, see, e.g. [7,14,16–19] or are announced [2]. The case $r > 2$ or $N > 2$ is still open.

The first real—and rational—counterexamples (i.e. counterexamples where the coordinates of all points are real numbers) come from [7,9] and [16]. They are modifications of Böröczky configuration of 12 lines. Böröczky configurations were introduced by Böröczky, they appeared in print probably for the first time in [6], and the construction of these configurations is described in the paper of Füredi and Palásti, see [13]. The non-existence of a rational counterexample among Böröczky configurations of 13, 14, 16, 18 and 24 lines is studied in [12]. Recently a new rational counterexample appeared, see [18].

In the paper of Bokowski and Pokora, [5], two non-isomorphic (and non-isomorphic to Böröczky configuration) examples of real configurations with 12 lines and 19 triple points are considered. They are named there C_2 and C_7.

In this paper, we consider the two configurations, C_2 and C_7. These configurations are realizable over the reals, and, what is interesting, they have the same arrangemental combinatorial features as Böröczky configuration of 12 lines. By "the same arrangemental combinatorial features", we mean that both configurations have the same number of lines, the same number of triple and double points, and that their distribution on lines is the same. So here the 12 lines intersect in 19 triple points, 9 lines have 5 triple points and one double point on them; and 3 lines have 4 triple points and 3 double points. However, the incidence matrices of these configurations are not equivalent, i.e. it is not possible to pass from one matrix to the other by permutations of rows or columns, so the configurations do not have the same *combinatorial data*, cf. [4].

In this paper, we describe the parameter spaces of configurations C_2 and C_7. It turns out, that one of them, C_2, is "rigid", this means that fixing some four out of 19 triple points (by a projective automorphism) to be (1:0:0), (0:1:0), (0:0:1), (1:1:1), the coordinates of other points can be computed, and these coordinates are non-rational. Moreover, for this configuration the containment $I_2^{(3)} \subset I_2^2$ holds, where I_2 is the radical ideal of the triple points of the configuration. The second configuration, namely C_7, turns out to have a one-dimensional projective space as a parameter space. Thus, we can take all the triple points of the configuration with rational coefficients. The radical ideal of these points, I_7, gives a new rational example of the non-containment $I_7^{(3)} \not\subseteq I_7^2$.

2 Configuration C_2

The real realization of the configuration C_2 is pictured in Fig. 1. Points P_1 and P_2 are "at infinity".

By a projective automorphism, we may move any four general points of $\mathbb{P}^2$ into other four general points. Thus, we may assume (with the notation as in the picture) that $P_1 = (1{:}0{:}0)$, $P_2 = (0{:}1{:}0)$, $P_3 = (0{:}0{:}1)$ and $P_4 = (1{:}1{:}1)$. We take the following lines:

$$\begin{aligned} L_{1,3} &: \ y = 0, \\ L_{2,4} &: \ x - z = 0, \\ L_{1,4} &: \ y - z = 0, \\ L_{3,4} &: \ x - y = 0, \\ L_{2,3} &: \ x = 0, \end{aligned}$$

where $L_{i,j}$, is the line passing through the points P_i and P_j. Then we obtain the points

$$\begin{aligned} P_5 &= L_{1,3} \cap L_{2,4} = (1, 0, 1), \\ P_6 &= L_{1,4} \cap L_{2,3} = (0, 1, 1) \end{aligned}$$

and the line

$$L_{5,6} : \ x + y - z = 0.$$

We need now to introduce a parameter to proceed with the construction. Thus, we take the point $P_7 = (0, 1, a) \in L_{2,3}$. Since all points and lines in the configuration should be distinct, we assume that $a \neq 1$ and $a \neq 0$. We obtain the remaining lines and points in the following order:

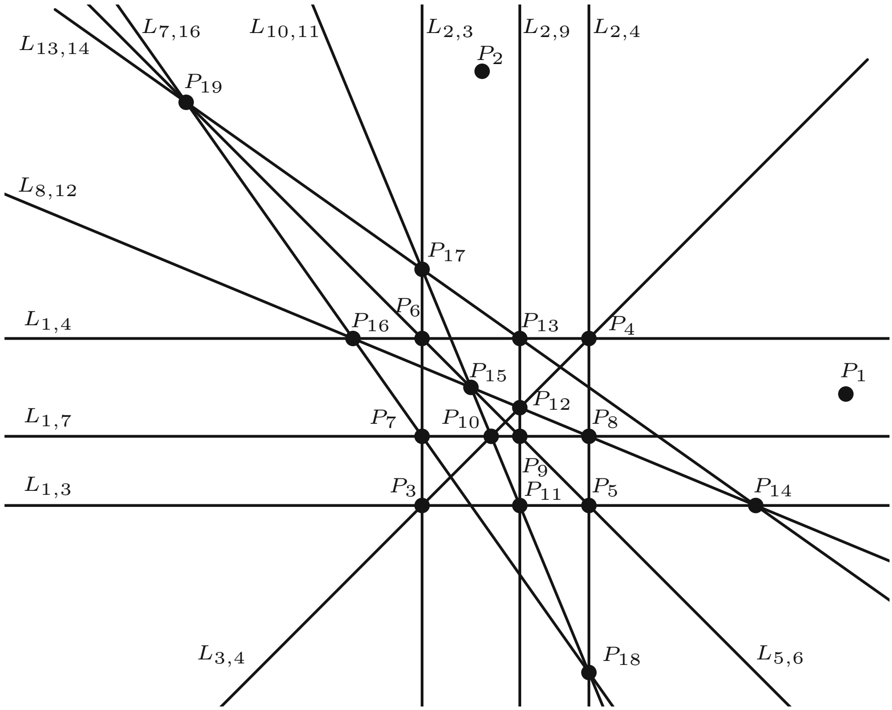

Fig. 1 The real realization of the configuration C_2

$$L_{1,7}: \ z - ay = 0,$$
$$P_8 = L_{1,7} \cap L_{2,4} = (a, 1, a),$$
$$P_9 = L_{1,7} \cap L_{5,6} = (a-1, 1, a),$$
$$P_{10} = L_{1,7} \cap L_{3,4} = (1, 1, a),$$
$$L_{2,9}: \ ax - (a-1)z = 0,$$
$$P_{11} = L_{2,9} \cap L_{1,3} = (a-1, 0, a),$$
$$P_{12} = L_{2,9} \cap L_{3,4} = (a-1, a-1, a),$$
$$P_{13} = L_{2,9} \cap L_{1,4} = (a-1, a, a),$$
$$L_{8,12}: \ a(2-a)x - ay + (a-1)^2 z = 0,$$
$$P_{14} = L_{8,12} \cap L_{1,3} = ((a-1)^2, 0, a(a-2)),$$
$$P_{15} = L_{8,12} \cap L_{5,6} = (a^2 - 3a + 1, -1, a(a-3)),$$
$$P_{16} = L_{8,12} \cap L_{1,4} = (a^2 - 3a + 1, a(a-2), a(a-2)),$$
$$L_{10,11}: \ ax + a(a-2)y - (a-1)z = 0,$$
$$P_{17} = L_{10,11} \cap L_{2,3} = (0, a-1, a(a-2)),$$
$$P_{18} = L_{10,11} \cap L_{2,4} = (a(2-a), 1, a(2-a)),$$
$$L_{13,14}: \ a(a-2)x + (a-1)y - (a-1)^2 z = 0,$$

$$L_{7,16}: \ a(a-1)(a-2)x - a(a^2-3a+1)y + (a^2-3a+1)z = 0,$$
$$P_{19} = L_{13,14} \cap L_{7,16} = (a^5-5a^4+7a^3-a^2-3a+1, a^3(a-2)^2, a^5 -4a^4+3a^3+3a^2-2a).$$

Almost all points in the configuration are triple directly from the construction. Only for four of them, i.e. P_{15}, P_{17}, P_{18} and P_{19}, we must verify this fact. We need to check the following incidences:

$$P_{15} = L_{8,12} \cap L_{5,6} \cap L_{10,11},$$
$$P_{17} = L_{10,11} \cap L_{2,3} \cap L_{13,14},$$
$$P_{18} = L_{10,11} \cap L_{2,4} \cap L_{7,16},$$
$$P_{19} = L_{13,14} \cap L_{7,16} \cap L_{5,6}.$$

By the determinant condition, we conclude that the lines $L_{8,12}$, $L_{5,6}$ and $L_{10,11}$ always meet at a point, but the remaining incidences occur under the algebraic condition

$$a^2 - 2a - 1 = 0.$$

Thus, the configuration has no rational realization.

Then, implementing, e.g. in Singular [8], the ideal I_2 of all the triple points, we check that $I_2^{(3)} \subset I_2^2$. This inclusion may be explained also more theoretically. From [3], we have that if $\alpha(I^{(m)}) \geq r \cdot \operatorname{reg} I$ (where $\alpha(J)$ denotes the least degree of a nonzero form in a homogeneous ideal J), then the containment $I^{(m)} \subset I^r$ holds. It may be computed (e.g. with Singular) that reg $I_2 = 6$ and $\alpha(I^{(3)}) = 12$. Thus, $I_2^{(3)} \subset I_2^2$.

There is an interesting phenomenon that for ideal I_2 the inclusion $I_2^{(3)} \subset I_2^2$ is true, while for other configurations of 12 lines, Böröczky and C_7, with the same arrangemental combinatorial features, the inclusion does not occur, see the next section for C_7 and [16] for Böröczky. Thus, the arrangemental combinatoric features of the configuration do not determine the containment.

3 Configuration C_7

The real realization of the configuration C_7 is shown in Fig. 2 (the points P_1, P_2, P_3 are "at infinity").

Here, using a projective automorphism, we may assume (with the notation as in the picture) that $P_1 = (1, 0, 0)$, $P_2 = (-1, 1, 0)$, $P_3 = (1, 1, 0)$ and $P_4 = (0, 0, 1)$. Then we have lines:

$$L_{2,4}: \ x + y = 0,$$
$$L_{3,4}: \ x - y = 0.$$

We need now to introduce the parameter to proceed with the construction, so take a point on the line $L_{3,4}$:

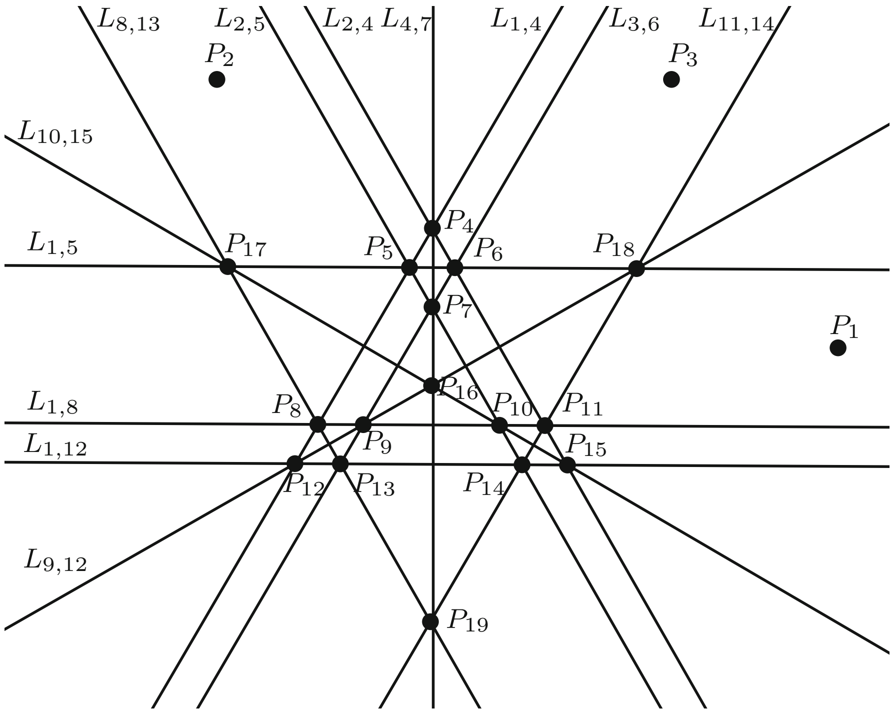

Fig. 2 The real realization of the configuration C_7

$$P_5 = (a, a, 1),$$

where $a \neq 0$. We get the lines

$$L_{1,5} : \; y - az = 0,$$
$$L_{2,5} : \; x + y - 2az = 0$$

and the point

$$P_6 = L_{2,4} \cap L_{1,5} = (-a, a, 1)$$

and then the line

$$L_{3,6} : \; x - y + 2az = 0.$$

To continue, we need to choose another point. We take a point on the line $L_{2,5}$.

$$P_7 = (b, -b + 2a, 1).$$

We get the line

$$L_{4,7} : \; 2ax - bx - by = 0.$$

The condition for the lines $L_{4,7}, L_{2,5}, L_{3,6}$ to meet at P_7 is

$$ba = 0.$$

As $a \neq 0$, we have to take $b = 0$. Thus, from now on:

$$P_7 = (0, 2a, 1)$$

and

$$L_{4,7} : \ 2ax = 0.$$

Again, we need a new parameter. Take a point on the line $L_{1,4}$

$$P_8 = (c, c, 1),$$

where $a \neq c, c \neq 0$. Then

$$\begin{aligned}
&L_{1,8} : \ y - cz = 0,\\
&P_9 = L_{1,8} \cap L_{3,6} = (-2a + c, c, 1),\\
&P_{10} = L_{1,8} \cap L_{2,5} = (2a - c, c, 1),\\
&P_{11} = L_{1,8} \cap L_{2,4} = (-c, c, 1).
\end{aligned}$$

Now, choose the last parameter by taking a point, again on the line $L_{3,4}$

$$P_{12} = (d, d, 1),$$

with d different from 0, a and c. Then

$$\begin{aligned}
&L_{1,12} : \ y - dz = 0,\\
&P_{13} = L_{1,12} \cap L_{3,6} = (-2a + d, d, 1),\\
&P_{14} = L_{1,12} \cap L_{2,5} = (2a - d, d, 1),\\
&P_{15} = L_{1,12} \cap L_{2,4} = (-d, d, 1),\\
&L_{10,15} : \ (c - d)x + (c - d - 2a)y + 2adz = 0,\\
&P_{16} = L_{4,7} \cap L_{9,12} = (0, 4a^2 d, -2a(-2a + c - d))\\
&P_{17} = L_{10,15} \cap L_{1,5} = (2a^2 - ac - ad, ac - ad, c - d),\\
&L_{9,12} : \ (c - d)x + (2a - c + d)y - 2adz = 0,\\
&P_{18} = L_{9,12} \cap L_{1,5} = (-2a^2 + ac + ad, ac - ad, c - d),\\
&L_{8,13} : \ (c - d)x - (2a + c - d)y + 2acz = 0,\\
&L_{11,14} : \ (c - d)x + (2a + c - d)y - 2acz = 0,
\end{aligned}$$

and finally

$$P_{19} = L_{8,13} \cap L_{11,14} = (0, 4ac^2 - 4acd, 4ac - 4ad + 2c^2 - 4cd + 2d^2).$$

Almost all points of the construction are triple without any additional conditions. Only P_2 and P_3 require an additional condition to be triple, namely:

$$4a(a + c - d) = 0.$$

As $a \neq 0$, we get $a+c-d = 0$. Thus, the parametrization space of this configuration is an affine plane and the configuration has a realization over $\mathbb{Q}$. It is not difficult to check (with help of, e.g. Singular) that the product of all twelve lines (which obviously is in $I_7^{(3)}$) does not belong to I_7^2. Thus, the triple points of this configuration give another rational example of the non-containment of the third symbolic power into the second ordinary power of an ideal.

For the convenience of the reader, we enclose the Singular script in Appendix.

Acknowledgements We would like to thank warmly Piotr Pokora for drawing our attention to the paper [5]. We thank Tomasz Szemberg for discussions and remarks. We also thank the knowledgeable referee for providing insightful comments, which helped us to improve our manuscript. The research of Lampa-Baczyńska was partially supported by National Science Centre, Poland, Grant 2016/23/N/ST1/01363, the research of Tutaj-Gasińska was partially supported by National Science Centre, Poland, Grant 2014/15/B/ST1/02197.

Appendix

To check that the product of all twelve lines of the configuration C_7 does not belong to I_7^2, and thus $I_7^{(3)} \nsubseteq I_7^2$, the following Singular script may be used.

```
LIB "elim.lib";
ring R=(0,a,d),(x,y,z),dp;
option(redSB);
proc rdideal(number p, number q, number r) {
   matrix m[2][3]=p,q,r,x,y,z;
   ideal I=minor(m,2);
   I=std(I);
   return(I);}
proc pline(list P1, list P2) {
   matrix A[3][3]=P1[1],P1[2],P1[3],P2[1],P2[2],P2[3],x,y,z;
   return(det(A));}
ideal P1=rdideal(1,0,0);
ideal P2=rdideal(-1,1,0);
ideal P3=rdideal(1,1,0);
ideal P4=rdideal(0,0,1);
ideal P5=rdideal(a,a,1);
ideal P6=rdideal(-a,a,1);
ideal P7=rdideal(0,2*a,1);
ideal P8=rdideal((d-a),(d-a),1);
ideal P9=rdideal(-2*a+(d-a),(d-a),1);
ideal P10=rdideal(2*a-(d-a),(d-a),1);
ideal P11=rdideal(-(d-a),(d-a),1);
ideal P12=rdideal(d,d,1);
ideal P13=rdideal(-2*a+d,d,1);
ideal P14=rdideal(2*a-d,d,1);
ideal P15=rdideal(-d,d,1);
ideal P16=rdideal(0,4*a*d,4*a-2*(d-a)+2*d);
ideal P17=rdideal(2*(a2)-a*(d-a)-a*d,a*(d-a)-a*d,(d-a)-d);
ideal P18=rdideal(-2*(a2)+a*(d-a)+a*d, a*(d-a)-a*d,(d-a)-d);
```

```
ideal P19=rdideal(0,4*a*((d-a)^2)-4*a*(d-a)*d,4*a*(d-a)-4*a*d+2*((d-a)^2)-4*
(d-a)*d+2*(d2));
poly pp=(2*(d*z-y))*((d-a)*z-y)*(a*z-y)*(2*a*d*z-2*a*y+(d-a)*x+(d-a)*y-d*x-d*y)*
(2*a*(d-a)*z-2*a*y+(d-a)*x-(d-a)*y-d*x+d*y)*(2*a*z-x-y)*(-x-y)*x*a*(2*a*z+x-y)*
(-2*a*(d-a)*z+2*a*y+(d-a)*x+(d-a)*y-d*x-d*y)*(-2*a*d*z+2*a*y+(d-a)*x-(d-a)
*y-d*x+d*y)*(-y+x);
ideal I=intersect(P1,P2,P3,P4,P5,P6,P7,P8,P9,P10,P11,P12,P13,P14,P15,P16,P17,
P18,P19);
I=std(I);
reduce(pp,std(I^2));
```

References

1. Bauer, Th., Di Rocco, S., Harbourne, B., Kapustka, M., Knutsen, A.L., Syzdek, W., Szemberg, T.: A primer on Seshadri constants. In: Interactions of Classical and Numerical Algebraic Geometry, Contemp. Math. 496, Amer. Math. 800, Providence, RI, 2009, pp. 33–70
2. Bauer, Th., Farnik, Ł., Harbourne, B., Tutaj-Gasińska, H.: A series of non-containment examples arising from nodal cubics **(in preparation)**
3. Bocci, C., Harbourne, B.: Comparing powers and symbolic powers of ideals. J. Algebr. Geom. **19**, 399–417 (2010)
4. Bogomolov, F., Kulikov, V.S.: On the diffeomorphic type of the complement to a line arrangement in a projective plane. Cent. Eur. J. Math. **10**, 521–529 (2012)
5. Bokowski, J., Pokora, P.: On the Sylvester-Gallai and the orchard problem for pseudoline arrangements. Period. Math. Hunger. (2017). https://doi.org/10.1007/s10998-017-0224-x
6. Crowe, D.W., McKee, T.A.: Sylvester's problem on collinear points. Math. Mag. **41**, 30–34 (1968)
7. Czapliński, A., Główka-Habura, A., Malara, G., Lampa-Baczyńska, M., Łuszcz-Świdecka, P., Pokora, P., Szpond, J.: A counterexample to the containment $I^{(3)} \subset I^2$ over the reals. Adv. Geom. **16**, 77–82 (2016)
8. Decker, W., Greuel, G.-M., Pfister, G., Schönemann, H.: SINGULAR 3-1-3—a computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2011). Accessed 18 Aug 2018
9. Dumnicki, M., Harbourne, B., Nagel, U., Seceleanu, A., Szemberg, T., Tutaj-Gasińska, H.: Resurgences for ideals of special point configurations in $\mathbb{P}^N$ coming from hyperplane arrangements. J. Algebra **443**, 383–394 (2015)
10. Dumnicki, M., Szemberg, T., Tutaj-Gasińska, H.: Counterexamples to the $I^{(3)} \subset I^2$ containment. J. Algebra **393**, 24–29 (2013)
11. Ein, L., Lazarsfeld, R., Smith, K.E.: Uniform bounds and symbolic powers on smooth varieties. Invent. Math. **144**, 241–252 (2001)
12. Farnik, Ł., Kabat, J., Lampa-Baczyńska, M., Tutaj-Gasińska, H.: On the parameter space of Böröczky configurations. arXiv:1706.09053
13. Füredi, Z., Palásti, I.: Arrangements of lines with a large number of triangles. Proc. Amer. Math. Soc. **92**, 561–566 (1984)
14. Harbourne, B., Seceleanu, A.: Containment counterexamples for ideals of various configurations of points in $\mathbb{P}^N$. J. Pure Appl. Algebra **219**(4), 1062–1072 (2015)
15. Hochster, M., Huneke, C.: Comparison of symbolic and ordinary powers of ideals. Invent. Math. **147**, 349–369 (2002)
16. Lampa-Baczyńska, M., Szpond, J.: From Pappus Theorem to parameter spaces of some extremal line point configurations and applications. Geom. Dedicata **188**, 103–121 (2017)
17. Malara, G., Szpond, J.: Fermat-type configurations of lines in $\mathbb{P}^3$ and the containment problem. J. Pure Appl. Algebra **222**(8), 2323–2329 (2018)
18. Malara, G., Szpond, J.: The containment problem and a rational simplicial arrangement. Electron. Res. Announc. Math. Sci. **24**, 123–128 (2017)
19. Szemberg, T., Szpond, J.: On the containment problem. Rend. Circ. Mat. Palermo (2) **66**, 233–245 (2017)