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A CERTAIN SUBCLASS OF UNIVALENT MEROMORPHIC FUNCTIONS DEFINED BY A LINEAR OPERATOR ASSOCIATED WITH THE HURWITZ-LERCH ZETA FUNCTION

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Abstract. In this paper, we study a linear operator related to Hurwitz-Lerch zeta function and hypergeometric function in the punctured unit disk. A certain subclass of meromorphically univalent functions associated with the above operator defined by the concept of subordination is also introduced, and its characteristic properties are studied.

1. Introduction and definitions

Here, Σ denotes the class of normalized meromorphic functions

$$\vartheta(z)=\frac{1}{z}+\sum_{\kappa=1}^{\infty}\eta_{\kappa}z^{\kappa}, \tag{1.1}$$

and they are regular in a punctured unit disk

$$\mathbb{D}^{*}=\{z: z\in\mathbb{C} \quad \text{and} \quad 0<|z|<1\}=\mathbb{D}\setminus\{0\}.$$

The subclasses of Σ are denoted as $\Sigma_{\mathcal{S}^{*}}(\zeta)$ and $\Sigma_{\mathcal{K}}(\zeta)$ $(\zeta\geq 0)$ and they consist of all the meromorphic functions that are starlike of order ζ and convex of order ζ in $\mathbb{D}^{*}$, respectively (see the recent works [21] and [19]).

If $\vartheta_{\jmath}$ $(\jmath=1,2)$ are given by

$$\vartheta_{\jmath}(z)=\frac{1}{z}+\sum_{\kappa=1}^{\infty}\eta_{\kappa_{\jmath}}z^{\kappa}, \tag{1.2}$$

the Hadamard (convolution) product of ϑ_1 and ϑ_2 is defined as

$$(\vartheta_1*\vartheta_2)(z)=\frac{1}{z}+\sum_{\kappa=1}^{\infty}\eta_{\kappa,1}\eta_{\kappa,2}z^{\kappa}. \tag{1.3}$$

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Some recent papers, see for example [5], [8], [9], [10], [12] and [16], utilized the Hadamard product for introducing the linear operator $\mathcal{J}^{*}_{\varrho,\upsilon}:\Sigma\to\Sigma$, defined on Σ as follows:

$$\mathcal{J}^{*}_{\varrho,\upsilon}\,\vartheta(z)=\Upsilon(\varrho,\upsilon;z)*\vartheta(z) \tag{1.4}$$

where

$$\Upsilon(\varrho,\upsilon;z)=\frac{\upsilon^{\varrho}\,\mathcal{H}(z,\varrho,\upsilon)}{z}, \qquad \left(z\in\mathbb{D}^{*};\ \upsilon\in\mathbb{C}\backslash\mathbb{Z}_{0}^{-}\right) \tag{1.5}$$

and the function $\mathcal{H}(z,\varrho,\upsilon)$ is the well-known Hurwitz-Lerch zeta function defined by (see, for example [17, p. 121], [15] and [18, p. 194])

$$\mathcal{H}(z,\varrho,\upsilon):=\sum_{\kappa=0}^{\infty}\frac{z^{\kappa}}{(\kappa+\upsilon)^{\varrho}} \tag{1.6}$$

$$(\upsilon\in\mathbb{C}\setminus\mathbb{Z}_{0}^{-};\ \varrho\in\mathbb{C}\quad\text{when}\quad|z|<1;\ \Re(\varrho)>1\quad\text{when}\quad|z|=1),$$

where $*$ refers to the Hadamard product of the regular functions. Furthermore, the function $\mathcal{J}^{*}_{\varrho,\upsilon}\,\vartheta(z)$ is described as:

$$\mathcal{J}^{*}_{\varrho,\upsilon}\,\vartheta(z)=\frac{1}{z}+\sum_{\kappa=0}^{\infty}\left(\frac{\upsilon}{\kappa+\upsilon+1}\right)^{\varrho}\eta_{\kappa}\,z^{\kappa} \tag{1.7}$$

$$\left(z\in\mathbb{D}^{*};\ \vartheta\in\Sigma;\ \upsilon\in\mathbb{C}\setminus\mathbb{Z}_{0}^{-};\ \varrho\in\mathbb{C}\right).$$

Remark 1.1. Note that

(1) $\mathcal{J}^{*}_{0,\upsilon}\vartheta(z)=\vartheta(z)$,

(2) $\mathcal{J}^{*}_{1,\frac{1}{c}-2}\vartheta(z)=\frac{1-2c}{cz^{\frac{1}{c}-1}}\int\limits_{0}^{z}t^{\frac{1}{c}-2}\vartheta(t)\,dt \qquad \left(0<c<\frac{1}{2}\right)$,

(3) $\mathcal{J}^{*}_{1,\upsilon}\vartheta(z)=\frac{\upsilon}{z^{\upsilon+1}}\int\limits_{0}^{z}t^{\upsilon}\vartheta(t)\,dt$,

(4) $\mathcal{J}^{*}_{a,b}\vartheta(z)=\frac{b^{a}}{\Gamma(a)z^{b+1}}\int\limits_{0}^{z}t^{b}\left(\log\frac{z}{t}\right)^{a-1}\vartheta(t)\,dt \qquad (a,b>0)$,

(5) $\mathcal{J}^{*}_{\varrho,1}\vartheta(z)=\frac{1}{z}+\sum\limits_{\kappa=0}^{\infty}\left(\frac{1}{\kappa+2}\right)^{\varrho}\eta_{\kappa}z^{\kappa}$,

(6) $\mathcal{J}^{*}_{-1,1}\vartheta(z)=-z\vartheta'(z)$,

(7) $\mathcal{J}^{*}_{-1,-2}\vartheta(z)=\frac{\vartheta(z)-z\vartheta'(z)}{2}$,

(8) $\mathcal{J}^{*}_{-m,-1}\vartheta(z)=\frac{1}{z}+\sum\limits_{\kappa=0}^{\infty}(\kappa)^{m}\eta_{\kappa}z^{\kappa} \qquad (m\in\mathbb{N})$,

(9) $\mathcal{J}^{*}_{-m,1}\vartheta(z)=\frac{1}{z}+\sum\limits_{\kappa=0}^{\infty}(\kappa+2)^{m}\eta_{\kappa}z^{\kappa} \qquad (m\in\mathbb{N})$,

The linear operator $\mathcal{J}^{*}_{1,\frac{1}{c}-2}\vartheta(z)$ was introduced by Cho et al. [5], $\mathcal{J}^{*}_{a,b}\vartheta(z)$ operators were studied previously by Lashin [12]. Moreover, the operator $\mathcal{J}^{*}_{\varrho,1}\vartheta(z)$ was introduced by Alhindi and Darus [1], the operator $\mathcal{J}^{*}_{-m,1}\vartheta(z)$ was defined by Uralegaddi and Somanatha [23] and $\mathcal{J}^{*}_{1,\upsilon}\vartheta(z)$ was derived from (in specific cases) the generalized Bernardi operator [3], when $\Re(b) > 0$; the operator $\mathcal{J}^{*}_{1,\upsilon}\vartheta\,(z)$ was introduced by Bajpai [2].

Let us consider the incomplete beta function $\widetilde{\wp}(\mu,\nu;z)$ defined by

$$\widetilde{\wp}(\mu,\nu;z) = \frac{1}{z} + \sum_{\kappa=0}^{\infty} \frac{(\mu)_{\kappa+1}}{(\nu)_{\kappa+1}}\, z^{\kappa} \tag{1.8}$$

$$\left(\nu \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mu \in \mathbb{C}\right),$$

where

$$\mathbb{Z}_0^- = \{0,-1,-2,\cdots\} = \mathbb{Z}^- \cup \{0\}.$$

Henceforth, throughout $(\varepsilon)_\kappa$ stands for the Pochhammer symbol which can be defined *via* the Gamma function

(1.9)

$$(\varepsilon)_\kappa := \frac{\Gamma(\varepsilon+\kappa)}{\Gamma(\varepsilon)} = \begin{cases} \varepsilon(\varepsilon+1)\cdots(\varepsilon+m-1) & (\kappa = m \in \mathbb{N};\ \varepsilon \in \mathbb{C}) \\ 1 & (\kappa = 0;\ \varepsilon \in \mathbb{C}\setminus\{0\}), \end{cases}$$

Conventionally, it is assumed that $(0)_0 := 1$. For further details refer to [22, p. 21 *et seq.*].

In addition, the relation between the functions of $\widetilde{\wp}(\mu,\nu;z)$ and the Gaussian hypergeometric function holds [14]:

$$\widetilde{\wp}(\mu,\nu;z) = \frac{1}{z}\ {}_2F_1(1,\mu;\nu;z), \tag{1.10}$$

where

$${}_2F_1\,(\epsilon,\mu,\nu;z) = \sum_{\kappa=0}^{\infty} \frac{(\epsilon)_\kappa (\mu)_\kappa}{(\nu)_\kappa} \frac{z^\kappa}{\kappa!}$$

is the well-known Gaussian hypergeometric function.

Let

$$\mathcal{J}^{*}_{\varrho,\upsilon}\,\omega\,(z) * \Lambda_{\varrho,\upsilon}\,(z) = \frac{z^{-1}}{1-z}.$$

Then we have

$$\Lambda_{\varrho,\upsilon}\,(z) = \frac{1}{z} + \sum_{\kappa=0}^{\infty} \left(\frac{\kappa+\upsilon+1}{\upsilon}\right)^{\varrho} \eta_\kappa\, z^\kappa. \tag{1.11}$$

Using the operator $\Lambda_{\varrho,\upsilon}\,(z)$, we define a linear operator $\Omega^{*}_{\varrho,\upsilon}\,(\mu,\nu) : \Sigma \to \Sigma$ in terms of the Hadamard product by:

(1.12)

$$\Omega^{*}_{\varrho,\upsilon}\,(\mu,\nu)\,(\vartheta)\,(z) = \widetilde{\wp}(\mu,\nu;z) * \Lambda_{\varrho,\upsilon}\,(z) = \frac{1}{z} + \sum_{\kappa=0}^{\infty} \frac{(\mu)_{\kappa+1}}{(\nu)_{\kappa+1}} \left(\frac{\kappa+\upsilon+1}{\upsilon}\right)^{\varrho} \eta_\kappa\, z^\kappa$$

$$\left(z \in \mathbb{D}^*;\ \vartheta \in \Sigma;\ \upsilon,\,\nu \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \varrho \in \mathbb{C};\ \mu \in \mathbb{C}\backslash\{0\}\right).$$

It can be shown from the definition of the operator $\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)$, that

$$\begin{aligned} &z\big(\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)\big)' \\ &= \mu\left(\Omega^*_{\varrho,\upsilon}(\mu+1,\nu)(\vartheta)(z)\right) - (\mu+1)\left(\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)\right) \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} &z\big(\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)\big)' \\ &= \nu\left(\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)\right) - (\nu+1)\left(\Omega^*_{\varrho,\upsilon}(\mu,\nu+1)(\vartheta)(z)\right). \end{aligned} \tag{1.14}$$

Now, with the help of the linear operator $\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)$, we introduce the subclass $\Sigma^{*,\lambda}_{\varrho,\upsilon}(\mu,\nu,T,S)$ of meromorphic functions as follows:

DEFINITION 1.2. For fixed parameters T,S $(-1 \le S < T \le 1)$ and $0 \le \lambda < 1$, the function of $\vartheta \in \Sigma$ belongs to the class $\Sigma^{*,\lambda}_{\varrho,\upsilon}(\mu,\nu,T,S)$ if it satisfies the following subordination condition:

$$\frac{1}{1-\lambda}\left(-\frac{z\big(\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)\big)'}{\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)} - \lambda\right) \prec \frac{1+Tz}{1+Sz} \qquad (z \in \mathbb{D}^*) \tag{1.15}$$

or,

(1.16)

$$\begin{aligned} &\Sigma^{*,\lambda}_{\varrho,\upsilon}(\mu,\nu,T,S) \\ &= \left\{\vartheta : \vartheta \in \Sigma \quad \text{and} \quad \left| \frac{\frac{z\left(\vartheta^*_{\varrho,\upsilon}(\mu,\nu)(\omega)(z)\right)'}{\vartheta^*_{\varrho,\upsilon}(\mu,\nu)(\omega)(z)} + 1}{S\,\frac{z\left(\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)\right)'}{\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)} + S + (T-S)(1-\lambda)} \right| < 1 \right\}. \end{aligned}$$

2. A set of lemmata

For establishing the main results in this study we need the following results.

LEMMA 2.1. (see [13]) *Let $-1 \le S < T \le 1$, $\alpha \neq 0$ and the complex number $\beta \in \mathbb{C}$ satisfies the inequality*

$$\Re\{\beta\} \ge -\frac{\alpha(1-T)}{1-S}.$$

Then the differential subordination

$$\phi(z) + \frac{z\phi'(z)}{\alpha\phi(z)+\beta} \prec \frac{1+Tz}{1+Sz} \qquad (z \in \mathbb{D})$$

has a univalent solution in $\mathbb{D}$,

$$\phi(z) = \begin{cases} \dfrac{z^{\alpha+\beta}(1+Sz)^{\alpha(T-S)/S}}{\alpha\int_0^z t^{\alpha+\beta-1}(1+St)^{\alpha(T-S)/S}\,dt} - \dfrac{\beta}{\alpha} & (S \neq 0) \\[2ex] \dfrac{z^{\alpha+\beta}\exp(\alpha Tz)}{\alpha\int_0^z t^{\alpha+\beta-1}\exp(\alpha Tt)\,dt} - \dfrac{\beta}{\alpha} & (S = 0). \end{cases} \tag{2.1}$$

If the function ψ

$$\psi(z) = 1 + b_1 z + b_2 z + \cdots$$

is holomorphic in $\mathbb{D}$ *and satisfies the subordination*

$$\psi(z) + \frac{z\psi'(z)}{\alpha\psi(z) + \beta} \prec \frac{1+Tz}{1+Sz} \qquad (z \in \mathbb{D}), \tag{2.2}$$

then

$$\psi(z) \prec \phi(z) \prec \frac{1+Tz}{1+Sz} \qquad (z \in \mathbb{D})$$

and ϕ *is the best dominant in (2.2).*

Lemma 2.2. (see [24]) *Suppose that* γ *is the positive measure in* $[0,1]$ *and* p *is a complex-valued function, defined in* $\mathbb{D} \times [0,1]$ *so that* $p(.,t)$ *ia analytic in* $\mathbb{D}$ *for every* $t \in [0,1]$*, while* $p(z,.)$ *is* γ*-integrable in* $[0,1]$ *for all* $z \in \mathbb{D}$. *Furthermore, assume that* $\Re\{p(z,t)\} > 0$, $p(-r,t)$ *is real, and*

$$\Re\left\{\frac{1}{p(z,t)}\right\} \geq \frac{1}{p(-r,t)} \qquad (|z| \leq r < 1;\ t \in [0,1]).$$

If

$$\mathfrak{p}(z) = \int_0^1 p(z,t)\, d\gamma(t),$$

then

$$\Re\left\{\frac{1}{\mathfrak{p}(z)}\right\} \geq \frac{1}{\mathfrak{p}(-r)} \qquad (|z| \leq r < 1).$$

Lemma 2.3. (see [25]) *For real numbers* σ, ς *and* δ $(\delta \neq 0, -1, -2, \cdots)$, *it holds:*

$$\int_0^1 t^{\sigma-1}(1-t)^{\delta-\sigma-1}(1-zt)^{-\varsigma}\, dt = \frac{\Gamma(\sigma)\Gamma(\delta-\sigma)}{\Gamma(\delta)}\ {}_2F_1(\varsigma,\sigma;\delta;z) \tag{2.3}$$

$$\big(\Re\{\delta\} > \Re\{\sigma\} > 0;\ z \in \mathbb{D}\big).$$

Moreover,

$${}_2F_1(\varsigma,\sigma;\delta;z) = {}_2F_1(\sigma,\varsigma;\delta;z) \tag{2.4}$$

and

$${}_2F_1(\varsigma,\sigma;\delta;z) = (1-z)^{-\varsigma}\ {}_2F_1\left(\varsigma,\delta-\sigma;\delta;\frac{z}{z-1}\right) \tag{2.5}$$

$$\big(\delta \neq 0,-1,-2,\cdots;\ |\arg(1-z)| < \pi\big).$$

Several methods were used to study the inclusion properties of the different classes of the holomorphic and meromorphic functions (see, [4], [6], [7], [11] and [20]). Here the authors have determined four inclusion theorems for studying the class $\Sigma^{*,\lambda}_{\varrho,\upsilon}(\mu,\nu,T,S)$ of meromorphic functions. Particularly, the authors have stated that increasing the parameter $\mu + 1 \mapsto \mu$ by one, $\mu + 1$, the class $\Sigma^{*;\lambda}_{\varrho,\upsilon}(\mu,\nu,T,S)$ narrows, while increasing ν to $\nu+1$ expands the class $\Sigma^{*;\lambda}_{\varrho,\upsilon}(\mu,\nu+1,T,S)$ of meromorphic functions.

3. Results and discussion

Throughout this study, the authors have assumed (unless mentioned otherwise) that:

$$-1 \le S < T \le 1,\ 0 \le \lambda < 1,\ \mu,\nu > 0,\ \upsilon \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ \varrho \in \mathbb{C} \quad \text{and} \quad z \in \mathbb{D}.$$

Initially, the inclusion relationships have been considered for the parameter μ for the class $\Sigma^{*;\lambda}_{\varrho,\upsilon}(\mu,\nu,T,S)$.

Theorem 3.1. *If $\vartheta(z) \in \Sigma^{*;\lambda}_{\varrho,\upsilon}(\mu+1,\nu,T,S)$ and*

$$\mu-\lambda+1 \ge \frac{(1-\lambda)(1-T)}{(1-S)}, \tag{3.1}$$

then

$$\frac{1}{1-\lambda}\left(-\frac{z\left(\Omega^{*}_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)\right)'}{\Omega^{*}_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)}-\lambda\right) \prec \frac{1}{1-\lambda}\left((\mu-\lambda+1)-\frac{1}{\Phi_1(z)}\right)$$
$$= \phi_1(z) \prec \frac{1+Tz}{1+Sz} \qquad (z \in \mathbb{D}), \tag{3.2}$$

where

$$\Phi_1(z) = \begin{cases} \int_0^1 y^{\mu-1}\left(\frac{1+Szy}{1+Sz}\right)^{-(1-\lambda)(T-S)/B} dy & (S \neq 0) \\ \int_0^1 y^{\mu-1} e^{-(1-\lambda)T(y-1)z}\, dy & (S = 0) \end{cases}$$

and ϕ_1 is the best dominant of (3.2). Furthermore,

$$\Sigma^{*;\lambda}_{\varrho,\upsilon}(\mu+1,\nu,T,S) \subseteq \Sigma^{*;\lambda}_{\varrho,\upsilon}(\mu,\nu,T,S). \tag{3.3}$$

Proof. Assume $\vartheta(z) \in \Sigma^{*;\lambda}_{\varrho,\upsilon}(\mu+1,\nu,T,S)$ and set

$$\psi(z) = \frac{1}{1-\lambda}\left(-\frac{z\left(\Omega^{*}_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)\right)'}{\Omega^{*}_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)}-\lambda\right). \tag{3.4}$$

It can be seen that $\psi(z)$ is holomorphic in $\mathbb{D}$ and $\psi(0)=1$. Applying the identity (1.13) to (3.4) we conclude:

$$-(1-\lambda)\psi(z)+(\mu-\lambda+1) = \mu\,\frac{\Omega^{*}_{\varrho,\upsilon}(\mu+1,\nu)(\vartheta)(z)}{\Omega^{*}_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)}. \tag{3.5}$$

After using a logarithmic differentiation on both the sides in (3.5) with respect to z it follows

$$\begin{aligned}\psi(z) + \frac{z\,\psi'(z)}{(\mu-\lambda+1)-(1-\lambda)\,\psi(z)} &= \frac{1}{1-\lambda}\Big(-\frac{z\big(\Omega^{*}_{\varrho,\upsilon}(\mu+1,\nu)(\vartheta)(z)\big)'}{\Omega^{*}_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)}-\lambda\Big)\\ &\prec \frac{1+Tz}{1+Sz} \qquad (z\in\mathbb{D}).\end{aligned}$$

Hence, after applying the Lemma 2.1 with

$$\alpha = -(1-\lambda) \qquad \text{and} \qquad \beta = \mu-\lambda+1,$$

we have

$$\psi(z) \prec \phi_1(z) \prec \frac{1+Tz}{1+Sz} \qquad (z\in\mathbb{D}),$$

wherein the best dominant of ϕ_1 was defined using (3.2). This proves Theorem 3.1. ∎

Theorem 3.2. *Let $\vartheta(z) \in \Sigma^{*;\lambda}_{\varrho,\upsilon}(\mu+1,\nu,T,S)$. If the added constraints $0<S<1$ and*

$$\mu+1 \geq \frac{(1-\lambda)(T-S)}{S} \tag{3.6}$$

are satisfied, then

$$\frac{1-|T|}{1-|S|} < \frac{1}{1-\lambda}\left(-\Re\left\{\frac{z\big(\Omega^{*}_{\varrho,\upsilon}(\mu+1,\nu)(\vartheta)(z)\big)'}{\Omega^{*}_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)}\right\}-\lambda\right) < \rho_1, \tag{3.7}$$

where

$$\rho_1 = \frac{1}{1-\lambda}\left\{(\mu-\lambda+1) - \frac{\mu}{{}_2F_1\left(1,\frac{(1-\lambda)(T-S)}{S};\mu+1;\frac{S}{S-1}\right)}\right\}. \tag{3.8}$$

The bound ρ_1 is the best possible solution.

Proof. For establishing (3.7) in Theorem 3.2, we apply the subordination principle in (1.15) to get

$$\frac{1-|T|}{1-|S|} < \frac{1}{1-\lambda}\left(-\Re\left\{\frac{z\big(\Omega^{*}_{\varrho,\upsilon}(\mu+1,\nu)(\vartheta)(z)\big)'}{\Omega^{*}_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)}\right\}-\lambda\right),$$

which is similar to the LHS inequality in (3.7). Furthermore, after making use the subordination principle in (3.2) there follows

$$\begin{aligned}
&\frac{1}{1-\lambda}\left(-\Re\left\{\frac{z\left(\Omega^{*}_{\varrho,\upsilon}\left(\mu+1,\nu\right)\left(\vartheta\right)\left(z\right)\right)'}{\Omega^{*}_{\varrho,\upsilon}\left(\mu,\nu\right)\left(\vartheta\right)\left(z\right)}\right\}-\lambda\right)\\
&\leq \sup_{z\in\mathbb{D}^{*}}\Re\left\{\phi_{1}\left(z\right)\right\}\\
&= \sup_{z\in\mathbb{D}}\left[\frac{1}{1-\lambda}\left(\mu-\lambda+1-\Re\left\{\frac{1}{\Phi_{1}\left(z\right)}\right\}\right)\right]\\
&= \frac{1}{1-\lambda}\left(\mu-\lambda+1-\inf_{z\in\mathbb{D}}\Re\left\{\frac{1}{\Phi_{1}\left(z\right)}\right\}\right).
\end{aligned}\tag{3.9}$$

In the remaining part of the proof we determine

$$\inf_{z\in\mathbb{D}}\Re\left\{\frac{1}{\Phi_{1}\left(z\right)}\right\}.$$

Based on the hypothesis $S\neq 0$. Hence, using (3.2)

$$\Phi_{1}\left(z\right)=\left(1+Sz\right)^{\varsigma}\int_{0}^{1}u^{\mu-1}\left(1-u\right)^{\delta-\mu-1}\left(1+Szu\right)^{-\varsigma}du,$$

where

$$\varsigma=\frac{\left(1-\lambda\right)\left(T-S\right)}{S}\qquad\text{and}\qquad\delta=\mu+1.$$

Also, as $\delta>\mu>0$, the use of (2.3) to (2.5) of Lemma 2.3 infers

$$\Phi_{1}\left(z\right)=\frac{\Gamma\left(\mu\right)}{\Gamma\left(\delta\right)}\,{}_{2}F_{1}\left(1,\varsigma;\delta;\frac{Sz}{Sz+1}\right).\tag{3.10}$$

Moreover, the condition

$$\mu+1>\frac{\left(1-\lambda\right)\left(T-S\right)}{S}\qquad\left(0<S<1\right)$$

indicates that $\delta>\varsigma>0$. When (2.5) of Lemma 2.3 is applied to (3.10), which gives

$$\Phi_{1}\left(z\right)=\int_{0}^{1}p\left(z,u\right)d\gamma(u),$$

where

$$p\left(z,u\right)=\frac{1+Sz}{1+\left(1-u\right)Sz}\qquad\left(0\leq u\leq 1\right)$$

and

$$d\gamma(u)=\frac{\Gamma\left(\mu\right)}{\Gamma\left(\varsigma\right)\Gamma\left(\delta-\varsigma\right)}u^{\varsigma-1}\left(1-u\right)^{\delta-\varsigma-1}du,$$

which is positive for $u\in[0,1]$. It could be quoted that

$$\Re\left\{p\left(z,u\right)\right\}>0\qquad\text{and}\qquad p\left(-r,u\right)$$

are real for $0 \leq r < 1$ and $u \in [0,1]$. Hence, the application of Lemma 2.2 results in:

$$\Re\left\{\frac{1}{\Phi_1(z)}\right\} \geq \frac{1}{\Phi_1(-r)} \qquad (|z| \leq r < 1),$$

so that

$$\begin{aligned}
\inf_{z\in\mathbb{D}} \Re\left\{\frac{1}{\Phi_1(z)}\right\} &= \sup_{0\leq r<1} \frac{1}{\Phi_1(-r)} \\
&= \sup_{0\leq r<1} \frac{1}{\int_0^1 p(-r,u)\,d\gamma} = \frac{1}{\int_0^1 p(-1,u)\,d\gamma} = \frac{1}{\Phi_1(-1)} \\
&= \frac{\mu}{{}_2F_1\left(1, \frac{(1-\lambda)(T-S)}{S}, \mu+1, \frac{S}{S-1}\right)}. \qquad (3.11)
\end{aligned}$$

Therefore, based on (3.9), the RHS inequality in (3.7) follows from (3.11).

This is the best possible result since the function $\phi_1(z)$ is the best dominant of (3.2). This proves Theorem 3.2. ∎

The following theorem describes the results concerning the parameter ν.

THEOREM 3.3. *If $\vartheta(z) \in \Sigma^{*,\lambda}_{\varrho,\upsilon}(\mu,\nu,T,S)$ and*

$$\nu - \lambda + 1 \geq \frac{(1-\lambda)(1-T)}{(1-S)}, \qquad (3.12)$$

then

$$\begin{aligned}
&\frac{1}{1-\lambda}\left(-\Re\left\{\frac{z\left(\Omega^*_{\varrho,\upsilon}(\mu,\nu+1)(\vartheta)(z)\right)'}{\Omega^*_{\varrho,\upsilon}(\mu,\nu+1)(\vartheta)(z)}\right\} - \lambda\right) \\
&\prec \frac{1}{1-\lambda}\left((\nu-\lambda+1) - \frac{1}{\Phi_2(z)}\right) \qquad (3.13) \\
&= \phi_2(z) \prec \frac{1+Tz}{1+Sz} \qquad (z \in \mathbb{D}),
\end{aligned}$$

where

$$\Phi_2(z) = \begin{cases} \int_0^1 u^{\nu-1}\left(\frac{1+Szu}{1+Sz}\right)^{-(1-\lambda)(T-S)/S} du & (S \neq 0) \\[2ex] \int_0^1 u^{\nu-1}\, e^{-(1-\lambda)T(u-1)z} du & (S = 0) \end{cases}$$

and $\phi_2(z)$ is the best dominant of (3.13). Moreover,

$$\Sigma(\mu,\nu,\lambda) \subseteq \Sigma(\mu,\nu+1,\lambda). \qquad (3.14)$$

PROOF. Assume $\vartheta(z) \in \Sigma(\mu,\nu,\lambda)$ and set

$$\psi(z) = \frac{1}{1-\lambda}\left(-\frac{z\left(\Omega^*_{\varrho,\upsilon}(\mu,\nu+1)(\vartheta)(z)\right)'}{\Omega^*_{\varrho,\upsilon}(\mu,\nu+1)(\vartheta)(z)} - \lambda\right). \qquad (3.15)$$

Applying (1.7) along with the logarithmic differentiation for (3.15) with respect to z we have

$$\psi(z) + \frac{z\,\psi'(z)}{-(1-\lambda)\,\psi(z) + (\nu+1-\lambda)} = \frac{1}{1-\lambda}\left(-\frac{z\big(\Omega^{*}_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)\big)'}{\Omega^{*}_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z)} - \lambda\right) \prec \frac{1+Tz}{1+Sz} \qquad (z\in\mathbb{D}).$$

Hence, applying Lemma 2.1 wherein

$$\alpha = -(1-\lambda) \qquad \text{and} \qquad \beta = \nu - \lambda + 1,$$

results in

$$\psi(z) \prec \phi_2(z) \prec \frac{1+Tz}{1+Sz} \qquad (z\in\mathbb{D}),$$

which shows that the best dominant of $q_2(z)$ is defined using (3.13). This proves Theorem 3.3. ∎

THEOREM 3.4. *Let $\vartheta(z) \in \Sigma^{*,\lambda}_{\varrho,\upsilon}(\mu,\nu,T,S)$. Furthermore, if we constrain $0<S<1$ and*

$$\nu + 1 \geq \frac{(1-\lambda)(T-S)}{S}, \tag{3.16}$$

then

$$\frac{1-|T|}{1-|S|} < \frac{1}{1-\lambda}\left(-\,\Re\left\{\frac{z\big(\Omega^{*}_{\varrho,\upsilon}(\mu,\nu+1)(\vartheta)(z)\big)'}{\Omega^{*}_{\varrho,\upsilon}(\mu,\nu+1)(\vartheta)(z)}\right\} - \lambda\right) < \rho_2, \tag{3.17}$$

where

$$\rho_2 = \frac{1}{1-\lambda}\left((\nu+1-\lambda) - \frac{\nu}{{}_2F_1\left(1, \frac{(1-\lambda)(T-S)}{S}; \nu+1; \frac{S}{S-1}\right)}\right). \tag{3.18}$$

The bound ρ_2 is the best possible solution.

PROOF. For establishing (3.17) in Theorem 3.4, we apply the subordination principle of (1.15). A similar technique as in Theorem 3.1 yields:

$$\begin{aligned}\Phi_2(z) &= (1+Sz)^{\varsigma}\int_0^1 u^{\nu-1}\,(1-u)^{\delta-\nu-1}(1+Szu)^{-\varsigma}\,du \\ &= \frac{\Gamma(\nu)}{\Gamma(\varsigma)}\,{}_2F_1\left(1,\varsigma;\delta;\frac{Sz}{Sz+1}\right)\end{aligned} \tag{3.19}$$

where $\varsigma = \frac{(1-\lambda)(T-S)}{S}$ and $\varsigma = \nu+1$.

Moreover, the condition

$$\nu + 1 > \frac{(1-\lambda)(T-S)}{S} \qquad (0<S<1)$$

implies that $\delta > \varsigma > 0$. Also, (2.5) applied to (3.19) in Lemma 2.3 results in:

$$\Phi_2(z) = \int_0^1 p(z,u)\ d\gamma(u),$$

where

$$p(z,u) = \frac{1+Sz}{1+(1-u)Sz}, \qquad (0 \leq u \leq 1)$$

and

$$d\gamma(u) = \frac{\Gamma(\nu)}{\Gamma(\delta)\Gamma(\delta-\varsigma)}\ u^{\varsigma-1}\ (1-u)^{\delta-\varsigma-1}\ du.$$

The use of Lemma 2.2 indicates:

$$\inf_{z\in\mathbb{D}} \Re\left\{\frac{1}{\Phi_2(z)}\right\} = \frac{\nu}{{}_2F_1\left(1, \frac{(1-\lambda)(T-S)}{S}; \nu+1; \frac{S}{S-1}\right)}. \tag{3.20}$$

Wherein, the RHS inequality in (3.17) results in (3.20).

The subordination principle sharpens the bound ρ_2, which proves the Theorem 3.4. □

4. Conclusions

In this study, the authors have investigated properties of a novel linear operator described in that was related to the Hurwitz-Lerch zeta function:

$$\Omega^*_{\varrho,\upsilon}(\mu,\nu)(\vartheta)(z) = \widetilde{\wp}(\mu,\nu;z) * \Lambda_{\varrho,\upsilon}(z) = \frac{1}{z} + \sum_{\kappa=0}^{\infty} \frac{(\mu)_{\kappa+1}}{(\nu)_{\kappa+1}} \left(\frac{\kappa+\upsilon+1}{\upsilon}\right)^{\varrho} \eta_\kappa\, z^\kappa$$

Different results and properties described in this study were seen to be associated to a particular subclass belonging to the class consisting of the (normalised) meromorphic univalent functions in a punctured unit disk $\mathbb{D}^*$. This has been described in this study using the Hadamard product (or convolutions). This study was able to derive several results which have been explained in Theorems 3.1, 3.2, 3.3 and 3.4.

References

[1] K. R. Alhindi and M. Darus, *A new class of meromorphic functions involving the polylogarithm function*, J. Complex Anal. **2014** (2014), Art. ID 864805, 5 pp.

[2] S. K. Bajpai, *A note on a class of meromorphic univalent functions*, Rev. Roum. Math. Pures Appl. **22** (1977), 295–297.

[3] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135** (1969), 429–449.

[4] K. A. Challab, M. Darus and F. Ghanim, *Inclusion relationships properties for certain subclasses of meromorphic functions associated with Hurwitz-Lerch zeta function*, Italian Journal of Pure and Applied Mathematics **39** (2018), 410-423.

[5] N. E. Cho, Y. S. Woo and S. Owa, *Argument estimates of certain meromorphic functions*, New extension of historical theorems for univalent function theory (Japanese) (Kyoto, 1999), Sūrikaisekikenkyūsho Kōkyūroku **1164** (2000), 1–11.

[6] R. M. El-Ashwah, *Inclusion properties regarding the meromorphic structure of Srivastava-Attiya operator*, Southeast Asian Bull. Math. **38** (2014), 501–512.

[7] R. M. El-Ashwah, *Inclusion relationships properties for certain subclasses of meromorphic functions associated with Hurwitz-Lerch zeta function*, Acta Univ. Apulensis **34** (2013), 191–205.

[8] F. Ghanim, *A study of a certain subclass of Hurwitz-Lerch zeta function related to a linear operator*, Abstr. Appl. Anal. **2013** (2013), Article ID 763756, 1–7.

[9] F. Ghanim, *Certain properties of classes of meromorphic functions defined by a linear operator and associated with Hurwitz-Lerch zeta function*, Advanced Studies in Contemporary Mathematics **27** (2017), 175–180.

[10] F. Ghanim and M. Darus, *Some properties on a certain class of meromorphic functions related to Cho-Kwon-Srivastava operator*, Asian-Eur. J. Math. **5(4)** (2012), Article ID 1250052, 1–9.

[11] F. Ghanim and R. M. El-Ashwah, *Inclusion properties of certain subclass of univalent meromorphic functions defined by a linear operator associated with Hurwitz–Lerch zeta function*, Asian-Eur. J. Math. **10(4)** (2017), 1–10.

[12] A. Y. Lashin, *On certain subclasses of meromorphic functions associated with certain integral operators*, Comput. Math. Appl. **59** (2010), 524–531.

[13] S. S. Miller and P. T. Mocanu, Differential Subordinations : Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics **225**, Marcel Dekker Incorporated, New York and Basel, 2000.

[14] E. D. Rainville, Special Functions, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[15] H. M. Srivastava, *Some formulas for the Bernoulli and Euler polynomials at rational arguments*, Math. Proc. Cambridge Philos. Soc. **129** (2000), 77–84.

[16] H. M. Srivastava and A. A. Attiya, *An integral operator associated with the Hurwitz-Lerch zeta function and diferential subordination*, Integral Transforms Spec. Funct. **18)** (2007), 207–216.

[17] H. M. Srivastava and J. Choi, Series Associated with Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[18] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[19] H. M. Srivastava, S. Gaboury and F. Ghanim, *Certain subclasses of meromorphically univalent functions defined by a linear operator associated with the λ-generalized Hurwitz-Lerch zeta function*, Integral Transforms Spec. Funct. **26** (2015), 258–272.

[20] H. M. Srivastava, S. Gaboury and F. Ghanim, *Some further properties of a linear operator associated with the λ-generalized Hurwitz-Lerch zeta function related to the class of meromorphically univalent functions*, Appl. Math. Comput. **259** (2015), 1019–1029.

[21] H. M. Srivastava, A. Y. Lashin and B. A. Frasin, *Starlikeness and convexity of certain classes of meromorphically multivalent functions*, Theory Appl. Math. Comput. Sci. **3** (2013), 93–102.

[22] H. M. Srivastava and H. L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwoord Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[23] B. A. Uralegaddi and C. Somanatha, *New criteria for meromorphic starlike univalent functions*, Bull. Austral. Math. Soc. **43** (1991), 137–140.

[24] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions, Fourth Edition, Cambridge University Press, Cambridge, London and New York, 1927.

[25] D. R. Wilken and J. Feng, *A remark on convex and starlike functions*, J. London Math. Soc. (2) **21** (1980), 287–290.

Jedna potklasa univalentnih meromorfnih funkcija definiranih pomoću linearnog operatora definiranog preko Hurwitz-Lerchove zeta funkcije

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Sažetak. Promatra se linearni operator povezan s Hurwitz-Lerchovom zeta funkcijom i Gaussovom hipergeometrijskom funkcijom u punktiranom jediničnom disku. Uvedena je nova potklasa meromorfnih univalentnih funkcija pridruženih tom operatoru pomoću koncepta subordinacije te su proučavana njezina karakteristična svojstva.

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