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CHARACTERIZATIONS OF $*$-LIE DERIVABLE MAPPINGS ON PRIME $*$-RINGS

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Abstract. Let $\mathcal{R}$ be a $*$-ring containing a nontrivial self-adjoint idempotent. In this paper it is shown that under some mild conditions on $\mathcal{R}$, if a mapping $d : \mathcal{R} \to \mathcal{R}$ satisfies

$$d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)]$$

for all $U, V \in \mathcal{R}$, then there exists $Z_{U,V} \in \mathcal{Z}(\mathcal{R})$ (depending on U and V), where $\mathcal{Z}(\mathcal{R})$ is the center of $\mathcal{R}$, such that $d(U+V) = d(U) + d(V) + Z_{U,V}$. Moreover, if $\mathcal{R}$ is a 2-torsion free prime $*$-ring additionally, then $d = \psi + \xi$, where ψ is an additive $*$-derivation of $\mathcal{R}$ into its central closure $\mathcal{T}$ and ξ is a mapping from $\mathcal{R}$ into its extended centroid $\mathcal{C}$ such that $\xi(U+V) = \xi(U) + \xi(V) + Z_{U,V}$ and $\xi([U, V]) = 0$ for all $U, V \in \mathcal{R}$. Finally, the above ring theoretic results have been applied to some special classes of algebras such as nest algebras and von Neumann algebras.

1. Introduction

Throughout this paper $\mathcal{R}$ will denote an associative ring with the center $\mathcal{Z}(\mathcal{R})$. Recall that a ring $\mathcal{R}$ is said to be n-torsion free, where $n > 1$ is an integer, if $nU = 0$ implies $U = 0$ for all $U \in \mathcal{R}$. A ring $\mathcal{R}$ is said to be prime if for any $U, V \in \mathcal{R}$, $U\mathcal{R}V = \{0\}$ implies $U = 0$ or $V = 0$. An additive mapping $x \mapsto x^*$ on a ring $\mathcal{R}$ is called involution in case $(UV)^* = V^*U^*$ and $(U^*)^* = U$ hold for all $U, V \in \mathcal{R}$. A ring equipped with an involution is called a ring with involution or $*$-ring (see [7]). An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivation on $\mathcal{R}$ if $d(UV) = d(U)V + Ud(V)$ for all $U, V \in \mathcal{R}$. In particular, derivation d is called an inner derivation if there exists some $X \in \mathcal{R}$ such that $d(U) = UX - XU$ for all $U \in \mathcal{R}$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a Lie derivation if $d([U, V]) = [d(U), V] + [U, d(V)]$ holds for all $U, V \in \mathcal{R}$, where $[U, V] = UV - VU$ is the usual Lie product. If the condition of additivity is dropped from the above definition, then the corresponding Lie derivation is

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called a Lie derivable map. Obviously, every derivation is a Lie derivation. However, the converse statements are not true in general.

Let $\mathcal{R}$ be a $*$-ring. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be an additive $*$-derivation on $\mathcal{R}$ if $d(UV) = d(U)V + Ud(V)$ and $d(U^*) = d(U)^*$ for all $U, V \in \mathcal{R}$. More generally, a mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a $*$-Lie derivable mapping if $d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)]$. Indeed, if $d(U^*) = d(U)^*$ for all $U \in \mathcal{R}$, then d is a Lie derivable mapping if and only if d is a $*$-Lie derivable mapping. An additive $*$-Lie derivable mapping is said to be a $*$-Lie derivation. It is not difficult to observe that any $*$-derivation is a $*$-Lie derivation but the converse is not true in general.

There has been a great interest in the study of characterizations of Lie derivations and $*$-Lie derivations for many years. The first quite surprising result is due to Martindale III who proved that every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive (see [14]). Miers [16] initially established that every Lie derivation d on a von Neumann algebra $\mathfrak{A}$ can be uniquely written as the sum $d = \psi + \xi$ where ψ is an inner derivation of $\mathfrak{A}$ and ξ is a linear mapping from $\mathfrak{A}$ into its center $Z(\mathfrak{A})$ vanishing on each commutator. Yu and Zhang [18] proved that every Lie derivable mapping of a triangular algebra is the sum of an additive derivation and a mapping from triangular algebra into its center sending commutators to zero. Mathieu and Villena [15] gave the characterizations of Lie derivations on C^*-algebras. W. Jing and F. Lu [8] showed that every Lie derivable mapping on a 2-torsion free prime ring $\mathcal{R}$ can be expressed as $d = \psi + \xi$, where $\psi : \mathcal{R} \to \mathcal{T}$ is an additive derivation and $\xi : \mathcal{R} \to \mathcal{C}$ is nearly additive i.e. $\xi(U + V) = \xi(U) + \xi(V) + Z_{U,V}$ where $Z_{U,V} \in \mathcal{Z}(\mathcal{R})$ (depending on U and V in $\mathcal{R}$) and vanishes on each commutator. Yu and Zhang [19] proved that every $*$-Lie derivable mapping from a factor von Neumann algebra into itself is an additive $*$-derivation. Also, Li, Chen and Wang [9] obtained the same result for $*$-Lie derivable mappings on von Neumann algebras and proved that every $*$-Lie derivable mapping on a von Neumann algebra with no central abelian projections can be expressed as the sum of an additive $*$-derivation and a mapping with image in the centre vanishing on commutators. In addition, the characterization of Lie derivations and $*$-Lie derivations on various algebras are considered in[1], [2], [5], [4],[6], [8], [12],[13], [17], [20].

Motivated by the results due to W. Jing & F. Lu [8] and C. Li et al. [9], in Section 2, we investigate the additivity of $*$-Lie derivable mappings on $*$-rings and show that every $*$-Lie derivable mapping on $\mathcal{R}$ is almost additive in the sense that for any $U, V \in \mathcal{R}$ there exists $Z_{U,V} \in \mathcal{Z}(\mathcal{R})$ (depending on U and V) such that $d(U + V) = d(U) + d(V) + Z_{U,V}$. In Section 3, we study the characterization of $*$-Lie derivable mappings on prime $*$-rings. Under some mild conditions on $\mathcal{R}$, we prove that, if d is an additive Lie derivable mapping on $\mathcal{R}$, then $d = \psi + \xi$, where ψ is an additive $*$-derivation of $\mathcal{R}$ into its central closure $\mathcal{T}$ and ξ is a mapping from $\mathcal{R}$ into its extended centroid $\mathcal{C}$ such that $\xi(U+V)=\xi(U)+\xi(V)+Z_{U,V}$ and $\xi([U,V])=0$ for all $U,V\in\mathcal{R}$. Finally, the above ring theoretic results have been applied to some special class of algebras such as nest algebras and von Neumann algebras.

2. Additivity of $*$-Lie derivable mappings on $*$-rings

In this section, we examine the additivity of $*$-Lie derivable mappings on rings. Let $\mathcal{R}$ be a $*$-ring with a nontrivial self-adjoint idempotent P. We write $Q=I-P$. It is to be noted that $\mathcal{R}$ may be without identity element. It is obvious that $PQ=QP=0$. By the Peirce decomposition of $\mathcal{R}$, we have $\mathcal{R}=\mathfrak{A}_{11}+\mathfrak{A}_{12}+\mathfrak{A}_{21}+\mathfrak{A}_{22}$, where $\mathfrak{A}_{11}=P\mathcal{R}P$, $\mathfrak{A}_{12}=P\mathcal{R}Q$, $\mathfrak{A}_{21}=Q\mathcal{R}P$ and $\mathfrak{A}_{22}=Q\mathcal{R}Q$. Throughout this paper, U_{ij} will denote an arbitrary element of $\mathfrak{A}_{\mathrm{ij}}$ and any element $U\in\mathcal{R}$ can be expressed as $U=U_{11}+U_{12}+U_{21}+U_{22}$.

The main result of this section starts as follows.

Theorem 2.1. *Let $\mathcal{R}$ be a $*$-ring containing a nontrivial self-adjoint idempotent P and satisfying the following conditions:*

(G_1) *If $U_{ii}V_{ij}=V_{ij}U_{jj}$ for all $V_{ij}\in\mathfrak{A}_{ij}$ and $1\leq i\neq j\leq 2$, then $U_{ii}+U_{jj}\in\mathcal{Z}(\mathcal{R})$.*

(G_2) *If $U_{ij}V_{jk}=0$ for all $V_{jk}\in\mathfrak{A}_{jk}$ and $1\leq i,j,k\leq 2$, then $U_{ij}=0$.*

If a mapping $d:\mathcal{R}\to\mathcal{R}$ satisfies

$$d([U^*,V])=[d(U)^*,V]+[U^*,d(V)],$$

for all $U,V\in\mathcal{R}$, then there exists $Z_{U,V}\in\mathcal{Z}(\mathcal{R})$ such that $d(U+V)=d(U)+d(V)+Z_{U,V}$.

Throughout assume that $\mathcal{R}$ satisfies the hypothesis of Theorem 2.1. The proof of the above theorem is given in a series of the following Lemmas.

Lemma 2.2. $d(0)=0$.

Proof. $d(0)=d([0^*,0])=[d(0)^*,0]+[0^*,d(0)]=0$. □

Lemma 2.3. *For any $U_{ii}\in\mathfrak{A}_{\mathrm{ii}}$, $V_{ij}\in\mathfrak{A}_{\mathrm{ij}}$, $1\leq i\neq j\leq 2$, there exists $Z_{U_{ii},V_{ij}}\in\mathcal{Z}(\mathcal{R})$ such that*

(*i*) $d(U_{ii}+V_{ij})=d(U_{ii})+d(V_{ij})+Z_{U_{ii},V_{ij}}$,

(*ii*) $d(U_{ii}+V_{ji})=d(U_{ii})+d(V_{ji})+Z_{U_{ii},V_{ji}}$.

Proof. (*i*) Let $A=d(U_{ii}+V_{ij})-d(U_{ii})-d(V_{ij})$. For any $U_{ii}\in\mathfrak{A}_{\mathrm{ii}}$, $V_{ij}\in\mathfrak{A}_{\mathrm{ij}}$, we have

$$\begin{aligned} d(V_{ij}) &= d([P^*,U_{ii}+V_{ij}]) \\ &= [d(P)^*,U_{ii}+V_{ij}]+[P^*,d(U_{ii}+V_{ij})]. \end{aligned}$$

On the other hand by Lemma 2.2, we have

$$\begin{aligned} d(V_{ij}) &= d([P^*, U_{ii}]) + d([P^*, V_{ij}]) \\ &= [d(P)^*, U_{ii} + V_{ij}] + [P^*, d(U_{ii}) + d(V_{ij})]. \end{aligned}$$

Comparing the above two identities, we get $[P, A] = 0$. Hence $A_{ij} = A_{ji} = 0$.

For any $W_{ji} \in \mathfrak{A}_{\mathrm{ji}}$, we compute

$$\begin{aligned} d(-U_{ii}W_{ji}^*) &= d([W_{ji}^*, U_{ii} + V_{ij}]) \\ &= [d(W_{ji})^*, U_{ii} + V_{ij}] + [W_{ji}^*, d(U_{ii} + V_{ij})]. \end{aligned}$$

Using Lemma 2.2, $d(-U_{ii}W_{ji}^*)$ can also be expressed as

$$\begin{aligned} d(-U_{ii}W_{ji}^*) &= d([W_{ji}^*, U_{ii}]) + d([W_{ji}^*, V_{ij}]) \\ &= [d(W_{ji})^*, U_{ii} + V_{ij}] + [W_{ji}^*, d(U_{ii}) + d(V_{ij})]. \end{aligned}$$

From the above two equations it follows that $[W_{ji}^*, A] = 0$. In other words $W_{ji}^*A = AW_{ji}^*$ for all $W_{ji} \in \mathfrak{A}_{\mathrm{ji}}$. By the condition (G_1), we see that $A_{ii}+A_{jj} \in \mathcal{Z}(\mathcal{R})$. Hence $d(U_{ii}+V_{ij}) = d(U_{ii})+d(V_{ij})+Z_{U_{ii},V_{ij}}$ for some $Z_{U_{ii},V_{ij}} \in \mathcal{Z}(\mathcal{R})$. Similarly, one can get (ii). ∎

Lemma 2.4. *For any* $U_{ij}, V_{ij} \in \mathfrak{A}_{\mathrm{ij}}$, $1 \leq i \neq j \leq 2$, *we have*

$$d(U_{ij} + V_{ij}) = d(U_{ij}) + d(V_{ij}).$$

Proof. By Lemma 2.3, we see that

$$\begin{aligned} d(U_{ij} + V_{ij}) &= d([(U_{ij}^* + P)^*, V_{ij} + Q]) \\ &= [d(U_{ij}^* + P)^*, V_{ij} + Q] + [(U_{ij}^* + P)^*, d(V_{ij} + Q)] \\ &= [d(U_{ij}^*)^* + d(P)^*, V_{ij} + Q] + [(U_{ij}^* + P)^*, d(V_{ij}) + d(Q)] \\ &= [d(U_{ij}^*)^*, V_{ij}] + [d(U_{ij}^*)^*, Q] + [d(P)^*, V_{ij}] + [d(P)^*, Q] \\ &\quad + [U_{ij}, d(V_{ij})] + [U_{ij}, d(Q)] + [P, d(V_{ij})] + [P, d(Q)] \\ &= d([(U_{ij}^*)^*, V_{ij}]) + d([(U_{ij}^*)^*, Q]) + d([P^*, V_{ij}]) + d([P^*, Q]) \\ &= d(U_{ij}) + d(V_{ij}). \end{aligned}$$

∎

Lemma 2.5. *For any* $U_{ii}, V_{ii} \in \mathfrak{A}_{\mathrm{ii}}$, $i = 1, 2$, *there exists* $Z_{U_{ii},V_{ii}} \in \mathcal{Z}(\mathcal{R})$ *such that*

$$d(U_{ii} + V_{ii}) = d(U_{ii}) + d(V_{ii}) + Z_{U_{ii},V_{ii}}.$$

Proof. Let $A = d(U_{11} + V_{11}) - d(U_{11}) - d(V_{11})$. For any $U_{11}, V_{11} \in \mathfrak{A}_{11}$, we have

$$\begin{aligned} 0 &= d([Q^*, U_{11} + V_{11}]) \\ &= [d(Q)^*, U_{11} + V_{11}] + [Q^*, d(U_{11} + V_{11})]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}0 &= d([Q^*, U_{11}]) + d([Q^*, V_{11}])\\ &= [d(Q)^*, U_{11} + V_{11}] + [Q^*, d(U_{11}) + d(V_{11})].\end{aligned}$$

Comparing the above two identities, we get $[Q, A] = 0$. Hence $A_{12} = A_{21} = 0$.

For any $W_{12} \in \mathfrak{A}_{12}$, we compute

$$\begin{aligned}d(W_{12}^*(U_{11} + V_{11})) &= d([W_{12}^*, U_{11} + V_{11}])\\ &= [d(W_{12})^*, U_{11} + V_{11}] + [W_{12}^*, d(U_{11} + V_{11})].\end{aligned}$$

On the other hand by using Lemma 2.4, we have

$$\begin{aligned}d(W_{12}^*(U_{11} + V_{11})) &= d(W_{12}^*U_{11}) + d(W_{12}^*V_{11})\\ &= d([W_{12}^*, U_{11}]) + d([W_{12}^*, V_{11}])\\ &= [d(W_{12})^*, U_{11} + V_{11}] + [W_{12}^*, d(U_{11}) + d(V_{11})].\end{aligned}$$

Comparing the above two equations, we have $[W_{12}^*, A] = 0$. Thus $W_{12}^*A_{11} = A_{22}W_{12}^*$ for all $W_{12} \in \mathfrak{A}_{12}$. By using the condition (G_1), we see that $A_{11} + A_{22} \in \mathcal{Z}(\mathcal{R})$. Therefore $d(U_{11} + V_{11}) = d(U_{11}) + d(V_{11}) + Z_{U_{11},V_{11}}$ for all $U_{11}, V_{11} \in \mathfrak{A}_{11}$ and for some $Z_{U_{11},V_{11}} \in \mathcal{Z}(\mathcal{R})$. Similarly, the result is true for the case when $i = 2$. ∎

Lemma 2.6. *For any $U_{12} \in \mathfrak{A}_{12}$ and $V_{21} \in \mathfrak{A}_{21}$, we have*

$$d(U_{12} + V_{21}) = d(U_{12}) + d(V_{21}).$$

Proof. Suppose $A = d(U_{12} + V_{21}) - d(U_{12}) - d(V_{21})$. For any $U_{12} \in \mathfrak{A}_{12}$ and $V_{21} \in \mathfrak{A}_{21}$, we compute

$$\begin{aligned}d(U_{12} + V_{21}) &= d([P^*, U_{12} - V_{21}])\\ &= [d(P)^*, U_{12} - V_{21}] + [P^*, d(U_{12} - V_{21})]\\ &= d([P^*, U_{12}]) - [P, d(U_{12})] + d([P^*, -V_{21}]) - [P^*, d(-V_{21})]\\ &\quad + [P^*, d(U_{12} - V_{21})]\\ &= d(U_{12}) + d(V_{21}) + [P^*, d(U_{12} - V_{21}) - d(U_{12}) - d(-V_{21})].\end{aligned}$$

Consequently $A = P(d(U_{12} - V_{21}) - d(U_{12}) - d(-V_{21})) - (d(U_{12} - V_{21}) - d(U_{12}) - d(-V_{21}))P$. Hence we see that $A_{11} = A_{22} = 0$.

For any $W_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned}d([W_{12}^*, U_{12}]) &= d([W_{12}^*, U_{12} + V_{21}])\\ &= [d(W_{12})^*, U_{12} + V_{21}] + [W_{12}^*, d(U_{12} + V_{21})].\end{aligned}$$

On the other hand, by Lemma, 2.2 we have

$$\begin{aligned}d([W_{12}^*, U_{12}]) &= d([W_{12}^*, U_{12}]) + d([W_{12}^*, V_{21}])\\ &= [d(W_{12})^*, U_{12} + V_{21}] + [W_{12}^*, d(U_{12}) + d(V_{21})].\end{aligned}$$

Comparing the above two identities, we get $[W_{12}^*, A] = 0$. This gives that $A_{12}W_{12}^* = 0$ for all $W_{12} \in \mathfrak{A}_{12}$. By the condition (G_2), we see that $A_{12} = 0$. Similarly, we obtain that $A_{21} = 0$. Thus we are done. ∎

LEMMA 2.7. *For any $U_{11} \in \mathfrak{A}_{11}$, $V_{12} \in \mathfrak{A}_{12}$ and $W_{22} \in \mathfrak{A}_{22}$, we have*

$$d(U_{11} + V_{12} + W_{22}) = d(U_{11}) + d(V_{12}) + d(W_{22}) + Z_{U_{11},V_{12},W_{22}}.$$

PROOF. Suppose $A = d(U_{11} + V_{12} + W_{22}) - d(U_{11}) - d(V_{12}) - d(W_{22})$. For any $U_{11} \in \mathfrak{A}_{11}$, $V_{12} \in \mathfrak{A}_{12}$ and $W_{22} \in \mathfrak{A}_{22}$, we compute

$$\begin{aligned} d(V_{12}) &= d([P^*, U_{11} + V_{12} + W_{22}]) \\ &= [d(P)^*, U_{11} + V_{12} + W_{22}] + [P^*, d(U_{11} + V_{12} + W_{22})]. \end{aligned}$$

On the other hand, by Lemma, 2.2 we have

$$\begin{aligned} d(V_{12}) &= d([P^*, U_{11}]) + d([P^*, V_{12}]) + d([P^*, W_{22}]) \\ &= [d(P)^*, U_{11} + V_{12} + W_{22}] + [P^*, d(U_{11}) + d(V_{12}) + d(W_{22})]. \end{aligned}$$

Comparing the above two identities, we get $[P^*, A] = 0$. This gives that $A_{12} = A_{21} = 0$.

Now for any $S_{21} \in \mathfrak{A}_{21}$, we see that

$$\begin{aligned} &d([S_{21}^*, U_{11} + V_{12} + W_{22}]) \\ &= [d(S_{21})^*, U_{11} + V_{12} + W_{22}] + [S_{21}^*, d(U_{11} + V_{12} + W_{22})]. \end{aligned}$$

On the other hand, by Lemmas 2.2 & 2.4 we have

$$\begin{aligned} d([S_{21}^*, U_{11} + V_{12} + W_{22}]) &= d([S_{21}^*, U_{11} + W_{22}]) + d([S_{21}^*, V_{12}]) \\ &= d(S_{21}^*W_{22} - U_{11}S_{21}^*) + d([S_{21}^*, V_{12}]) \\ &= d(S_{21}^*W_{22}) + d(-U_{11}S_{21}^*) + d([S_{21}^*, V_{12}]) \\ &= d([S_{21}^*, W_{22}]) + d([S_{21}^*, U_{11}]) + d([S_{21}^*, V_{12}]) \\ &= [d(S_{21})^*, U_{11} + V_{12} + W_{22}] \\ &\quad + [S_{21}^*, d(U_{11}) + d(V_{12}) + d(W_{22})]. \end{aligned}$$

Comparing the above two identities, we get $[S_{21}^*, A] = 0$. This gives that $S_{21}^*A_{22} = A_{11}S_{21}^*$ for all $S_{21} \in \mathfrak{A}_{21}$. By the condition (G_1), we get $A_{11} + A_{22} \in \mathcal{Z}(\mathcal{R})$. Thus we have obtained that $d(U_{11} + V_{12} + W_{22}) = d(U_{11}) + d(V_{12}) + d(W_{22}) + Z_{U_{11},V_{12},W_{22}}$ for some $Z_{U_{11},V_{12},W_{22}} \in \mathcal{Z}(\mathcal{R})$. ∎

LEMMA 2.8. *For any $U_{11} \in \mathfrak{A}_{11}$, $V_{12} \in \mathfrak{A}_{12}$, $W_{21} \in \mathfrak{A}_{21}$ and $X_{22} \in \mathfrak{A}_{22}$, we have*

$$\begin{aligned} &d(U_{11} + V_{12} + W_{21} + X_{22}) \\ &= d(U_{11}) + d(V_{12}) + d(W_{21}) + d(X_{22}) + Z_{U_{11},V_{12},W_{21},X_{22}}. \end{aligned}$$

PROOF. Assume $A = d(U_{11} + V_{12} + W_{21} + X_{22}) - d(U_{11}) - d(V_{12}) - d(W_{21}) - d(X_{22})$. For any $U_{11} \in \mathfrak{A}_{11}$, $V_{12} \in \mathfrak{A}_{12}$, $W_{21} \in \mathfrak{A}_{21}$ and $X_{22} \in \mathfrak{A}_{22}$, we see that

$$\begin{aligned} d(V_{12} - W_{21}) &= d([P^*, U_{11} + V_{12} + W_{21} + X_{22}]) \\ &= [d(P)^*, U_{11} + V_{12} + W_{21} + X_{22}] \\ &\quad + [P^*, d(U_{11} + V_{12} + W_{21} + X_{22})]. \end{aligned}$$

On the other hand, by using Lemmas 2.2 & 2.6, we have

$$\begin{aligned} d(V_{12} - W_{21}) &= d([P^*, U_{11}]) + d([P^*, V_{12}]) + d([P^*, W_{21}]) + d([P^*, X_{22}]) \\ &= [d(P)^*, U_{11} + V_{12} + W_{21} + X_{22}] \\ &\quad + [P^*, d(U_{11}) + d(V_{12}) + d(W_{21}) + d(X_{22})]. \end{aligned}$$

Comparing the above two equations, we have $[P, A] = 0$. This gives that $A_{12} = A_{21} = 0$.

Now for any $S_{12} \in \mathfrak{A}_{12}$, we compute

$$\begin{aligned} &d([S_{12}^*, U_{11} + V_{12} + W_{21} + X_{22}]) \\ &= [d(S_{12})^*, U_{11} + V_{12} + W_{21} + X_{22}] + [S_{12}^*, d(U_{11} + V_{12} + W_{21} + X_{22})]. \end{aligned}$$

On the other hand, by using Lemma 2.7, we have

$$\begin{aligned} &d([S_{12}^*, U_{11} + V_{12} + W_{21} + X_{22}]) \\ &= d([S_{12}^*, U_{11} + V_{12} + X_{22}]) + d([S_{12}^*, W_{21}]) \\ &= [d(S_{12})^*, U_{11} + V_{12} + X_{22}] + [S_{21}^*, d(U_{11}) + d(V_{12}) + d(X_{22})] \\ &\quad + [d(S_{12})^*, W_{21}] + [S_{12}^*, d(W_{21})] \\ &= [d(S_{12})^*, U_{11} + V_{12} + W_{21} + X_{22}] \\ &\quad + [S_{21}^*, d(U_{11}) + d(V_{12}) + d(W_{21}) + d(X_{22})]. \end{aligned}$$

Comparing the above two identities, we get $[S_{12}^*, A] = 0$. This gives that $S_{12}^* A_{11} = A_{22} S_{12}^*$ for all $S_{12} \in \mathfrak{A}_{12}$. By using condition (G_1), we see that $A_{11} + A_{22} \in \mathcal{Z}(\mathcal{R})$. Thus we have obtained that $d(U_{11} + V_{12} + W_{21} + X_{22}) = d(U_{11}) + d(V_{12}) + d(W_{21}) + d(X_{22}) + Z_{U_{11}, V_{12}, W_{21}, X_{22}}$ for some $Z_{U_{11}, V_{12}, W_{21}, X_{22}} \in \mathcal{Z}(\mathcal{R})$. ◻

PROOF OF THEOREM 2.1. Now take $U = U_{11} + U_{12} + U_{21} + U_{22}$ and $V = V_{11} + V_{12} + V_{21} + V_{22}$. By using Lemmas 2.4, 2.5 & 2.8, we see that

$$\begin{aligned} d(U+V) &= d(U_{11} + U_{12} + U_{21} + U_{22} + V_{11} + V_{12} + V_{21} + V_{22}) \\ &= d((U_{11} + V_{11}) + (U_{12} + V_{12}) + (U_{21} + V_{21}) + (U_{22} + V_{22})) \\ &= d(U_{11} + V_{11}) + d(U_{12} + V_{12}) + d(U_{21} + V_{21}) \\ &\quad + d(U_{22} + V_{22}) + Z_1 \\ &= d(U_{11}) + d(V_{11}) + Z_2 + d(U_{12}) + d(V_{12}) + d(U_{21}) \\ &\quad + d(V_{21}) + d(U_{22}) + d(V_{22}) + Z_3 + Z_1 \\ &= (d(U_{11}) + d(U_{12}) + d(U_{21}) + d(U_{22})) + (d(V_{11}) \\ &\quad + d(V_{12}) + d(V_{21}) + d(V_{22})) + Z_1 + Z_2 + Z_3 \\ &= d(U_{11} + U_{12} + U_{21} + U_{22}) - Z_4 + d(V_{11} + V_{12} + V_{21} + V_{22}) \\ &\quad - Z_5 + Z_1 + Z_2 + Z_3 \\ &= d(U) + d(V) + (Z_1 + Z_2 + Z_3 - Z_4 - Z_5). \end{aligned}$$

Take $Z_{U,V} = Z_1 + Z_2 + Z_3 - Z_4 - Z_5$. Thus we see that $d(U+V) = d(U) + d(V) + Z_{U,V}$ for some $Z_{U,V} \in \mathcal{Z}(\mathcal{R})$. This completes the proof of our main theorem. □

Now we apply Theorem 2.1 to prime $*$-rings and nest algebras. We begin with the following important lemma.

LEMMA 2.9. *Let $\mathcal{R}$ be a prime $*$-ring containing a nontrivial self-adjoint idempotent P with centre $\mathcal{Z}(\mathcal{R})$.*

(i) *If $U_{ij}V_{jk} = 0$ for all $V_{jk} \in \mathfrak{A}_{\mathfrak{jk}}$ and $1 \leq i, j, k \neq 2$ then $U_{ij} = 0$.*
(ii) *If $U_{11}V_{12} = V_{12}U_{22}$ for all $V_{12} \in \mathfrak{A}_{12}$, then $U_{11} + U_{22} \in \mathcal{Z}(\mathcal{R})$.*

PROOF. (i) is the direct consequence of the primeness of $\mathcal{R}$.

(ii) For any $V_{11} \in \mathfrak{A}_{11}$ and $V_{12} \in \mathfrak{A}_{12}$, we get $U_{11}V_{11}V_{12} = V_{11}V_{12}U_{22} = V_{11}U_{11}V_{12}$ for all $V_{12} \in \mathfrak{A}_{12}$. As $\mathcal{R}$ is prime, we have $U_{11}V_{11} = V_{11}U_{11}$.

For any $V_{12} \in \mathfrak{A}_{12}$ and $V_{22} \in \mathfrak{A}_{22}$, we get $V_{12}V_{22}U_{22} = U_{11}V_{12}V_{22} = V_{12}U_{22}V_{22}$ for all $V_{12} \in \mathfrak{A}_{12}$. It follows by the primeness of $\mathcal{R}$ that $V_{22}U_{22} = U_{22}V_{22}$.

For any $V_{12} \in \mathfrak{A}_{12}$ and $V_{21} \in \mathfrak{A}_{21}$, we get $U_{22}V_{21}V_{12} = V_{21}V_{12}U_{22} = V_{21}U_{11}V_{12}$ for all $V_{12} \in \mathfrak{A}_{12}$. It follows that $U_{22}V_{21} = V_{21}U_{11}$.

For any $V \in \mathcal{R}$, we have

$$\begin{aligned} (U_{11} + U_{22})V &= (U_{11} + U_{22})(V_{11} + V_{12} + V_{21} + V_{22}) \\ &= U_{11}V_{11} + U_{11}V_{12} + U_{22}V_{21} + U_{22}V_{22} \\ &= V_{11}U_{11} + V_{12}U_{22} + V_{21}U_{11} + V_{22}U_{22} \\ &= (V_{11} + V_{12} + V_{21} + V_{22})(U_{11} + U_{22}) \\ &= V(U_{11} + U_{22}). \end{aligned}$$

Hence it follows that $U_{11} + U_{22} \in \mathcal{Z}(\mathcal{R})$. □

It follows from Lemma 2.9 that every prime ∗-ring with nontrivial self-adjoint idempotent satisfies the conditions (G_1) and (G_2) of Theorem 2.1. So we have the following immediate corollary.

Corollary 2.10. *Let $\mathcal{R}$ be a prime ∗-ring containing a nontrivial self-adjoint idempotent P. If a mapping $d : \mathcal{R} \to \mathcal{R}$ satisfies*

$$d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)],$$

for all $U, V \in \mathcal{R}$, then there exists $Z_{U,V} \in \mathcal{Z}(\mathcal{R})$ such that $d(U + V) = d(U) + d(V) + Z_{U,V}$.

Let $\mathcal{H}$ be a complex Hilbert space. Recall that a nest $\mathcal{N}$ of projections on $\mathcal{H}$ is a chain of orthogonal projections on $\mathcal{H}$ containing zero operator 0 and the identity operator I and is closed in the strong operator topology. By $\mathcal{B}(\mathcal{H})$, we mean the algebra of all bounded linear operators on $\mathcal{H}$. The nest algebra $\mathcal{T}(\mathcal{N})$ corresponding to the nest $\mathcal{N}$ is the set of all operators U in $\mathcal{B}(\mathcal{H})$ such that $UP = PUP$ for all $P \in \mathcal{N}$. It is to be noted that $\mathcal{T}(\mathcal{N})$ is a weak ∗- closed operator algebra. A nest is said to be nontrivial if it contains at least one nontrivial projection. The centre of the nest algebra $\mathcal{T}(\mathcal{N})$ is $\mathbb{C}I$, where $\mathbb{C}$ is the complex field. It is to be noted that by every nest algebra $\mathcal{T}(\mathcal{N})$ with non trivial projection P satisfies the conditions (G_1) and (G_2) of Theorem 2.1 (see [10, Lemma 2.6]). Thus we have the following immediate corollary.

Corollary 2.11. *Let $\mathcal{N}$ be a nontrivial nest on a complex Hilbert space $\mathcal{H}$ and $\mathcal{T}(\mathcal{N})$ be the associated nest algebra. If a mapping $d : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ satisfies*

$$d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)]$$

for all $U, V \in \mathcal{T}(\mathcal{N})$, then there exists $\lambda_{U,V} \in \mathbb{C}$ such that $d(U + V) = d(U) + d(V) + \lambda_{U,V} I$.

3. Characterization of ∗-Lie derivable mappings on Prime ∗-rings

In this section, we list some notations and results which will be used frequently to prove our results. Let $\mathcal{R}$ be a prime ∗-ring containing a nontrivial self-adjoint idempotent P with the centre $\mathcal{Z}(\mathcal{R})$. The maximal right ring of quotients is denoted by $\mathcal{Q}_{mr}(\mathcal{R})$ and the two-sided right ring of quotients of $\mathcal{R}$ by $\mathcal{Q}_r(\mathcal{R})$. The centre of $\mathcal{Q}_r(\mathcal{R})$ is called the extended centroid of $\mathcal{R}$ and is denoted by $\mathcal{C}$. It is to be noted that $\mathcal{C}$ of any prime ring is a field. The subring $\mathcal{RC}$ of $\mathcal{Q}_{mr}(\mathcal{R})$ is called the central closure of $\mathcal{R}$ which is also prime for any prime ring. We denote the central closure of $\mathcal{R}$ by $\mathcal{T}$.

We facilitate our discussion with the following known results.

LEMMA 3.1 ([3, Theorem 2.3.4]). *If $\mathcal{R}$ is a prime ring and $U, V \in \mathcal{Q}_{mr}(\mathcal{R})$ such that $UXV = VXU$ for all $X \in \mathcal{R}$, then $U = CV$ for some $C \in \mathcal{C}$. In otherwords U and V are $\mathcal{C}$-dependent.*

LEMMA 3.2 ([11, Lemma 2 (ii)]). *For $U = U_{11} + U_{12} + U_{21} + U_{22} \in \mathcal{R}$. If $U_{ij}V_{jk} = 0$ for every $U_{ij} \in \mathfrak{A}_{\rm ij}$, $1 \leq i, j, k \leq 2$, then $V_{jk} = 0$. Dually, if $V_{ki}U_{ij} = 0$ for every $U_{ij} \in \mathfrak{A}_{\rm ij}$, $1 \leq i, j, k \leq 2$, then $V_{ki} = 0$.*

THEOREM 3.3. *Let $\mathcal{R}$ be a 2-torsion free prime $*$-ring containing a non-trivial self-adjoint idempotent P. If a mapping $d : \mathcal{R} \to \mathcal{R}$ satisfies*

$$d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)], \tag{3.1}$$

for all $U, V \in \mathcal{R}$, then there exists $Z_{U,V} \in \mathcal{Z}(\mathcal{R})$ such that $d(U + V) = d(U) + d(V) + Z_{U,V}$ and $d = \psi + \xi$, where ψ is an additive $*$-derivation from $\mathcal{R}$ into its central closure $\mathcal{T}$ and ξ is a mapping from $\mathcal{R}$ into its extended centriod $\mathcal{C}$ such that $\xi(U + V) = \xi(U) + \xi(V) + Z_{U,V}$ and $\xi([U, V]) = 0$ for all $U, V \in \mathcal{R}$.

Now we shall use the hypothesis of Theorem 3.3 freely without any specific mention in proving the following lemmas.

LEMMA 3.4. *For any non trivial self-adjoint idempotents P and $Q = I - P$, we have*

(i) $Pd(P)P + Qd(P)Q \in \mathcal{Z}(\mathcal{R})$,
(ii) $Pd(P)Q = Pd(P)^*Q$, $Qd(P)P = Qd(P)^*P$.

PROOF.

(i) For any $U_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned} d(U_{12}) &= d([P^*, U_{12}]) \\ &= [d(P)^*, U_{12}] + [P^*, d(U_{12})] \\ &= d(P)^*U_{12} - U_{12}d(P)^* + P^*d(U_{12}) - d(U_{12})P^*. \end{aligned}$$

Multiplying the above identity from the left by P and from the right by Q, we arrive at

$$Pd(P)^*PU_{12} = U_{12}Qd(P)^*Q.$$

By using Lemma 2.9, it follows that $Pd(P)P + Qd(P)Q \in \mathcal{Z}(\mathcal{R})$.

(ii) We compute

$$\begin{aligned} 0 &= d([P^*, P]) \\ &= [d(P)^*, P] + [P^*, d(P)] \\ &= d(P)^*P - Pd(P)^* + Pd(P) - d(P)P. \end{aligned}$$

Multiplying the above identity from the left by P and from the right by Q, we arrive at $Pd(P)Q = Pd(P)^*Q$. Similarly, we can also obtain $Qd(P)P = Qd(P)^*P$. □

In the sequel, we define $\phi : \mathcal{R} \to \mathcal{R}$ by

$$\phi(U) = d(U) + [S, U] \text{ for all } U \in \mathcal{R}$$

where $S = Pd(P)Q - Qd(P)P$. It is to be noted that by Lemma 3.4, we have $S^* = -S$.

Lemma 3.5.

(*i*) $\phi([U^*, V]) = [\phi(U)^*, V] + [U^*, \phi(V)]$,
(*ii*) $\phi(P) \in \mathcal{Z}(\mathcal{R})$,
(*iii*) $\phi(Q) \in \mathcal{Z}(\mathcal{R})$,
(*iv*) $\phi(U + V) = \phi(U) + \phi(V) + \mathcal{Z}_{U,V}$, $\mathcal{Z}_{U,V} \in \mathcal{Z}(\mathcal{R})$,
(*v*) ϕ *is additive on* $\mathfrak{A}_{ij}$, $1 \le i \ne j \le 2$.

Proof. Since (*i*), (*iv*) and (*v*) are easy to verify, we prove only (*ii*) and (*iii*).

(*ii*) By the definition of ϕ, we see that

$$\begin{aligned}\phi(P) &= d(P) + [S, P] \\ &= d(P) - Qd(P)P - Pd(P)Q \\ &= d(P)P + d(P)Q - Qd(P)P - Pd(P)Q \ \{\text{since } P + Q = I\} \\ &= Pd(P)P + Qd(P)Q \in \mathcal{Z}(\mathcal{R}).\end{aligned}$$

(*iii*) In order to prove that $\phi(Q) \in \mathcal{Z}(\mathcal{R})$, we first show that $\phi(PUQ + QUP) = P\phi(U)Q + Q\phi(U)P$ for all $U \in \mathcal{R}$. Since $[P^*, [P^*, U]] = PU - 2PUP + UP = PUQ + QUP$, it follows, applying (*i*) twice,

$$\begin{aligned}\phi(PUQ + QUP) &= \phi([P^*, [P^*, U]]) = [P^*, [P^*, \phi(U)]] \\ &= P\phi(U)Q + Q\phi(U)P.\end{aligned} \tag{3.2}$$

By Lemma 3.4(i), $Pd(Q)P + Qd(Q)Q \in \mathcal{Z}(\mathcal{R})$. By the definition of ϕ, we see that

$$\phi(Q) = d(Q) + [S, Q] = d(Q) + Pd(P)Q + Qd(P)P.$$

The above equation gives us that $Pd(Q)P = P\phi(Q)P$ and $Qd(Q)Q = Q\phi(Q)Q$ and hence $Pd(Q)P + Qd(Q)Q = P\phi(Q)P + Q\phi(Q)Q$.

Now we know that $\phi(Q) = P\phi(Q)P + P\phi(Q)Q + Q\phi(Q)P + Q\phi(Q)Q$, by (3.2), we have

$$P\phi(Q)Q + Q\phi(Q)P = \phi(PQQ + QQP) = 0.$$

Consequently, we get $\phi(Q) = P\phi(Q)P + Q\phi(Q)Q \in \mathcal{Z}(\mathcal{R})$. □

Lemma 3.6. $\phi(\mathfrak{A}_{ij}) \subseteq \mathfrak{A}_{ij}$, $1 \le i \ne j \le 2$.

Proof. For $U_{12} \in \mathfrak{A}_{12}$, we have $U_{12} = [P^*, U_{12}]$. Compute

$$\phi(U_{12}) = \phi([P^*, U_{12}]) = [P, \phi(U_{12})] = P\phi(U_{12}) - \phi(U_{12})P,$$

and hence we see that $P\phi(U_{12})P = Q\phi(U_{12})P = Q\phi(U_{12})Q = 0$. This implies that $\phi(\mathfrak{A}_{12}) \subseteq \mathfrak{A}_{12}$. Similarly, $\phi(U_{21}) = Q\phi(U_{21})P \in \mathfrak{A}_{21}$ for each $U_{21} \in \mathfrak{A}_{21}$ and therefore $\phi(\mathfrak{A}_{21}) \subseteq \mathfrak{A}_{21}$. ◻

Lemma 3.7. *There is a functional $f_i : \mathfrak{A}_{ii} \to \mathcal{C}$ such that $\phi(U_{ii}) - f_i(U_{ii}) \in \mathcal{T}_{ii}$ for all $U_{ii} \in \mathfrak{A}_{ii}$, $i = 1, 2$.*

Proof. For $U_{11} \in \mathfrak{A}_{11}$, by Lemma 3.5$(ii)$, we have

$$0 = \phi([P^*, U_{11}]) = [P^*, \phi(U_{11})] = P\phi(U_{11}) - \phi(U_{11})P,$$

and hence we see that $P\phi(U_{11})Q = Q\phi(U_{11})P = 0$. Thus, it can be assumed that $\phi(U_{11}) = A_{11} + A_{22}$ and similarly, $\phi(U_{22}) = B_{11} + B_{22}$, here $A_{ii}, B_{ii} \in \mathfrak{A}_{ii}, i = 1, 2$. Since $[U_{11}^*, U_{22}] = 0$, a simple calculation gives $[A_{22}^*, U_{22}] = 0$ for all $U_{22} \in \mathfrak{A}_{22}$; $[U_{11}^*, B_{11}] = 0$ for all $U_{11} \in \mathfrak{A}_{11}$. Since $[A_{22}^*, U_{22}] = 0$ for all $U_{22} \in \mathfrak{A}_{22}$, we see that $A_{22}^* X Q = Q X A_{22}^*$ for any $X \in \mathcal{R}$. As both $A_{22}^*, Q \in \mathcal{Q}_{mr}(\mathcal{R})$, by Lemma 3.1, $A_{22}^* = QC$ for some $C \in \mathcal{C}$. A simple calculation gives us that $\phi(U_{11}) \in \mathcal{T}_{11} + \mathcal{C}$. Similarly one can see that $\phi(\mathfrak{A}_{22}) \in \mathcal{T}_{22} + \mathcal{C}$. Therefore, there exist scalars $f_1(U_{11})$ and $f_2(U_{22})$ such that $A_{22} = f_1(U_{11})Q$ and $B_{11} = f_2(U_{22})P$. Hence $\phi(U_{11}) - f_1(U_{11})I \in \mathcal{T}_{11}$ and $\phi(U_{22}) - f_2(U_{22})I \in \mathcal{T}_{22}$. ◻

Now for any $U \in \mathcal{R}$, we define a mapping $\Delta : \mathcal{R} \to \mathcal{T}$ by $\Delta(U) = \phi(PUP) + \phi(PUQ) + \phi(QUP) + \phi(QUQ) - (f_1(PUP) + f_2(QUQ))I$. Further, by the definitions of $\phi(U)$ and $\Delta(U)$ and by Corollary 2.10, it is clear that the difference $\phi(U) - \Delta(U) \in \mathcal{C}$. So, define a mapping $\xi : \mathcal{R} \to \mathcal{C}$ by $\xi(U) = \phi(U) - \Delta(U)$ for all $U \in \mathcal{R}$. By Lemmas 3.6 and 3.7, Δ has the following properties.

Lemma 3.8. *Let $U_{ij} \in \mathfrak{A}_{ij}$, $1 \leq i, j \leq 2$. Then*

(i) $\Delta(U_{ij}) \in \mathcal{T}_{ij}$, $1 \leq i \neq j \leq 2$,
(ii) $\Delta(U_{12}) = \phi(U_{12})$ *and* $\Delta(U_{21}) = \phi(U_{21})$,
(iii) $\Delta(U_{ii}) \in \mathfrak{A}_{ii}$, $i = 1, 2$,
(iv) $\Delta(U_{11} + U_{12} + U_{21} + U_{22}) = \Delta(U_{11}) + \Delta(U_{12}) + \Delta(U_{21}) + \Delta(U_{22})$.

Now, we shall show that Δ is an additive $*$-derivation. First, we shall prove the additivity of Δ.

By Lemma 2.4 and Lemma 3.8(ii), we get the following result.

Lemma 3.9. *Δ is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.*

Lemma 3.10. *Let $U_{ii} \in \mathfrak{A}_{ii}$, $U_{ij} \in \mathfrak{A}_{ij}$, $1 \leq i \neq j \leq 2$. Then*

(i) $\Delta(U_{ij}^*) = \Delta(U_{ij})^*$,
(ii) $\Delta(U_{ii}V_{ij}) = \Delta(U_{ii})V_{ij} + U_{ii}\Delta(V_{ij})$,
(iii) $\Delta(V_{ij}U_{jj}) = \Delta(V_{ij})U_{jj} + V_{ij}\Delta(U_{jj})$,
(iv) $\Delta(P) = \Delta(Q) = 0$.

Proof.

(*i*) By Lemmas 3.5 & 3.8, for any $V_{21} \in \mathfrak{A}_{21}$, we compute

$$\begin{aligned}\Delta(V_{21}^*) &= \Delta([V_{21}^*, Q]) \\ &= [\phi(V_{21}), Q] + [V_{21}, \phi(Q)] \\ &= \Delta(V_{21})^*.\end{aligned}$$

Similarly, it is easy to prove the other case.

(*ii*) Since $[V_{21}^*, U_{11}] = -U_{11}V_{21}^*$, by Lemmas 3.7 & 3.8, we have

$$\begin{aligned}-\Delta(U_{11}V_{21}^*) &= -\phi(U_{11}V_{21}^*) = \phi([V_{21}^*, U_{11}]) \\ &= [\phi(V_{21})^*, U_{11}] + [V_{21}^*, \phi(U_{11})] \\ &= [\Delta(V_{21})^*, U_{11}] + [V_{21}^*, \Delta(U_{11})] \\ &= -\Delta(U_{11})V_{21}^* - U_{11}\Delta(V_{21})^*.\end{aligned}$$

Thus, we have $\Delta(U_{11}V_{21}^*) = \Delta(U_{11})V_{21}^* + U_{11}\Delta(V_{21})^*$. Hence $\Delta(U_{11}V_{12}) = \Delta(U_{11}(V_{12}^*)^*) = \Delta(U_{11})V_{12} + U_{11}\Delta(V_{12}^*)^* = \Delta(U_{11})V_{12} + U_{11}\Delta(V_{12})$. Similarly, it is easy to prove the other identity.

(*iii*) Proof is same as that of part (*ii*).

(*iv*) Since $\Delta(V_{12}) = \Delta(PV_{12}) = \Delta(P)V_{12} + P\Delta(V_{12})$, multiplying above expression by P from the left we have $P\Delta(P)PV_{12} = 0$, which implies $P\Delta(P)P = 0$ because $\mathcal{R}$ is prime. By Lemma 3.8, $\Delta(P) \in \mathfrak{A}_{11}$, hence $\Delta(P) = P\Delta(P)P = 0$. Similarly, $\Delta(Q) = 0$. □

Lemma 3.11. *Δ is additive on $\mathfrak{A}_{11}$ and $\mathfrak{A}_{22}$.*

Proof. Let $U_{11}, V_{11} \in \mathfrak{A}_{11}$. For any $W_{12} \in \mathfrak{A}_{12}$, by Lemma 3.10, we have

$$\Delta((U_{11} + V_{11})W_{12}) = \Delta(U_{11} + V_{11})W_{12} + (U_{11} + V_{11})\Delta(W_{12}).$$

On the other hand, by Lemmas 3.9 & 3.10, we have

$$\begin{aligned}&\Delta((U_{11} + V_{11})W_{12}) \\ &= \Delta(U_{11}W_{12} + V_{11}W_{12}) = \Delta(U_{11}W_{12}) + \Delta(V_{11}W_{12}) \\ &= \Delta(U_{11})W_{12} + U_{11}\Delta(W_{12}) + \Delta(V_{11})W_{12} + V_{11}\Delta(W_{12}).\end{aligned}$$

Comparing the above two identities, we get $(\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))W_{12} = 0$. In other words $(\Delta(U_{11}+V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))P\mathcal{R}Q = 0$. Since $\mathcal{R}$ is prime, it follows that $(\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))P = 0$. Hence, $\Delta(U_{11} + V_{11}) = \Delta(U_{11}) + \Delta(V_{11})$ as $\Delta(\mathfrak{A}_{11}) \subseteq \mathfrak{A}_{11}$. Similarly, Δ is additive on $\mathfrak{A}_{22}$. □

Lemma 3.12. *Δ is additive.*

Proof. Let $U = \sum_{i,j=1}^{2} U_{ij}$, $V = \sum_{i,j=1}^{2} V_{ij}$ be in $\mathcal{R}$. By Lemmas 3.8, 3.9 & 3.11, we have

$$\begin{aligned}\Delta(U+V) &= \Delta\Big(\sum_{i,j=1}^{2}(U_{ij}+V_{ij})\Big)\\ &= \sum_{i,j=1}^{2}\Delta(U_{ij}+V_{ij}) = \sum_{i,j=1}^{2}(\Delta(U_{ij})+\Delta(V_{ij}))\\ &= \Delta\Big(\sum_{i,j=1}^{2}U_{ij}\Big)+\Delta\Big(\sum_{i,j=1}^{2}V_{ij}\Big) = \Delta(U)+\Delta(V).\end{aligned}$$

□

In the sequel, we shall prove that Δ is a derivation.

Lemma 3.13. *Let $U_{ii}, V_{ii} \in \mathfrak{A}_{\rm ii}, i = 1, 2$. Then $\Delta(U_{ii}V_{ii}) = \Delta(U_{ii})V_{ii} + U_{ii}\Delta(V_{ii})$ and $\Delta(U_{ii}^*) = \Delta(U_{ii})^*$.*

Proof. For any $U_{11}, V_{11} \in \mathfrak{A}_{11}$ and $W_{12} \in \mathfrak{A}_{12}$, we have by Lemma 3.10 that

$$\Delta(U_{11}V_{11}^*W_{12}) = \Delta(U_{11}V_{11}^*)W_{12} + U_{11}V_{11}^*\Delta(W_{12}).$$

On the other hand by Lemmas 3.5, 3.7, 3.8 & 3.10 we have,

$$\begin{aligned}&\Delta(U_{11}V_{11}^*W_{12})\\ &= \Delta(U_{11})V_{11}^*W_{12} + U_{11}\Delta(V_{11}^*W_{12})\\ &= \Delta(U_{11})V_{11}^*W_{12} + U_{11}\phi([V_{11}^*, W_{12}])\\ &= \Delta(U_{11})V_{11}^*W_{12} + U_{11}([\phi(V_{11})^*, W_{12}]) + U_{11}([V_{11}^*, \phi(W_{12})])\\ &= \Delta(U_{11})V_{11}^*W_{12} + U_{11}([\Delta(V_{11})^*, W_{12}]) + U_{11}([V_{11}^*, \Delta(W_{12})])\\ &= \Delta(U_{11})V_{11}^*W_{12} + U_{11}\Delta(V_{11})^*W_{12} + U_{11}V_{11}^*\Delta(W_{12}).\end{aligned}$$

Comparing the above two identities, we get $\big(\Delta(U_{11}V_{11}^*) - \Delta(U_{11})V_{11}^* - U_{11}\Delta(V_{11})^*\big)W_{12} = 0$. In other words

$$\big(\Delta(U_{11}V_{11}) - \Delta(U_{11})V_{11} - U_{11}\Delta(V_{11})\big)P\mathcal{R}Q = 0.$$

Since $\mathcal{R}$ is prime, it follows that $\big(\Delta(U_{11}V_{11}^*) - \Delta(U_{11})V_{11}^* - U_{11}\Delta(V_{11})^*\big)P = 0$. Hence, $\Delta(U_{11}V_{11}^*) = \Delta(U_{11})V_{11}^* + U_{11}\Delta(V_{11})^*$ as $\Delta(\mathfrak{A}_{11}) \subseteq \mathfrak{A}_{11}$. Since $U_{11}^* = PU_{11}^*$, we see that $\Delta(U_{11}^*) = \Delta(PU_{11}^*) = \Delta(U_{11})^*$. Thus $\Delta(U_{11}V_{11}) = \Delta(U_{11})V_{11} + U_{11}\Delta(V_{11})$. Similarly, $\Delta(U_{22}V_{22}) = \Delta(U_{22})V_{22} + U_{22}\Delta(V_{22})$. □

Lemma 3.14. *Let $U_{12} \in \mathfrak{A}_{12}$ and $W_{21} \in \mathfrak{A}_{21}$. Then $\Delta(U_{12}W_{21}) = \Delta(U_{12})W_{21} + U_{12}\Delta(W_{21})$ and $\Delta(U_{21}W_{12}) = \Delta(U_{21})W_{12} + U_{21}\Delta(W_{12})$.*

Proof. For any $W_{21} \in \mathfrak{A}_{21}$, by Lemmas 3.8 & 3.10, we compute

$$\begin{aligned}\phi([[V_{12}^*, U_{12}], W_{12}^*]) &= \phi(V_{12}^* U_{12} W_{12}^* + W_{12}^* U_{12} V_{12}^*)\\ &= \Delta(V_{12}^* U_{12} W_{12}^* + W_{12}^* U_{12} V_{12}^*)\\ &= \Delta(V_{12}^* U_{12} W_{12}^*) + \Delta(W_{12}^* U_{12} V_{12}^*)\\ &= \Delta(V_{12})^* U_{12} W_{12}^* + V_{12}^* \Delta(U_{12} W_{12}^*)\\ &\quad + \Delta(W_{12}^* U_{12}) V_{12}^* + W_{12}^* U_{12} \Delta(V_{12})^*.\end{aligned}$$

On the other hand, by Lemmas 3.5 & 3.8 we have

$$\begin{aligned}&\phi([[V_{12}^*, U_{12}], W_{12}^*])\\ &= [[\phi(V_{12})^*, U_{12}], W_{12}^*] + [[V_{12}^*, \phi(U_{12})], W_{12}^*] + [[V_{12}^*, U_{12}], \phi(W_{12})^*]\\ &= [[\Delta(V_{12})^*, U_{12}], W_{12}^*] + [[V_{12}^*, \Delta(U_{12})], W_{12}^*] + [[V_{12}^*, U_{12}], \Delta(W_{12})^*]\\ &= \Delta(V_{12})^* U_{12} W_{12}^* + W_{12}^* U_{12} \Delta(V_{12})^* + V_{12}^* \Delta(U_{12}) W_{12}^*\\ &\quad + W_{12}^* \Delta(U_{12}) V_{12}^* + V_{12}^* U_{12} \Delta(W_{12})^* + \Delta(W_{12})^* U_{12} V_{12}^*.\end{aligned}$$

Comparing the above two identities, we arrive at

$$\begin{aligned}&V_{12}^*\big(\Delta(U_{12} W_{12}^*) - \Delta(U_{12}) W_{12}^* - U_{12}\Delta(W_{12})^*\big)\\ &= \big(-\Delta(W_{12}^* U_{12}) + \Delta(W_{12})^* U_{12} + W_{12}^* \Delta(U_{12})\big) V_{12}^*.\end{aligned}$$

By using Lemma 2.9, we see that

$$\begin{aligned}&\Delta(U_{12} W_{12}^*) - \Delta(U_{12}) W_{12}^* - U_{12}\Delta(W_{12})^* - \Delta(W_{12}^* U_{12})\\ &\quad + \Delta(W_{12})^* U_{12} + W_{12}^* \Delta(U_{12}) = C \in \mathcal{C}.\end{aligned}$$

From the later relation we obtain the two identities

$$\Delta(U_{12} W_{12}^*) - \Delta(U_{12}) W_{12}^* - U_{12}\Delta(W_{12})^* = PC$$

and

$$\Delta(W_{12}^* U_{12}) - \Delta(W_{12})^* U_{12} - W_{12}^* \Delta(U_{12}) = -QC.$$

Since $\Delta(W_{21}^*) = \Delta(W_{21})^*$, we have

$$\begin{aligned}&\Delta(U_{12} W_{21}) - \Delta(U_{12}) W_{21} - U_{12}\Delta(W_{21})\\ &= \Delta(U_{12}(W_{21}^*)^*) - \Delta(U_{12}) W_{21} - U_{12}\Delta(W_{21}^*)^* = PC.\end{aligned} \tag{3.3}$$

Similarly, we obtain the other identity as

$$\Delta(W_{21} U_{12}) - \Delta(W_{21}) U_{12} - W_{21}\Delta(U_{12}) = -QC. \tag{3.4}$$

Now it is sufficient to show that $C = 0$. Assume $C \neq 0$. Then by using equations (3.3) and (3.4) together with Lemma 3.10, we have

$$\begin{aligned}&\Delta(U_{12} W_{21} U_{12})\\ &= \Delta(U_{12}) W_{21} U_{12} + U_{12}\Delta(W_{21} U_{12})\\ &= \Delta(U_{12}) W_{21} U_{12} + U_{12}\Delta(W_{21}) U_{12} + U_{12} W_{21}\Delta(U_{12}) - C U_{12},\end{aligned}$$

and

$$\begin{aligned}
&\Delta(U_{12}W_{21}U_{12})\\
&= \Delta(U_{12}W_{21})U_{12} + U_{12}W_{21}\Delta(U_{12})\\
&= \Delta(U_{12})W_{21}U_{12} + U_{12}\Delta(W_{21})U_{12} + U_{12}W_{21}\Delta(U_{12}) + CU_{12}.
\end{aligned}$$

Comparing the above two identities, we obtain $CU_{12} = 0$. Since $\mathcal{C}$ is a field, we have $U_{12} = 0$, a contradiction. Consequently, $\Delta(U_{12}W_{21}) = \Delta(U_{12})W_{21} + U_{12}\Delta(W_{21})$ and $\Delta(U_{21}W_{12}) = \Delta(U_{21})W_{12} + U_{21}\Delta(W_{12})$. □

Proof of Theorem 3.3. Let $U, V \in \mathcal{R}$. Assume that $U = U_{11} + U_{12} + U_{21} + U_{22}$ and $V = V_{11} + V_{12} + V_{21} + V_{22}$. By Lemmas 3.8-3.14, we see that

$$\begin{aligned}
\Delta(UV) &= \Delta((U_{11} + U_{12} + U_{21} + U_{22})(V_{11} + V_{12} + V_{21} + V_{22}))\\
&= \Delta(U_{11}V_{11} + U_{11}V_{12} + U_{12}V_{21} + U_{12}V_{22} + U_{21}V_{11}\\
&\quad + U_{21}V_{12} + U_{22}V_{21} + U_{22}V_{22})\\
&= \Delta(U_{11}V_{11} + U_{12}V_{21}) + \Delta(U_{11}V_{12} + U_{12}V_{22})\\
&\quad + \Delta(U_{21}V_{11} + U_{22}V_{21}) + \Delta(U_{21}V_{12} + U_{22}V_{22})\\
&= \Delta(U_{11}V_{11}) + \Delta(U_{12}V_{21}) + \Delta(U_{11}V_{12}) + \Delta(U_{12}V_{22})\\
&\quad + \Delta(U_{21}V_{11}) + \Delta(U_{22}V_{21}) + \Delta(U_{21}V_{12}) + \Delta(U_{22}V_{22})\\
&= \Delta(U_{11})V_{11} + U_{11}\Delta(V_{11}) + \Delta(U_{12})V_{21} + U_{12}\Delta(V_{21})\\
&\quad + \Delta(U_{11})V_{12} + U_{11}\Delta(V_{12}) + \Delta(U_{12})V_{22} + U_{12}\Delta(V_{22})\\
&\quad + \Delta(U_{21})V_{11} + U_{21}\Delta(V_{11}) + \Delta(U_{22})V_{21} + U_{22}\Delta(V_{21})\\
&\quad + \Delta(U_{21})V_{12} + U_{21}\Delta(V_{12}) + \Delta(U_{22})V_{22} + U_{22}\Delta(V_{22})\\
&= (U_{11} + U_{12} + U_{21} + U_{22})\Delta(V_{11} + V_{12} + V_{21} + V_{22})\\
&\quad + \Delta(U_{11} + U_{12} + U_{21} + U_{22})(V_{11} + V_{12} + V_{21} + V_{22})\\
&= U\Delta(V) + \Delta(U)V.
\end{aligned}$$

It is easy to show that $\Delta(U^*) = \Delta(U)^*$. Hence, Δ is an additive $*$-derivation.

Now using the definition of ξ, we see that

$$\begin{aligned}
\xi(U + V) &= \phi(U + V) - \Delta(U + V)\\
&= \phi(U) + \phi(V) + Z_{U,V} - \Delta(U) - \Delta(V)\\
&= \xi(U) + \xi(V) + Z_{U,V}.
\end{aligned}$$

and

$$\begin{aligned}
\xi([U, V]) &= \phi([U, V]) - \Delta([U, V])\\
&= [\phi(U^*)^*, V] + [U, \phi(V)] - \Delta([U, V])\\
&= [\Delta(U), V] + [U, \Delta(V)] - \Delta([U, V]) = 0.
\end{aligned}$$

Finally, let us define $\psi(U) = \Delta(U) - (SU - US)$ for all $U \in \mathcal{R}$, where $S = Pd(P)Q - Qd(P)P$. It is easy to check that ψ is an additive $*$-derivation on $\mathcal{R}$. By the definitions of Δ and ϕ, ψ is an additive $*$-derivation and $d(U) = \psi(U) + \xi(U)$ for all $U \in \mathcal{R}$. ∎

We conclude this section by the following result. Recall that a von Neumann algebra $\mathcal{M}$ is called a factor if its centre is $\mathbb{C}I$. It is to be noted that every factor von Neumann algebra is prime. So we have the following immediate corollary.

Corollary 3.15. *Let $\mathcal{M}$ be a factor von Neumann algebra. Suppose that a mapping $d : \mathcal{M} \to \mathcal{M}$ satisfies*

$$d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)]$$

for all $U, V \in \mathcal{M}$. Then there exists $\lambda_{U,V} \in \mathbb{C}$ such that $d(U+V) = d(U) + d(V) + \lambda_{U,V}$ and $d = \psi + \xi$, where ψ is an additive $*$-derivation on $\mathcal{M}$ and ξ is a mapping from $\mathcal{M}$ into $\mathbb{C}$ such that $\xi(U+V) = \xi(U) + \xi(V) + \lambda_{U,V}$ and $\xi([U,V]) = 0$ for all $U, V \in \mathcal{M}$.

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Karakterizacije $*$-Liejevih derivabilnih preslikavanja na prostim $*$-prstenima

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Sažetak. Neka je $\mathcal{R}$ $*$-prsten koji sadrži netrivijalni samoadjungirajući idempotentni element. U ovom članku se pokazuje da uz izvjesne pretpostavke na $\mathcal{R}$, ako preslikavanje $d:\mathcal{R}\to\mathcal{R}$ zadovoljava

$$d([U^*,V]) \quad = \quad [d(U)^*,V]+[U^*,d(V)]$$

za sve $U,V\in\mathcal{R}$, tada postoji $Z_{U,V}\in\mathcal{Z}(\mathcal{R})$ (koji ovisi o U i V), gdje je $\mathcal{Z}(\mathcal{R})$ u centru od $\mathcal{R}$, tako da vrijedi $d(U+V)=d(U)+d(V)+Z_{U,V}$. Štoviše, ako je $\mathcal{R}$ slobodan od 2-torzije prosti $*$-prsten, tada je $d=\psi+\xi$, gdje je ψ aditivna $*$-derivacija od $\mathcal{R}$ u njegov centralni zatvarač $\mathcal{T}$ i ξ je preslikavanje s $\mathcal{R}$ u njegov prošireni centroid $\mathcal{C}$ tako da $\xi(U+V)=\xi(U)+\xi(V)+Z_{U,V}$ i $\xi([U,V])=0$ za sve $U,V\in\mathcal{R}$. Naposljetku, gornji rezultati iz teorije prstena primijenjeni su na neke specijalne klase algebri kao što su ugniježđene algebre i von Neumannove algebre.

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