On idempotent n-ary uninorms

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in collaboraton with Gergely Kiss and Jean-Luc Marichal

Part I: Ultrabisymmetry

Associativity and symmetry

Definition.

$F\colon X^3 \to X$ is said to be

- *associative* if for all $x_1, x_2, x_3, x_4, x_5 \in X$

$$F(F(x_1, x_2, x_3), x_4, x_5) = F(x_1, F(x_2, x_3, x_4), x_5) = F(x_1, x_2, F(x_3, x_4, x_5))$$

- *symmetric* if $F(x_1, x_2, x_3)$ is invariant under any permutation of x_1, x_2, x_3

Example. $F(x, y, z) = x + y + z$ on $X = \mathbb{R}$

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Bisymmetry

Definition. (Aczél, 1946)

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Associativity and bisymmetry

Proposition

- associativity + symmetry $\implies$ ultrabisymmetry $\implies$ bisymmetry
- bisymmetry + symmetry $\implies$ ultrabisymmetry

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- associativity $+$ symmetry $\implies$ ultrabisymmetry $\implies$ bisymmetry
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Quasitriviality

Definition

$F\colon X^3 \to X$ is said to be

- *quasitrivial* (or *conservative*) if

$$F(x,y,z) \in \{x,y,z\} \qquad (x,y,z \in X)$$

- *idempotent* if

$$F(x,x,x) = x \qquad (x \in X)$$

Fact. quasitriviality $\implies$ idempotency

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$$\text{bisymmetry} + \text{symmetry} \implies \text{ultrabisymmetry}$$

Proposition

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Corollary

$$\text{quasitriviality} + \text{symmetry}$$

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$$\text{associativity} \iff \text{bisymmetry}$$

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$$\text{associativity} \quad \Longleftrightarrow \quad \text{bisymmetry}$$

Part II: Idempotent n-ary uninorms

Uninorm

Definition

$e \in X$ is said to be a *neutral element* of $F\colon X^3 \to X$ if

$$F(x,e,e) = F(e,x,e) = F(e,e,x) = x \qquad x \in X$$

Definition. (Kiss et al., 2018)

A *ternary uninorm on* $(X, \leq)$ is an operation $F\colon X^3 \to X$ that

- has a neutral element $e \in X$

and is

- associative
- symmetric
- $\leq$-nondecreasing

First characterization

Proposition

$F\colon X^3 \to X$ is an idempotent ternary uninorm if and only if there exists an idempotent binary uninorm $U\colon X^2 \to X$ such that

$$F(x,y,z) \;=\; U(\min(x,y,z),\max(x,y,z)) \qquad x,y,z \in X.$$

Single-peaked orderings

Definition. (Black, 1948)
Let $\leq$ and $\preceq$ be total orderings on X.
Then $\preceq$ is said to be *single-peaked for* $\leq$ if for all $a, b, c \in X$

$$a < b < c \quad \Longrightarrow \quad b \prec a \quad \text{or} \quad b \prec c$$

Example. On $X = \{1, 2, 3, 4\}$ consider $\leq$ and $\preceq$ defined by

$$1 < 2 < 3 < 4 \qquad \text{and} \qquad 2 \prec 3 \prec 1 \prec 4$$

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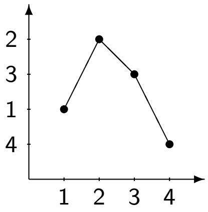


Alternative characterization

Theorem

Let $F\colon X^3 \to X$ be an operation. The following assertions are equivalent.

(i) F is associative, quasitrivial, symmetric, and $\leq$-nondecreasing.

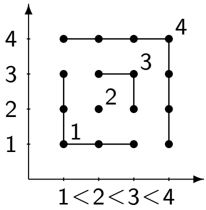
(ii) F is bisymmetric, quasitrivial, symmetric, and $\leq$-nondecreasing.

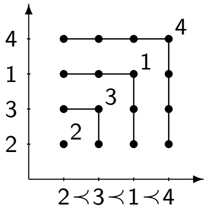
(iii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on X that is single-peaked for $\leq$

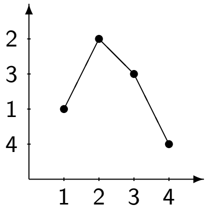
If F has a neutral element, then (i)–(iii) are equivalent to

(iv) F is an idempotent ternary uninorm.

Example







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