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On the operators D^3 and D^4 whose spectrum fills the entire complex plane

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It is well known, perhaps, only an example for differential operator of any even order for which the spectrum fills the entire complex plane [1] (see also [2] and [3]). In this example the boundary conditions have the following form

$$U_j(y) = y^{(j-1)}(0) + (-1)^{j-1}\, y^{(j-1)}(1) = 0, \qquad j = 1, 2, 3, 4.$$

All eigenvalue boundary problems for the operators D^3 and D^4 whose spectrum fills the entire complex plane are described. There are only finitely many such third-order differentiation operators. A characteristic determinant for D^3 is identically equal to zero if and only if the matrix of coefficients consists of two diagonal submatrices, on one of the diagonals of which there are units, and on the other are roots of minus one. But fourth-order differential operators, whose spectrum fills the whole complex plane, is infinitely many (continuum). For operator D^4 it is found 12 examples for which the spectrum fills the entire complex plane. All examples contains arbitrary constant.

Consider the following problems for the operators D^3 and D^4:

$$y'''(x) = \lambda\, y(x) = s^3\, y(x), \qquad x \in [0, 1] \tag{1}$$

$$U_j(y) = \sum_{k=0}^{3} a_{jk}\, y^{(k-1)}(0) + \sum_{k=0}^{3} a_{j\,k+3}\; y^{(k-1)}(1) = 0, \qquad j, k = 1, 2, 3 \tag{2}$$

$$y^{(4)}(x) = \lambda\, y(x) = s^4\, y(x), \qquad x \in [0, 1] \tag{3}$$

$$U_j(y) = \sum_{k=0}^{4} a_{jk}\, y^{(k-1)}(0) + \sum_{k=0}^{n} a_{j\,k+4}\; y^{(k-1)}(1) = 0, \qquad j, k = 1, 2, 3, 4 \tag{4}$$

It is known [4, P. 26] that if the coefficients of an ordinary linear differential equation are continuous on [0,1], then for the spectrum of the problem (3), (4) the following two possibilities occur: 1) there exist at most a countable number of eigenvalues such that do not have limit points in $\mathbb{C}$; 2) every $\lambda \in \mathbb{C}$ is an eigenvalue.

Direct and inverse problems with nonseparated boundary conditions for case 1) have been fairly well studied (see, for example, [5, 6, 7, 8, 9]). The degenerate case 2) has been studied little (The boundary conditions are called degenerate if the characteristic determinant of corresponding eigenvalue problem is constant [10, p. 29]). It is well known, perhaps, only an example for differential operator of any even order for which the spectrum fills the entire complex plane [1] (see also [2]). In this example the boundary conditions (4) have the followng form

$$U_j(y) = y^{(j-1)}(0) + (-1)^{j-1}\, y^{(j-1)}(1) = 0, \qquad j = 1, 2, 3, 4. \tag{5}$$

Recently in [11] it is shown that there exist similar differential operators of any odd order. However, in connection with this, another question arises: are there other examples of such operators? In the present paper, for the operators D^3 and D^4 we find other examples of such operators and describe all boundary value problems for the operators D^3 and D^4 whose spectrum fills the entire complex plane. The form of degenerate boundary conditions is found too.

The question of describing all boundary value problems with degenerate boundary conditions is related to a description of all Volterra problems. The problem for operator L is called Volterra problem if inverse operator L^{-1} is Volterra operator (see [12, p. 208]). In the case of nondegenerate boundary conditions for an arbitrary continuous function $q(x)$, the system of eigen-vectors of the operator L is complete in $L_2(0,\pi)$ (see [10, p. 29]). Therefore, Volterra problems are among problems with degenerate boundary conditions.

In [13] it is shown, that all Volterra problems for operator D^2 with comon boundary conditions have the form

$$y(0) \mp a\, y(\pi) = 0, \quad y'(0) \pm a\, y'(\pi) = 0, \tag{6}$$

where $a \neq 1$. A similar result is obtained in [14] for Sturm-Liouville problems with differential equation $-y'' + q(x)\, y = \lambda\, y$ and symmetric potential $(q(x) = q(\pi - x))$.

In [16] it is discribed all degenerate boundary conditions for D^2. In [17] a similar result is obtained for Sturm-Liouville problems (see also [18] and [15]).

In [16, p. 556] and [2] it is shown that there can not exist example for the operators D^2 and D^4 with finite (but not empty) spectrum. In [19] it is shown that the spectrum of common nth order linear differential operators generated by regular boundary conditions is either empty or infinite.

For operator D^3 we denote the matrix consisting of the coefficients a_{lk} in the boundary conditions (2) by A and the minor consisting of the i_1th, i_2th and i_3th columns of this matrix A by A_{i_1,i_2,i_3},

$$A = \left\| \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{matrix} \right\|. \tag{7}$$

$$A_{i_1,i_2,i_3} = \begin{vmatrix} a_{1,i_1} & a_{1,i_2} & a_{1,i_3} \\ a_{2,i_1} & a_{2,i_2} & a_{2,i_3} \\ a_{3,i_1} & a_{3,i_2} & a_{3,i_3} \end{vmatrix}. \tag{8}$$

In what follows, we assume that the rank of the matrix A is equal to 4,

$$\operatorname{rank} A = 3. \tag{9}$$

For operator D^4 we denote the matrix consisting of the coefficients a_{lk} in the boundary conditions (4) by A and the minor consisting of the i_1th, i_2th, i_3th and i_4th columns of this matrix A by A_{i_1,i_2,i_3,i_4},

$$A = \begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \end{Vmatrix}. \tag{10}$$

$$A_{i_1,i_2,i_3,i_4} = \begin{vmatrix} a_{1,i_1} & a_{1,i_2} & a_{1,i_3} & a_{1,i_4} \\ a_{2,i_1} & a_{2,i_2} & a_{2,i_3} & a_{2,i_4} \\ a_{3,i_1} & a_{3,i_2} & a_{3,i_3} & a_{3,i_4} \\ a_{4,i_1} & a_{4,i_2} & a_{4,i_3} & a_{4,i_4} \end{vmatrix}. \tag{11}$$

In what follows, we assume that the rank of the matrix A is equal to 4,

$$\operatorname{rank} A = 4. \tag{12}$$

The aim of this paper is to prove the following theorems:

Theorem 1.

For operator D^3 Matrix (7) for coefficients of degenerate boundary conditions (2) has the following form:

$$A_1 = \begin{Vmatrix} 1 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & a_3 \end{Vmatrix} \tag{13}$$

or

$$A_2 = \begin{Vmatrix} a_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 1 \end{Vmatrix}, \tag{14}$$

where a_i $(i = 1, 2, 3)$ are some numbers.

For operator D^4 Matrix (10) for coefficients of degenerate boundary conditions (4) has the following form:

$$A_1 = \begin{Vmatrix} 1 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_4 \end{Vmatrix} \tag{15}$$

or

$$A_2 = \begin{Vmatrix} a_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 1 \end{Vmatrix}, \tag{16}$$

where a_i $(i = 1, 2, 3, 4)$ are some numbers.

Theorem 2.

For operator D^3 the characteristic determinant of problem (1),(2) is identically equal to zero if and only if matrix (7) of coefficients of boundary conditions (4) has form (13) or (14), where $\{a_i\}$ $(i = 1, 2, 3)$ are roots of minus one.

For operator D^4 the characteristic determinant of problem (3),(4) is identically equal to zero if and only if matrix (10) of coefficients of boundary conditions (4) has form (15) or (16), where $\{a_i\}$ $(i = 1, 2, 3, 4)$ are one of the following 12 sets:

$$\begin{array}{l} 1.\quad a_1 = C_1,\quad a_2 = -1,\quad a_3 = C_1^{-1},\quad a_4 = 1,\\ 2.\quad a_1 = C_2,\quad a_2 = 1,\quad a_3 = C_2^{-1},\quad a_4 = -1,\\ 3.\quad a_1 = C_3,\quad a_2 = -1,\quad a_3 = 1,\quad a_4 = -1,\\ 4.\quad a_1 = C_4,\quad a_2 = 1,\quad a_3 = -1,\quad a_4 = 1,\\ 5.\quad a_1 = -1,\quad a_2 = C_5,\quad a_3 = -1,\quad a_4 = 1,\\ 6.\quad a_1 = -1,\quad a_2 = C_6,\quad a_3 = 1,\quad a_4 = C_6^{-1},\\ 7.\quad a_1 = 1,\quad a_2 = C_7,\quad a_3 = -1,\quad a_4 = C_7^{-1},\\ 8.\quad a_1 = 1,\quad a_2 = C_8,\quad a_3 = 1,\quad a_4 = -1,\\ 9.\quad a_1 = -1,\quad a_2 = 1,\quad a_3 = C_9,\quad a_4 = 1,\\ 10.\quad a_1 = 1,\quad a_2 = -1,\quad a_3 = -1,\quad a_4 = -1,\\ 11.\quad a_1 = -1,\quad a_2 = 1,\quad a_3 = -1\quad a_4 = C_{11},\\ 12.\quad a_1 = 1,\quad a_2 = -1,\quad a_3 = 1,\quad a_4 = C_{12}, \end{array} \tag{17}$$

where C_j $(j = 1, 2, \ldots, 12)$ are arbitrary constants.

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