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Woven Fusion frames in Hilbert spaces and some of their properties

[a]Rahimi A., [b]Samadzadeh Z.S

[a]*Department of Mathematics, University of Maragheh, Maragheh, P. O. Box 55136-553, Iran*
[b]*Department of Mathematics, University of Maragheh, Maragheh, P. O. Box 55136-553, Iran*
(rahimi@maragheh.ac.ir)

Abstract. A new notion in frame theory has been introduced recently that called woven frames. Woven and weaving frames are powerful tools for preprocessing signals and distributed data processing. Also, the purpose of introducing fusion frame or frame of subspace is to first construct local components and then build a global frame from these. This type of frames behaves as a generalization of frames. Motivating by the concepts of fusion and weaving frames, we investigate the notion woven fusion frames and study some of their features.

Keywords: Frame; Fusion frame; Weaving frame; Woven frame; Woven fusion frame.

1 Introduction

Frames are generalizations of orthonormal bases in Hilbert spaces. A frame, as well as an orthonormal basis, allows each element in Hilbert space to be written as an in finite linear combination of the frame elements so that unlike the bases conditions, the coefficients might not be unique. Also fusion frames is a generalization which were introduced by Cassaza and Kutyniok [3] in 2004 and were studied in variety of papers. The significance of fusion frame is the construction of global frames from local frames in Hilbert space. In recent years, Bemrose et al. introduced weaving frames [1]. This notion studied by researchers in a lot of papers [1, 2, 5]. From the point of view of its introducers, weaving frames are powerful tools for preprocessing signals and distributed data processing. By the concepts of fusion frames and weaving frames, we investigate the notion of woven fusion frames and study some behaviors in this paper, that is, we review some properties of fusion frames on weaving and conversely.

Throughout this paper $\mathbb{H}$ is a sparable Hilbert space. Also, $[m]$ shows the natural numbers $\{1, 2, 3, ..., m\}$.

Definition. A family $\{f_i\}_{i\in\mathbb{N}} \subset \mathbb{H}$ is a frame for $\mathbb{H}$, if there exist constants $\mathcal{A}, \mathcal{B} > 0$ such that:

$$\mathcal{A}\|f\|^2 \leq \sum_{i\in\mathbb{N}} |\langle f, f_i\rangle|^2 \leq \mathcal{B}\|f\|^2; \quad \forall f \in \mathbb{H}. \tag{1}$$

The constants $\mathcal{A}$ and $\mathcal{B}$ are called the lower and upper frame bounds respectively. The frame $\{f_i\}_{i\in\mathbb{N}}$ is tight if $\mathcal{A} = \mathcal{B}$ and Parseval frame if $\mathcal{A} = \mathcal{B} = 1$. The sequence $\{f_i\}_{i\in\mathbb{N}}$ is called Bessel sequence, if satisfies the upper inequality in (1). For the Bessel sequences $\{f_i\}_{i\in\mathbb{N}}$, the analysis operator $U : \mathbb{H} \longrightarrow l^2(\mathbb{N})$ is defined by $U(f) = \{\langle f, f_i\rangle\}_{i\in\mathbb{N}}$ for all $f \in \mathbb{H}$. The adjoint of analysis operator named synthesis operator and

$$T : l^2(\mathbb{N}) \longrightarrow \mathcal{H} \quad s.t \quad T\{c_i\} = \sum_{i\in\mathbb{N}} c_i f_i; \quad \forall \{c_i\} \in l^2(\mathbb{N}).$$

By combination of synthesis and analysis operators, we define the frame operator S for all $f \in \mathbb{H}$:

$$S : \mathbb{H} \longrightarrow \mathbb{H} \quad s.t \quad S(f) = TU(f) = \sum_{i\in\mathbb{N}} \langle f, f_i\rangle f_i, \quad \forall f \in \mathbb{H}.$$

The operator S is bounded, positive, self-adjoint and invertible [4].

Definition. Let $\{\nu_i\}_{i\in\mathbb{N}}$ be a family of weights such that $\nu_i > 0$ for all $i \in \mathbb{N}$. A family of closed subspaces $\{W_i\}_{i\in\mathbb{N}}$ of a Hilbert space $\mathbb{H}$ is called a fusion frame for $\mathbb{H}$ with respect to $\{\nu_i\}_{i\in\mathbb{N}}$, if there exist constants $\mathcal{A}, \mathcal{B} > 0$ such that:

$$\mathcal{A}\|f\|^2 \leq \sum_{i\in\mathbb{N}} \nu_i^2 \|P_{W_i}(f)\|^2 \leq \mathcal{B}\|f\|^2; \quad \forall f \in \mathbb{H}. \tag{2}$$

P_{W_i} is the orthogonal projection of a Hilbert space $\mathbb{H}$ onto subspace W_i. Constants $\mathcal{A}$ and $\mathcal{B}$ are called the lower and upper fusion frame bounds respectively. $\{W_i\}_{i\in\mathbb{N}}$ is called tight fusion frame with respect to $\{\nu_i\}_{i\in\mathbb{N}}$, if in (2) the constants $\mathcal{A}$ and $\mathcal{B}$ are equal and is called a Parseval fusion frame if $\mathcal{A} = \mathcal{B} = 1$. We say $\{W_i\}_{i\in\mathbb{N}}$ an orthonormal fusion basis for $\mathbb{H}$ if $\mathcal{H} = \bigoplus_{i\in\mathbb{N}} W_i$. If we have only the upper bound, we call $\{W_i\}_{i\in\mathbb{N}}$ a Bessel sequence of subspaces with respect to $\{\nu_i\}_{i\in\mathbb{N}}$ with Bessel bound $\mathcal{B}$.

2 Woven-Weaving frames

In this section, we mention to definition of woven and weaving frames in Hilbert spaces.

Definition. Let $F := \{f_{ij}\}_{i\in\mathbb{N}}$ for $j \in [m]$ be a family of frames for the separable Hilbert space $\mathbb{H}$. If there exist universal constants $\mathcal{A}$ and $\mathcal{B}$ such that for every partition $\{\sigma_j\}_{j\in[m]}$, the family $F_j := \{f_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for $\mathbb{H}$ with bounds $\mathcal{A}$ and $\mathcal{B}$. Then F is said a woven frame and for every $j \in [m]$, the frame F_j is called weaving frame.

Example. Let $\{e_i\}_{i=1}^2$ be standard basis for Euclidean space $\mathbb{R}^2$. Also F and G are given by:

$$F = \{f_1 = (1, 2), f_2 = (2, 1), f_3 = (2, 3)\}$$

and

$$G = \{g_1 = (1, 0), g_2 = (0, 1), g_3 = (3, 1)\}.$$

By a simple calculation, it is easy to see that both F and G are frames and

$$9\|f\|^2 \leq \sum_{i=1}^{3} |\langle f, f_i\rangle|^2 \leq 14\|f\|^2; \quad \forall f \in \mathbb{R}^2$$

and

$$2\|f\|^2 \leq \sum_{i=1}^{3} |\langle f, g_i\rangle|^2 \leq 10\|f\|^2; \quad \forall f \in \mathbb{R}^2.$$

By getting $\sigma = \{1, 2\}$, the family $\{f_i\}_{i\in\sigma=\{1,2\}} \bigcup \{g_i\}_{i\in\sigma^c=\{3\}}$ is frame with lower and upper bounds 6 and 14, respectively. Similarly for every $\sigma \subset \{1, 2, 3\}$, $\{f_i\}_{i\in\sigma} \bigcup \{g_i\}_{i\in\sigma^c}$ is frame. Then $\{f_i\}_{i=1}^3$ and $\{g_i\}_{i=1}^3$ are woven frames.

3 Woven-Weaving fusion frames

By using the ideas of fusion and woven frames, we define the notion of woven fusion frames. Also we review some of its properties. **Definition.** A family of fusion frames $\{W_{ij}\}_{i=1}^{\infty}$, $j \in [m]$ with respect to weights $\{\nu_{ij}\}_{i\in\mathbb{N},j\in[m]}$ is said to be woven fusion frames if there are universal constant $\mathcal{A}$ and $\mathcal{B}$ so that for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{N}$, the family $\{W_{ij}\}_{i\in\sigma_j,j\in[m]}$ is a fusion frame for $\mathbb{H}$ with lower and upper frame bounds $\mathcal{A}$ and $\mathcal{B}$. Each family $\{W_{ij}\}_{i\in\sigma_j,j\in[m]}$ is called a weaving of fusion frame.

[$\star$] From hereafter, we use briefly W.F.F instead of the statement of woven fusion frame.

Theorem. For each $i \in \mathbb{N}$, let $\nu_i, \mu_i > 0$ and $\{f_{ij}\}_{j\in\mathbb{J}_i}$ and $\{g_{ij}\}_{j\in\mathbb{J}_i}$ be frames in $\mathbb{H}$ with frame bounds $(\mathcal{A}_{f_i}, \mathcal{B}_{f_i})$ and $(\mathcal{A}_{g_i}, \mathcal{B}_{g_i})$ respectively. Define:

$$W_i = \overline{span}_{j\in\mathbb{J}_i} \{f_{ij}\} \qquad , \qquad V_i = \overline{span}_{j\in\mathbb{J}_i} \{g_{ij}\} \qquad \forall i \in \mathbb{N}$$

and choose an orthonormal basis $\{e_{ij}\}_{j\in\mathbb{J}_i}$ for each subspaces W_i and V_i. Suppose that:

$$0 < \mathcal{A}_f = \inf_{i\in\mathbb{N}} \mathcal{A}_{f_i} \le \mathcal{B}_f = \sup_{i\in\mathbb{N}} \mathcal{B}_{g_i} < \infty$$

and

$$0 < \mathcal{A}_g = \inf_{i\in\mathbb{N}} \mathcal{A}_{g_i} \le \mathcal{B}_g = \sup_{i\in\mathbb{N}} \mathcal{B}_{g_i} < \infty.$$

Then the following conditions are equivalent:

(i) $\{\nu_i f_{ij}\}_{i\in\mathbb{N},j\in\mathbb{J}_i}$ and $\{\mu_i g_{ij}\}_{i\in\mathbb{N},j\in\mathbb{J}_i}$ are woven frames.

(ii) $\{\nu_i e_{ij}\}_{i\in\mathbb{N},j\in\mathbb{J}_i}$ and $\{\mu_i e_{ij}\}_{i\in\mathbb{N},j\in\mathbb{J}_i}$ are woven frames.

(iii) $\{W_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ are W.F.F with respect to weights $\{\nu_i\}_{i\in\mathbb{N}}$, $\{\mu_i\}_{i\in\mathbb{N}}$ respectively.

Theorem Assume that $\{W_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be fusion frames with weights $\{\mu_i\}_{i\in\mathbb{N}}$ and $\{\nu_i\}_{i\in\mathbb{N}}$ respectively. If $\{W_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ are W.F.F and T is a self-adjoint and invertible operator on $\mathbb{H}$, such that $T^*T(W) \subset W$, for every closed subspace W of $\mathbb{H}$. Then for every $\sigma \subset \mathbb{N}$, the sequence $\{TW_i\}_{i\in\sigma} \bigcup \{TV_i\}_{i\in\sigma^c}$ is a fusion frame with frame operator $TS_\sigma T^{-1}$ where S_σ is frame operator of $\{TW_i\}_{i\in\sigma} \bigcup \{TV_i\}_{i\in\sigma^c}$, i.e. $\{TW_i\}_{i\in\mathbb{N}}$ and $\{TV_i\}_{i\in\mathbb{N}}$ are W.F.F.

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