# OPTIMIZATION OF DAMPING POSITIONS IN A MECHANICAL SYSTEM

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Abstract. This paper deals with damping optimization of the mechanical system based on the minimization of the so-called "average displacement amplitude". Further, we propose three different approaches to solving this minimization problems, and present their performance on two examples.

## 1. Introduction

The theory of linear damped oscillations has been studied for more than hundred years by researchers in control theory, optimization, and computational aspects, but numerous problems connected to linear damped oscillations still remain open. The main issue, this paper will be focused on, is a damping optimization problem of the mechanical system described by the following second order differential equation

$$(1.1) \qquad M\ddot{u}(t) + C(c)\dot{u}(t) + Ku(t) = \hat{f}(t),$$
$$u(0) = 0 \quad \text{and} \quad \dot{u}(0) = 0,$$

where $M$, $K \in \mathbb{R}^{n \times n}$ are positive definite symmetric matrices, vector $\hat{f}(t) \in \mathbb{R}^{n \times 1}$ presents the external force and $C(c) \in \mathbb{R}^{n \times n}$ presents a damping (positive semidefinite) symmetric matrix depending on viscosities $c$ (in general. $c = (c_1, \ldots, c_m)^\top \in \mathbb{R}_+^m$, where $m \leq n$). The structure of the damping matrix is given by

$$(1.2) \qquad C(c) = \sum_{i=1}^{m} c_i h_i h_i^\top.$$

Here, $h_1, \ldots, h_m \in \mathbb{R}^n$ are constant vectors, and each $h_i$ determines the geometry of the $i$th damping position, while $c_i$ is the corresponding viscosity.

Thus, the problem which will be considered in this paper can be stated as follows: "what are the best positions as well as corresponding damping viscosities such that the objective function of interest becomes minimal."

In this paper, we will focus on minimization of an objective function which will be closely related to:

$$(1.3) \qquad\qquad \frac{1}{\tau} \int_0^\tau \|u(t)\|^2 dt \,.$$

The above approach is usually called the "average displacement amplitude" minimization and has been widely used in different areas, like for example in aeronautics and astronautics [28] or [27] or in physics [11] and engineering [16] and applied mathematics in [36], [34] and [23]. Except this criterion, one can also consider the "average energy amplitude" minimization that in general gives different optimal values. Initially, it is hard to say which criterion is better since it strongly depends on applications but also (1.3) depends on the right hand side of (1.1). There exists a vast literature in this field of research. A nice overview of some results related to the considered problem can be found in [36]. But, for a brief insight, one can study some older references, i.e., [8, 18, 25], as well as some more recent ones, i.e., [4, 14, 26]. Critical damping and other modal damping were considered in [9, 22]. Optimization of positions, for the particular case of minimizing the total average energy, was considered in [5, 32]. This criterion of the average total energy is also an important criterion, (for more details, see, e.g., [5–7,35]), but it corresponds to a different problem setting including stationary systems. On the other hand, optimization of positions for a shear building model with discrete variables was considered in [19].

In general, the problem of determining optimal damping is very demanding, because numerous function evaluations need to be performed. In particular, one approach to determining optimal damping positions is the "direct" approach, which includes viscosity optimization for all possible damping configurations. This means that the function given by (1.3) needs to be optimized for all different configurations of damping positions. Then the optimal positions are those corresponding to the minimal objective function value.

Obviously, this approach is computationally infeasible for dimensions larger than $n \sim O(10)$, thus in this paper we will consider several different approaches that can efficiently optimize damping. For that purpose, we propose a novel approach which combines approaches from [34], [32] and [5] with the one from [19].

The first novelty in the newly proposed approaches will be handling a discrete set of possible viscosity, that is, we assume that $c_i$ can be chosen among predetermined candidates $\bar{c}, 2\bar{c}, \ldots, p\bar{c}$, where $\bar{c} > 0$ is a constant, and $p$ is a predetermined number of available candidates.

As the second novelty, we propose three different approaches to solving the optimization problem, which will be introduced in Section 2 by (2.8). The first heuristic approach will be called the "multigrid-like" approach. This approach is based on the idea that we first calculate optimal viscosities for some grid of damping positions, that is, we consider a grid sparsely distributed over the set of admissible damping positions, and, then, around the best positions we refine the grid and continue searching for optimal positions.

The second heuristic will be called the "discrete to continuous" optimization approach. This approach is "direct" in a sense that it uses some standard optimization method for continuous functions like the Nelder–Mead method (implemented in the MATLAB function `fminsearch`) or Newton-like methods (implemented in the MATLAB function `fmincon` or `fminunc`), or any other method (for example, one can use a genetic algorithm implemented in the MATLAB function `ga`), where positions will be treated as continuous variables, that is, "obtained position $i_j$" will be the damping position $[i_j]$ with the corresponding viscosity $c_j$. A similar approach has been considered in [32,33], but for a different penalty function (in particular, the goal was to minimize the total energy of the system). The third approach will be called the "mixed-integer programming formulation". This approach is based on the fact that the problem, which will be introduced in Section 2 by (2.8), can be recast as a mixed-integer second-order cone programming (MISOCP) problem [19]. This formulation was adopted to minimize transfer functions of interstory drifts evaluated at the undamped natural frequency of the building model [29–31]. MISOCP is a minimization problem of a linear objective function under the integrality constraints of some of variables and some second-order cone constraints. If there exists no integrality constraint, then the problem is called a second-order cone programming (SOCP) problem. SOCP is a convex optimization problem, and its solution can be computed with a primal-dual interior-point method [2, 38]. MISOCP can be solved globally with, e.g., a branch-and-bound method, because its continuous relaxation is SOCP; see, e.g., [3, 13, 37] for more details. Applications of MISOCP in engineering can be found in, e.g., [20, 21]. The mixed-integer programming formulation can be used to provide benchmark examples for evaluating the performance of the proposed heuristic methods.

The paper is organized as follows. In Section 2, we precisely define the problem and its setting, while in Section 3, we present two heuristics for solving our problem. The mixed-integer programming formulation approach is described in Section 4. The efficiency and accuracy of the proposed approaches are illustrated in Section 5.

## 2. Description of the problem

In this section, we will describe the problem of interest in more detail. For that purpose, recall that we consider the model described by the following second order differential equation given by (1.1). It should be emphasized in (1.1) that vector $\hat{f} \in \mathbb{R}^{n \times 1}$ denoting the external force has a special structure. Specifically, we assume that

$$\hat{f}(t) = \sum_{j=1}^{q} f_j^a \cos(\hat{\omega}_j t) + f_j^b \sin(\hat{\omega}_j t), \quad t \in [0, T],$$

with $\hat{\omega}_j = \frac{2\pi j}{T}$ $(j = 1, \ldots, p)$, where $T$ corresponds to the least common multiple of all periods. The main motivation for this assumption is related with a Fourier series which decomposes a function into the sum of a set of simple oscillating functions. Thus, we can write equation (1.1) as

$$(2.1) \qquad M\ddot{u}(t) + C(c)\dot{u}(t) + Ku(t) = \sum_{j=1}^{q} \left( f_j^a \cos(\hat{\omega}_j t) + f_j^b \sin(\hat{\omega}_j t) \right),$$

with

$$u(0) = 0 \quad \text{and} \quad \dot{u}(0) = 0.$$

Using the standard results from the theory of ODE, we can write the solution of (2.1) as

$$(2.2) \qquad u(t) = \sum_{j=1}^{q} u_j(t),$$

where $u_j$ satisfies

$$(2.3) \qquad M\ddot{u}_j(t) + C(c)\dot{u}_j(t) + Ku_j(t) = f_j^a \cos(\hat{\omega}_j t) + f_j^b \sin(\hat{\omega}_j t),$$
$$u_j(0) = 0 \quad \text{and} \quad \dot{u}_j(0) = 0.$$

As mentioned in Section 1, the so-called "average displacement amplitude" criterion minimizes

$$(2.4) \qquad \frac{1}{\tau} \int_0^\tau \|u(t)\|^2 dt.$$

Moreover, it can be shown that the average displacement amplitude criterion can be written in a more appropriate form, with $\tau$ equal to the above stated parameter $T$, that is, it can be shown that

$$\frac{1}{T} \int_0^T \|u(t)\|^2 dt$$

$$(2.5)$$

$$= \sum_{j=1}^{q} (f_j^0)^* \left( -\hat{\omega}_j^2 M + \mathrm{i}\hat{\omega}_j C(c) + K \right)^{-*} \left( -\hat{\omega}_j^2 M + \mathrm{i}\hat{\omega}_j C(c) + K \right)^{-1} f_j^0,$$

where

$$(2.6) \qquad f_j^0 = f_j^a - \mathrm{i} f_j^b \quad \text{for} \quad j = 1, \dots, q.$$

For more details, see, e.g., [23, 34, 36]. This problem was considered in [34], where the authors provided an explicit formula for the objective function for the structured system, in particular, all dampers have the same damping viscosities and internal damping is neglected. Moreover, a more general case, which includes the arbitrary time $\tau$ in criterion (2.4), was considered in [23].

As already emphasized in Section 1, the problem of determining optimal damping is very demanding, because numerous function evaluations have to be conducted. This approach is feasible only for small problems, that is, problems of dimension up to $n \sim O(10)$, but our intention is to increase the dimension of optimization problems, such that we can treat problems of dimension $n \sim O(100)$, for example $n = 200$, and the number of dampers $r \sim O(10)$, for example 4 dampers.

In order to perform that properly,what follows, we will first slightly reformulate our problem such that we can consider particular damping settings.

In the frequency domain, (2.1) reads

$$(2.7) \qquad (-\hat{\omega}_j^2 M + \mathrm{i} \hat{\omega}_j C(c) + K) v(\hat{\omega}_j) = f_j^a + \mathrm{i} f_j^b, \quad j = 1, \dots, q,$$

for more details see, e.g., [36]. For notational simplicity, we write $v_j = v(\hat{\omega}_j)$ $(j = 1, \dots, q)$.

Since we consider a particular external force in our system with parameter $\tau$ equal to the period of the considered system, we can write our problem setting in a particular form, that allows us to formulate our setting within a mixed-integer programming framework. This means that within considered setting we are able to reformulate the minimization problem of (2.4) such that we obtain the problem formulation given by (2.8).

Suppose that $c_i$ is chosen among predetermined candidates $\bar{c}, 2\bar{c}, \dots, p\bar{c}$, where $\bar{c} > 0$ is a constant, and $p$ is a predetermined number of available candidates. Then the optimization problem is formulated as

$$(2.8a) \ \text{Minimize} \quad \sum_{j=1}^{q} \|v_j\|^2$$

$$(2.8b) \ \text{subject to} \quad (-\hat{\omega}_j^2 M + \mathrm{i} \hat{\omega}_j C(c) + K) v_j = f_j^a + \mathrm{i} f_j^b, \quad j = 1, \dots, q,$$

$$(2.8c) \qquad\qquad\quad \sum_{i=1}^{m} c_i \leq c_{\text{sum}}^{\max},$$

$$(2.8d) \qquad\qquad\quad c_i \in \{0, \bar{c}, 2\bar{c}, \dots, p\bar{c}\}, \qquad\qquad i = 1, \dots, m.$$

Optimization problem (2.8) is very similar to the so-called drift optimization problems which are considered, inter alia, in [29–31].

## 3. Damping optimization by using approximate approaches

Since we are particularly interested in optimization of viscosities as well as damping positions, in this section we will introduce an auxiliary notation which allows us a formulation suitable for describing the optimization algorithms presented later.

Let external damping be given by

$$(3.1) \qquad C(c) = c_1 h_{i_1} h_{i_1}^\top + c_2 h_{i_2} h_{i_2}^\top + \cdots + c_m h_{i_m} h_{i_m}^\top,$$

where $i_j$ $(j = 1, \ldots, m)$ corresponds to the damping position with viscosity $c_j$. It follows directly from equation (3.1) that it is sufficient to find the optimal positions such that $1 \le i_1 < i_2 < \ldots < i_m \le n$. Since we are interested in the determination of optimal damping positions and viscosities, we will use a new notation for the objective function, which is now a function of the damping positions $(i_1, \ldots, i_m)$ and the corresponding viscosities $(c_1, \ldots, c_m)$. Thus, for given viscosities $(c_1, \ldots, c_m)$ and positions $(i_1, \ldots, i_m)$ the objective function will be denoted by

$$(3.2) \qquad f(c_1, \ldots, c_m; i_1, \ldots, i_m),$$

where we are particularly interested in the objective function given by (14a), with constraints (2.8b–2.8d).

Damping optimization can be also considered using a different objective function, in particular, similar approximation strategies were considered in [5,32], where the authors considered minimization of the total average energy (for more details, see, e.g., [5–7, 35]).

Recall that the main problem is, for given mass matrix $M$ and stiffness matrix $K$, to find optimal positions $(i_1^{opt}, \ldots, i_m^{opt})$ and corresponding viscosities $(c_1^{opt}, \ldots, c_m^{opt})$ such that $f(c_1, \ldots, c_m; i_1, \ldots, i_m)$ is minimal. Since comparison of all possible damping configurations is computationally infeasible, we will propose two approaches to computation of approximate optimal parameters to avoid combinatorial complexity. The first one is the so-called "multigrid-like" approach and it will be introduced in the next subsection.

3.1. *"Multigrid-like" optimization approach.* In this section, we will present the "multigrid-like" approach to determining optimal damping positions.

As we already mentiond, the basic idea of the "multigrid-like" is to first calculate optimal viscosities for some grid of damping positions, and then, around the best positions we refine the grid and continue the search for optimal positions.

Algorithm 4 describes the "multigrid-like" approach. There, parameter $d_1$ denotes the step size that determines how fine the first grid is (the first grid is defined in Steps 1 to 4 of Algorithm 4). Similarly, parameter $d_2$ denotes the step size that determines the second grid (the second grid is defined in Steps

12 to 15 of Algorithm 4). In Steps 12 to 15, we have min and max terms in order to ensure that $1 \leq i_1 < i_2 < \ldots < i_m \leq n$.

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**Algorithm 4** ("Multigrid-like" approach for determination of optimal damping)

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**Require:** $d_1$, $d_2$ — parameters that determine the first and the second grid;
**Ensure:** Optimal damper positions $i_1^{opt}, \ldots, i_m^{opt}$ and optimal viscosities $c_1^{opt}, \ldots, c_m^{opt}$.

  1: **for** $i_1 = d_2 : d_1 : n$ **do**
  2:    **for** $i_2 = i_1 + d_2 : d_1 : n$ **do**
  3:      $\cdots$
  4:      **for** $i_m = i_{m-1} + d_2 : d_1 : n$ **do**
  5:        calculate optimal viscosities
$$(c_1^{opt}, \ldots, c_m^{opt}) = \underset{(c_1, \ldots, c_m) \in \mathbb{R}_+^m}{\arg\min} f(c_1, \ldots, c_m; i_1, \ldots, i_m)$$
  6:        calculate $f(c_1^{opt}, \ldots, c_m^{opt}; i_1, \ldots, i_m)$
  7:      **end for**
  8:      $\cdots$
  9:    **end for**
10: **end for**
11: Denote positions that correspond to the minimal value calculated in Step 6 with $(\hat{i}_1, \ldots, \hat{i}_m)$.
12: **for** $i_1 = \max\{\hat{i}_1 - d_2, 1\} : \min\{\hat{i}_1 + d_2, n\}$ **do**
13:    **for** $i_2 = \max\{\hat{i}_2 - d_2, i_1 + 1\} : \min\{\hat{i}_2 + d_2, n\}$ **do**
14:      $\cdots$
15:      **for** $i_m = \max\{\hat{i}_{m-1} - d_2, i_{m-1} + 1\} : \min\{\hat{i}_{m-1} + d_2, n\}$ **do**
16:        calculate optimal viscosities
$$(c_1^{opt}, \ldots, c_m^{opt}) = \underset{(c_1, \ldots, c_m) \in \mathbb{R}_+^m}{\arg\min} f(c_1, \ldots, c_m; i_1, \ldots, i_m)$$
17:        calculate $f(c_1^{opt}, \ldots, c_m^{opt}; i_1, \ldots, i_m)$
18:      **end for**
19:      $\cdots$
20:    **end for**
21: **end for**
22: Optimal positions $i_1^{opt}, \ldots, i_m^{opt}$ are the positions that correspond to the minimal value calculated in Step 17 (corresponding viscosities are the optimal ones).

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We would like to emphasize that optimal parameters $d_1$ and $d_2$ should be determined so that the number of function evaluations is reduced. For this purpose, we additionally assume the following:

- At given damper positions, we have a constant number of function evaluations in order to determine an optimal viscosity (in our setting, this number is equal to $m$);
- We take that $d_2 = \lfloor p_d \cdot d_1 \rfloor$, where $p_d$ is a ratio that determines $d_2$ in terms of $d_1$, for example $p_d = \frac{2}{3}$;
- We take that $\hat{i}_1, \ldots, \hat{i}_m$ are not too close to the edge of the area where optimization is performed. This assumption is used in order to estimate the number of function evaluations needed for generating the second grid.

Now, for a given number of masses $n$, a number of dampers $m$ and $p_d$, we can determine $d_1$ such that a number of function evaluations becomes minimal. Moreover, if we want to additionally determine how many dampers $m$ we would like to include, we first start with smaller number (e.g., $k$ damping parameters) and increase parameter $k$ (up to the number $m$) until we observe a significant decrease in the objective function. One can vary parameter $k$ throughout all admissible configurations and repeat the optimization process for all of them, but this will require many optimization processes, while in numerical experiments, we have observed that starting with small $k$ and increasing it (up to the number $m$) until we observe a smaller value of the objective function, yields satisfactory results.

In the next subsection, we propose an approach to efficient damping optimization based on the idea that we consider variables that determine damping positions as a continuous variable.

3.2. *"Discrete to continuous" optimization approach.* In this section, we consider another approach to damping optimization, which relies on the optimization of continuous real functions. This approach can benefit from using of standard minimization procedures constructed for real functions, for example, Newton-like methods, or derivative-free methods like the Nelder–Mead method.

The approach we propose here is similar to the heuristic optimization approach to determination of optimal damping positions presented in [33]. In this paper, the authors propose to group the candidates for optimal damping positions into small groups, which enables the optimization of viscosities with respect to a much smaller number of damping positions.

Similarly, first we define an auxiliary objective function which will be used in optimization procedure. Our goal is to determine optimal damping with $k$ dampers with different viscosities. Thus, for $\mathcal{D} \subset \mathbb{R}^{2m}$, we define a function $\overline{f} : \mathcal{D} \to \mathbb{R}$ by

$$(3.3) \qquad \overline{f}(c_1, \ldots, c_m; i_1, \ldots, i_m) = f(c_1, \ldots, c_m; [i_1], \ldots, [i_m]),$$

where $[\cdot]$ stands for the rounding (we use the MATLAB® function `round`). Function $\overline{f}$ is similar to the one from (3.2) and it represents the objective

function given by (2.8). Here $i_m$ represents a continuous variable and the corresponding damping positions are $[i_1], [i_2], \ldots, [i_m]$, which together with viscosities $c_1, c_2, \ldots, c_m$ determine the damping matrix.

Now, we reduce our optimization problem to the minimization of function (3.3) with a continuous domain. For the minimization of these functions we can use standard methods like the Nelder–Mead method [24] or, for example, Newton-like methods (e.g., see [12]). Once we determine the minimum of function (3.3), we will denote the point where the minimum is achieved by $(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_m; \hat{i}_1, \hat{i}_2, \ldots, \hat{i}_m)$. Then the optimal positions are $[\hat{i}_1], [\hat{i}_2], \ldots, [\hat{i}_m]$ with the corresponding optimal viscosities equal to $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_m$.

In the optimization process, we will use the aforementioned Newton-like methods, which are much more robust than the other methods mentioned in relation to our minimization problem.

A further question in the minimization procedure is the choice of good starting points. For this purpose, we will define a grid of starting points for damping positions, which will correspond to the starting points generated in Steps 1 to 4 of Algorithm 5. Some fixed values $c_1^s, c_2^s, \ldots, c_m^s$ will be taken as starting viscosities. First, we have to choose parameters $d_3$ and $d_4$ which determine the grid of starting points $(i_1^s, \ldots, i_m^s)$. As can be seen from Algorithm 5, parameter $d_3$ determines the difference between points inside the region, while parameter $d_4$ defines the distance to the edge of the region where the optimal position is to be found.

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**Algorithm 5** ("Discrete to continuous" approach to determining of optimal positions)

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**Require:** $d_3$, $d_4$ — parameters which determine the first and the second grid;
**Ensure:** Optimal damper positions $[i_1^{opt}], \ldots, [i_m^{opt}]$ with optimal viscosities $c_1^{opt}, \ldots, c_m^{opt}$.

1: **for** $i_1^s = d_4 : d_3 : n - d_4$ **do**
2:   **for** $i_2^s = i_1^s + d_4 : d_3 : n - d_4$ **do**
3:     $\cdots$
4:     **for** $i_m^s = i_{m-1}^s + d_4 : d_3 : n - d_4$ **do**
5:       Using starting points $(c_1^s, \ldots, c_m^s; i_1^s, \ldots, i_m^s)$   calculate
$$\min_{\substack{(c_1, \ldots, c_m) \in \mathbb{R}_+^m \\ 1 \le i_1 < i_2 < \ldots < i_m \le n}} f(c_1, \ldots, c_m; i_1, \ldots, i_m).$$
6:     **end for**
7:     $\cdots$
8:   **end for**
9: **end for**
10: The parameters which correspond to the minimal value calculated in Step 5 are returned as optimal parameters $(c_1^{opt}, \ldots, c_m^{opt}; i_1^{opt}, \ldots, i_m^{opt})$.

Note that if one uses an unconstrained multidimensional optimization method, then one should take special care if the optimization process returns an optimal setting that corresponds to feasible parameters. In particular, in numerical experiments, the optimization procedure could require an evaluation at the points that are not in the domain (for example, viscosities may become negative). Thus, at points outside the domain where the optimization is performed, in our optimization procedure, one can set the function value to some constant large enough. With this, our optimization procedure will always return a minimum which is in the feasible domain. On the other hand, an optimization method like Matlab's `fmincon` can take into account constrains on variables directly.

We would like to emphasize that in this paper the first and the second approach will be used in order to optimize the criterion of the average displacement amplitude. Thus, we can make a comparison with the mixed-integer approach which will be described in the next section. On the other hand, "discrete to continuous" and "multigrid-like" approaches can also be applied to the above-mentioned criterion of the average energy amplitude.

## 4. Mixed-integer programming formulation

In this section, we show that problem (2.8) can be recast as a mixed-integer second-order cone programming (MISOCP) problem. The essential idea of this reformulation can be found in [19]. An MISOCP problem can be solved globally with, e.g., a branch-and-cut method. Several software packages, e.g., CPLEX [17], Gurobi Optimizer [15], and SCIP [1], are available for this purpose.

Constraint (2.8d) can be rewritten by introducing 0-1 variables as

$$(4.1) \qquad c_i = \bar{c} \sum_{l=1}^{r} 2^{l-1} t_{il},$$

$$(4.2) \qquad c_i \leq p\bar{c},$$

$$(4.3) \qquad t_{il} \in \{0, 1\}, \quad l = 1, \ldots, r,$$

where $r = \lfloor \log_2 p \rfloor + 1$ (see, e.g. [10]). Define $q_{ij}$ $(i = 1, \ldots, m; j = 1, \ldots, q)$ by

$$(4.4) \qquad q_{ij} = c_i h_i^\top v_j$$

to obtain

$$(4.5) \qquad C(c) v_j = \sum_{i=1}^{m} h_i q_{ij}.$$

We further rewrite $q_{ij}$ in (4.4) as

$$(4.6) \qquad q_{ij} = \bar{c} \sum_{l=1}^{r} 2^{l-1} w_{ijl}$$

with

$$(4.7) \qquad w_{ijl} = \begin{cases} h_i^\top v_j & \text{if } t_{il} = 1, \\ 0 & \text{if } t_{il} = 0. \end{cases}$$

Then (4.7) can be rewritten as

$$(4.8) \qquad |w_{ijl}| \le \mu t_{il},$$

$$(4.9) \qquad |w_{ijl} - h_i^\top v_j| \le \mu(1 - t_{il}),$$

where $\mu \gg 0$ is a sufficiently large constant.

Consequently, problem (2.8) is reduced to the following form:

$$(4.10\text{a}) \quad \text{Minimize} \quad \sum_{j=1}^{q} y_j$$

$$(4.10\text{b}) \quad \text{subject to} \quad y_j \ge \|v_j\|^2, \qquad\qquad\qquad\qquad \forall j,$$

$$(4.10\text{c}) \qquad (K - \hat{\omega}_j^2 M)v_j + \mathrm{i}\hat{\omega}_j \sum_{i=1}^{m} h_i q_{ij} = f_j^a + \mathrm{i} f_j^b, \quad \forall j,$$

$$(4.10\text{d}) \qquad q_{ij} = \bar{c} \sum_{l=1}^{r} 2^{l-1} w_{ijl}, \qquad\qquad\qquad \forall i; \ \forall j,$$

$$(4.10\text{e}) \qquad |w_{ijl}| \le \mu t_{il}, \qquad\qquad\qquad\qquad \forall i; \ \forall j; \ \forall l,$$

$$(4.10\text{f}) \qquad |w_{ijl} - h_i^\top v_j| \le \mu(1 - t_{il}), \qquad\qquad \forall i; \ \forall j; \ \forall l,$$

$$(4.10\text{g}) \qquad c_i = \bar{c} \sum_{l=1}^{r} 2^{l-1} t_{il}, \qquad\qquad\qquad\quad \forall i,$$

$$(4.10\text{h}) \qquad c_i \le p\bar{c}, \qquad\qquad\qquad\qquad\qquad\quad \forall i,$$

$$(4.10\text{i}) \qquad \sum_{i=1}^{m} c_i \le c_{\text{sum}}^{\max},$$

$$(4.10\text{j}) \qquad t_{il} \in \{0, 1\}, \qquad\qquad\qquad\qquad\quad\ \forall i; \ \forall l.$$

Problem (4.10) includes complex variables $v_j$, $q_j$, and $w_{ijl}$. In practice, when we solve problem (4.10), we convert these complex variables to real variables. The resulting optimization problem is MISOCP.

## 5. NUMERICAL ILLUSTRATION

In this section, we will illustrate the performance of all three approaches presented in Sections 3.1, 3.2 and 4. For this, we will construct two examples

which show differences in computational time, optimal damping positions and optimal damping values between Algorithms 4 and 5 and the one described in Section 4.

The first example is of small dimension and it illustrates behaviour of all three approaches, while the second example is of larger dimension in which we can see a significant difference in time needed for optimization, optimal damping positions and optimal damping values. In both examples, for the minimization process we used Matlab's `fmincon` with tolerance $10^{-8}$. The mixed-integer programming formulation was solved with CPLEX ver. 12.6.3 [17]. In numerical experiments, we adopt the indicator constraint formulation of CPLEX to avoid the large constant $\mu$ in problem (4.10).

EXAMPLE 5.1. In this example, we consider the system given by (1.1) with dimension $n = 6$, where the mass matrix is defined as:

$$M = \mathrm{diag}(80, 80, 80, 80, 80, 80).$$

The geometry of stiffness and damping matrices is defined by

$$(5.1) \qquad\qquad C(c) = \sum_{l=1}^{m} c_{i_l} h_{i_l} h_{i_l}^{\top},$$

$$(5.2) \qquad\qquad K = \sum_{i=1}^{n} s_i h_i h_i^{\top},$$

where vectors $h_i$, $h_{i_l}$ are given as

$$(5.3) \qquad\qquad h_1 = e_1, \quad h_j = e_j - e_{j-1}, \quad j = 2, \ldots, n,$$

where $e_i$ are cannonical vectors in $\mathbb{R}^n$ and $s_i$ is the $i$-th element of vector $s = [51.31, 40.1, 11.6, 20.76, 44.44, 51]^{\top}$.
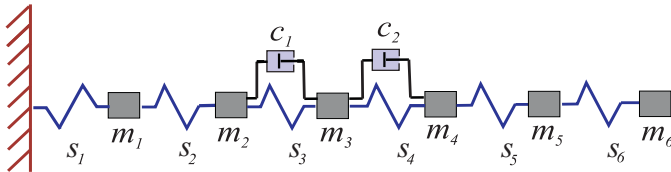


FIGURE 1. 6-mass oscillator

The geometry of stiffness is shown in Figure 1. We took $\bar{c} = 5$, $p = 15$, $c_{\mathrm{sum}}^{\max} = 90$ and the number of frequencies $q = 2$. All $f_j$ from equation (2.7) are equal to $-M \cdot [1, 1, 1, 1, 1, 1]^{\top}$. For grid parameters we took $d_1 = 2$, $d_2 = 1$, $d_3 = 2$ and $d_4 = 1$. In the "discrete to continuous" approach and the "multigrid-like" approach, we took 2 times the vector of ones and the

vector of ones, respectively, both in correct dimension, as the starting point for optimization.

In all three approaches we obtained that the optimal number of dampers is two.

TABLE 1. Results obtained for Example 5.1 by each approach

|  | "discrete to continuous" | "multigrid-like" | approach described in Section 4 |
|---|---|---|---|
| $(c_{i_1}, c_{i_2})$ | (75,15) | (75,15) | (75,15) |
| $(i_1, i_2)$ | (3,4) | (3,4) | (3,4) |
| $f_{value}$ | 91320.8436 | 91320.8436 | 91320.8436 |
| time | 0.09 $sec$ | 0.28 $sec$ | 6.75 $sec$ |

Table 1 summarizes all approaches, with $f_{value} = \sum_{j=1}^{n} \|v_j\|^2$. One can see that we have obtained the same results with all approaches and the only (important) difference is in the time needed for calculation. Figure 1 also shows the geometry of the damping matrix for the obtained results.

EXAMPLE 5.2. In this example, we consider the system from (1.1) with dimension $n = 200$, where the mass matrix is defined as:

$$M = 0.8 \cdot I \in \mathbb{R}^{200 \times 200}.$$

The geometry of stiffness and damping matrices is defined by

$$(5.4) \qquad\qquad C(c) = \sum_{l=1}^{m} c_{i_l} h_{i_l} h_{i_l}^{\top},$$

$$(5.5) \qquad\qquad K = \sum_{i=1}^{n} 400 \, h_i h_i^{\top},$$

where vectors $h_i$ are given in (5.3). We took $\bar{c} = 5$, $p = 7$, $c_{\text{sum}}^{\max} = 40$ and the number of frequencies $q = 2$. All $f_j$ from equation (2.7) are equal to $-M \cdot [1, \ldots, 1]^{\top}$. Grid parameters are $d_1 = 60$, $d_2 = 1$, $d_3 = 60$ and $d_4 = 1$. In the "discrete to continuous" approach and the "multigrid-like" approach, we took 2 times the vector of ones and the vector of ones, respectively, both in correct dimension, as the starting point for optimization.

By means of the "discrete to continuous" and "multigrid-like" approaches we obtained that the optimal number of dampers is two. In the mixed-integer programming approach described in Section 4, CPLEX terminated abnormally with error code 3019, which means that a routine in CPLEX failed to solve one of the subproblems in the branch-and-cut tree [17]. The solution reported in Table 2 is the best feasible solution found by CPLEX before termination, and hence it is not a global optimal solution. The number of dampers in this solution is three.

TABLE 2.  Results obtained for Example 5.2 by each approach

|  | "discrete to continuous" | "multigrid-like" | approach described in Section 4 |
|---|---|---|---|
| $(c_{i_1}, c_{i_2})$ | (25,15) | (25,15) | (5,10,25) |
| $(i_1, i_2)$ | (1,2) | (1,2) | (7,144,190) |
| $f_{value}$ | $5.653170639 \cdot 10^{12}$ | $5.653170639 \cdot 10^{12}$ | $1.8258 \cdot 10^{14}$ |
| time | 3.07 $sec$ | 2125.74 $sec$ | 34017.16 $sec$ |

In a manner similar to Example 5.1, Table 2 summarizes all approaches, where $f_{value} = \sum_{j=1}^{n} \|v_j\|^2$.

We can see that the "discrete to continuous" approach is the fastest approach; it is almost 700 times faster than the "multigrid-like" approach, although we obtained the same result. Also, we can see that the objective value obtained by the "discrete to continuous" and "multigrid-like" approaches is less than the best objective value obtained by the approach described in Section 4.

We would like to note that generally the objective function is non-convex, with more local minima. Since, in general, standard iterative minimization methods converge only to the local minimum, in general, one could add several starting points in order to improve robustness of the proposed approach. In this particular example, we have used only one starting vector since with additional numerical tests using various starting points a better solution was not found.

We would like to note that, in general, the third approach can guarantee the calculation of the global minimum (with appropriate terminal tolerances), but as we have shown, the first two approaches can very efficiently calculate the approximation of the minimum.

## 6. CONCLUSION

We have presented three different approaches for damping optimization in mechanical system, including the number and positions of dampers. For that purpose, we have considered the minimization of the "average displacement amplitude". Two of the above-mentioned approaches are approximate, in particular, "multigrid-like" and "discrete to continuous" approaches and the third approach is based on the mixed-integer programming formulation. We have illustrated that, although the mixed-integer programming formulation can guarantee calculation of optimal parameters (with appropriate termination tolerances), we can even more efficiently calculate approximation of optimal parameters by an approximate approach. The efficiency and performance of all three approaches are illustrated on numerical examples.

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# Optimizacija pozicija prigušivača u mehaničkom sustavu

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SAŽETAK.    Rad se bavi optimizacijom prigušenja u mehaničkim sustavima s obzirom na minimizaciju sume funkcija prijenosa otklona od početnog položaja. Pokazano je da je pod određenim dodatnim pretpostavkama minimizacija tzv. "amplitude prosječnog otklona" ekvivalentna minimizaciji sume funkcija prijenosa. Nadalje, predložena su tri pristupa za rješavanje minimizacijskog problema te je njihova učinkovitost prikazana na dvama primjerima.

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